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INVERSE SCATTERING PROBLEM FOR THE WAVE EQUATION WITH DISCONTINUOUS WAVESPEED

Tuncay Aktosun[1], Martin Klaus[2], and Cornelis van der Mee[3]

[1]Department of Mathematics, North Dakota State University, Fargo, ND 58105, USA
[2]Department of Mathematics, Virginia Tech, Blacksburg, VA 24061, USA
[3]Department of Mathematics, University of Cagliari, Cagliari, Italy

Abstract: The inverse problem on the line is studied for the generalized Schrödinger equation $d^2\psi/dx^2 + k^2H(x)^2\psi = Q(x)\psi$, where k is the wavenumber, $1/H(x)$ is the wavespeed, and $Q(x)$ is the restoring force per unit length. $H(x)$ is a positive, piecewise continuous function having limits $H_\pm$ as $x \to \pm\infty$, and $Q(x)$ satisfies certain integrability conditions. This equation describes wave propagation in a nonhomogeneous medium in which the wavespeed is allowed to change abruptly at certain interfaces. The inverse problem considered here consists in determining the function $H(x)$ from a suitable set of scattering data and for a given $Q(x)$. At the heart of the solution are a Riemann-Hilbert problem and a related singular integral equation. The solvability of the integral equation is discussed, and the solution method is illustrated by some explicitly solved examples.

Keywords: 1-D Schrödinger equation, Inverse scattering, 1-D wave equation, Energy-dependent potential, Acoustics, Discontinuous wavespeed, Wave propagation, Singular integral equation

1. Introduction

In this article we report on recent work concerning the inverse scattering problem for the one-dimensional generalized Schrödinger equation

$$\psi''(k,x) + k^2H(x)^2\psi(k,x) = Q(x)\psi(k,x), \qquad x \in \mathbf{R}, \tag{1.1}$$

where the prime denotes the x-derivative. The functions $H(x)$ and $Q(x)$ obey certain conditions that will be detailed below. In the context of this article (1.1) describes the propagation of waves in a nonhomogeneous medium where k is the wavenumber, $1/H(x)$ is the wavespeed, and $Q(x)$ is the restoring force (per unit length). The function $H(x)$ may have jump discontinuities, that is, we allow for the possibility of the physical properties of the medium to change abruptly at certain interfaces. In the time domain (1.1) is equivalent to

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c(x)^2}\frac{\partial^2 u}{\partial t^2} = Q(x)\,u(x,t), \tag{1.2}$$

where $u(x,t)$ is the wave amplitude and $c(x) = 1/H(x)$ is the wavespeed. Conversely, (1.1) is the frequency domain version of (1.2). The conditions that $H(x)$ and $Q(x)$ need to satisfy are as follows:

(H1) $H(x)$ is strictly positive and piecewise continuous with jump discontinuities at x_n for $n = 1, \cdots, N$.

(H2) $H(x) \to H_\pm$ as $x \to \pm\infty$, where $H_\pm$ are positive constants.

(H3) $H - H_\pm \in L^1(\mathbf{R}^\pm)$, where $\mathbf{R}^- = (-\infty, 0)$ and $\mathbf{R}^+ = (0, +\infty)$.

(H4) $H'(x)$ is absolutely continuous on every interval (x_n, x_{n+1}) and $2H''\,H - 3\,(H')^2$ belongs to $L^1_1(x_n, x_{n+1})$ for $n = 0, \cdots, N$, where $x_0 = -\infty$ and $x_{N+1} = +\infty$.

(H5) $Q \in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha \in [0,1]$, where $L^1_\beta(I)$ is the Banach space of complex-valued measurable functions $f(x)$ on I such that $\int_I dx\,(1+|x|)^\beta\,|f(x)| < +\infty$.

Under these conditions, (1.1) has two linearly independent solutions, so-called Jost solutions, satisfying the boundary conditions

$$f_l(k,x) = \begin{cases} e^{ikH_+x} + o(1), & x \to +\infty, \\ \dfrac{1}{T_l(k)}e^{ikH_-x} + \dfrac{L(k)}{T_l(k)}e^{-ikH_-x} + o(1), & x \to -\infty, \end{cases} \tag{1.3}$$

$$f_r(k,x) = \begin{cases} \dfrac{1}{T_r(k)}e^{-iH_+x} + \dfrac{R(k)}{T_r(k)}e^{ikH_+x} + o(1), & x \to +\infty, \\ e^{-ikH_-x} + o(1), & x \to -\infty. \end{cases} \tag{1.4}$$

Here $T_r(k)$ and $T_l(k)$ are the transmission coefficients from the right and from the left, respectively, and $R(k)$ and $L(k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrix is defined by

$$\mathbf{S}(k) = \begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{bmatrix}.$$

The scattering matrix will not play a direct role in this article; we introduce it mainly as a convenient notational device. For certain negative values of k^2 (1.1) may have a solution belonging to $L^2(\mathbf{R})$; such values of k^2 will be referred to as bound state energies and we put $k = i\kappa$ with $\kappa > 0$. Our assumptions on $H(x)$ and $Q(x)$ guarantee that the number of bound states is finite. We will denote the number of bound states by $\mathcal{N}$. Associated with each eigenvalue $-\kappa_j^2$ ($k = i\kappa_j$, $j = 1, \cdots, \mathcal{N}$) is the norming constant ν_j in (3.17).

There are several inverse problems that can be studied in the context of (1.1). For example:

1. The classical inverse problem, where $H(x) \equiv 1$ and one is asked to determine the function $Q(x)$ from the scattering data consisting of either $R(k)$ or $L(k)$ for $k \in \mathbf{R}$, the eigenvalues $-\kappa_j^2$ and their norming constants ν_j, $j = 1, \cdots, N$.
2. The same as problem 1, but with a given $H(x)$ from a suitable class of functions.
3. The problem where $Q(x)$ is given and one is asked to determine $H(x)$ from an appropriate set of scattering data.

The first problem is well understood [1,2,3]. When $Q \in L_1^1(\mathbf{R})$ one has a complete characterization of the scattering data and there is a one-to-one correspondence between the scattering data and the potentials in $L_1^1(\mathbf{R})$. The second problem was solved in [4] along the lines of problem 1. The aim of this article is to study the third problem. The case when $H(x)$ is continuous was studied in [5]; the case when $Q(x) \equiv 0$ and $H(x)$ has jump discontinuities was considered by Grinberg [6,7]. Here we will consider the general case when $Q(x) \neq 0$ and $H(x)$ has jump discontinuities. The present article is based on [8], where more details can be found. The main difference between the case $Q(x) = 0$ and $Q(x) \neq 0$ is that in the former we have $|R(k)| < 1$ for $k \in \mathbf{R}$, while in the latter we may have $R(0) = -1$. This difference makes the case $Q(x) \neq 0$ more difficult to study. It turns out that for problem 3 "essentially" the same set of scattering data as for problem 1 is appropriate. We say "essentially" because, as we will see in the examples, it may be necessary also to know either the value of H_+ or H_- in order to determine $H(x)$ uniquely. So we may consider H_+ or H_- to be part of the scattering data. However, we also present an example (Example 2, Section 4), where H_+ cannot be chosen freely, but is determined by $Q(x)$ and $R(k)$. This suggests that for problem 3 the characterization of the scattering data is more difficult than for problem 1 and needs to be investigated further.

