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83

Simple Characteristics of Scattering Matrices

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Abstract

An overview is given of simple properties of the scattering matrix of an arbitrary particle and of a general assembly of such particles. Matrices that are linearly related to the scattering matrix, such as the coherency matrix, are also considered. Tests are provided to establish whether a particular calculated or measured real 4 x 4 matrix can be a scattering matrix.

1 Introduction

The scattering of radiation by a particle may be described by means of a 4 x 4 scattering matrix **F** satisfying

$$\mathbf{I}_2 = \mathbf{F}\,\mathbf{I}_1. \tag{1}$$

Here $\mathbf{I}_2$ and $\mathbf{I}_1$ are the Stokes vectors of the scattered and incident wave, respectively. A Stokes vector is a column vector whose elements are the Stokes parameters I, Q, U and V [1,2]. Equation (1) also holds for scattering by an assembly of independently scattering particles, but in this case the scattering matrix of the assembly is the sum of the scattering matrices of the individual particles. A large number of relationships (scalar as well as matrix equalities and inequalities) have been reported for the elements of both types of scattering matrices [3-8].

Many of the relationships reported are rather complicated. Moreover, it is often not clear whether a particular collection of relationships is equivalent to another one or not. The first purpose of this paper is to present and discuss an overview of simple properties of the scattering matrix of an arbitrary particle and of an assembly of such particles. We will mainly consider properties that hold for arbitrary directions of the incident and scattered beams. As an application we will give examples of other relationships that can easily be derived from our simple ones. The second purpose of this paper is to provide tests for investigating whether a particular calculated or measured real 4 x 4 matrix can be a scattering matrix. These tests are important because in practice many types of errors may play a role, both in experiments and in computations.

2 Main results

2.1

Suppose F_{ij}, with i, j = 1, 2, 3, 4, is the element on the i-th row and j-th column of the scattering matrix **F** of an arbitrary particle. We then have the following simple properties.

A) All sums of the rows and columns of the matrix

$$\begin{bmatrix} F_{11}^2 & -F_{12}^2 & -F_{13}^2 & -F_{14}^2 \\ -F_{21}^2 & F_{22}^2 & F_{23}^2 & F_{24}^2 \\ -F_{31}^2 & F_{32}^2 & F_{33}^2 & F_{34}^2 \\ -F_{41}^2 & F_{42}^2 & F_{43}^2 & F_{44}^2 \end{bmatrix}$$

are the same and equal d^2, where d is the absolute value of the determinant of the amplitude matrix from which **F** is derived. By eliminating d^2 we thus find seven equations for the squares of the elements of **F**.

B) Thirty relations involving products of different elements of **F**. A quick overview of these relations may be obtained by means of a graphical code. Let a 4 x 4 array of dots in a pictogram represent the elements of **F**, a solid curve or line connecting two elements a positive product and a dashed curve or line a negative product. Let us further adopt the convention that all positive and negative products in a pictogram have to be added to get zero. The result is shown in Figs. 1 and 2. For example, the pictogram in the upper left corner of Fig.1 means

$$F_{11}F_{12} - F_{21}F_{22} - F_{31}F_{32} - F_{41}F_{42} = 0 \tag{2}$$

and the pictogram in the upper left corner of Fig. 2 stands for

$$F_{11}F_{22} - F_{12}F_{21} - F_{33}F_{44} + F_{34}F_{43} = 0. \tag{3}$$

Together all 120 possible products of two distinct elements appear in the thirty relations and each such product occurs only once. The thirty relations subdivide into the following two types. The 12 equations shown in Fig. 1 carry corresponding products of any two chosen rows and columns. The 18 equations shown in

