ON THE NUMBER OF BOUND STATES FOR THE 1-D SCHRÖDINGER EQUATION

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Abstract: The number of bound states of the one-dimensional Schrödinger equation is analyzed in terms of the number of bound states corresponding to "fragments" of the potential. When the potential is integrable and has a finite first moment, the sharp inequalities $1-n+\sum_{j=1}^n N_j \le N \le \sum_{j=1}^n N_j$ are proved, where n is the number of fragments, N is the total number of bound states, and N_j is the number of bound states for the j-th fragment. An illustrative example is provided.

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1. INTRODUCTION

Consider the one-dimensional Schrödinger equation

$$\psi''(k,x) + k^2\psi(k,x) = V(x)\,\psi(k,x), \tag{1.1}$$

where the potential $V(x)$ is real valued and belongs to $L_1^1(\mathbf{R})$, the class of functions for which $\int_{-\infty}^{\infty} dx\,(1+|x|)\,|V(x)|$ is finite. Let us partition the real axis as $\mathbf{R} = \cup_{j=1}^{n}(x_{j-1}, x_j)$, with $x_{j-1} < x_j$ for $j = 1, \cdots, n$. Here we use the convention $x_0 = -\infty$ and $x_n = +\infty$. We obtain a fragmentation of the potential by setting $V(x) = \sum_{j=1}^{n} V_j(x)$, where

$$V_j(x) = \begin{cases} V(x), & x \in (x_{j-1}, x_j), \\ 0, & \text{elsewhere.} \end{cases} \tag{1.2}$$

In this paper we analyze the relationship between the number of bound states of $V(x)$ and the number of bound states of its fragments. In Section 2 we prove a pair of sharp inequalities relating these numbers (Theorem 2.1). In Section 3 we give another proof of Theorem 2.1 by using a factorization formula for the scattering matrix and exploiting its small-k asymptotics. We also briefly discuss what happens if we increase the separation distance between two fragments (Theorem 3.1). In Section 4 we give an example which illustrates various aspects of our results.

The inequality (2.3) in Theorem 2.1 has been proved before by different methods and under stronger assumptions on the potential. In [Kl81], (2.3) was proved when $n = 2$ and the fragments have compact support. In [Sa94] and [Sa95] special cases of (2.3) were proved for parity invariant compactly supported fragments, but, as already mentioned in those references, the parity invariance is not an essential restriction. The method used in [Kl81] was based on the nodal properties of the zero-energy solutions of the Schrödinger equation but was fairly contrived, while the method used in [Sa94] and [Sa95] relied on a factorization formula for the scattering matrix [Ak92] and the small-k behavior of the scattering coefficients. Our proof of inequality (2.3) uses certain properties of the Jost solutions, especially the interlacing property of zeros, in a very straightforward way. As a result, we are able to establish the connection with the factorization method used in

Section 3. Furthermore, no additional technical restrictions are imposed on the potential besides $V \in L_1^1(\mathbf{R})$.

At various places in this paper we need to distinguish between "generic" and "exceptional" potentials. Recall that a potential is called generic if the corresponding transmission coefficient $T(k)$ vanishes at $k=0$, and that a potential is called exceptional if $T(0) \neq 0$. Equivalently, a potential is generic (exceptional) if for $k=0$ the two Jost solutions are linearly independent (dependent) [Fa64,DT79,CS89].

2. INEQUALITY FOR THE NUMBER OF BOUND STATES

In preparation of this section we first establish some notation and collect some results about the Jost solutions and their nodal properties. Let $f_{l;j}(k,x)$ and $f_{r;j}(k,x)$ denote the Jost solutions from the left and from the right, respectively, for the fragment $V_j(x)$. Recall that $f_{l;j}(k,x) = e^{ikx}[1+o(1)]$ as $x \to +\infty$ and $f_{r;j}(k,x) = e^{-ikx}[1+o(1)]$ as $x \to -\infty$. Furthermore, let n_j denote the number of zeros of $f_{r;j}(0,x)$ lying in $(-\infty, x_j)$, m_j the number of zeros of $f_{l;j}(0,x)$ lying in $(x_{j-1}, +\infty)$, and N_j the number of bound states of the fragment $V_j(x)$. Since N_j is equal to the number of the zeros of either $f_{l;j}(0,x)$ or $f_{r;j}(0,x)$ (cf. [RS78, AKV97]), we conclude that

$$N_j = \begin{cases} n_j & \text{if} \quad f_{r;j}(0,x_j)\, f'_{r;j}(0,x_j) > 0, \\ n_j+1 & \text{if} \quad f_{r;j}(0,x_j)\, f'_{r;j}(0,x_j) < 0, \\ n_j & \text{if} \quad f'_{r;j}(0,x_j) = 0, \\ n_j+1 & \text{if} \quad f_{r;j}(0,x_j) = 0, \end{cases} \tag{2.1}$$

