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MATHEMATICAL STUDY OF THE RUNAWAY PROCESS IN PARTICLE SWARMS

GIOVANNI FROSALI - CORNELIS VAN DER MEE

Abstract

In this article we study two linear Boltzmann equations which describe the time evolution of the electron distribution in a weakly ionized gas, both of them one-dimensional in the velocity and any spatial variable, and one of them spatially homogeneous. We present two mathematical definitions of electron runaway, one based on travelling wave phenomena and the other one involving the average speed asymptotics. Under suitable assumptions on the collision frequency, we prove electron runaway according to the average speed definition in the spatially homogeneous as well as the spatially dependent case. For constant collision frequency ν_0, the average speed is shown to relax to a/ν_0 where a is the electrostatic acceleration.

1. Introduction.

In this paper we study the linear integro-differential equation which describes the time evolution of the space-velocity electron distribution $f(x, v, t)$ in a weakly inonized host medium. This

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equation reads

$$(1.1)\qquad \begin{aligned} &\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} + a\frac{\partial f}{\partial v} + \nu(x,v)f(x,v,t) = \\ &= \int_{-\infty}^{+\infty} k(x,v,v')\nu(x,v')f(x,v',t)dv' \end{aligned}$$

and is studied under the initial condition

$$(1.2)\qquad f(x,v,0) = f_0(x,v).$$

In this one dimensional model the electrostatic acceleration is assumed to be uniform in time and position, i.e. a is constant and positive. If e and m are the electron charge and mass and E is the electric field, we have $a = |e|E/m$. The collision frequency $\nu(x,v)$ and the scattering kernel $k(x,v,v')$ are independent of the temporal variable t and satisfy the following assumptions:

ASSUMPTION 1. *There exists a measurable even function $\bar{\nu} = \bar{\nu}(v)$ in $L_{1,\mathrm{loc}}(\mathbb{R}, dv)$ such that the collision frequency $\nu(x,v)$ satisfies*

$$0 < \nu(x,v) \le \bar{\nu}(v) \quad \textit{for a.e.} \quad (x,v) \in \mathbb{R}^2.$$

ASSUMPTION 2. *The collision kernel $k(x,v,v')$ appearing in the integral operator is nonnegative and satisfies the normalization condition*

$$\int_{-\infty}^{+\infty} k(x,v,v')dv = 1, \quad (x,v') \in \mathbb{R}^2 \quad \textit{a.e.}$$

and the reciprocity condition $k(x,-v,-v') = k(x,v,v')$, $(x,v,v') \in \mathbb{R}^3$ a.e. The normalization condition expresses the balance between ionization and recombination effects.

Physical considerations suggest that, as the solution $f(x,v,t)$ and its initial condition $f_0(x,v)$ represent electron densities, they must be nonnegative and have a finite integral in position-velocity space which stands for the total number of electrons at time t and at time $t = 0$, respectively. Thus it is natural from the physical point of view to introduce $L_1(\mathbb{R}^2, dxdv)$ endowed with the norm

$$||f||_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(x,v)|dxdv$$

as the functional setting where the problem (1.1)-(1.2) is to be studied. On the other hand, it is sometimes convenient to have a measure for the total number of collisions at a certain moment in time. This measure is given by the integral of the product of collision frequency and electron density in position-velocity space. Hence, let us also introduce the Banach space $L_1(\mathbb{R}^2, \nu dxdv)$ with the norm

$$||f||_\nu = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \nu(x,v)|f(x,v)|dxdv$$

as a tool to measure the total number of collisions.

Let us denote by T_0 the operator on $L_1(\mathbb{R}^2, dxdv)$ defined by

$$\begin{cases} D(T_0) = \{f \in L_1(\mathbb{R}^2, dxdv) : v\dfrac{\partial f}{\partial x} + a\dfrac{\partial f}{\partial v} \in L_1(\mathbb{R}^2, dxdv)\} \\ T_0 f = -v\dfrac{\partial f}{\partial x} - a\dfrac{\partial f}{\partial v} \end{cases}$$

with the partial derivatives being distributional, and by A and K the operators with domain $L_1(\mathbb{R}^2, dxdv) \cap L_1(\mathbb{R}^2, \nu dxdv)$ defined by

$$(Af)(x,v) = -\nu(x,v)f(x,v),$$

$$(Kf)(x,v) = \int_{-\infty}^{+\infty} k(x,v,v')\nu(x,v')f(x,v')dv'.$$

Then $-A$ and K extend to bounded linear operators from the space $L_1(\mathbb{R}^2, \nu dxdv)$ into $L_1(\mathbb{R}^2, dxdv)$ which are positive and satisfy

$$||-Af||_1 = ||Kf||_1 = ||f||_\nu, \text{ for all } f \in L_1(\mathbb{R}^2, \nu dxdv) \text{ and } f \geq 0.$$

Using the preceding definitions, problem (1.1)-(1.2) can be put in the abstract form

$$\frac{df}{dt} = T_0 f(t) + Af(t) + Kf(t), \quad t > 0, \tag{1.3}$$

$$\lim_{t \to 0^+} f(t) = f_0, \tag{1.4}$$

where d/dt is a strong derivative on $L_1(\mathbb{R}^2, dxdv)$, $f(t) = f(\cdot, t)$ is a function from $\mathbb{R}^+$ into $L_1(\mathbb{R}^2, dxdv)$, the limit as $t \to 0^+$ is taken in the strong sense and f_0 is the initial datum. It is known [1] that $T_0 + A + K$ generates a strongly continuous, positive contraction semigroup $\{S(t)\}_{t \geq 0}$ on $L_1(\mathbb{R}^2, dxdv)$ which, under very mild extra hypotheses, satisfies

$$||S(t)g||_1 = ||g||_1, \; g \geq 0 \text{ in } L_1(\mathbb{R}^2, dxdv). \tag{1.5}$$

Hence, $f(t) = S(t)f_0$ is the unique solution of problem (1.3)-(1.4) and, as (1.5) indicates, the total number of electrons with distribution fuction $f(x, v, t)$ is preserved in time.