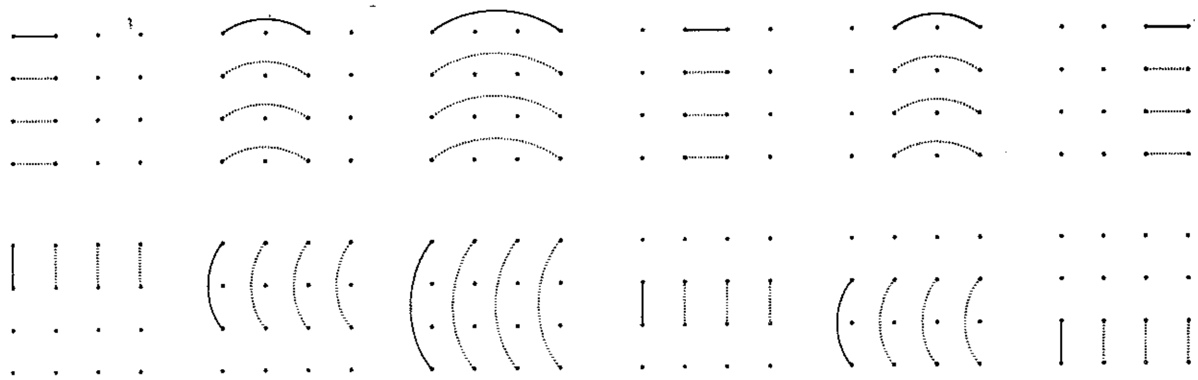

Fig. 1. The 16 dots in each pictogram represent the elements of the scattering matrix of an arbitrary particle. A solid line or curve connecting two elements stands for a positive product and a dashed line or curve for a negative product. In each pictogram the sum of all positive and negative products vanishes.

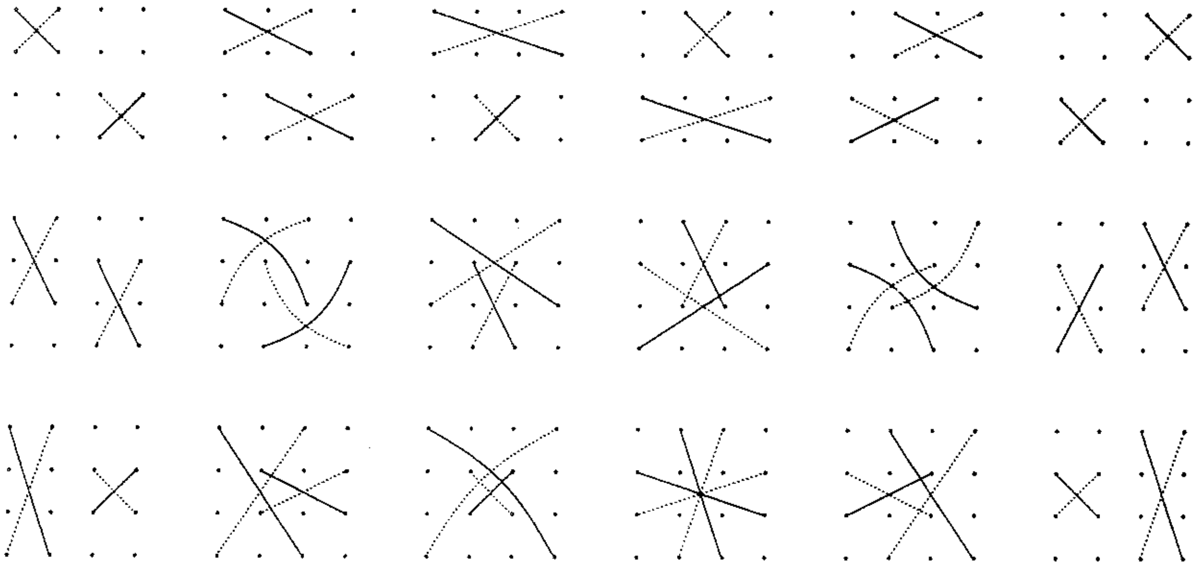

Fig. 2. As Fig. 1

Fig. 2 express that the sum or difference of any chosen pair of complementary subdeterminants vanishes. Here the word "complementary" refers to the remaining rows and columns. Sums and differences of subdeterminants alternate in each column and row of the logical arrangement of pictograms shown in Fig. 2. Keeping the signs in mind for the first pictograms in Figs. 1 and 2 one should have no trouble reproducing all pictograms, and thus all 30 equations, by heart. Figs. 1 and 2 are based on Fig. 2 of Hovenier et al [5] but they present a clearer and more systematic overview.

C)

$$F_{11} \geq | F_{ij} | \qquad (4)$$

which implies that F_{11} is non-negative.

The simple properties mentioned above can be used to derive a lot of corollaries. To clarify this point we give the following examples.

(i) Figs. 1 and 2 show that sums and differences of the elements in the first and second column obey the two equations

$$(F_{11} \pm F_{12})^2 - (F_{21} \pm F_{22})^2 - (F_{31} \pm F_{32})^2 - (F_{41} \pm F_{42})^2 = 0 \qquad (5)$$

and similarly for all six combinations involving the first row or column. However, the sums and differences of the elements in the third and fourth column obey

$$(F_{13} \pm F_{14})^2 - (F_{23} \pm F_{24})^2 - (F_{33} \pm F_{34})^2 - (F_{43} \pm F_{44})^2 = -2d^2 \qquad (6)$$

and similarly for all 6 combinations of rows or columns.