$$N_j = \begin{cases} m_j & \text{if} \quad f_{l;j}(0,x_{j-1})\, f'_{l;j}(0,x_{j-1}) < 0, \\ m_j+1 & \text{if} \quad f_{l;j}(0,x_{j-1})\, f'_{l;j}(0,x_{j-1}) > 0, \\ m_j & \text{if} \quad f'_{l;j}(0,x_{j-1}) = 0, \\ m_j+1 & \text{if} \quad f_{l;j}(0,x_{j-1}) = 0. \end{cases} \tag{2.2}$$

Note that on $(x_j, +\infty)$ the function $f_{r;j}(0,x)$ is equal to $f'_{r;j}(0,x_j)\,(x-x_j) + f_{r;j}(0,x_j)$ and that this linear function has the root $x = x_j - f_{r;j}(0,x_j)/f'_{r;j}(0,x_j)$ which lies in $[x_j, +\infty)$ precisely if $f'_{r;j}(0,x_j) \neq 0$ and $f_{r;j}(0,x_j)\, f'_{r;j}(0,x_j) \leq 0$; in this case we have $N_j = n_j + 1$. On the other hand, if $f'_{r;j}(0,x_j) = 0$ or $f_{r;j}(0,x_j)\, f'_{r;j}(0,x_j) > 0$, then $f_{r;j}(0,x)$ has no zeros in $[x_j,+\infty)$, i.e. all its zeros are in $(-\infty,x_j)$; thus $N_j=n_j$. This proves (2.1). We obtain (2.2) by applying a similar argument to $f_{l;j}(0,x)$.

Theorem 2.1 Suppose that $V\in L_1^1(\mathbf{R})$. Let N denote the number of bound states of $V(x)$. Then

$$1-n+\sum_{j=1}^n N_j\le N\le\sum_{j=1}^n N_j,\qquad n=1,2,\cdots,\tag{2.3}$$

where both inequalities are sharp.

PROOF: It suffices to prove (2.3) for $n=2$ because the general case follows by induction. Let $f_r(k,x)$ and $f_l(k,x)$ denote the Jost solutions from the right and left, respectively, associated with $V(x)$. In order to determine N we count the zeros of $f_r(0,x)$ and note that on $(-\infty,x_1]$ we have $f_r(0,x)=f_{r;1}(0,x)$. We already know that n_1 zeros lie in $(-\infty,x_1)$, where n_1 is related to N_1 by (2.1). Thus we need to count the zeros of $f_r(0,x)$ that lie in $[x_1,+\infty)$. We do this by using the interlacing property of the zeros of $f_r(0,x)$ and $f_l(0,x)$, noting that $f_l(0,x)=f_{l;2}(0,x)$ on $[x_1,+\infty)$ and $f_r(0,x)=f_{r;1}(0,x)$ on $(-\infty,x_1]$.

We distinguish four cases:

(a) $f'_{r;1}(0,x_1)=f'_{l;2}(0,x_1)=0$: Then $f_r(0,x)$ and $f_l(0,x)$ are linearly dependent, i.e. we are in the exceptional case, and $f_l(0,x_1)\ne0\ne f_r(0,x_1)$. Hence from (2.1) and (2.2) it follows that $N=n_1+m_2=N_1+N_2$.

(b) $f'_{l;2}(0,x_1)=0$, $f'_{r;1}(0,x_1)\ne0$: Then there are nonzero constants α and β such that the functions $\varphi_{l;2}(0,x)=\alpha\,f_{l;2}(0,x)$ and $\varphi_{r;1}(0,x)=\beta\,f_{r;1}(0,x)$ obey $\varphi_{l;2}(0,x_1)>0$ and $\varphi'_{r;1}(0,x_1)>0$. We also let $\varphi_l(0,x)=\alpha\,f_l(0,x)$ and $\varphi_r(0,x)=\beta\,f_r(0,x)$. Of course, $\varphi_l(0,x)$, $\varphi_{l;2}(0,x)$, $\varphi_r(0,x)$, and $\varphi_{r;1}(0,x)$ have the same number of zeros as $f_l(0,x)$, $f_{l;2}(0,x)$, $f_r(0,x)$, and $f_{r;1}(0,x)$, respectively. We will determine N by counting the zeros of $\varphi_r(0,x)$. The Wronskian $W[\varphi_l,\varphi_r](x)=\varphi_l(0,x)\,\varphi'_r(0,x)-\varphi'_l(0,x)\,\varphi_r(0,x)$, which is of course a constant, satisfies

$$W[\varphi_l,\varphi_r](x_1)=\varphi_{l;2}(0,x_1)\,\varphi'_{r;1}(0,x_1)>0.$$

First suppose that $\varphi_{l;2}(0,x)$ has no zeros in $(x_1,+\infty)$, and thus no zeros at all, since