This article is organized as follows. In Section 2 we present some results concerning the asymptotic behavior of $\mathbf{S}(k)$ as $k \to 0$ and $k \to \pm\infty$. These results are essential for Section 3, where we formulate a key Riemann-Hilbert problem and solve it by converting it into a singular integral equation. We also discuss the unique solvability of this integral equation. In Section 4 we consider three examples. Examples 1 and 2 have been worked out in detail in [8] and are included here for illustrative purposes. Example 3 is new and we give most of the details.

Except in a few instances it is not possible to give detailed proofs in this article. So we will often refer the reader to [8] for more information.

2. Small-k and Large-k behavior of $\mathbf{S}(k)$

In this section we determine the asymptotic behavior of $\mathbf{S}(k)$ as $k \to 0$ and $k \to \pm\infty$. We let $\mathbf{C}^+$ denote the upper-half complex plane and $\overline{\mathbf{C}^+} = \mathbf{C}^+ \cup \mathbf{R}$ its closure. The transmission coefficients can be extended meromorphically to $\mathbf{C}^+$, and we will analyze their behavior as $k \to 0$ and $k \to \infty$ in $\overline{\mathbf{C}^+}$.

Let $[f;g] = fg' - f'g$ denote the Wronskian. From (1.3) and (1.4) we have

$$[f_l(k,x); f_r(k,x)] = -2ik\frac{H_+}{T_r(k)} = -2ik\frac{H_-}{T_l(k)}, \tag{2.1}$$

$$[f_l(k,x); f_r(-k,x)] = 2ikH_-\frac{L(k)}{T_l(k)} = -2ikH_+\frac{R(-k)}{T_r(-k)}. \tag{2.2}$$

Moreover, we have $f_l(-k,x) = \overline{f_l(k,x)}$ and $f_r(-k,x) = \overline{f_r(k,x)}$ when $k \in \mathbf{R}$. The analyticity properties of the Jost solutions and the asymptotic properties of $\mathbf{S}(k)$ are studied by using the integral equations

$$\begin{aligned} f_{l,r}(k,x) =& e^{\pm ikH_\pm x} \\ & \pm \frac{1}{kH_\pm}\int_x^{\pm\infty} dz\ [\sin kH_\pm(z-x)]\ [k^2\{H_\pm^2 - H(z)^2\} + Q(z)]\ f_{l,r}(k,z), \end{aligned} \tag{2.3}$$

where the subscripts l and $+$, respectively r and $-$, correspond to one another. Based on (2.3) the following results, Theorems 2.1-2.3 below, can be proved. For details we refer the reader to [8] (Theorems 2.1, 4.1, and 4.2).

Theorem 2.1 Assume $Q \in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha \in [0,1)$ and $H - H_\pm \in L^1(\mathbf{R}^\pm)$. Then, for each x, $f_l(k,x)$ and $f_r(k,x)$ and their x-derivatives are analytic functions on $\mathbf{C}^+$ and continuous on $\overline{\mathbf{C}^+}$. Moreover, as $k \to 0$ in $k \in \overline{\mathbf{C}^+}$

$$f_{l,r}(k,x) - f_{l,r}(0,x) = o(|k|^\alpha), \qquad f'_{l,r}(k,x) - f'_{l,r}(0,x) = o(|k|^\alpha), \tag{2.4}$$

uniformly in x on any finite interval.

We can allow $\alpha = 1$ in (2.4), but then the error terms are $O(k)$. In that case $f_l(k,x)$ and $f_r(k,x)$ are differentiable at $k = 0$. The quantities $\tau(k)$, $\rho(k)$, and $\ell(k)$ defined next will play a crucial role in the discussion of the large-k behavior of the scattering matrix, and they are also convenient to use for small k. We define

$$\tau(k) = \sqrt{\frac{H_+}{H_-}}\, T_l(k) e^{ikA} = \sqrt{\frac{H_-}{H_+}}\, T_r(k) e^{ikA}, \qquad (2.5)$$

$$\rho(k) = R(k) e^{2ikA_+}, \qquad \ell(k) = L(k) e^{2ikA_-}, \qquad (2.6)$$

where

$$A_\pm = \pm \int_0^{\pm\infty} ds\,[H_\pm - H(s)], \qquad A = A_+ + A_-. \qquad (2.7)$$

We will call $\tau(k)$ the reduced transmission coefficient and $\rho(k)$ and $\ell(k)$ the reduced reflection coefficients from the right and left, respectively. In the second equation in (2.5) we have used the relation $H_+ T_l(k) = H_- T_r(k)$, which follows from (2.1). We remark that the scattering matrix $\mathbf{S}(k)$ is not unitary unless $H_+ = H_-$, but that the reduced scattering matrix is unitary. In particular, we have that

$$|\tau(k)|^2 + |\rho(k)|^2 = |\tau(k)|^2 + |\ell(k)|^2 = 1. \qquad (2.8)$$

Moreover, for $k \in \mathbf{R}$ we have $\tau(k) = \overline{\tau(-k)}$, $\rho(k) = \overline{\rho(-k)}$, and $\ell(k) = \overline{\ell(-k)}$.

Theorem 2.2 Assume $Q \in L_1^1(\mathbf{R})$ and $H - H_\pm \in L^1(\mathbf{R}^\pm)$. Then:

(i) $k/\tau(k)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$; $\tau(k)$ is continuous at $k = 0$, and either $\tau(0) \neq 0$ or $\tau(k)$ vanishes linearly as $k \to 0$ in $\overline{\mathbf{C}^+}$. The bound states for (1.1) correspond to the (simple) zeros of $k/\tau(k)$ in $\mathbf{C}^+$ and can only occur on the imaginary axis in $\mathbf{C}^+$. There is never a bound state at $k = 0$.

(ii) $\rho(k)$ and $\ell(k)$ are continuous for $k \in \mathbf{R}$. Either $|\rho(k)| = |\ell(k)| < 1$ for all $k \in \mathbf{R}$, or $|\rho(k)| = |\ell(k)| < 1$ for $k \neq 0$ and $\rho(0) = \ell(0) = -1$.

We will refer to the case when $\tau(0) = 0$ ($\tau(0) \neq 0$) as the generic (exceptional) case. By (2.1), the exceptional case occurs if and only if the Jost solutions $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent, i.e. if

$$f_l(0,x) = \gamma\, f_r(0,x) \qquad (2.9)$$

for some nonzero constant γ. In the exceptional case $f_l(0,x)$ and $f_r(0,x)$ are bounded, while in the generic case $f_l(0,x)$ ($f_r(0,x)$) behaves linearly as $x \to -\infty$ ($x \to +\infty$). When $Q(x) \equiv 0$, which is the case considered in [6,7], we have $f_l(0,x) = f_r(0,x) = 1$ and this corresponds to the exceptional case with $\gamma = 1$.

The next theorem describes the small-k behavior of $\mathbf{S}(k)$.

Theorem 2.3 Assume $H - H_\pm \in L^1(\mathbf{R}^\pm)$ and $Q \in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha \in [0,1)$. Then:

(i) In the generic case

$$\rho(k) = -1 + o(|k|^\alpha), \qquad \ell(k) = -1 + o(|k|^\alpha), \qquad k \to 0 \text{ in } \mathbf{R},$$

$$\tau(k) = ick + o(|k|^{1+\alpha}), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{2.10}$$

where

$$c = -\frac{2\sqrt{H_+ H_-}}{[f_l(0,x); f_r(0,x)]}. \tag{2.11}$$

(ii) In the exceptional case

$$\tau(k) = \frac{2\sqrt{H_+ H_-}\,\gamma}{H_-\gamma^2 + H_+} + o(|k|^\alpha), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{2.12}$$

$$\rho(k) = \frac{H_+ - H_-\gamma^2}{H_-\gamma^2 + H_+} + o(|k|^\alpha), \qquad k \to 0 \text{ in } \mathbf{R}, \tag{2.13}$$

$$\ell(k) = \frac{H_-\gamma^2 - H_+}{H_-\gamma^2 + H_+} + o(|k|^\alpha), \qquad k \to 0 \text{ in } \mathbf{R},$$

where γ is the constant defined in (2.9). Both (i) and (ii) remain valid for $\alpha = 1$, provided we replace the error terms by $O(k)$.