(ii) The well-known relation

$$\sum_{i=1}^{4} \sum_{j=1}^{4} F_{ij}^2 = 4\, F_{11}^2 \qquad (7)$$

is easily obtained from the simple property A) considered above.

(iii) As shown by Barakat [6] and Simon [7] we have the matrix equation

$$\tilde{F}\, G\, F = d^2\, G \qquad (8)$$

where a tilde stands for the matrix transpose and

$$G = \mathrm{diag}\,(1, -1, -1, -1). \qquad (9)$$

By writing out Eq. (8) for each element of G we readily verify that Eq. (8) follows from properties A) and B) whereby the relations for the non-diagonal elements correspond to the top six pictograms in Fig. 1.

2.2

We now consider a scattering matrix **F** of an arbitrary assembly of arbitrary particles that scatter independently. We then have the following properties:

A) Six inequalities [4,5] which may be written in the form

$$(F_{11} \pm F_{22})^2 \geq (F_{12} \pm F_{21})^2 + (F_{33} \pm F_{44})^2 + (F_{34} \mp F_{43})^2 \qquad (10)$$

$$(F_{11} \pm F_{12})^2 \geq (F_{21} \pm F_{22})^2 + (F_{31} \pm F_{32})^2 + (F_{41} \pm F_{42})^2 \qquad (11)$$

$$(F_{11} \pm F_{21})^2 \geq (F_{12} \pm F_{22})^2 + (F_{13} \pm F_{23})^2 + (F_{14} \pm F_{24})^2. \qquad (12)$$

It is important to note that Eq. (10) does not follow from the requirement that the degree of polarization of the scattered beam cannot exceed 100% [9]. Although this requirement must be met by any physically meaningful matrix that transforms Stokes parameters, additional constraints apparently exist for scattering matrices.

B) As in the case of scattering by a single particle we have

$$F_{11} \geq | F_{ij} |. \qquad (13)$$

The simple properties A) and B) can be used to derive many corollaries, such as

(i)

$$\sum_{i=1}^{4} \sum_{j=1}^{4} F_{ij}^2 \leq 4\, F_{11}^2 \qquad (14)$$

(ii)

$$| F_{33} + F_{44} | + | F_{34} - F_{43} | \leq (F_{11} + F_{22})\sqrt{2} \qquad (15)$$

(iii)

$$| F_{33} + F_{44} | + | F_{34} - F_{43} | + | F_{12} + F_{21} | \leq (F_{11} + F_{22})\sqrt{3} \qquad (16)$$

(iv)

$$\sum_{i=1}^{4} \sum_{j=1}^{4} |F_{ij}| \leq F_{11}\,(1 + \sqrt{45}) < 8\, F_{11}. \qquad (17)$$

2.3

Let **E** be a real 4 x 4 matrix whose elements may be numbers or functions of one or more variables. Evidently, if the elements of **E** do not obey one or more of the simple properties or corollaries of Section 2.1 then **E** cannot be a scattering matrix of a single particle. Similarly **E** cannot be a scattering matrix of an assembly of independently scattering particles if the elements of **E** violate at least one of the simple properties or corollaries given in Section 2.2. In particular, Eq. (4) provides a simple eyeball test, especially for a matrix consisting of numbers, to judge whether such a matrix can be a scattering matrix [See also Eq. (13)].

As an example of a matrix of functions we consider

$$\mathbf{E} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & q\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & \cancel{0}\,1 \end{pmatrix} \qquad (18)$$

where θ is an angle and q is a real number. Clearly, for arbitrary θ, property A) of Section 2.1 is only fulfilled if $| q | = 1$, whereas property B) of that section requires $q = -1$, as may be seen from the pictograms in Fig. 2. Similarly, for an arbitrary value of θ, Eqs. (10) and (11) are fulfilled for arbitrary values of q, but Eq. (12) only if $| q | \leq 1$.