$\varphi_{l;2}(0,x_1)\neq 0$. If $\varphi_{r;1}(0,x_1)\geq 0$, then $\varphi_r(0,x)$ cannot have any zeros in $(x_1,+\infty)$, for if ξ were the first zero of $\varphi_r(0,x)$ in $(x_1,+\infty)$, then $W[\varphi_l,\varphi_r](\xi)=\varphi_{l;2}(0,\xi)\,\varphi_r'(0,\xi)<0$, which is a contradiction. Then, if $\varphi_{r;1}(0,x_1)>0$, we have $n_1=N_1$, $m_2=N_2=0$ (by (2.1) and (2.2)) and thus $N=n_1=N_1+N_2$. If $\varphi_{r;1}(0,x_1)=0$, then $n_1=N_1-1$, $m_2=N_2=0$, and thus $N=n_1+1=N_1=N_1+N_2$. If $\varphi_{r;1}(0,x_1)<0$, then $\varphi_r(0,x)$ has a zero in $(x_1,+\infty)$. To see this recall that the following asymptotic relations hold as $x\to+\infty$:

$$f_{l;2}(0,x)=f_l(0,x)=1+o(1),\qquad f_{l;2}'(0,x)=f_l(0,x)=o(1/x),$$

$$f_r(0,x)=c_r x+o(x),\qquad f_r'(0,x)=c_r+o(1),$$

with some constant $c_r\neq 0$; note that we are in the generic case. Hence $0<W[\varphi_l,\varphi_r](x)=\alpha\,\beta\,c_r$ under the present assumptions. Moreover, since $\varphi_{l;2}(0,x)$ has no zeros on $(x_1,+\infty)$, we have $\alpha>0$ and hence $\beta\,c_r>0$. Because $\varphi_r(0,x)=\beta\,c_r\,x+o(x)$ as $x\to+\infty$ and $\varphi_r(0,x_1)=\varphi_{r;1}(0,x_1)<0$, it follows that $\varphi_r(0,x)$ must have a zero in $(x_1,+\infty)$ and, by the interlacing property, this is the only zero on this interval. Consequently, we have $n_1=N_1-1$, $m_2=N_2=0$, and thus $N=n_1+1=N_1+N_2$. Now suppose that $\varphi_{l;2}(0,x)$ has its zeros at z_j $(j=1,\cdots,m_2)$, where $x_1<z_1<z_2<\cdots<z_{m_2}$ and that $\varphi_{r;1}(0,x_1)\geq 0$. Then the zeros of $\varphi_r(0,x)$ on $(x_1,+\infty)$ occur in the intervals (z_1,z_2), (z_2,z_3), $\cdots$, $(z_{m_2},+\infty)$ and each such interval contains exactly one zero of $\varphi_r(0,x)$. As a result, if $\varphi_{r;1}(0,x_1)>0$, then the zeros of $\varphi_r(0,x)$ are counted as follows: $n_1=N_1$, $m_2=N_2$, and hence $N=n_1+m_2=N_1+N_2$. If $\varphi_{r;1}(0,x_1)=0$, then we have $n_1=N_1-1$, $m_2=N_2$ and thus $N=n_1+1+m_2=N_1+N_2$. If $\varphi_{r;1}(0,x_1)<0$, then $\varphi_r(0,x)$ has a zero in each of the intervals (x_1,z_1), (z_1,z_2), $\cdots,(z_{m_2},+\infty)$. Thus $\varphi_r(0,x)$ has m_2+1 zeros on $(x_1,+\infty)$. Consequently, $n_1=N_1-1$, $m_2=N_2$, and $N=n_1+(m_2+1)=N_1+N_2$.

(c) $f_{l;2}'(0,x_1)\neq 0$, $f_{r;1}'(0,x_1)=0$: This case is analogous to (b), so $N=N_1+N_2$.