We remark that (i) follows directly from (2.1), (2.2), and (2.4)-(2.6). The proof of (ii) is more involved [8].

Next we consider the large-k behavior of $\mathbf{S}(k)$. We use the fact that although $H(x)$ is discontinuous at x_j $(j = 1, \cdots, N)$, we can, on each interval (x_j, x_{j+1}) $(j = 0, \cdots, N)$, perform a Liouville transformation of the form

$$y = y(x) = \int_0^x ds\, H(s), \qquad \psi(k,x) = \frac{1}{\sqrt{H(x)}}\phi(k,y). \tag{2.14}$$

Under this transformation, in each interval (x_j, x_{j+1}), (1.1) is transformed into the standard Schrödinger equation

$$\frac{d^2\phi(k,y)}{dy^2} + k^2\phi(k,y) = V(y)\,\phi(k,y),$$

where for $x \in \mathbf{R} \setminus \{x_1, \cdots, x_N\}$

$$V(y) = V(y(x)) = \frac{H''(x)}{2H(x)^3} - \frac{3}{4}\frac{H'(x)^2}{H(x)^4} + \frac{Q(x)}{H(x)^2}. \tag{2.15}$$

Note that since $H(x)$ is strictly positive we have that $y_j < y_{j+1}$, $y_0 = -\infty$, and $y_{N+1} = +\infty$. Here $y_j = y(x_j)$ for $j = 0, \cdots, N+1$.

Let $V_{j,j+1}(y)$ be the potential defined by

$$V_{j,j+1}(y) = \begin{cases} V(y), & y \in (y_j, y_{j+1}), \\ 0, & y \in \mathbf{R} \setminus \{y_1, \cdots, y_N\}, \end{cases} \tag{2.16}$$

where $V(y)$ is given by (2.15). By hypothesis (H4) we have that $V_{j,j+1}(y) \in L^1_1(\mathbf{R})$, $j = 0, \cdots, N$. Let $g_{l;j,j+1}(k,y)$ and $g_{r;j,j+1}(k,y)$ denote the Jost solutions from the left and right, respectively, associated with the potential $V_{j,j+1}(y)$. Define

$$\eta_{j,j+1}(k,x) = \frac{1}{\sqrt{H(x)}}\, g_{l;j,j+1}(k,y), \qquad \xi_{j,j+1}(k,x) = \frac{1}{\sqrt{H(x)}}\, g_{r;j,j+1}(k,y).$$

Then $\eta_{j,j+1}(k,x)$ and $\xi_{j,j+1}(k,x)$ are two linearly independent solutions of (1.1) in the interval (x_j, x_{j+1}) for $j = 0, \cdots, N$. Hence, on this interval, they can be related to the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ of (1.1). It is shown in [8] (Section 3) that for $x \in (x_j, x_{j+1})$ with $0 \le j \le N-1$

$$\begin{bmatrix} f_l(k,x) \\ f_l'(k,x) \end{bmatrix} = \Gamma_{j,j+1}(k,x)\, \mathcal{G}_j(k) \begin{bmatrix} \sqrt{H_+}\, e^{ikA_+} \\ 0 \end{bmatrix},$$

where

$$\mathcal{G}_j(k) = \prod_{n=j}^{N-1} \Gamma_{n,n+1}(k, x_{n+1} - 0)^{-1} \Gamma_{n+1,n+2}(k, x_{n+1} + 0),$$

$$\Gamma_{j,j+1}(k,x) = \begin{bmatrix} \eta_{j,j+1}(k,x) & \xi_{j,j+1}(k,x) \\ \eta'_{j,j+1}(k,x) & \xi'_{j,j+1}(k,x) \end{bmatrix}, \qquad j = 0, \cdots, N,$$

with A_+ as defined in (2.7). Let $t_{j,j+1}(k)$, $r_{j,j+1}(k)$, and $l_{j,j+1}(k)$ denote the transmission and reflection coefficients corresponding to the potential $V_{j,j+1}(y)$ given in (2.16). Then the following relations are the basis for studying the large-k behavior of $\mathbf{S}(k)$:

$$\frac{1}{\tau(k)} = \frac{1}{t_{0,1}(k)} [1 \quad 0]\, \mathcal{G}_0(k) \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{t_{N,N+1}(k)} [0 \quad 1]\, \mathcal{G}_0(k)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{2.17}$$

$$\frac{\rho(k)}{\tau(k)} = \begin{bmatrix} 1 & \dfrac{r_{N,N+1}(k)}{t_{N,N+1}(k)} \end{bmatrix} \mathcal{G}_0(k)^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.18}$$

One can show that (see [8], Section 4) as $k \to \infty$ in $\overline{\mathbf{C}^+}$

$$\begin{aligned} &\Gamma_{n,n+1}(k, x_{n+1} - 0)^{-1} \Gamma_{n+1,n+2}(k, x_{n+1} + 0) \\ &\quad = \begin{bmatrix} \alpha_{n+1}(1 + o(1)) & \beta_{n+1} e^{-2iky_{n+1}}(1 + o(1)) \\ \beta_{n+1} e^{2iky_{n+1}}(1 + o(1)) & \alpha_{n+1}(1 + o(1)) \end{bmatrix}, \end{aligned} \tag{2.19}$$

where

$$\alpha_n = \frac{1}{2}\frac{H(x_n-0)+H(x_n+0)}{\sqrt{H(x_n-0)H(x_n+0)}}, \qquad \beta_n = \frac{1}{2}\frac{H(x_n-0)-H(x_n+0)}{\sqrt{H(x_n-0)H(x_n+0)}}. \tag{2.20}$$

Define

$$E(k,x_n) = \begin{bmatrix} \alpha_n & \beta_n e^{-2iky_n} \\ \beta_n e^{2iky_n} & \alpha_n \end{bmatrix}, \tag{2.21}$$

and note that $E(-k,x_n) = \mathbf{q}\,E(k,x_n)\,\mathbf{q}$, where $\mathbf{q} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and $\overline{E(-k,x_n)} = E(k,x_n)$ for $k \in \mathbf{R}$. Hence $\prod_{n=1}^{N} E(k,x_n)$ is of the form

$$\prod_{n=1}^{N} E(k,x_n) = \begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix}, \tag{2.22}$$

such that $a(-k) = \overline{a(k)}$ and $b(-k) = \overline{b(k)}$ for $k \in \mathbf{R}$. By (2.20)-(2.22), $\det E(k,x_n) = \alpha_n^2 - \beta_n^2 = 1$, and hence

$$\det\left(\prod_{n=1}^{N} E(k,x_n)\right) = |a(k)|^2 - |b(k)|^2 = 1, \qquad k \in \mathbf{C}.$$

By using induction on n it follows from (2.21) and (2.22) that

$$a(k) = \prod_{n=1}^{N} \alpha_n + \sum_s \gamma_s e^{2ikc_s}, \tag{2.23}$$

where the summation runs over a finite number of terms, and where γ_s and c_s are real constants. Moreover, $c_s > 0$ owing to the fact that each c_s is a sum of terms of the form $y_j - y_i$ with $j > i$. If $N \le 1$, the summation in (2.23) is absent and $a(k)$ is constant.

