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# An Abstract Approach to Nonlinear Boltzmann-Type Equations.

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## 1. – Introduction.

In the past 15 years there has been considerable progress on the existence of a (unique) solution of nonlinear Boltzmann-type equations, although the main problem of proving the global in time existence of a (unique) solution of the nonlinear Boltzmann equation for arbitrary initial-boundary data remains unsolved. Both various cases of the nonlinear Boltzmann equation and simplifications such as spatially homogeneous media and discrete velocity models were considered. Since the global in time solvability proof for the spatially homogeneous case by ARKERYD [1] we have seen the development of roughly three schools of thought.

The first method consists of solving the nonlinear Boltzmann equation within the framework of nonstandard analysis and thus reducing the original existence problem to the regularity problem of proving the obtained nonstandard solution to be standard for standard initial-boundary data. This method was widely used by ARKERYD [2, 3] and ELMROTH [9] and has led to a nonstandard existence result for the nonlinear Boltzmann equation with arbitrary data [2] and a standard existence result for the Enskog equation [3].

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The second method was introduced by KANIEL and SHINBROT [15]. They approximated the solution of the nonlinear Boltzmann equation from above and below by the unique solutions of suitable linear kinetic equations. For small time intervals or for small initial-boundary data the approximating sequences are monotone and the existence of a unique solution of the nonlinear Boltzmann equation can be concluded. The scope of the method has since been expanded, in particular through an increased sophistication of the estimates occurring in the monotonicity proof. In this respect we mention the work of ILLNER and SHINBROT [14], TOSCANI [19], HAMDACHE [13], BELLOMO and TOSCANI [8, 20] and TOSCANI and PROTOPOPESCU [21]. The latter work addressed a Boltzmann equation on a parallelepiped with partially absorbing boundary. This method will be one of the two pillars of our study.

The third method was developed by UKAI [22], ASANO [5], UKAI and ASANO [24] and NISHIDA and IMAI [17] and consists of obtaining a number of intricate Sobolev estimates that allow one to derive local in time and, for small data, global in time solutions and a proof of their eventual decay to equilibrium. Recently, PALCZEWSKI [18] obtained global existence for initial data close to a Maxwellian in an $L_\infty$-setting. More recently, KAWASHIMA [16] and ARKERYD et al. [4] combined the third method with the results in the spatially homoheneous case to solve the nonlinear Boltzmann equation for initial data close to a spatially homogeneous distribution.

In spite of the results on the nonlinear equation, there has been much more progress on its linear counterpart, which is easier to handle. Recently BEALS and PROTOPOPESCU [7] have obtained a fairly complete existence and uniqueness theory of time dependent kinetic equations with partially absorbing boundary conditions and suitable force terms. Rather than the application of the method of characteristics to deal with the trace problem arising from the initial-boundary conditions in an $L_p$-space setting, however general their trace theorems may be, the novelty of their approach has been the use of one phase space for the position, velocity and time variables which allowed them to treat equations with time dependent collision and external interaction processes and moving boundaries. As a matter of fact, the method of characteristics was before by BARDOS [6] in a general study of first order hyperbolic equations of linear Boltzmann type while a less general trace theorem was derived before by VOIGT [25] for the force free equation in $L_1$-space. Parallel to the Beals-Protopopescu study special trace theorems were also derived by UKAI [23]. A comprehensive theory of the time dependent linear theory may be found in Chapters XI and XII of [12]. In any case, the symmetric usage of time, spatial and velocity variables makes the Beals-Protopopescu approach the natural vehicle for treating the nonlinear Boltzmann equation, where the estimates always involve the time variable.



In the present article we shall study the abstract boundary value problem

$$\frac{\partial u}{\partial t} + \boldsymbol{v}\cdot\frac{\partial u}{\partial \boldsymbol{x}} + u(\boldsymbol{x}, \boldsymbol{v}, t)(Ru)(\boldsymbol{x}, \boldsymbol{v}, t) = Q[u, u](\boldsymbol{x}, \boldsymbol{v}, t)\,, \tag{1.1}$$

$$u(\boldsymbol{x}, \boldsymbol{v}, t = 0) = g_0(\boldsymbol{x}, \boldsymbol{v})\,, \tag{1.2}$$