(d) $f_{l;2}'(0,x_1)\neq 0$, $f_{r;1}'(0,x_1)\neq 0$: Similarly as in (b), upon multiplying $f_{l;2}(0,x)$ and $f_{r;1}(0,x)$ by suitable constants α and β, we can achieve that $\varphi_{l;2}(0,x)=\alpha\,f_{l;2}(0,x)$ and $\varphi_{r;1}(0,x)=\beta\,f_{r;1}(0,x)$ satisfy $\varphi_{l;2}'(0,x_1)=\varphi_{r;1}'(0,x_1)>0$. It turns out that whether we have $N = N_1 + N_2$ or $N = N_1 + N_2 - 1$ is determined by the sign of the expression

$$Z(x_1) = \frac{f_{l;2}(0,x_1)}{f'_{l;2}(0,x_1)} - \frac{f_{r;1}(0,x_1)}{f'_{r;1}(0,x_1)} = \frac{\varphi_{l;2}(0,x_1)}{\varphi'_{l;2}(0,x_1)} - \frac{\varphi_{r;1}(0,x_1)}{\varphi'_{r;1}(0,x_1)}. \tag{2.4}$$

First suppose that $Z(x_1) > 0$, which is equivalent to assuming $W[\varphi_l, \varphi_r](x_1) > 0$, respectively $\varphi_{l;2}(x_1) > \varphi_{r;1}(x_1)$. We only consider the case where $\varphi_{l;2}(0,x)$ has at least one zero on $(x_1, +\infty)$; the special case where $\varphi_{l;2}(0,x)$ has no zeros on $(x_1, +\infty)$ is dealt with analogously. If $\varphi_{l;2}(0,x_1) > \varphi_{r;1}(0,x_1) > 0$, then $\varphi_r(0,x)$ has m_2 zeros in $(x_1, +\infty)$ because, by a Wronskian argument, there are no zeros in (x_1, z_1), where z_1 is as in (b), and there is exactly one zero in each of the intervals (z_1, z_2), (z_2, z_3), $\cdots$, $(z_{m_2}, +\infty)$. Hence $n_1 = N_1$, $m_2 = N_2 - 1$, and $N = n_1 + m_2 = N_1 + N_2 - 1$. If $\varphi_{l;2}(0,x_1) > \varphi_{r;1}(0,x_1) = 0$, then $n_1 = N_1 - 1$, $m_2 = N_2 - 1$, and $N = n_1 + 1 + m_2 = N_1 + N_2 - 1$. If $\varphi_{l;2}(0,x_1) > 0$ and $\varphi_{r;1}(0,x_1) < 0$, then $\varphi_r(0,x)$ has $m_2 + 1$ zeros on $(x_1, +\infty)$ because one zero lies in (x_1, z_1), and thus $n_1 = N_1 - 1$, $m_2 = N_2 - 1$, and $N = n_1 + m_2 + 1 = N_1 + N_2 - 1$. If $0 = \varphi_{l;2}(0,x_1) > \varphi_{r;1}(0,x_1)$, then $n_1 = N_1 - 1$, $m_2 = N_2 - 1$, and $N = n_1 + m_2 + 1 = N_1 + N_2 - 1$. If $0 > \varphi_{l;2}(0,x_1) > \varphi_{r;1}(0,x_1)$, then $n_1 = N_1 - 1$, $m_2 = N_2$, and $N = n_1 + m_2 = N_1 + N_2 - 1$ because $\varphi_r(0,x)$ has no zeros in (x_1, z_1). All the possibilities with $Z(x_1) > 0$ have now been exhausted. If $Z(x_1) < 0$, we can apply similar arguments and find that $N = N_1 + N_2$. Finally, if $Z(x_1) = 0$ because $\varphi_{l;2}(0,x_1) = \varphi_{r;1}(0,x_1) > 0$, then $n_1 = N_1$, $m_2 = N_2 - 1$, and $N = n_1 + m_2 = N_1 + N_2 - 1$. If $Z(x_1) = 0$ because $\varphi_{l;2}(0,x_1) = \varphi_{r;1}(0,x_1) = 0$, then $n_1 = N_1 - 1$, $m_2 = N_2 - 1$, and $N = n_1 + m_2 + 1 = N_1 + N_2 - 1$. If $Z(x_1) = 0$ because $\varphi_{l;2}(0,x_1) = \varphi_{r;1}(0,x_1) < 0$, then $n_1 = N_1 - 1$, $m_2 = N_2$, and $N = n_1 + m_2 = N_1 + N_2 - 1$. This concludes the proof of (2.3).