Let AP^W stand for the algebra of all functions $f(k)$ on $\mathbf{R}$ which are of the form

$$f(k) = \sum_{j=-\infty}^{\infty} f_j e^{ik\lambda_j}, \qquad k \in \mathbf{R},$$

where $f_j \in \mathbf{C}$ and $\lambda_j \in \mathbf{R}$ for all j and $\sum_j |f_j| < +\infty$. Then the closure of AP^W in $L^\infty(\mathbf{R})$ is the algebra AP of almost periodic functions. The next theorem is proved by using (2.8), (2.17)-(2.19), (2.23), and by exploiting the analyticity properties of $a(k)$ and using the growth properties of entire functions. It summarizes several results given in Section 4 of [8].

Theorem 2.4 (i) $|a(k)| \ge 1$ and $|\tau(k)| \le 1$ on $\overline{\mathbf{C}^+}$.

(ii) $\limsup_{k\to\pm\infty} |\rho(k)| < 1$.

(iii) $\dfrac{1}{\tau(k)} = a(k)\,[1+o(1)]$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$; $\rho(k) = \dfrac{-b(k)}{a(k)} + o(1)$ as $k \to \pm\infty$ in $\mathbf{R}$.

(iv) $a(k)$, $b(k)$, $\dfrac{1}{a(k)}$, and $\dfrac{b(k)}{a(k)}$ belong to AP^W.

Theorem 2.4 plays a crucial role in proving the solvability of a key Riemann-Hilbert problem studied in the next section. The reader is referred to [8] (Section 7) for the details.

3. Solution of the inverse problem

First we assume that there are no bound states. The formulation of the Riemann-Hilbert problem will involve the functions $Z_l(k,y)$ and $Z_r(k,y)$ which are defined by

$$f_l(k,x) = \sqrt{\frac{H_+}{H(x)}}e^{iky+ikA_+}Z_l(k,y), \qquad f_r(k,x) = \sqrt{\frac{H_-}{H(x)}}e^{-iky+ikA_-}Z_r(k,y), \quad (3.1)$$

where y is defined in (2.14) and $A_\pm$ are the constants from (2.7). Let us introduce the vector function

$$Z(k,y) = \begin{bmatrix} Z_l(k,y) \\ Z_r(k,y) \end{bmatrix}. \quad (3.2)$$

Theorem 3.1 Under assumptions (H1)-(H5) the vector $Z(k,y)$ defined in (3.2) satisfies

$$Z(-k,y) = \mathbf{g}(k,y)\,\mathbf{q}\,Z(k,y), \qquad k \in \mathbf{R}, \quad y \in \mathbf{R} \setminus \{y_1, \cdots, y_N\}, \quad (3.3)$$

where $\mathbf{J} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $\mathbf{q}$ is the matrix introduced below (2.21), and

$$\mathbf{g}(k,y) = e^{i\mathbf{J}ky}\sigma(k)e^{-i\mathbf{J}ky} = \begin{bmatrix} \tau(k) & -\rho(k)e^{2iky} \\ -\ell(k)e^{-2iky} & \tau(k) \end{bmatrix}.$$

PROOF: The solutions $\psi_l(k,x) = T_l(k)\,f_l(k,x)$ and $\psi_r(k,x) = T_r(k)\,f_r(k,x)$ of (1.1) satisfy

$$\begin{bmatrix} \psi_l(k,x) \\ \psi_r(k,x) \end{bmatrix} = \begin{bmatrix} T_l(k) & L(k) \\ R(k) & T_r(k) \end{bmatrix} \begin{bmatrix} \psi_r(-k,x) \\ \psi_l(-k,x) \end{bmatrix}, \qquad k \in \mathbf{R}, \quad (3.4)$$

and hence using (2.5), (2.6), and (3.1) in (3.4) we obtain (3.3). ∎

Eq. (3.3) constitutes a Riemann-Hilbert problem for the vector function $Z(k,y)$; however, it is not a standard Riemann-Hilbert problem because $Z(k,y)$ does not converge to a constant vector as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Our goal is to recast (3.3) as an integral equation. For each fixed $y \in \mathbf{R} \setminus \{y_1, \cdots, y_N\}$, from (3.3) we have

$$\tau(k)\,Z_r(k,y) = Z_l(-k,y) + \rho(k)\,e^{2iky}Z_l(k,y), \qquad k \in \mathbf{R}, \quad (3.5)$$

$$\tau(0)\,Z_r(0,y)=Z_l(0,y)+\rho(0)\,Z_l(0,y). \tag{3.6}$$

Define

$$F_+(k,x,y)=\frac{1}{k\sqrt{H(x)}}\left[\tau(k)\,Z_r(k,y)-\tau(0)\,Z_r(0,y)\right], \tag{3.7}$$

$$F_-(k,x,y)=\frac{1}{k\sqrt{H(x)}}\left[Z_l(-k,y)-Z_l(0,y)\right]. \tag{3.8}$$

Using

$$\frac{Z_l(0,y)}{\sqrt{H(x)}}=\frac{f_l(0,x)}{\sqrt{H_+}},\qquad \frac{Z_r(0,y)}{\sqrt{H(x)}}=\frac{f_r(0,x)}{\sqrt{H_-}},$$

from (3.5)-(3.8), for $k\in\mathbf{R}$ and $y\in\mathbf{R}\setminus\{y_1,\cdots,y_N\}$ we obtain

$$\begin{aligned}F_+(k,x,y)-F_-(k,x,y)=&-\rho(k)\,e^{2iky}F_-(-k,x,y)\\&+\frac{1}{k}\left[\rho(k)\,e^{2iky}-\rho(0)\right]\frac{f_l(0,x)}{\sqrt{H_+}}.\end{aligned} \tag{3.9}$$

Since we are considering the case without bound states, for $x\in\mathbf{R}\setminus\{x_1,\cdots,x_N\}$ and $y\in\mathbf{R}\setminus\{y_1,\cdots,y_N\}$, $F_\pm(k,x,y)$ have analytic extensions in k to $\mathbf{C}^\pm$, and $F_\pm(k,x,y)\to 0$ as $k\to\infty$ in $\overline{\mathbf{C}^\pm}$. The detailed justification for this conclusion is given in [8] (Theorems 4.4 and 5.2). The behavior of $F_\pm(k,x,y)$ at $k=0$ depends on the decay of $Q(x)$ and $H(x)-H_\pm$ as $x\to\pm\infty$. If $Q\in L^1_{1+\alpha}(\mathbf{R})$ with $\alpha\in(0,1)$ and $H(x)-H_\pm\in L^1(\mathbf{R}^\pm)$, then, by Theorem 2.1 and (3.1), we have

$$Z_l(k,y)-Z_l(0,y)=o(|k|^\alpha),\qquad Z_r(k,y)-Z_r(0,y)=o(|k|^\alpha),\qquad k\to 0 \text{ in } \overline{\mathbf{C}^+}.$$

Also $\tau(k)-\tau(0)=o(|k|^\alpha)$ by (2.10) and (2.12). It follows that $F_\pm(k,x,y)$ belong to the Hardy spaces $\mathbf{H}^p_\pm(\mathbf{R})$ for $1<p<1/(1-\alpha)$; if $\alpha=1$ a similar argument shows that $F_\pm(k,x,y)$ belong to $\mathbf{H}^p_\pm(\mathbf{R})$ for all $p\in(1,+\infty)$. Recall that the Hardy spaces $\mathbf{H}^p_\pm(\mathbf{R})$ are the spaces of analytic functions $f(k)$ on $\mathbf{C}^\pm$ for which $\sup_{\epsilon>0}\int_{-\infty}^{+\infty}dk|f(k\pm i\epsilon)|^p$ is finite. Associated with these spaces are the projection operators $\Pi_\pm$ which project $L^p(\mathbf{R})$ onto $\mathbf{H}^p_\pm(\mathbf{R})$ given by

$$(\Pi_\pm f)(k)=\pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s-k\mp i0}\,f(s). \tag{3.10}$$