2.4

A general procedure to exclude a real 4 x 4 matrix **E** from the set of all possible scattering matrices may be obtained from an analogous problem in the field of radar polarimetry. Following Cloude [10,11] we may transform a scattering matrix **F** into a 4 x 4 coherency matrix **T** by the linear transformation

$$\left.\begin{aligned} T_{11} &= \tfrac{1}{4}(F_{11} + F_{22} + F_{33} + F_{44}) \\ T_{22} &= \tfrac{1}{4}(F_{11} + F_{22} - F_{33} - F_{44}) \\ T_{33} &= \tfrac{1}{4}(F_{11} - F_{22} + F_{33} - F_{44}) \\ T_{44} &= \tfrac{1}{4}(F_{11} - F_{22} - F_{33} + F_{44}) \end{aligned}\right\} \qquad (19)$$

$$\left.\begin{aligned} T_{14} &= \tfrac{1}{4}(F_{14} - iF_{23} + iF_{32} + F_{41}) \\ T_{23} &= \tfrac{1}{4}(iF_{14} + F_{23} + F_{32} - iF_{41}) \\ T_{32} &= \tfrac{1}{4}(-iF_{14} + F_{23} + F_{32} + iF_{41}) \\ T_{41} &= \tfrac{1}{4}(F_{14} + iF_{23} - iF_{32} + F_{41}) \end{aligned}\right\} \quad (20)$$

$$\left.\begin{aligned} T_{12} &= \tfrac{1}{4}(F_{12} + F_{21} - iF_{34} + iF_{43}) \\ T_{21} &= \tfrac{1}{4}(F_{12} + F_{21} + iF_{34} - iF_{43}) \\ T_{34} &= \tfrac{1}{4}(iF_{12} - iF_{21} + F_{34} + F_{43}) \\ T_{43} &= \tfrac{1}{4}(-iF_{12} + iF_{21} + F_{34} + F_{43}) \end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned} T_{13} &= \tfrac{1}{4}(F_{13} + F_{31} + iF_{24} - iF_{42}) \\ T_{31} &= \tfrac{1}{4}(F_{13} + F_{31} - iF_{24} + iF_{42}) \\ T_{24} &= \tfrac{1}{4}(-iF_{13} + iF_{31} + F_{24} + F_{42}) \\ T_{42} &= \tfrac{1}{4}(iF_{13} - iF_{31} + F_{24} + F_{42})\,. \end{aligned}\right\} \quad (22)$$

Note that the coherency matrix may have complex elements, but is always Hermitian, so that its eigenvalues are real. Using fairly elementary mathematics we may show that for scattering by particles all eigenvalues of the coherency matrix are non-negative and that only one non-zero eigenvalue exists for scattering by a single particle. The same results may be obtained by using group theory [10,11]. Thus we can exclude a real 4x4 matrix **E** from the set of all possible scattering matrices if its coherency matrix has a negative eigenvalue. Further, **E** cannot be a scattering matrix of a single particle if its coherency matrix has more than one non-zero eigenvalue.

2.5

Simon [7,8] used a matrix **N** which is easily seen to be unitarily equivalent to twice the coherency matrix, i.e.

$$\mathbf{N} = 2\,\mathbf{U}\,\mathbf{T}\,\mathbf{U}^{-1} \qquad (23)$$

where **U** and its inverse $\mathbf{U}^{-1}$ are the unitary matrices given by

$$\mathbf{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 0 & 0 & 1 & i \\ 1 & -1 & 0 & 0 \end{bmatrix}, \quad \mathbf{U}^{-1} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \end{bmatrix}. \qquad (24)$$

The eigenvalues of **N** are twice the eigenvalues of **T** and

$$\mathrm{Tr}\,\mathbf{N} = 2\,\mathrm{Tr}\,\mathbf{T} = 2\,F_{11} \qquad (25)$$

where Tr denotes the trace of a matrix.

Clearly the eigenvalues of **N** can be used instead of those of **T** when testing a given real 4 x 4 matrix **E**. It also follows from the work of Simon [7,8] that **E** cannot be a scattering matrix of a single particle unless its **N**-matrix satifies

$$\mathbf{N}^2 = (\mathrm{Tr}\,\mathbf{N})\,\mathbf{N} \text{ and } \mathrm{Tr}\,\mathbf{N} \geq 0. \qquad (26)$$

3 Conclusions

The scattering matrix of an arbitrary particle in an arbitrary orientation with respect to the incident and scattered beams has many simple characteristics. This is also true for the scattering matrix of an arbitrary assembly of particles. These simple characteristics are the sources of many other properties of scattering matrices. Apart from their intrinsic value for the theory of light scattering each property can be used to unmask a given matrix as not being a proper scattering matrix for either one or more particles, depending on the property considered. This goal can also be achieved by analyzing the number and signs of the eigenvalues of the coherency matrix or Simons **N** matrix. Thus many errors can be detected in scattering matrices that are computed or experimentally determined.

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