To see that (2.3) is sharp, note that a square-well potential of depth $-H^2$ and width w has exactly N bound states, where N is the positive integer satisfying

$$(N-1)\pi < wH \leq N\pi. \tag{2.5}$$

Choose $V(x)$ to be the square-well potential of depth $-\pi^2$ with support $(0,1)$. Then $N = 1$. Let us partition the interval $(0,1)$ into n nonempty subintervals and hence obtain a fragmentation of $V(x)$; each fragment still contains exactly one bound state and hence the lower bound in (2.3) becomes equal to N. On the other hand, consider the square-well potential of depth $-\pi^2$ with support $(0,n)$, and partition $(0,n)$ into the n subintervals $(j-1,j)$ for $j=1,\cdots,n$. Then $N_j=1$ and $N=n$, and hence the upper bound in (2.1) becomes equal to N. ∎

3. FURTHER OBSERVATIONS

In this section we analyze the result of Theorem 2.1 in conjunction with the scattering matrices corresponding to the fragments of this potential. For simplicity let us consider the fragmentation of $V(x)$ as $V(x)=V_1(x)+V_2(x)$, where $V_1(x)$ has support in $(-\infty,x_1]$ and $V_2(x)$ has support in $[x_1,+\infty)$. The analysis for three or more fragments can be carried out by using induction. Let $\mathbf{S}_1(k)$, $\mathbf{S}_2(k)$, and $\mathbf{S}(k)$ be the scattering matrices corresponding to the potentials $V_1(x)$, $V_2(x)$, and $V(x)$, respectively. The scattering coefficients appear in the scattering matrix as follows:

$$\mathbf{S}(k)=\begin{bmatrix} T(k) & R(k) \\ L(k) & T(k) \end{bmatrix}, \tag{3.1}$$

where $T(k)$ is the transmission coefficient, and $L(k)$ and $R(k)$ are the reflection coefficients from the left and from the right, respectively. Similarly, $T_j(k)$, $R_j(k)$, and $L_j(k)$ denote the corresponding entries of $\mathbf{S}_j(k)$ for $j=1,2$. Let us define the so-called transition matrix associated with $\mathbf{S}(k)$ as follows:

$$\Lambda(k)=\begin{bmatrix} \dfrac{1}{T(k)} & -\dfrac{R(k)}{T(k)} \\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(k)^*} \end{bmatrix}=\begin{bmatrix} \dfrac{1}{T(k)} & \dfrac{L(k)^*}{T(k)^*} \\ \dfrac{L(k)}{T(k)} & \dfrac{1}{T(k)^*} \end{bmatrix}, \tag{3.2}$$

where $*$ denotes complex conjugation. Similarly, let $\Lambda_1(k)$ and $\Lambda_2(k)$ be the transition matrices corresponding to $\mathbf{S}_1(k)$ and $\mathbf{S}_2(k)$. It is known [Ak92] that

$$\Lambda(k)=\Lambda_1(k)\,\Lambda_2(k). \tag{3.3}$$

From the $(1,1)$ entry of the matrix product in (3.3) we get

$$\frac{1}{T(k)}=\frac{1-R_1(k)\,L_2(k)}{T_1(k)\,T_2(k)}. \tag{3.4}$$

Let $\mathbf{R}^+ = (0, +\infty)$. For $k \in \mathbf{R}^+$, let us define the phases of the transmission coefficients:

$$T(k) = |T(k)|\, e^{i\phi(k)}, \qquad T_1(k) = |T_1(k)|\, e^{i\phi_1(k)}, \qquad T_2(k) = |T_2(k)|\, e^{i\phi_2(k)}, \tag{3.5}$$

where it is understood that $\phi(k)$, $\phi_1(k)$, and $\phi_2(k)$ are continuous in $k \in \mathbf{R}^+$ and normalized such that

$$\phi(+\infty) = \phi_1(+\infty) = \phi_2(+\infty) = 0. \tag{3.6}$$

Similarly, let

$$1 - R_1(k)\, L_2(k) = |1 - R_1(k)\, L_2(k)|\, e^{i\omega(k)}, \tag{3.7}$$

where $\omega(k)$ is assumed continuous in $k \in \mathbf{R}^+$ and to satisfy $\omega(+\infty) = 0$. From (3.4), we obtain

$$\phi(k) = \phi_1(k) + \phi_2(k) - \omega(k), \qquad k \in \mathbf{R}^+. \tag{3.8}$$

From Levinson's theorem [Ne80] we have

$$\phi(0+) = \left[N - \frac{d}{2}\right]\pi, \qquad \phi_1(0+) = \left[N_1 - \frac{d_1}{2}\right]\pi, \qquad \phi_2(0+) = \left[N_2 - \frac{d_2}{2}\right]\pi, \tag{3.9}$$

where N, N_1, and N_2 denote the number of bound states corresponding to the potentials $V(x)$, $V_1(x)$, and $V_2(x)$, respectively; $d = 1$ if $V(x)$ is a generic potential and $d = 0$ if $V(x)$ is exceptional; in a similar manner, d_1 and d_2 take values 1 or 0 depending on whether $V_1(x)$ and $V_2(x)$ are generic or exceptional. Using (3.9) in (3.8) we obtain

$$N = N_1 + N_2 + \frac{1}{2}\left[d - d_1 - d_2\right] - \frac{1}{\pi}\omega(0+). \tag{3.10}$$

Now let us analyze $\omega(k)$ further. Note that $R_1(k)$ and $L_2(k)$ are continuous and nonzero and strictly less than one in absolute value for $k \in \mathbf{R}^+$ and that, as $k \to +\infty$, both $R_1(k)$ and $L_2(k)$ vanish.