When the spatial variable x is dropped and $\nu = \nu(v)$ and $k = k(v, v')$ no longer depend on X, the Banach spaces $L_1(\mathbb{R}^2, dxdv)$ and $L_1(\mathbb{R}^2, \nu dxdv)$ must be replaced by $L_1(\mathbb{R}, dv)$ and $L_1(\mathbb{R}, \nu dv)$, respectively. Modifying the definitions of T_0, A and K in the obvious way, we arrive at problem (1.3)-(1.4) in the setting of $L_1(\mathbb{R}, dv)$. As shown in [2], $T_0 + A + K$ generates a strongly continuous, positive contraction semigroup $\{S(t)\}_{t \geq 0}$ on $L_1(\mathbb{R}, dv)$ which, under very mild extra hypotheses, satisfies (1.5). Moreover, in the spatially independent case usually one of the following two situations occurs: 1. The collision frequency $\nu(v)$ satisfies $\int_{-\infty}^{+\infty} \nu(v)dv = +\infty$ and there is a stationary solution $f_\infty \in L_1(\mathbb{R}, dv)$ such that the solution $f(t)$ satisfies

$$\lim_{t \to +\infty} ||f(t) - f_\infty||_1 = 0.$$

This situation is characterized as relaxation to equilibrium. It occurs, for instance [2], if $\nu(v)$ is bounded but satisfies $\int_{-\infty}^{+\infty} \nu(v)dv = +\infty$ and K is a weakly compact operator from $L_1(\mathbb{R}, \nu dv)$ into $L_1(\mathbb{R}, dv)$.

2. The collision frequency $\nu(v)$ satisfies $\int_{-\infty}^{+\infty} \nu(v)dv < +\infty$. If we write $[W_0(t)g](v) = g(v - at)$, the strong limit $\Omega^- g = \lim_{t\to+\infty} W_0(-t)S(t)g$ exists in $L_1(\mathbb{R}, dv)$ and the solution $f(t)$ of problem (1.3)-(1.4) satisfies

$$\lim_{t\to+\infty} \int_{-\infty}^{+\infty} |f(v + at, t) - [\Omega^- f_0](v)|dv = 0.$$

In this situation, as shown in [3], the solution behaves asymptotically as a travelling wave in velocity space with «velocity» a. In this case there always is a stationary solution f_∞ in $L_1(\mathbb{R}, \nu dv)$, but this function does not belong to $L_1(\mathbb{R}, dv)$ and does not occur as a limit of $f(t)$ as $t \to +\infty$.

In the sequel we will often deal with the so-called BGK model where BGK stands for Bhatnagar-Gross- Krook. In this simplified model, the collision term has the special form

$$\int_{-\infty}^{+\infty} k(v, v')\nu(v')f(v', t)dv' = \frac{\nu(v)f_m(v)}{\int_{-\infty}^{+\infty} \nu(v')f_m(v')dv'}$$

$$\int_{-\infty}^{+\infty} \nu(v')f(v', t)dv'$$

where $f_m(v)$ is a Maxwellian distribution or some other nonnegative fucntion in $L_1(\mathbb{R}, \nu dv)$. In this case the assumption $\int_{-\infty}^{+\infty} \nu(v)dv = +\infty$ always leads to the existence of a stationary solution in $L_1(\mathbb{R}, dv)$ and to relaxation to equilibrium, irrespective of other properties of ν.

In the physical literature of the linear Boltzmann equation (1.1) and it Fokker-Planck approximations, runaway of electrons is one of the major topics of interest. As early as 1949, Giovanelli [4] has argued in a heuristic manner that for an electric field exceeding a critical field there always is some fraction of electrons that run away. In 1972, for the spatially independent case, in the general linear Boltzmann context, Cavalleri and Paveri-Fontana [5] have pointed out $\int_{-\infty}^{+\infty} \nu(v)dv = +\infty$ as a necessary (but not necessarily sufficient) condition for suppression of runaways. Many heuristic results for a variety of electron drift model equations have been reported [6, 7]. A stricIty mathematical definition of runaway, however, was missing in these and similar studies.

For the spatially independent case, a useful characterization of electron runaway appeared to be the behaviour of the electron distribution $f(v,t)$ as a travelling wave of the form $g(v+at)$ for large time [2, 3]. Characterizations as above have been generalized to the three-dimensional (in $\vec{v}$), spatially independent case by Arlotti [8] and Poupaud [9, 10].

For the spatially dependent equation it is not clear how to define electron runaway in a mathematically meaningful way, since one might in principle have travelling waves in either position space or velocity space. Also, as well-known from the theory of the Korteweg-De Vries equation and related equations, the concept of travelling wave has defied a satisfactory generalization to higher dimensional equations. In the next section, we will discuss two ways to operationalize electron runaway as a viable concept of mathematics. One hinges on the definition for the corresponding spatially independent equation. The other one makes use of the average velocity for large time. In Section 4 and 5 we will work out the theory in more detail and specialize it to the BGK model, first in the spatially independent case and then with the spatial variable taken into account.

2. Characterization of runaway in spatially dependent systems.

As pointed out in the introduction, it is not obvious how to define the concept of electron runaway for the spatially dependent problem (1.1)-(1.2) where the collision frequency ν and the collision kernel k may depend on position x in addition to their dependence on the velocity variables. However, starting from the solution $f(x,v,t)$ and the initial datum $f_0(x,v)$ we introduce the velocity electron distributions

$$\mathcal{F}_0(v) = \int_{-\infty}^{+\infty} f_0(x,v)dx, \quad \mathcal{F}(v,t) = \int_{-\infty}^{+\infty} f(x,v,t)dx. \tag{2.1}$$

Since, as functions of (x,v), f_0 and $f(t)$ belong to $L_1(\mathbb{R}^2, dxdv)$ it is clear that, as functions of v, $\mathcal{F}_0$ and $\mathcal{F}(t)$ belong to $L_1(\mathbb{R}, dv)$. Via (2.1), we can then adopt the definition of runaway in the spatially independent case.

DÉFINITION 1. *The physical process modelled by problem (1.1)-(1.2) gives rise to runaway if there exists a functions* $\mathcal{F}_\infty = \mathcal{F}_\infty(v)$ *in* $L_1(\mathbb{R}, dv)$ *such that*

$$\lim_{t\to+\infty} \mathcal{F}(v+at,t) = \mathcal{F}_\infty(v)$$

in the limit of $L_1(\mathbb{R}, dv)$.

Using this definition, the theory of initial-value problem (1.3)-(1.4) in the spatially dependent case can largely be developed as in the spatially independent case. If we suppose Assumptions 1 and 2 stated in the introduction to hold, then we have again the following results [1]:

1. A necessary condition for the existence of a nontrivial nonnegative stationary solution to problem (1.1)-(1.2) in $L_1(\mathbb{R}^2, dxdv) \cap L_1(R^2, \nu dxdv)$ is that

$$\int_{-\infty}^{+\infty} \bar{\nu}(v)dv = +\infty, \tag{2.2}$$

for any $\bar{\nu}$ which satisfies Assumption 1.