It is known that $\Pi_\pm$ are bounded and complementary projections on $L^p(\mathbf{R})$ when $1<p<\infty$. Applying Π_+ and Π_- to (3.9) and using $\Pi_\pm F_\pm(k,x,y)=F_\pm(k,x,y)$, we obtain for $k\in\mathbf{R}$, $x\in\mathbf{R}\setminus\{x_1,\cdots,x_N\}$, and $y\in\mathbf{R}\setminus\{y_1,\cdots,y_N\}$

$$\begin{aligned}&F_\pm(k,x,y)\\&=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{ds}{s-k\mp i0}\left[\frac{\rho(s)\,e^{2isy}-\rho(0)}{s}\,\frac{f_l(0,x)}{\sqrt{H_+}}-\rho(s)\,e^{2isy}\,F_-(-s,x,y)\right].\end{aligned}$$

Hence $F_-(k,x,y)$ obeys the singular integral equation

$$\begin{aligned} F_-(k,x,y) =& \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{ds}{s-k+i0} \frac{\rho(s)\,e^{2isy}-\rho(0)}{s} \frac{f_l(0,x)}{\sqrt{H_+}} \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{ds}{s+k-i0}\rho(-s)\,e^{-2isy}\,F_-(s,x,y). \end{aligned} \tag{3.11}$$

Defining $\hat{F}_-(k,y) = \dfrac{\sqrt{H_+}}{f_l(0,x)}\,F_-(k,x,y)$ we can write (3.11) in the form

$$\hat{F}_-(k,y) = X_0(k,y) + (\mathcal{O}_y\hat{F}_-)(k,y), \tag{3.12}$$

where

$$X_0(k,y) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{ds}{s-k+i0} \frac{\rho(s)\,e^{2isy}-\rho(0)}{s}, \tag{3.13}$$

$$(\mathcal{O}_y\hat{F}_-)(k,y) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{ds}{s+k-i0}\rho(-s)\,e^{-2isy}\,\hat{F}_-(s,y).$$

Note that, since there are no bound states, $f_l(0,x) > 0$ for all $x \in \mathbf{R}$. The next theorem establishes the connection between $\hat{F}_-(k,y)$ and $y(x)$. Let an overdot denote the derivative with respect to k.

Theorem 3.2 Suppose that assumptions (H1)-(H5) hold with $\alpha = 1$ in (H5). Then $f_l(0,x)$ is determined by $Q(x)$ alone and $\dot{f}_l(0,x)$ is determined by $Q(x)$ and H_+ alone. Furthermore, we have

$$-i\,\hat{F}_-(0,y) = i\,\frac{\dot{f}_l(0,x)}{f_l(0,x)} + y + A_+. \tag{3.14}$$

PROOF: From (2.3) it follows that $f_l(0,x)$ and $\dot{f}_l(0,x)$ obey the integral equations

$$f_l(0,x) = 1 + \int_x^\infty dz\,(z-x)\,Q(z)\,f_l(0,z), \tag{3.15}$$

$$\dot{f}_l(0,x) = iH_+x + \int_x^\infty dz\,(z-x)\,Q(z)\,\dot{f}_l(0,z). \tag{3.16}$$

Eqs. (3.15) and (3.16) can be solved by iteration and the first assertion follows. Eq. (3.14) follows on taking $k \to 0$ in (3.8) and using (3.1). ∎

Note that $\dot{f}_l(0,x)$ is purely imaginary and hence the right-hand side of (3.14) is real. Hence $\hat{F}_-(0,y)$ must be purely imaginary. This can also be seen from (3.12) using the fact that $\overline{\rho(k)} = \rho(-k)$ for $k \in \mathbf{R}$. In order to find $\hat{F}_-(k,y)$ we need to know $X_0(k,y)$ first. We see from (3.13) that $X_0(k,y)$ is completely determined by $\rho(k)$. Provided (3.12) has a unique solution, $\hat{F}_-(k,y)$ is also completely determined by $\rho(k)$. However, there is the possibility that a restriction on H_+ arises from the solution of (3.14), since $y(x)$

must also be such that $y(x) \to \pm\infty$ as $x \to \pm\infty$. As we will see this situation occurs in Example 2, Section 4. Once $\hat{F}_-(k, y)$ has been obtained, the value of A_+ is determined by setting $x = 0$ and $y = 0$ in (3.14), so that

$$A_+ = -i\left(\hat{F}_-(0,0) + \frac{\dot{f}_l(0,0)}{f_l(0,0)}\right).$$

Then $y(x)$ is found by solving (3.14) for y in terms of x. Finally, $H(x)$ can be obtained by using $H(x) = dy/dx$.

We remark that if, in addition to $\rho(k)$ and $Q(x)$, H_- is known instead of H_+, then we can first compute H_+ as follows. In the exceptional case (i.e. if $-1 < \rho(0) < 1$), we get from (2.13)

$$H_+ = \frac{\gamma^2[1+\rho(0)]}{1-\rho(0)}\, H_-.$$

In the generic case (i.e. if $\rho(0) = -1$), using (2.8) we first compute $|\tau(k)| = \sqrt{1-|\rho(k)|^2}$ for $k \in \mathbf{R}$, and then find $|c| = \lim_{k\to 0} |\tau(k)|/|k|$, where c is the constant given in (2.11). Thus, by (2.11),

$$H_+ = \frac{|c|^2\, |[f_l(0,x); f_r(0,x)]|^2}{4H_-}.$$

Theorem 3.2 no longer applies if we only have $Q \in L^1_1(\mathbf{R})$, but $Q \notin L^1_2(\mathbf{R})$. Then $F_-(k,x,y)$ in (3.8) will in general diverge as $k \to 0$. In this case we have a partial result under specific assumptions on the fall-off of $Q(x)$ at either $+\infty$ or $-\infty$ (see [8], Theorem 5.5). An alternate inversion method that works in the generic case when $Q \in L^1_{1+\alpha}(\mathbf{R})$ ($\alpha > 0$), and in the exceptional case when $Q \in L^1_2(\mathbf{R})$, is available [9].