In the following we need to distinguish between the generic and the exceptional case. When $V_1(x)$ and $V_2(x)$ are both generic we have

$$R_1(k) = -1 - 2ika_{r;1} + o(k), \qquad L_2(k) = -1 - 2ika_{l;2} + o(k), \qquad k \to 0, \tag{3.11}$$

where

$$a_{r;1} = \frac{1 - \displaystyle\int_{-\infty}^{x_1} dx\, x\, V_1(x)\, f_{r;1}(0,x)}{\displaystyle\int_{-\infty}^{x_1} dx\, V_1(x)\, f_{r;1}(0,x)}, \tag{3.12}$$

$$a_{l;2} = \frac{1 + \displaystyle\int_{x_1}^{\infty} dx\, x\, V_2(x)\, f_{l;2}(0,x)}{\displaystyle\int_{x_1}^{\infty} dx\, V_2(x)\, f_{l;2}(0,x)}. \tag{3.13}$$

In the exceptional case we define

$$\gamma_1 = \frac{f_{l;1}(0,x)}{f_{r;1}(0,x)} = \frac{1}{f_{r;1}(0,x_1)}, \qquad \gamma_2 = \frac{f_{l;2}(0,x)}{f_{r;2}(0,x)} = f_{l;2}(0,x_1), \tag{3.14}$$

and note that, if $V_1(x)$, resp. $V_2(x)$, is exceptional, then

$$R_1(k) = -b_1 + o(1), \qquad \text{resp.} \qquad L_2(k) = b_2 + o(1), \qquad k \to 0, \tag{3.15}$$

where

$$b_j = \frac{\gamma_j^2 - 1}{\gamma_j^2 + 1}, \qquad j = 1, 2. \tag{3.16}$$

The relations (3.11)-(3.13) follow from [DT79, p. 146]; (3.15) was proved in [Kl88]. We remark that the validity of (3.11) depends on the property that $V_1(x)$ and $V_2(x)$ are each supported on a semi-infinite interval; this guarantees the convergence of the integrals in the numerators in (3.12) and (3.13). In general, for potentials in $L_1^1(\mathbf{R})$ one can only conclude that the reflection coefficients behave like $-1 + o(1)$ as $k \to 0$ in the generic case [Kl88].

When both $V_1(x)$ and $V_2(x)$ are generic we have

$$1 - R_1(k)\, L_2(k) = -2ik[a_{r;1} + a_{l;2}] + o(k), \qquad k \to 0. \tag{3.17}$$

When both $V_1(x)$ and $V_2(x)$ are exceptional we get

$$1 - R_1(k)\, L_2(k) = 1 + b_1\, b_2 + o(1), \qquad k \to 0. \tag{3.18}$$

When $V_1(x)$ is generic and $V_2(x)$ is exceptional we have

$$1 - R_1(k)\, L_2(k) = 1 + b_2 + o(1), \qquad k \to 0, \tag{3.19}$$

and finally, when $V_1(x)$ is exceptional and $V_2(x)$ is generic, we have

$$1 - R_1(k)\, L_2(k) = 1 - b_1 + o(1), \qquad k \to 0. \tag{3.20}$$

From (3.15) and (3.18)-(3.20) we see that if at least one of $V_1(x)$ and $V_2(x)$ is exceptional, then $[1 - R_1(0)\, L_2(0)]$ is strictly positive, and hence $\omega(0+) = 0$.

If both $V_1(x)$ and $V_2(x)$ are generic, the analysis is slightly more complicated: If $a_{r;1} < -a_{l;2}$, then $\omega(0+) = \pi/2$; if $a_{r;1} > -a_{l;2}$, then $\omega(0+) = -\pi/2$. If $a_{r;1} = -a_{l;2}$, then, as $k \to 0$, we get $1 - R_1(k)\, L_2(k) = o(k)$, where we used (3.17). As a result, (3.4) implies that $k/T(k) = o(1)$ as $k \to 0$, and this, in turn, implies that $V(x)$ is exceptional. Therefore, the left-hand side of (3.4) has a limit as $k \to 0$, which means that in fact $1 - R_1(k)\, L_2(k) = O(k^2)$, from which we obtain $\omega(0+) = 0$.