2. If we write $[W_0(t)g](x,v) = g\left(x - vt + \frac{1}{2}at^2, v - at\right)$ and the collision frequency $\nu(x,v)$ satisfies the additional assumption

$$\int_{-\infty}^{+\infty} \bar{\nu}(v)dv = M < +\infty, \tag{2.3}$$

then the limits

$$\Omega^- = \lim_{t\to+\infty} W_0(-t)S(t) \text{ and } \Omega^+ = \lim_{t\to-\infty} S(-t)W_0(t) \tag{2.4}$$

exist in the strong operator topology of $L_1(\mathbb{R}^2, dxdv)$ and are bounded positive operators. Furthermore, problem (1.1)-(1.2) gives rise to runaway according to Definition 1.

It has not been shown so far that if a stationary solution $f_\infty(x,v)$ exists in $L_1(\mathbb{R}^2, dxdv)$, the solution $f(x,v,t)$ of problem (1.1)-(1.2) approaches $f_\infty(x,v)$ as $t \to +\infty$ in the sense that

$$\lim_{t\to+\infty} ||f(t) - f_\infty||_1 = 0. \tag{2.5}$$

However, if $\bar{\nu}(v)$ is bounded and K is a weakly compact operator from $L_1(\mathbb{R}^2, \nu dxdv)$ into $L_1(\mathbb{R}^2, dxdv)$, then the reasoning of [2] could be repeated to prove (2.5) and hence relaxation to equilibrium if a non-trivial stationary solution $f_\infty \in L_1(\mathbb{R}^2, dxdv)$ exists.

It might be instructive to give the proof of the second statement, i.e. the proof of the occurrence of runaway according to Definition 1. Indeed, let f_0 be the initial condition and let us define $\mathcal{F}_\infty(v) = \int_{-\infty}^{+\infty} [\Omega^- f_0](x,v)dx$. Let us denote by $f(x,v,t)$ the solution $S(t)f_0$ and by $\mathcal{F}(v,t)$ the corresponding velocity electron distribution $\int_{-\infty}^{+\infty} f(x,v,t)dx$. We find that

$$\int_{-\infty}^{+\infty} |\mathcal{F}(v+at,t)-\mathcal{F}_\infty(v)|dv \leq$$

$$\leq \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |f(x+vt+\frac{1}{2}at^2, v+at, t)-[\Omega^- f_0](x,v)|dxdv =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |[W_0(-t)S(t)f_0](x,v)-[\Omega^- f_0](x,v)|dxdv,$$

which proves the assertion.

Another physical interpretation of this result can be given in terms of travelling waves. In velocity space, by a travelling wave with «velocity» a, we mean a function of the form $g(v-at)$. Then the previous result can be read in the sense that $\mathcal{F}(v,t)$ behaves like a travelling wave, i.e.

$$\mathcal{F}(v,t)= \int_{-\infty}^{+\infty} [S(t)f_0](x,v)dx \simeq$$

$$\simeq \int_{-\infty}^{+\infty} [\Omega^- f_0](x,v-at)dx = \mathcal{F}_\infty(v-at),$$

as $t \to +\infty$.

A different, more hydrodynamic, characterization of runaway can be given by making use of the concept of average speed of the charge carriers. The particle current density divided by the particle number density gives the drift velocity

$$\langle v\rangle(t) = \frac{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dv v f(x,v,t)}{\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dv f(x,vt)}. \tag{2.6}$$

DÉFINITION 2. *The physical process modelled by problem (1.1)-(1.2) gives rise to runaway if*

$$\langle v \rangle_\infty \stackrel{\text{def}}{=} \lim_{t \to +\infty} \langle v \rangle (t) = \infty.$$

Note that, in general, $\langle v \rangle_\infty$ is a positive real number which can be either finite or infinite. In physical terms, it is proportional to the D.C. conductivity of the gaseous medium. When $\langle v \rangle_\infty \neq \infty$, the D.C. conductivity is finite and, according to Definition 2, the physical process modelled by Eq. (1.1) does not give rise to runaway, but we cannot say if the process decays to equilibrium.

Definitions 1 and 2 are not necessarily equivalent. In fact, physical folk wisdom tells us that the physical process governed by Eq. (1.1) either displays relaxation to equilibrium or gives rise to runaway. In practice, there may exist situations in which the distinction between those two phenomena gets blurred. For instance, a nontrivial stationary solution in $L_1(\mathbb{R}^2, dxdv)$ may fail to exist, even though (2.2) is fulfilled, or there is relaxation to the equilibrium solution $f_\infty(x, v)$ satisfying the condition $\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v f_\infty(x, v) dxdv = \infty$ and hence there is runaway according to Definition 2.

3. The Bhatnagar-Gross-Krook (BGK) model: spatially independent case.

A highly popular model in gas kinetics is the BGK model (e.g., [11], Sec. II.10). If we limit our discussion to the spatially independent case, then the basic assumption of the BGK model is that the average effect of collisions is to provide a «source» which is proportional to the deviation of the distribution function $f(v, t)$ from a Maxwellian $f_m(v)$. The collision term in (1.1) then takes the simple form

$$\int_{-\infty}^{+\infty} k(v,v')\nu(v')f(v',t)dv' = \nu(v)F_m(v)\int_{-\infty}^{+\infty} \nu(v')f(v',t)dv',$$

where

$$F_m(v) = \frac{f_m(v)}{\displaystyle\int_{-\infty}^{+\infty} \nu(v')f_m(v')dv'}$$

and $f_m(v) = (\beta/\pi)^{1/2}\exp(-\beta v^2)$ is the Maxwellian with $\langle v^2\rangle = \int_{-\infty}^{+\infty} v^2 f_m(v)dv = (1/2\beta)$. The simplified model is represented by the kinetic equation

$$\text{(3.1)} \qquad \frac{\partial f}{\partial t}(v,t) + a\frac{\partial f}{\partial v}(v,t) = \nu(v)\{c(t)f_m(v) - f(v,t)\},\ v \in \mathbb{R},\ t \geq 0,$$

where

$$c(t) = \frac{\displaystyle\int_{-\infty}^{+\infty} \nu(v)f(v,t)dv}{\displaystyle\int_{-\infty}^{+\infty} \nu(v)f_m(v)dv}$$

is a normalization parameter.