$$u_-(\boldsymbol{x}, \boldsymbol{v}, t) = \big(K(u_+)\big)(\boldsymbol{x}, \boldsymbol{v}, t) + g_-(\boldsymbol{x}, \boldsymbol{v}, t)\,, \tag{1.3}$$

where $R$ is a positive linear operator, $Q[\cdot, \cdot]$ is a positive and symmetric bilinear form and $K$ is a positive and contractive (linear or nonlinear) operator. Throughout positivity of operators refers to the property of mapping nonnegative functions into nonnegative functions. Equations (1.2) and (1.3) represent the initial and the boundary conditions, respectively. The representation of the collision term as the difference

$$J[u, u] = Q[u, u] - u(Ru)$$

requires a cut-off of the intermolecular potential in the nonlinear Boltzmann equation. By first considering the case of vacuum boundary conditions ($K = 0$) we shall prove the existence of a unique nonnegative solution of eqs. (1.1)-(1.3) for nonnegative initial-boundary data $(g_0, g_-)$ belonging to a suitable order ideal in $L_p$-space, provided either the time $t \in (0, T)$ with $T$ small enough or the initial-boundary data are small enough in ideal norm. Here we assume that the bilinear form $\int_0^t Q[\cdot, \cdot]\,dt'$ is continuous with respect to the ideal norm. The method of proof will be the Kaniel-Shinbrot type approximation by solutions of suitable linear equations. An analysis will be made of lower bounds for the size of the time interval of existence of the solution in relationship with the functional dependence of the ideal norm on the lenght of the time interval. Having accomplished this, we shall treat the case of nonvacuum boundary conditions with the help of a contraction mapping argument, since, no matter how small the time interval is, unique solvability of the problem with vacuum boundary conditions allows one to reduce the problem with nonvacuum boundary conditions to a nonlinear vector equation in the position-velocity-time domain. When using the method repeatedly, one obtains an existence proof on an extended time interval. For an arbitrarily large number of successive applications of the method we will prove this time interval to be bounded above by an absolute bound depending on the ideal norm of the initial-boundary datum. Hence, the Kaniel-Shinbrot method as expounded in the present article is not suitable for proving global existence for arbitrary initial-boundary data.

On having an abstract theory, we will obtain more specific existence results, actually by deriving the ideal norm estimate of the nonlinear collision term for the specific problem under consideration. To this purpose we shall discuss the two-dimensional Broadwell model, the nonlinear Boltzmann equation in the full space, and the nonlinear Boltzmann equation in a parallelepiped. For the nonlinear Boltzmann equation in the full space we will prove the existence of local solutions for arbitrarily large data and global solutions for small data if the initial data are bounded by a multiple of $h(|\boldsymbol{x}|)m(\boldsymbol{v})$, where $h \in L_1(\mathbb{R}^+)$ is decreasing and $m$ is a global Maxwellian. These results will be derived for soft, Maxwellian and hard interactions with angular cut-off, including the hard sphere model.

## 2. – Preliminaries from the linear theory.

In this section we shall introduce the part of the linear theory needed for the analysis of the nonlinear problem. The material is a variation of the material presented in [7] and Chapters XI and XII of [12]. Let $\Lambda_t$, $t>0$, be an open subset of $\mathbb{R}^{2d}$ that coincides with the interior of its closure, and put $\Sigma_T = \{(\boldsymbol{x}, \boldsymbol{v}, t)\colon (\boldsymbol{x}, \boldsymbol{v}) \in \Lambda_t,\ t \in (0, T)\}$, where for every $T > 0$ the set $\Sigma_T$ coincides with the interior of its closure in $\mathbb{R}^{2d+1}$. We endow $\Sigma_T$ with a Borel measure $\mu_T$ that is finite on every bounded Borel set of $\Sigma_T$. Similarly, $\Lambda_t$ is equipped with a Borel measure $\varrho_t$ that is finite on every bounded Borel set of $\Lambda_t$. The connection between the measures is given by $\mu_T(E) = \int_0^T \varrho_t(E_t)\, dt$ where $E_t = \Lambda_t \times \{t\}$. Let us now consider the vector fields $Y = \partial/\partial t + \boldsymbol{v}\cdot\partial/\partial\boldsymbol{x}$ on $\Sigma_T$ and $X = \boldsymbol{v}\cdot\partial/\partial\boldsymbol{x}$ on each slice $\Lambda_t$. We further assume that each measure $\varrho_t$ is the product of the Lebesgue measure in $\boldsymbol{x}$ and a Borel measure in $\boldsymbol{v}$, so that $\int_{\Sigma_T} Yw\, d\mu_T = 0$ for all $w \in C_0^1(\Sigma_T)$, the continuously differentiable functions on $\Sigma_T$ of compact support. Then through every point of $\Sigma_T$ there passes exactly one integral curve of $Y$, i.e. precisely one solution of the characteristic equations