Now we consider the case when there are bound states at energies $-\kappa_j^2$ with $j = 1, \cdots, \mathcal{N}$. Then the reduced transmission coefficient $\tau(k)$ has simple poles on the positive imaginary axis at $k = i\kappa_j$. Let

$$\nu_j = \left(\int_{-\infty}^{\infty} dx\, f_l(i\kappa_j, x)^2\, H(x)^2\right)^{-1}, \qquad j = 1, \cdots, \mathcal{N}, \tag{3.17}$$

denote the norming constants. The norming constants are part of the scattering data. We only list here the main steps of the inversion procedure, so that we can apply it in the next section. A detailed derivation is given in Section 8 of [8]. Let

$$w(k) = (-1)^{\mathcal{N}} \prod_{j=1}^{\mathcal{N}} \frac{k + i\kappa_j}{k - i\kappa_j}, \qquad \tilde{\rho}(k) = \rho(k)\, w(k)^{-1},$$

$$(\mathcal{L}_y X)(k) = \frac{w(k)}{2\pi i} \int_{-\infty}^{+\infty} ds\, \frac{\tilde{\rho}(-s)\, e^{-2isy}}{s + k - i0} X(s), \tag{3.18}$$

$$X(k) = G_-(k,x,y) = \frac{1}{k\sqrt{H(x)}}\left[Z_l(-k,y) - Z_l(0,y)\right], \tag{3.19}$$

where $G_-(k,x,y)$ is the analog of $F_-(k,x,y)$ in (3.8). In analogy to (3.12) we have the singular integral equation

$$X(k) = B(k) + (\mathcal{L}_y X)(k), \qquad k \in \mathbf{R}, \tag{3.20}$$

where

$$\begin{aligned} B(k) = B(k,x,y) &= \frac{P_{\mathcal{N}-1}(k,x,y)}{\prod_{j=1}^{\mathcal{N}}(k - i\kappa_j)} \\ &+ \frac{w(k)}{2\pi i}\int_{-\infty}^{+\infty} \frac{ds}{s-k+i0}[\tilde{\rho}(s)\,e^{2isy} - \tilde{\rho}(0) - 1 + w(s)^{-1}]\frac{f_l(0,x)}{s\sqrt{H_+}}. \end{aligned} \tag{3.21}$$

Here $P_{\mathcal{N}-1}(k,x,y)$ is a function of the form

$$P_{\mathcal{N}-1}(k,x,y) = \sum_{n=0}^{\mathcal{N}-1} p_n(x,y)\,k^n,$$

with $p_n(x,y) = (-1)^{\mathcal{N}+n+1}\overline{p_n(x,y)}$, which follows from an application of Liouville's theorem ([8], Section 8). The functions $p_j(x,y)$ for $j = 0, \cdots, \mathcal{N}-1$ are to be determined. In order to accomplish this we introduce the function

$$\begin{aligned} \Omega(k,x,y) = &-(-1)^{\mathcal{N}}\tilde{\rho}(k)\,e^{2iky}G_-(-k,x,y) + \frac{(-1)^{\mathcal{N}}}{k}\left[\tilde{\rho}(k)\,e^{2iky} - \tilde{\rho}(0)\right]\frac{f_l(0,x)}{\sqrt{H_+}} \\ &+ \frac{(-1)^{\mathcal{N}}}{k}\left[w(k)^{-1} - 1\right]\frac{f_l(0,x)}{\sqrt{H_+}}. \end{aligned} \tag{3.22}$$

Let $\Omega_+(k,x,y) = (\Pi_+\Omega(\cdot,x,y))(k)$, where Π_+ is the projection operator defined in (3.10). Then we have [8]

$$\begin{aligned} \Omega_+(i\kappa_n,x,y) + (-i)^{\mathcal{N}}\frac{P_{\mathcal{N}-1}(i\kappa_n,x,y)}{\prod_{j=1}^{\mathcal{N}}(\kappa_n+\kappa_j)} &= -i^{\mathcal{N}}H_+\frac{\nu_n}{2\kappa_n}e^{-2\kappa_n(y+A_+)}\frac{P_{\mathcal{N}-1}(-i\kappa_n,x,y)}{\prod_{j=1}^{\mathcal{N}}(\kappa_n+\kappa_j)} \\ &+ \frac{1}{i\kappa_n\sqrt{H_-}}\left[\sqrt{H_+H_-}\frac{\nu_n}{2\kappa_n}e^{-2\kappa_n(y+A_+)}f_l(0,x) - (-1)^{\mathcal{N}}\tau(0)f_r(0,x)\right]. \end{aligned} \tag{3.23}$$

The equations (3.23) constitute a set of $\mathcal{N}$ equations for the $\mathcal{N}$ unknowns $p_0(x,y), \cdots, p_{\mathcal{N}-1}(x,y)$. Note, however, that $\Omega_+(i\kappa_n,x,y)$ also depends on $p_0(x,y), \cdots, p_{\mathcal{N}-1}(x,y)$ via $B(k,x,y)$ in (3.21) and $G_-(-k,x,y)$ in (3.19). Once the polynomial $P_{\mathcal{N}-1}(k,x,y)$ has been found, $H(x)$ can be obtained by using $\frac{\sqrt{H_+}}{f_l(0,x)}X(0)$ on the left hand side of

(3.14). The zeros of $f_l(0,x)$ give rise to singularities in (3.14). This need not cause any trouble as Example 3, Section 4, will show. Note that in the generic case $\tau(0)=0$ and in the exceptional case $\tau(0)=(-1)^{\mathcal{N}}\sqrt{1-\rho(0)^2}$ by (2.12) and Proposition 4.6 of [8].

We conclude this section with a theorem on the solvability of the singular integral equation (3.12). A similar theorem can also be given for (3.20). It is clear that $\mathcal{O}_y$ is a bounded operator on $\mathbf{H}^p_-(\mathbf{R})$ for $1<p<+\infty$. Using the results of Theorem 2.3 we deduce that $X_0(k,y)$ in (3.13) belongs to $\mathbf{H}^p_-(\mathbf{R})$ provided $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha\in(0,1]$ satisfying $\alpha>1-(1/p)$. Indeed, this is immediate since

$$\frac{\rho(k)e^{2iky}-\rho(0)}{k}=\begin{cases} o(|k|^{\alpha-1}), & k\to 0,\\ O(1/k), & k\to\pm\infty.\end{cases}$$

So it is natural to study (3.12) in $\mathbf{H}^p_-(\mathbf{R})$.

Theorem 3.3 For $1<p<+\infty$, (3.12) has a unique solution $X\in\mathbf{H}^p_-(\mathbf{R})$ for every $X_0\in\mathbf{H}^p_-(\mathbf{R})$. This solution is given by $X(k)=\sum_{n=0}^{\infty}[\mathcal{O}_y^nX_0](k)$, where the series converges absolutely in the norm of $\mathbf{H}^p_-(\mathbf{R})$.

The proof of this theorem is given in [8] (Theorem 7.1). We add a few remarks about the proof. When $p=2$ the result follows from a contraction argument. That $\mathcal{O}_y$ is a strict contraction is obvious in the exceptional case, since $\|\mathcal{O}_y\|\le\sup_{k\in\mathbf{R}}|\rho(k)|<1$. Moreover, by using $\rho(k)=\overline{\rho(-k)}$ we see that $\mathcal{O}_y$ is self-adjoint. As shown in [8], $\mathcal{O}_y$ is also a strict contraction in the generic case (when $\rho(0)=-1$). To deal with $p\neq 2$ we derive a (vector) Riemann-Hilbert problem satisfied by any solution of (3.12) which is in $\mathbf{H}^p_-(\mathbf{R})$. The accompanying Riemann-Hilbert problem, where only the asymptotic part $-\dfrac{b(k)}{a(k)}$ of $\rho(k)$ is retained, can be shown to be uniquely solvable by factorization of an almost periodic 2×2 matrix function. It is here where Theorem 2.4, in particular (iv), enters. As a result, (3.12) is a Fredholm integral equation of index zero in $\mathbf{H}^p_-(\mathbf{R})$. A Fredholm argument then leads to the unique solvability of (3.12) in $\mathbf{H}^p_-(\mathbf{R})$, where $1<p<+\infty$. As a further result it follows that the spectral radius of $\mathcal{O}_y$ is strictly less than one in any space $\mathbf{H}^p_-(\mathbf{R})$ $(1<p<\infty)$.