In order to study when runaway, according to either Definition 1 or Definition 2, occurs, we solve the kinetic equation (3.1) as well as the integro-differential equation

$$\text{(3.2)} \qquad \frac{\partial f}{\partial t}(v,t) + a\frac{\partial f}{\partial v}(v,t) + \nu(v)f(v,t) = 0,\ v \in \mathbb{R},\ t \geq 0,$$

under the initial condition (1.2). Under this initial condition, Eqs. (3.1) and (3.2) have the solutions

$f(v,t) = [S(t)f_0](v)$ for problem (3.1), and

$f(v,t) = [S_0(t)f_0](v)$ for problem (3.2), where $\{S(t)\}_{t\geq 0}$ and $\{S_0(t)\}_{t\geq 0}$ are strongly continuous, positive contraction semigroup on $L_1(\mathbb{R}, dv)$ related via one the Duhamel formulas

$$S(t)f_0 = S_0(t)f_0 + \int_0^t S_0(t-\sigma)KS(\sigma)f_0 d\sigma, \tag{3.3}$$
$$t \geq 0, \text{ if } \int_{-\infty}^{+\infty} \nu(v)dv < \infty,$$

$$S(t)f_0 = S_0(t)f_0 + \int_0^t S(t-\sigma)KS_0(\sigma)f_0 d\sigma, \tag{3.4}$$
$$t \geq 0, \text{ for general } \nu(v),$$

where $(Kg)(v) = \nu(v)F_m(v)\int_{-\infty}^{+\infty} \nu(v')g(v')dv'$ and

$$[S_0(t)f_0](v) = \exp\left(-\int_0^t \nu(v - at + a\tau)d\tau\right) \tag{3.5}$$
$$f_0(v - at), \ v \in \mathbb{R}, \ t \geq 0.$$

We note that (3.3) and (3.4) are valid for general K and not just for K as in the BGK model. it is readily verified that (1.5) holds for every $g \geq 0$ in $L_1(\mathbb{R}, dv)$. Moreover, for every $g \geq 0$ in $L_1(\mathbb{R}, dv)$. We have (cf. [3])

$$\begin{aligned}
\int_0^t ||S_0(t')g||_\nu dt' &= \int_0^t \int_{-\infty}^{+\infty} \nu(v)\cdot \\
&\cdot \exp\left(-\int_0^{t'} \nu(v - at' + a\tau)d\tau\right) g(v - at')dvdt' = \\
&= \int_{-\infty}^{+\infty} \int_0^t \nu(v + at')\exp\left(-\int_0^{t'} \nu(v + a\tau)d\tau\right) g(v)dt'dv = \\
&= \int_{-\infty}^{+\infty} \left(1 - \exp\left(-\int_0^t \nu(v + a\tau)d\tau\right)\right) g(v)dv \leq ||g||_1,
\end{aligned} \tag{3.6}$$

so that $\int_0^t ||S_0(t')g||_\nu dt' \leq ||g||_1$. Introducing $Z_0(t) = \int_0^t S_0(t')dt'$ and $Z(t) = \int_0^t S(t')dt'$, we easily derive from (3.3) and (3.6) for $f_0 \geq 0$ in $L_1(\mathbb{R}, dv)$

$$\begin{aligned}||Z(t)f_0||_\nu &= ||Z_0(t)f_0||_\nu + \int_{-\infty}^{+\infty} \nu(v) \int_0^t \int_0^{t-\sigma} S_0(\tau)[KS(\sigma)f_0](v)d\tau d\sigma \leq \\ &\leq ||Z_0(t)f_0||_\nu + \int_0^t \int_{-\infty}^{+\infty} \\ &\cdot \left(1 - \exp\left(-\int_0^{t-\sigma} \nu(v + a\tau)d\tau\right)\right) [KS(\sigma)f_0](v)dv d\sigma \leq \\ &\leq ||Z_0(t)f_0||_\nu + \left(1 - \exp\left(-\frac{M}{a}\right)\right) ||Z(t)f_0||_\nu,\end{aligned}$$

where $M = ||\nu||_1 = \int_{-\infty}^{+\infty} \nu(v)dv$. Hence, if the collision frequency $\nu(v)$ is integrable, we have

$$(3,7) \qquad ||Z(t)f_0||_\nu \exp\left(\frac{M}{a}\right) ||Z_0(t)f_0||_\nu \leq \exp\left(\frac{M}{A}\right) ||f_0||_1,$$

because from (3.6) $||Z_0(t)g||_\nu \leq ||g||_1$, for every $g \geq 0$ in $L_1(\mathbb{R}, dv)$.

Let us recall the definition (2.6) for the drift velocity $\langle v\rangle(t)$. For a spatially independent problem, we shall now modify (2.6) and add the following definition for the uncollided flux $[q_0](t)$ for the model governed by Eq. (3.2), for which the total number of electrons is not preserved in time:

$$(3.8) \qquad \langle v\rangle(t) = \frac{\int_{-\infty}^{+\infty} v[S(t)f_0](v)dv}{\int_{-\infty}^{+\infty} [S(t)f_0](v)dv} \text{ and } [q_0](t) = \int_{-\infty}^{+\infty} v[S_0(t)f_0](v)dv.$$

In case of confusion, we write $\langle v\rangle_g(t)$ and $[q_0]_g(t)$ to indicate that $g(v)$ is the initial datum. Quite naturally, for the drift velocities to make sense at $t = 0$ we must require that the initial datum $g(v)$ has finite first moment, i.e. that $\int_{-\infty}^{+\infty}(1+|v|)|g(v)|dv < +\infty$.

To begin with n elementary situation, let us consider the BGK model equation (3.1) for constant $\nu(v) \equiv \nu_0$. Then, using the nonnegativity of its solution, Eq. (3.1) reduces to

$$\frac{\partial f}{\partial t}(v,t) + a\frac{\partial f}{\partial v}(v,t) = \nu_0(||f(t)||_1 f_m(v) - f(v,t)), v \in \mathbb{R},\ t \geq 0.$$

Assuming an initial datum f_0 with finite first moment, we multiply the above equation by v and eliminate the partial derivative with respect to v by partial integration. Because $[vf(v,t)]_{-\infty}^{+\infty} = 0$, see Appendix, we obtain

$$\frac{d}{dt}(||f(t)||_1\langle v\rangle(t)) - a||f(t)||_1 =$$
$$= \nu_0||f(t)||_1\left(\int_{-\infty}^{+\infty} vf_m(v)dv - \langle v\rangle(t)\right),\ t \geq 0.$$

Let us now make use of the following three easily recognized facts:

(i) the Maxwellian f_m has average speed zero,

(ii) $||f(t)||_1 = ||f_0||_1$, and

(iii) $\langle v\rangle(0) = \left[\int_{-\infty}^{+\infty} vf_0(v)dv\right]/||f_0||_1$.