It is known [AKV96] that when $V_1(x)$ and $V_2(x)$ are both exceptional, then $V(x)$ is exceptional. If exactly one of $V_1(x)$ or $V_2(x)$ is exceptional, then $V(x)$ is generic. If both $V_1(x)$ and $V_2(x)$ are generic, then $V(x)$ can be exceptional or generic. By using these facts along with the value of $\omega(0+)$ and (3.10), we arrive at the following conclusions:

(i) If both $V_1(x)$ and $V_2(x)$ are exceptional, then $N = N_1 + N_2$.

(ii) If exactly one of $V_1(x)$ and $V_2(x)$ is exceptional and the other is generic, then $N = N_1 + N_2$.

(iii) If both $V_1(x)$ and $V_2(x)$ are generic and $V(x)$ is also generic, then $\omega(0+) = \pm\pi/2$. In this case, we have $N = N_1 + N_2 - 1$ if $\omega(0+) = \pi/2$, and this happens if $a_{r;1} < -a_{l;2}$ in (3.17); or we have $N = N_1 + N_2$ if $\omega(0+) = -\pi/2$, and this happens if $a_{r;1} > -a_{l;2}$.

(iv) If both $V_1(x)$ and $V_2(x)$ are generic and $V(x)$ is exceptional, then we must have $\omega(0+) = 0$ and $N = N_1 + N_2 - 1$. This happens if $a_{r;1} = -a_{l;2}$ in (3.17).

Summarizing, if $a_{r;1} \le -a_{l;2}$ in (3.17) and both $a_{r;1}$ and $a_{l;2}$ are finite, then we have $N = N_1 + N_2 - 1$; if at least one of $a_{r;1}$ and $a_{l;2}$ is infinite or if $a_{r;1} > -a_{l;2}$, then we have $N = N_1 + N_2$.

There is a direct connection between cases (i)-(iv) above and cases (a)-(d) in the proof of Theorem 2.1 because the coefficients $a_{r;1}$ and $a_{l;2}$ are related to the quantity $Z(x_1)$ defined in (2.4). To see this recall that $f_{r;1}(0,x)$ and $f_{l;2}(0,x)$ obey the integral equations

$$f_{r;1}(0,x) = 1 + \int_{-\infty}^{x} dy\,(x-y)\,V_1(y)\,f_{r;1}(0,y), \tag{3.21}$$

$$f_{l;2}(0,x) = 1 + \int_{x}^{\infty} dy\,(y-x)\,V_2(y)\,f_{l;2}(0,y). \tag{3.22}$$

Hence, from (3.21) and (3.22) we obtain

$$f_{r;1}(0,x) = c_{r;1}\,x + d_{r;1}, \qquad x > x_1,$$

$$f_{l;2}(0,x) = -c_{l;2}\,x + d_{l;2}, \qquad x < x_1,$$

with

$$c_{r;1} = \int_{-\infty}^{x_1} dy\,V_1(y)\,f_{r;1}(0,y), \qquad d_{r;1} = 1 - \int_{-\infty}^{x_1} dy\,y\,V_1(y)\,f_{r;1}(0,y), \tag{3.23}$$

$$c_{l;2} = \int_{x_1}^{\infty} dy\,V_2(y)\,f_{l;2}(0,y), \qquad d_{l;2} = 1 + \int_{x_1}^{\infty} dy\,y\,V_2(y)\,f_{l;2}(0,y). \tag{3.24}$$

Thus, from (3.12), (3.13), (3.23), and (3.24) we conclude that

$$a_{r;1} = \frac{d_{r;1}}{c_{r;1}}, \qquad a_{l;2} = \frac{d_{l;2}}{c_{l;2}}. \tag{3.25}$$

Moreover,

$$\frac{f_{r;1}(0,x_1)}{f'_{r;1}(0,x_1)} = x_1 + a_{r;1}, \qquad \frac{f_{l;2}(0,x_1)}{f'_{l;2}(0,x_1)} = x_1 - a_{l;2},$$

and hence

$$Z(x_1) = -a_{r,1} - a_{l;2}.$$

Thus, (i)-(ii) above correspond to (a)-(c) in Section 2 and (iii) and (iv) correspond to (d); in particular, (iv) corresponds to (d) with $Z(x_1) = 0$.