4. Examples

In this section we consider three examples. Since the first two examples have been worked out in detail in [8], we will only state the main results here. The third example is new and we give most of the details. We also comment on the spectrum of $\mathcal{O}_y$ in the first two examples. The spectral properties of $\mathcal{O}_y$ in another example can be found in [9]. Here we confine ourselves to constructing $H(x)$ from a given reduced reflection coefficient $\rho(k)$, H_+, and bound state data. The problem where one starts from $R(k)$ requires some additional steps, which are outlined in [8]. In all three examples it is assumed that $Q \in L^1_2(\mathbf{R})$, so that Theorem 3.2 applies, and we are allowed to consider the singular integral equations (3.12) and (3.20) in the space $\mathbf{H}^2_-(\mathbf{R})$.

Example 1 Suppose that

$$\rho(k) = \rho_0 \, e^{i\beta k}, \qquad \rho_0, \beta \in \mathbf{R}, \qquad |\rho_0| < 1.$$

Since $\rho(0) = \rho_0 \neq -1$, we are in the exceptional case. We also assume that there are no bound states. It turns out that the spectrum of $\mathcal{O}_y$ consists of the three points $-\rho_0$, 0, and ρ_0, each of which is an eigenvalue of infinite multiplicity. So, $\mathcal{O}_y$ is bounded and self-adjoint, but not compact. The function $H(x)$ is given by

$$H(x) = \begin{cases} \dfrac{H_+}{f_l(0,x)^2}, & x > x_1, \\ \dfrac{1-\rho_0}{1+\rho_0}\,\dfrac{H_+}{f_l(0,x)^2}, & x < x_1, \end{cases}$$

where x_1 is such that $y(x_1) = -\beta/2$. A more explicit equation for determining x_1 is given in [8]. It can also be verified that $H(x)$ satisfies the conditions (H1)-(H5).

Example 2 Suppose that there are no bound states and

$$\rho(k) = \frac{\mu + i\xi k}{-\mu + ik}\, e^{ik\beta}, \quad -1 < \xi < 1, \quad \beta \in \mathbf{R}, \quad \mu > 0.$$

Since $\rho(0) = -1$, we are in the generic case. In this case the spectrum of $\mathcal{O}_y$ consists of the eigenvalue zero and two infinite sequences of eigenvalues that converge to $+\xi$ and $-\xi$, respectively. For $y < -\beta/2$, by solving (3.14) we find

$$y(x) = \frac{2H_+(\xi - 1 + \mu\beta)\,\varphi(x) - 2\beta\,\xi - \mu\beta^2}{2[1 + \xi + \mu\beta - 2\mu H_+\,\varphi(x)]}, \tag{4.1}$$

where $\varphi(x) = \int_x^0 dz\, f_l(0,z)^{-2}$. Now the denominator in (4.1) must be nonzero and $y(x)$ must behave linearly as $x \to -\infty$. Since $f_l(0,x)$ behaves linearly as $x \to -\infty$, $\varphi(-\infty) = \lim_{x\to-\infty} \varphi(x)$ is finite. Hence, in order for $y(x)$ to be unbounded as $x \to -\infty$, it is necessary and sufficient that $1 + \xi + \mu\beta - 2\mu H_+\varphi(-\infty) = 0$. This says that H_+ is determined by $\rho(k)$ and $Q(x)$, namely

$$H_+ = \frac{1 + \xi + \mu\beta}{2\mu\,\varphi(-\infty)}.$$

The function $H(x)$ is given by

$$H(x) = \begin{cases} \dfrac{H_+}{f_l(0,x)^2}, & x > x_1, \\ \dfrac{1-\xi^2}{4\mu^2 H_+} \dfrac{[f_l(0,x); f_r(0,x)]^2}{f_r(0,x)^2}, & x < x_1, \end{cases}$$

and x_1 is determined such that $y(x_1) = -\beta/2$.

Example 3 Suppose that

$$\rho(k) = \rho_0 \frac{k+i\kappa}{k-i\kappa} e^{ik\beta}, \qquad \rho_0, \beta \in \mathbf{R}, \quad |\rho_0| < 1, \quad \kappa > 0.$$

We also assume that there is a bound state at $-\kappa^2$ with norming constant ν. Evaluating the integral in (3.21) we have

$$B(k,x,y) = \begin{cases} \dfrac{p_0(x,y)}{k-i\kappa}, & 2y+\beta \geq 0, \\ \dfrac{p_0(x,y)}{k-i\kappa} - \rho_0 \dfrac{k+i\kappa}{k-i\kappa} \left[e^{ik(2y+\beta)} - 1\right] \dfrac{1}{k} \dfrac{f_l(0,x)}{\sqrt{H_+}}, & 2y+\beta < 0. \end{cases}$$

Using the Fourier transform $(\mathcal{F}g)(t) = \int_{-\infty}^{\infty} dk\, e^{itk}\, g(k)$, we obtain

$$(\mathcal{F}B(\cdot,x,y))(t,x,y) = 2\pi i p_0(x,y) e^{-\kappa t}, \qquad 2y+\beta \geq 0,$$

and when $\beta + 2y < 0$,

$$(\mathcal{F}B(\cdot,x,y))(t,x,y) = \begin{cases} 2\pi i p_0(x,y)e^{-\kappa t} + D_1, & 0 < t < -(2y+\beta), \\ 2\pi i p_0(x,y)e^{-\kappa t} + D_2, & t > -(2y+\beta), \end{cases}$$

where

$$D_1 = 4\pi i \rho_0 e^{-\kappa t} \frac{f_l(0,x)}{\sqrt{H_+}} - 2\pi i \rho_0 \frac{f_l(0,x)}{\sqrt{H_+}},$$

$$D_2 = 4\pi i \rho_0 e^{-\kappa t} (1 - e^{-\kappa(2y+\beta)}) \frac{f_l(0,x)}{\sqrt{H_+}}.$$

Let $h(t) = (\mathcal{F}G_-(\cdot,x,y))(t,x,y) = (\mathcal{F}X(\cdot))(t)$. For the operator $\mathcal{L}_y$ in (3.18) we have that $\mathcal{L}_y = 0$, when $2y+\beta \geq 0$, and when $2y+\beta < 0$

$$(\mathcal{F}\mathcal{L}_y\mathcal{F}^{-1}h)(t) = \begin{cases} -2\rho_0\kappa e^{-\kappa t}\, e^{-\kappa(2y+\beta)} \displaystyle\int_{-2y-\beta-t}^{-2y-\beta} du\, e^{-\kappa u} h(u) \\ \qquad + \rho_0\, h(-t-2y-\beta), \quad 0 < t < -(2y+\beta), \\ -2\rho_0\kappa e^{-\kappa t}\, e^{-\kappa(2y+\beta)} \displaystyle\int_0^{-2y-\beta} du\, e^{-\kappa u} h(u), \quad t > -(2y+\beta). \end{cases}$$