Then we are left with a calculus exercise, yielding

$$\langle v\rangle(t) = \frac{a}{\nu_0}(1 - e^{-\nu_0 t}) + \frac{e^{-\nu_0 t}}{||f_0||_1}\int_{-\infty}^{+\infty} vf_0(v)dv,$$

which approaches $\frac{a}{\nu_0}$ exponentially as $t \to +\infty$. Consequently, the BGK model with constant collision frequency does not display runaway according to Definition 2. Runaway does not according to Definition 1 either, because one can easily prove relaxation to equilibrium in this case ([2], Theorem 9).

It turns out that, for the BGK model without spatial dependence, the criterion for runaway according to Definition 1, namely the integrability of $\nu(v)$, is also a criterion for runaway according to Definition 2.

THEOREM 1. *Suppose* $\int_{-\infty}^{+\infty} \nu(v)dv = M < \infty$. *Then for nonnegative initial data* f_0 *with the property* $\int_{-\infty}^{+\infty} (1+|v|)f_0(v)dv < +\infty$ *the solution* $f(v,t)$ *of the BGK model equation (3.1) satisfies the asymptotic formula*

(3.9)
$$\lim_{t\to+\infty} \frac{\langle v\rangle(t)}{at} = \frac{1}{||f_0||_1} \cdot \left\| \Omega_0^- \left(f_0 + \nu F_m \int_0^{+\infty} ||S(\sigma)f_0||_\nu d\sigma \right) \right\|_1 ,$$

which implies runaway according to Definition 2. Here $\Omega^-{}_{+0}$ *is the wave operator defined by*

$$\Omega_0^- g = \lim_{t\to+\infty} W_0(-t)S_0(t)g. \tag{3.10}$$

Proof. Using (3.5) and (3.8) we have

$$\begin{aligned}
[q_0](t) &= \int_{-\infty}^{+\infty} \{at + (v-at)\}[S_0(t)f_0](v)dv = \\
&= at\int_{-\infty}^{+\infty} \exp\left(-\int_0^t \nu(v+a\tau)d\tau\right) f_0(v)dv + \\
&+ \int_{-\infty}^{+\infty} v\exp\left(-\int_0^t \nu(v+a\tau)d\tau\right) f_0(v)dv,
\end{aligned}$$

whence

$$\lim_{t\to+0} \frac{[q_0](t)}{at} = \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{a}\int_v^{+\infty} \nu(v')dv'\right) f_0(v)dv,$$

which is a finite positive number.

Let us now shed some light on $\langle v\rangle(t)$ itself. From (3.3) we have immediately

$$[S(t)f_0](v) = [S_0(t)f_0](v) + \int_0^t [S_0(t-\sigma)\nu F_m](v)\|S(\sigma)f_0\|_\nu d\sigma, \ t \geq 0, \tag{3.11}$$

where $\nu F_m \in L_1(\mathbb{R}, dv)$. Multiplying the above equation by v, integrating with respect to v and using (3.8) we obtain

$$\langle v\rangle_{f_0}(t) = \frac{1}{\|f_0\|_1}\left\{[q_0]_{f_0}(t) + \int_0^t [q_0]_{\nu F_m}(t-\sigma)\|S(\sigma)f_0\|_\nu d\sigma\right\}, \quad t \geq 0, \tag{3.12}$$

We now note that both f_0 and νF_m have finite first moments, the latter being true because ν is integrable and $vF_m(v)$ is bounded. Further, as $[q_0]_{\nu F_m}(t)/(at)$ is bounded for $t > 0$ and (3.6) holds true, we may divide (3.12) by at and apply the theorem of dominated convergence to pull the limit as $t \to +\infty$ under the integral sign. As a result, we find

$$\begin{aligned}\lim_{t\to+\infty}\frac{\langle v\rangle_{f_0}(t)}{at} &= \frac{1}{\|f_0\|_1}\left\{\lim_{t\to+\infty}\frac{[q_0]_{f_0}(t)}{at} + \int_0^{+\infty}\lim_{t\to+\infty}\frac{[q_0]_{\nu F_m}(t-\sigma)}{a(t-\sigma)}\|S(\sigma)f_0\|_\nu d\sigma\right\} = \\ &= \frac{1}{\|f_0\|_1}\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{a}\int_v^{+\infty}\nu(v')dv'\right)\cdot \\ &\cdot f_0(v)dv + \frac{1}{\|f_0\|_1}\int_{-\infty}^{+\infty}\exp\left(-\frac{1}{a}\int_v^{+\infty}\nu(v')dv'\right)\cdot \\ &\cdot \nu(v)F_m(v)dv \cdot \int_0^{+\infty}\|S(\sigma)f_0\|_\nu d\sigma.\end{aligned} \tag{3.13}$$

Starting from the other side, we have

$$[W_0(-t)S_0(t)g](v) = \exp\left(-\frac{1}{a}\int_v^{v+at} \nu(v')dv'\right) g(v).$$

As a consequence, (2.4) implies

$$[\Omega_0^- f_0](v) = \exp\left(-\frac{1}{a}\int_v^{+\infty} \nu(v')dv'\right) g(v).$$

Comparing this identity with (3.13) we obtain (3.9), which completes the proof. ∎

Theorem 1 can, in fact, be generalized to non-BGK models without spatial dependence. Assuming a nonnegative initial datum f_0 with the property $\int_{-\infty}^{+\infty} (1+|v|)f_0(v)dv < +\infty$, the proof of Theorem 1 can be repeated until we reach (3.11), which must be replaced by (3.3), and (3.12), which must be replaced by

$$\langle v\rangle_{f_0}(t) = \frac{1}{||f_0||_1}\left\{[q_0]_{f_0}(t) + \int_0^t [q_0]_{KS(\sigma)f_0}(t-\sigma)d\sigma\right\}, \quad t \geq 0.$$

An important technical issue at this point is our need to have the existence of the first moment of the vector $KS(\sigma)f_0$. However, since we assume $\nu(v)$ to be integrable, $KS(\sigma f_0 \in L_1(\mathbb{R}, dv)$ for almost every $\sigma \geq 0$. If we now assume K to be a bounded linear operator from $L_1(\mathbb{R}, \nu dv)$ into $L_1(\mathbb{R}, (1+|v|)dv)$, $KS(\sigma)f_0 \in L_1(\mathbb{R}, (1+|v|)dv)$ for almost every $\sigma \geq 0$, (see Apppendix). Repeating the calculations of (3.13) with the help of (3.10), we eventually obtain

$$(3.14)\quad \lim_{t\to+\infty} \frac{\langle v\rangle_{f_0}(t)}{at} = \frac{1}{||f_0||_1}\cdot\left\|\Omega_0^-\left(f_0 + \int_0^{+\infty} KS(\sigma)f_0 d\sigma\right)\right\|_1 .$$

Here we note that, due to (3.7) and the integrability of $\nu(v)$, the operator $f_0 \to \int_0^{+\infty} S(\sigma) f_0 d\sigma$ can be defined as a bounded operator from $L_1(\mathbb{R}, dv)$ into $L_1(\mathbb{R}, \nu dv)$, so that the right-hand side of (3.14) is well-defined.