We conclude this section with a brief look at families of potentials of the form

$$V_\xi(x) = V_1(x) + V_2(x-\xi), \tag{3.26}$$

where ξ is a nonnegative parameter and $V_1(x)$ and $V_2(x)$ are the two fragments of $V(x)$. In other words, the parameter ξ controls the separation distance between the two fragments. The next result shows that the number of bound states can only increase if ξ is increased. By virtue of (2.3) it can only increase by one. Since the proof is short we present two versions, one using the method of Section 2 and the other using the method of this section. In the case of compactly supported fragments the result is already known from [Kl81] and [Sa95].

Theorem 3.1 Let N_ξ denote the number of bound states of $V_\xi(x)$. Then either $N_\xi = N_1 + N_2$ for all $\xi \geq 0$ or there is a unique $\xi_0 \geq 0$ such that $N_\xi = N_1 + N_2 - 1$ for $0 \leq \xi \leq \xi_0$ and $N_\xi = N_1 + N_2$ for $\xi > \xi_0$.

PROOF: (a) First, if one of the fragments is exceptional, then we have $N_\xi = N_1 + N_2$ for all $\xi \geq 0$. If both fragments are generic, then we let $f_{l;2;\xi}(k,x)$ denote the Jost solution from the left for the potential $V_2(x-\xi)$. Then $f_{l;2;\xi}(0,x) = -c_{l;2}\,(x-\xi) + d_{l;2}$ for $x < x_1 + \xi$, and thus, by using (2.4) and (3.26), we obtain $Z_\xi(x_1) = -\xi - a_{l;2} - a_{r;1}$. Thus if $Z_0(x_1) < 0$, then, for all $\xi \geq 0$, $Z_\xi(x_1) < 0$ and hence $N_\xi = N_1 + N_2$. If $Z_0(x_1) \geq 0$, then $Z_{\xi_0}(x_1) = 0$ when $\xi_0 = Z_0(x_1) = -a_{l;2} - a_{r;1}$ and the assertion follows.

(b) Replacing $L_2(k)$ by $e^{2ik\xi}\,L_2(k)$ in (3.18) we obtain

$$1 - R_1(k)\,L_{2;\xi}(k) = -2ik[a_{r;1} + a_{l;2} + \xi] + o(k), \qquad k \to 0.$$

Now the conclusion follows using (iii) and (iv) above. ∎

4. AN EXAMPLE

The following example illustrates Theorems 2.1 and 3.1. Let

$$V(x) = \begin{cases} A^2, & x \in (0,1), \\ -B^2, & x \in (1,2), \\ 0, & \text{elsewhere}, \end{cases} \tag{4.1}$$

where A and B are some positive constants. We can fragment $V(x)$ as $V_1(x) + V_2(x)$, where $V_1(x)$ is a square potential barrier of height A^2 with support $(0,1)$ and $V_2(x)$ is a square well of depth $-B^2$ with support $(1,2)$. Then a straightforward computation using (3.23)-(3.25) yields $c_{r;1} = A\ \sinh A$, $d_{r;1} = \cosh A - A\ \sinh A$, $c_{l;2} = -B\ \sin B$, $d_{l;2} = \cos B - B\ \sin B$, and thus

$$a_{r;1} = \frac{1}{A}\ \coth A - 1, \qquad a_{l;2} = -\frac{1}{B}\ \cot B + 1.$$

Let us demonstrate that by choosing A and B suitably, we can have $N_1 = 0$, $N_2 = 1$, $N = 0$. In other words, the positive fragment $V_1(x)$ may cancel the bound state caused by the negative fragment $V_2(x)$, resulting in no bound states for $V(x)$. Unless B is a multiple of π, both $V_1(x)$ and $V_2(x)$ are generic. If we let, for example, $B = \pi/4$, then from (2.4) we get $Z(x_1) = -a_{r;1} - a_{l;2} \geq 0$ whenever $A \geq A_0$, where A_0 satisfies $A_0 \tanh A_0 = \pi/4$ i.e. $A_0 = 1.0201\overline{1}$. For $A = A_0$ the potential $V(x)$ is exceptional with no bound state and for $A > A_0$ it is generic with no bound state. Now let us consider the family $V_\xi(x)$ defined in (3.26). If $A < A_0$ and $B = \pi/4$, then $Z_0(x_1) < 0$ and we have $N_\xi = N_1 + N_2 = 1$ for all ξ. If $A = A_0$, then we have $N_0 = 0$ but $N_\xi = 1$ for $\xi > 0$, i.e. $\xi_0 = 0$. If $A > A_0$, then ξ_0 is given by

$$\xi_0 = \frac{1}{B\ \tan B} - \frac{1}{A\ \tanh A}$$

and we have $N_\xi = 0$ for $\xi \leq \xi_0$ and $N_\xi = 1$ for $\xi > \xi_0$.

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