Then $h(t)$ obeys the equations

$$h(t) = 2\pi i p_0(x,y)\, e^{-\kappa t}, \qquad 2y+\beta \geq 0, \tag{4.2}$$

and when $2y+\beta<0$

$$\begin{aligned} h(t) =& 2\pi i p_0(x,y)e^{-\kappa t} + 4\pi i\rho_0 e^{-\kappa t}(1-e^{-\kappa(2y+\beta)})\frac{f_l(0,x)}{\sqrt{H_+}} \\ &- 2\rho_0\kappa e^{-\kappa t}e^{-\kappa(2y+\beta)}\int_0^{-2y-\beta} du\, e^{-\kappa u}\, h(u) \end{aligned} \tag{4.3}$$

if $t>-(2y+\beta)$, and

$$\begin{aligned} h(t) =& 2\pi i p_0(x,y)e^{-\kappa t} + 4\pi i\rho_0 e^{-\kappa t}\frac{f_l(0,x)}{\sqrt{H_+}} - 2\pi i\rho_0\frac{f_l(0,x)}{\sqrt{H_+}} \\ &- 2\rho_0\kappa e^{-\kappa t}e^{-\kappa(2y+\beta)}\int_{-2y-\beta-t}^{-2y-\beta} du\, e^{-\kappa u}\, h(u) + \rho_0 h(-t-2y-\beta) \end{aligned} \tag{4.4}$$

if $0<t<-(2y+\beta)$. We first solve (4.4) and then use the result in (4.3). Following [8] (Example 6.2) we can solve (4.4) by converting it to a second-order differential equation for the function $\int_0^t du\, e^{-\kappa u}\, h(u)$. The solution is

$$h(t) = -2\pi i\frac{\rho_0}{1+\rho_0}\frac{f_l(0,x)}{\sqrt{H_+}} + \frac{1}{1-\rho_0 e^{\kappa(2y+\beta)}}\left[2\pi i p_0(x,y) + \frac{4\pi i\rho_0}{\rho_0+1}\frac{f_l(0,x)}{\sqrt{H_+}}\right]e^{-\kappa t}. \tag{4.5}$$

From (4.2)-(4.5), evaluating the inverse Fourier transform of $h(t)$, we get

$$G_-(k,x,y) = \frac{p_0(x,y)}{k-i\kappa}, \qquad 2y+\beta\geq 0, \tag{4.6}$$

$$\begin{aligned} G_-(k,x,y) =& \frac{p_0(x,y)}{k-i\kappa}\frac{1-\rho_0 e^{ik(2y+\beta)}}{1-\rho_0 e^{\kappa(2y+\beta)}} - \frac{\rho_0}{1+\rho_0}\frac{k+i\kappa}{k-i\kappa}\frac{e^{ik(2y+\beta)}-1}{k}\frac{f_l(0,x)}{\sqrt{H_+}} \\ &+ \frac{2\rho_0^2}{(1+\rho_0)(1-\rho_0 e^{\kappa(2y+\beta)})}\frac{e^{\kappa(2y+\beta)}-e^{ik(2y+\beta)}}{k-i\kappa}\frac{f_l(0,x)}{\sqrt{H_+}}, \qquad 2y+\beta<0. \end{aligned} \tag{4.7}$$

Inserting (4.6) and (4.7) in (3.22), projecting $\Omega(k,x,y)$ onto $H^2_-(\mathbf{R})$ and putting $k=i\kappa$, after tedious computations we obtain

$$\begin{aligned} \Omega_+(i\kappa,x,y) =& -i\,\frac{\rho_0 p_0(x,y)}{2\kappa}\,e^{-\kappa(2y+\beta)} \\ &- i\,\frac{\rho_0}{\kappa}(e^{-\kappa(2y+\beta)}-1)\frac{f_l(0,x)}{\sqrt{H_+}} - \frac{i}{\kappa}\frac{f_l(0,x)}{\sqrt{H_+}}, \qquad 2y+\beta\geq 0, \end{aligned} \tag{4.8}$$

$$\begin{aligned} \Omega_+(i\kappa,x,y) =& -i\,\frac{\rho_0 p_0(x,y)}{2\kappa}\,\frac{e^{\kappa(2y+\beta)}-\rho_0}{1-\rho_0 e^{\kappa(2y+\beta)}} \\ &- i\frac{(1-\rho_0)(1+\rho_0-\rho_0 e^{\kappa(2y+\beta)})}{\kappa(1-\rho_0 e^{\kappa(2y+\beta)})}\,\frac{f_l(0,x)}{\sqrt{H_+}}, \qquad 2y+\beta<0. \end{aligned} \tag{4.9}$$

Using (3.23), (4.8), (4.9), and $1-\rho_0 = 1+\rho(0) = \dfrac{\tau(0)\sqrt{H_+}}{\sqrt{H_-}\gamma}$ we get

$$p_0(x,y) = \begin{cases} \dfrac{2[d\,e^{-2\kappa(y+A_+)} - \rho_0 e^{-\kappa(2y+\beta)}]}{\rho_0 e^{-\kappa(2y+\beta)} + 1 - d\,e^{-2\kappa(y+A_+)}}\, \dfrac{f_l(0,x)}{\sqrt{H_+}}, & 2y+\beta \geq 0, \\ -2\dfrac{U_1}{U_2}\, \dfrac{f_l(0,x)}{\sqrt{H_+}}, & 2y+\beta < 0, \end{cases}$$

where $d = \dfrac{\nu H_+}{2\kappa}$ and

$$U_1 = \frac{(1-\rho_0)\rho_0}{1-\rho_0 e^{\kappa(2y+\beta)}} - \frac{\nu H_+}{2\kappa} e^{-2\kappa(y+A_+)},$$

$$U_2 = \frac{1-\rho_0^2}{1-\rho_0 e^{\kappa(2y+\beta)}} - \frac{\nu H_+}{2\kappa} e^{-2\kappa(y+A_+)}.$$

Note that $\gamma < 0$, since there is one bound state [8]. To solve (3.14) we let $z = e^{\kappa(2y+\beta)}$ and $a = e^{\kappa(2A_+ - \beta)}$. Then (3.14) can be written in the form

$$\frac{2[d-\rho_0 a]}{\kappa[az+\rho_0 a - d]} - \frac{1}{2\kappa}\ln z = i\,\frac{\dot f_l(0,x)}{f_l(0,x)} - \frac{\beta}{2} + A_+, \qquad 2y+\beta \geq 0, \tag{4.10}$$

$$\begin{aligned} \frac{-2\rho_0}{(1+\rho_0)\kappa} + \frac{2d}{\kappa}\frac{1-\rho_0}{1+\rho_0}\, \frac{1}{z[a(1-\rho_0^2)+d\rho_0]-d} - \frac{1-\rho_0}{2(1+\rho_0)\kappa}\ln z \\ = i\,\frac{\dot f_l(0,x)}{f_l(0,x)} - \frac{\beta}{2} + A_+, \qquad 2y+\beta < 0. \end{aligned} \tag{4.11}$$

So, in terms of the variable z, (4.10) holds when $z \geq 1$ and (4.11) holds when $0 < z < 1$. Differentiating (4.10) and (4.11), using $dz/dx = 2\kappa H(x) z$ and solving for $H(x)$ we get

$$H(x) = \begin{cases} \dfrac{H_+}{f_l(0,x)^2}\left(\dfrac{az+\rho_0 a - d}{az - \rho_0 a + d}\right)^2, & 2y+\beta \geq 0, \\ \dfrac{1-\rho_0}{1+\rho_0}\,\dfrac{H_+}{f_l(0,x)^2}\left(\dfrac{z[a(1-\rho_0^2)+d\rho_0]-d}{z[a(1-\rho_0^2)+d\rho_0]+d}\right)^2, & 2y+\beta < 0. \end{cases} \tag{4.12}$$

Note that $f_l(0,x)$ has one zero. It can be seen from (4.10) and (4.11) that this zero is canceled by the zero of the numerator in (4.12).

The jump in $H(x)$ occurs at x_1, where $y = -\beta/2$, i.e. $z = 1$. Then, by (4.12) we have

$$\frac{H(x_1 - 0)}{H(x_1 + 0)} = \frac{1-\rho_0}{1+\rho_0}.$$

When $\rho_0 = 0$ we see that $H(x)$ is continuous. This special case is worked out in [8] (Example 8.1).

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