4. Generalization to the spatially dependent case.

In this section we geenralize our results on electron runaway from the spatially independent case to the spatially dependent situation. In the latter situation we have, in Section 2, given two definitions of electron runaway. As we have seen in [3] and Section 3, for the spatially independent problem runaway occurs according to both definitions if the collision frequency $\nu(v)$ is integrable, the initial datum $f_0 \geq 0$ satisfies $\int_{-\infty}^{+\infty} (1 + |v|) f_0(v) dv < +\infty$ and K is a bounded operator from $L_1(\mathbb{R}, \nu dv)$ into $L_1(\mathbb{R}, (1 + |v|) dv)$. In the present section we will aim at analogous results.

Let us first introduce the strongly continuous, positive contraction semigroups $\{W_0(t)\}_{t \geq 0}$, $\{S_0(t)\}_{t \geq 0}$ and $\{S(t)\}_{t \geq 0}$ on $L_1(\mathbb{R}^2, dx dv)$ which govern the time evolution of the unique solutions of problem (1.1)-(1.2) with $\nu \equiv 0$ and $k \equiv 0$, problem (1.1)-(1.2) with ν arbitrary and $k \equiv = 0$, and problem (1.1)-(1.2) with arbitrary ν and k. We have

$$[W_0(t)g](x, v) = g\left(x - vt + \frac{1}{2} a t^2, v - at\right); \tag{4.1}$$

$$[S_0(t)g](x, v) = \exp\left(-\int_0^t \nu(x - vs + \frac{1}{2} a s^2, v - as) ds\right) g\left(x - vt + \frac{1}{2} a t^2, v - at\right); \tag{4.2}$$

$$
\begin{aligned}
[S(t)g](x,v) =& [S_0(t)g](x,v)+ \\
& + \int_0^t [S_0(t-\sigma)KS(\sigma)g](x,v)d\sigma, \\
& t \geq 0, \text{ if } \int_{-\infty}^{+\infty} \nu(v)dv < \infty,
\end{aligned} \tag{4.3}
$$

$$
\begin{aligned}
[S(t)g](x,v) =& [S_0(t)g](x,v)+ \\
& + \int_0^t [S(t-\sigma)KS_0(\sigma)g](x,v)d\sigma, \\
& t \geq 0, \text{ for general } \nu(v).
\end{aligned} \tag{4.4}
$$

If (2.3) holds true, then runaway occurs according to Definition 1, [1]. More precisely, as $t \to +\infty$ the solution $[S(t)g](x,v)$ behaves as

$$[S(t)f_0](x,v) \sim [W_0(t)\Omega^- f_0](x,v),$$

where the asymptotic equivalence is to be considered in the strong operator topology of $L_1(\mathbb{R}^2, dxdv)$ and

$$
\begin{aligned}
[\Omega^- g](x,v) = & \exp\left(-\int_0^{+\infty} \nu(x+v\tau+\frac{1}{2}a\tau^2, v+a\tau)d\tau\right) \cdot \\
& \cdot \left[g(x,v) + \int_0^{+\infty} [S_0(-s)KS(s)g](x,v)ds\right].
\end{aligned}
$$

Let us now find a criterion for the occurrence of runaway according to Definition 2. Let us recal the definition of $\langle v\rangle(t)$ and define $[q_0](t)$ as follows:

$$
\begin{aligned}
\langle v\rangle(t) &= \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v[S(t)f_0](x,v)dxdv}{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} [S(t)f_0](x,v)dxdv} \quad \text{and} \\
[q_0](t) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v[S_0(t)f_0](x,v)dxdv
\end{aligned} \tag{4.5}
$$

and define

$$[\Omega_0^- g](x,v) = \exp\left(-\int_0^{+\infty} \nu(x+v\tau+\frac{1}{2}a\tau^2, v+a\tau)d\tau\right) g(x,v).$$

We then have

THEOREM 2. *Suppose* $\int_{-\infty}^{+\infty} \bar{\nu}(v)dv = M < +\infty$ *and* K *is a bounded linear operator from* $L_1(\mathbb{R}^2, \nu dxdv)$ *into* $L_1(\mathbb{R}^2, (1+|v|)dxdv)$*. Then for nonnegative initial data* f_0 *which have the integrability property* $\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (1+|v|)f_0(v)dxdv < +\infty$ *the solution* $f(x,v,t)$ *of the initial-value problem (1.1)-(1.2) satisfies the asymptotic formula (3.14), which implies runaway according to Definition 2. Here* Ω_0^- *is the wave operator defined by (3.10), with* W_0 *and* S_0 *given by (4.1) and (4.2).*

Proof. Using (4.1), (4.2) and (4.5) we have

$$\begin{aligned}[q_0](t) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \{at+(v-at)\}[S_0(t)f_0](x,v)dxdv = \\ &= at\|S_0(t)f_0\|_1 + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v \exp \cdot \\ &\cdot \left(-\int_0^t \nu(x+(v-at)s+\frac{1}{2}as^2, v+asds\right) f_0(x,v)dxdv,\end{aligned}$$

whence

$$\begin{aligned}\lim_{t\to+\infty} \frac{[q_0](t)}{at} &= \lim_{t\to+\infty} \|W_0(t)[W_0(-t)S_0(t) - \Omega_0^-]f_0 + \\ &+ W_0(t)\Omega_0^- f_0\|_1 = \|\Omega_0^- f_0\|_1,\end{aligned}$$

which is a finite positive number.

Let us now consider on $\langle v\rangle(t)$ itself. From (4.3) we readily have

$$(4.6)\quad \langle v\rangle_{f_0}(t)=\frac{1}{\|f_0\|_1}\left\{[q_0]_{f_0}(t)+\int_0^t [q_0]_{KS(\sigma f_0}(t-\sigma)d\sigma\right\},t\geq 0.$$

Since K is a bounded linear operator from $L_1(\mathbb{R}^2,\nu dxdv)$ into $L_1(\mathbb{R}^2,(1+|v|)dxdv)$ and the operator $f_0\to\int_0^t S(\sigma)f_0$ is bounded from $L_1(\mathbb{R}^2,dxdv)$ into $L_1(\mathbb{R}^2,\nu dxdv)$ [which one proves by repeating (3.6) and (3.7)], we may divide (4.6) by at and apply the theorem of dominated convergence for vector-valued functions ([12], Theorem II 2.3) to pull the limit as $t\to+\infty$ under the integral sign. As a result, we find in analogy with (3.14)

$$\lim_{t\to+\infty}\frac{\langle v\rangle_{f_0}(t)}{at}=\frac{1}{\|f_0\|_1}\cdot\left\|\Omega_0^-\left(f_0+\int_0^{+\infty}KS(\sigma)f_0d\sigma\right)\right\|_1,$$

which completes the proof. ■

If condition (2.2) is fulfilled, one expects from physical considerations that the average velocity $\langle v\rangle(t)$ approaches a positive constant as $t\to+\infty$. Related physical considerations suggest in this case the existence of a nontrivial steady-state solution $f_\infty(x,v)$ of Eq. (1.1) in $L_1(\mathbb{R}^2,dxdv)$ and the relaxation of the time dependent solution $f(x,v,t)$ of problem (1.1)-(1.2) to this equilibrium. It is then to be expected that, since the L_1-norm of the solution of problem (1.1)-(1.2) does not depend on time or, in physical terms, the total number of electrons is preserved in time,

$$(4.7)\qquad \langle v\rangle_\infty=\lim_{t\to+\infty}\langle v\rangle(t)=\frac{1}{\|f_\infty\|_1}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}vf_\infty(x,v)dxdv.$$

Of course, as exemplified by the difficulties encountered in deriving the results of [2], there are many technical problems, some of which are of a physical nature, because the distinction between relaxation to equilibrium and runaway may get blurred. To mention just a few of these «technicalities»: The nontrivial stationary solution $f_\infty(x,v)$ may not exist or, when it does exist, it may not have a finite first velocity moment so that the right-hand side of (4.7) does not make sense. At this occasion we will not dwell on these problems much longer but attempt to illustrate them by working out the example of problem (1.1)-(1.2) without spatial dependence and with constant collision frequency $\nu(v) \equiv \nu_0$.

Let us consider Eq. (1.1) without spatial dependence and with collision frequency $\nu(v) \equiv \nu_0$:

$$\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial v} + \nu_0 f(v,t) = \nu_0 \int_{-\infty}^{+\infty} k(v,v')f(v',t)dv'. \tag{4.8}$$

Multiplying (4.8) by v, integrating with respect to v and eliminating the partial derivate with respect to v by partial integration, we obtain

$$\begin{aligned}\frac{d}{dt}(||f(t)||_1\langle v\rangle(t)) - a||f(t)||_1 + \nu_0||f(t)||_1\langle v\rangle(t) = \\ \nu_0\int_{-\infty}^{+\infty} v[Kf(t)](v)dv.\end{aligned} \tag{4.9}$$

If we assume that K is a bounded linear operator from continuous $L_1(\mathrm{IR}, \nu dv)$ into $L_1(\mathrm{IR}, (1+|v|)dv)$, then the integral in the right-hand side of (4.9) is a bounded continuous functions of t on IR^+ which we will denote as $G(t)$, see (A.1) in the Appendix. Then for nonnegative initial data f_0 which have the integrability property $\int_{-\infty}^{+\infty}(1+|v|)f_0(v)dv < +\infty$, we have $||f(t)||_1 = ||f_0|_1$ and $\langle v\rangle(0) = \left[\int_{-\infty}^{+\infty} vf_0(v)dv\right]/||f_0||_1$. If we then assume the right-hand side of (4.9) as known [which, of course, it is not], we can solve the differential equation (4.9) by elementary means and find the expression

$$\langle v\rangle(t) = \frac{a}{\nu_0}(1 - e^{-\nu_0 t}) + \\ + \frac{e^{-\nu_0 t}}{||f_0||_1}\int_{-\infty}^{+\infty} v f_0(v)dv + \frac{\nu_0}{||f_0||_1}\int_0^t e^{-\nu_0(t-s)}G(s)ds.$$

Since $G(t)$ is bounded and continuous in t, the last term on the right-hand side is a bounded continuous fucntion of t on $\mathbb{R}^+$. Hence, $\langle v\rangle(t)$ remains bounded as $t \to +\infty$.

Appendix (prepared by C. v.d.M.).

In this appendix we prove the following result.

THEOREM *Suppose $N \geq 0$. Let K be a bounded operator from $L_1(\mathbb{R}, \nu dv)$ into the weigted L_1-space $L - 1(\mathbb{R}, (1+|v|)^N dv)$, and let $\nu(v)$ be integrable on $\mathbb{R}$. Then the operators $S_0(t)$ and $S(t)$ are bounded on $L_1(\mathbb{R}, (1+|v|^N)dv)$ for every $t \geq 0$. Moroever, $\{S_0(t)\}_{t\geq 0}$ and $\{S(t)\}_{t\geq 0}$ are strongly continuous, positive semigroups on $L_1(\mathbb{R}, (1+|v|)^N dv)$.*

Proof. Consider $g \in L_1(\mathbb{R}, (1+|v|)^N dv)$ and put $||g||_{1,N} = \int_{-\infty}^{+\infty}(1+v^2)^{N/2}|g(v)|dv$, noting that this norm is equivalent to the natural norm on $L_1(\mathbb{R}, (1+|v|)^N dv)$. For $t \geq 0$ we get

$$||W_0(t)g||_{1,N} \leq \sup_{v\in\mathbb{R}}\left(\frac{1+(v+at)^2}{1+v^2}\right)^{N/2}||g||_{1,N} \leq \\ \leq \left\{\frac{1}{2}\left(at + (a^2t^2+4)^{1/2}\right)\right\}^N ||g||_{1,N}$$

and hence

$$||S_0(t)g||_{1,N} \leq ||W_0(t)g||_{1,N} \leq \left\{\frac{1}{2}(at + (a^2t^2+4)^{1/2})\right\}^N ||g||_{1,N}.$$

We wil write $\Sigma_N(t) = \left\{ \frac{1}{2}(at + (a^2t^2+4)^{1/2}) \right\}^N$ and we note that $(at)^N \leq \Sigma_N(t) \leq (1+at)^N$. Thus $\Sigma_N(t)$ is an upper bound for the norm of $S_0(t)$.

Now recall the Duhamel formula (3.3)

$$S(t)g = S_0(t)g + \int_0^t S_0(t-\sigma)KS(\sigma)g d\sigma,\ t \geq 0.$$

Applying the norm $g \to ||g||_{1,N}$ to both sides of this identity for arbitrary $g \geq 0$ in $L-1(\mathbb{R}, (1+|v|^N dv)$ and denoting the norm of K as an operator from $L_1(\mathbb{R}, \nu dv)$ into $L_1(\mathbb{R}, (1+|v|)^N dv)$ by $\kappa(\geq 1)$, we obtain

$$\begin{aligned} ||S(t)g||_{1,N} &\leq \Sigma_N(t)||g||_{1,N} + \kappa \int_0^t \Sigma_N(t-\sigma)||S(\sigma)g||_\nu d\sigma \leq \\ &\leq \Sigma_N(t)\left(||g||_{1,N} + \kappa \int_0^t ||S(\sigma)g||_\nu d\sigma \right) \leq \\ &\leq \Sigma_N(t)(1+\kappa \exp(||\nu||_1/a))||g||_{1,N}, \end{aligned}$$

where we have used (3.7), the integrability of $\nu(v)$ and the inequality $||g||_1 \leq ||g||_{1,N}$. Hence the proof of the boundedness of $S(t)$ is complete.

To establish the strong continuity, we estimate for every $g \in L_1(\mathbb{R}, (1+|v|)^N dv)$

$$\begin{aligned} \int_{-\infty}^{+\infty} |[S(t)g](v) - [S(\tau)g](v)|(1+v^2)^{N/2} dv &\leq \\ &\leq \Sigma_N((t-\tau)||g||_{1,N} \leq \Sigma_N(2T)||g||_{1,N} \end{aligned}$$

for $t, \tau \in [0,T]$. Thus we may apply the theorem of dominated convergence to the integral

$$\int_{-\infty}^{+\infty} |[S(t)g](v) - [S(\tau)g](v)|(1+v^2)^{N/2} dv$$

and prove that $||S(t)g - S(\tau)g||_{1,N} \to 0$ as $\tau \to t$. Thus $\{S_0(t)\}_{t\geq 0}$ is a strongly continuous semigroup on $L_1(\mathbb{R}, (1+|v|)^N dv)$.

For $g \in L_1(\mathbb{R}, (1+|v|)^n dv)$ and $0 \leq t \leq T$, we estimate, under the assumption that $M = \int_{-\infty}^{+\infty} \nu(v)dv < \infty$ and that $\kappa(\geq 1)$ is the norm of $K : L_1(\mathbb{R}, \nu dv) \to L_1(\mathbb{R}, (1+|v|)^N dv)$,

$$\left\| \int_0^t S_0(t-\sigma)KS(\sigma)g d\sigma \right\|_{1,N} \leq \kappa \int_0^t \Sigma_N(t-\sigma)||S(\sigma)g||_\nu d\sigma \leq$$

$$\leq \kappa \Sigma_N(T) \int_0^t ||S(\sigma)g||_\nu d\sigma \leq$$

$$\leq \kappa \Sigma_N(T) \exp\left(\frac{M}{a}\right) ||g||_{1,N}.$$

As a result $S(t) - S_0(t)$, and hence $S(t)$, is bounded on $L_1(\mathbb{R}, (1+|v|)^N dv)$. Next, using the theorem of dominated convergence for Bochner integrals ([12], Theorem II 2.3), we see that

$$\left\| \int_0^t S_0(t-\sigma)KS(\sigma)g d\sigma - \int_0^\tau S_0(\tau-\sigma)KS(\sigma)g d\sigma \right\|_{1,N} \to 0$$

as $t \to \tau$. From Duhamel's formula (3.3) it follows that $[S(t) - S_0(t)]g$, and hence $S(t)g$, is continuous in t in the norm of $L_1(\mathbb{R}, (1+|v|)^N dv)$. In fact, for $0 \leq t \leq T$ we have

$$||S(t)g||_{1,N} \leq \left[\Sigma_N(t) + \kappa \int_0^t \Sigma_N(t-\sigma)d\sigma \right] ||g||_{1,N} \leq$$
$$(1+\kappa T)\Sigma_N(T)||g||_{1,N}. \qquad \blacksquare$$

In the spatially dependent case, the above theorem is true if we replace the weighted L_1-space $L_1(\mathbb{R}, (1+|v|)^N dv)$ by $L_1(\mathbb{R}^2, (1+|v|)^N dx dv)$. With the same modification, the proof can be repeated verbatin, even to the point that the norm estimates remain valid.

Finally, let us consider the case of constant collision frequency $\nu(v) \equiv \nu_0$. Then for every $g \geq 0$ in $L_1(R, (1+|v|)^N dv)$ we have

$$||S_0(t)g||_{1,N} \leq e^{-\nu_0 t}||W_0(t)g||_{1,N} \leq \Sigma_N(t)e^{-\nu_0 t}||g||_{1,N}.$$

Assuming that K is a bounded operator from $L_1(\mathbb{R}, dv)$ into $L_1(\mathbb{R}, (1+|v|)^N dv)$ of norm λ, we obtain from the Duhamel formula (3.3) (also valid if $\nu(v) \equiv \nu_0$)

$$S(t)g = S_0(t)g + \int_0^t S_0(\sigma)KS(t-\sigma)g\,d\sigma, \ t \geq 0,$$

the estimate

$$||S(t)g||_{1,N} \leq (\Sigma_N(t)e^{-\nu_0 t} + \lambda \int_0^t \Sigma_N(\sigma)e^{-\nu_0\sigma}d\sigma)||g||_{1,N}.$$

Here we have used that $||KS(t-\sigma)g||_{1,N} \leq \lambda||S(t-\sigma)g||_1 = \lambda||g||_1 \leq \lambda||g||_{1,N}$. Hence the operator $S(t)$ is bounded on $L_1(\mathbb{R}, (1+|v|)^N dv)$ and its norm there is bounded above by

$$\Sigma_N(t)e^{-\nu_0 t} + \lambda \int_0^t \Sigma_N(\sigma)e^{-\nu_0\sigma}d\sigma,$$

which has a finite limit as $t \to +\infty$. Moreover, $S(t)g$ depends continuously on t in the strong operator topology of $L_1(\mathbb{R}, (1+|v|)^N dv)$; this can be proved by repeating the correpsonding argument for integrable $\nu(v)$. Hence, $\{S(t)\}_{t\geq 0}$ is a bounded strongly continuous semigroup on $L_1(\mathbb{R}, (1+|v|)^N dv)$. As a result,

$$G(t) = \int_{-\infty}^{+\infty} v[KS(t)g](v)dv \tag{A.1}$$

is a bounded continuous function on $\mathbb{R}^+$.

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Giovanni Frosali
Dipartimento di Matematica
"V. Volterra"
Università di Ancona
Via Brecce Bianche - I-60131 Ancona (Italy)

Cornelis van der Mee
Università di Firenze
Dipartimento di Matematica Applicata
"G. Sansone"
Via S. Marta, 3 - I-50139 Firenze (Italy)