$$\frac{dt}{ds} = 1\,, \quad \frac{d\boldsymbol{x}}{ds} = \boldsymbol{v}\,, \quad \frac{d\boldsymbol{v}}{ds} = 0\,,$$

which when extended maximally within $\Sigma_T$ has a left and a right endpoint belonging to $\partial\Sigma_T$. We denote these sets of left and right endpoints by $D^-$ and $D^+$, respectively. By the length of an integral curve we denote the length of its $s$-interval of definition, which in fact is the travel time along the curve. Since $(dt/ds) = 1$ and $\Sigma_T \subset \mathbb{R}^{2d} \times (0, T)$, the length of a maximal integral curve of $Y$ is the travel time along its entire length, which is bounded above by $T$. Similarly, on each slice $\Lambda_t$ we define the sets $D_{-,t}$ and $D_{+,t}$ of left and right endpoints of all maximal integral curves of $X$, not counting $(0, 0)$ in either set whenever $(0, 0) \in \Lambda_t$. Then $D^\pm$ can be decomposed as

$$D^- = D_0 \cup D_-\,, \qquad D_- = \{(\boldsymbol{x}, \boldsymbol{v}, t)\colon (\boldsymbol{x}, \boldsymbol{v}) \in D_{-,t},\ t \in (0, T)\}\,, \tag{2.1}$$

$$D^+ = D_T \cup D_+\,, \qquad D_+ = \{(\boldsymbol{x}, \boldsymbol{v}, t)\colon (\boldsymbol{x}, \boldsymbol{v}) \in D_{+,t},\ t \in (0, T)\}\,, \tag{2.2}$$

where

$$D_0 = \Lambda_0 \times \{0\}\,, \qquad D_T = \Lambda_T \times \{T\}\,.$$

Since the sets $D^\pm$, $D_0$, $D_T$, $D_\pm$ and $D_{\pm,t}$ are Borel sets and the length $l(\mathbf{z})$ of a maximal integral curve of $Y$ is a Borel function of its left endpoint $\mathbf{z} \in D^-$, we may parametrize $\Sigma_T \cup D^- \cup D^+$ (with the points of $D^- \cap D^+$ counted twice) as

$$\Sigma_T \cup D^- \cup D^+ = \{(\mathbf{z}, s)\colon \mathbf{z} \in D^-,\ 0 < s < l(\mathbf{z})\}\,.$$

Then $Y$ reduces to $d/ds$. Since $\int_{\Sigma_T} Yv\, d\mu_T = 0$, we may decompose $\mu_T$ along the integral curves of $Y$ as

$$d\mu_T = d\nu^-\, ds = d\nu^+\, ds\,,$$

where $\nu^\pm$ are positive Borel measures on $D^\pm$. Similarly we may decompose $\varrho_t$ along the integral curves of $X$ as

$$d\varrho_t = d\nu_{-,t}\, ds = d\nu_{+,t}\, ds\,,$$

where $\nu_{\pm,t}$ are positive Borel measure on $D_{\pm,t}$. We then define $\Phi_T$ (resp. $\Psi_t$) as the space of all Borel functions $w$ on $\Sigma_T$ (resp. $\Lambda_t$) with the following properties:

(i) Every $w$ is continuously differentiable along $\nu^\pm$- (resp. $\nu_{\pm,t}$-) almost every integral curve of $Y$ (resp. $X$).

(ii) $w$ and its directional derivative $Yw$ (resp. $Xw$) are bounded.

(iii) The support of $w$ is bounded with $l(\mathbf{z})^{-1}$ bounded on the support of $w$.

If $(0,0)\in\Lambda_t$, we also require:

(iv) $(0,0)$ does not belong to the support of $w$ in $\Lambda_t$.

Then every $w\in\Phi_T$ (resp. $\Lambda_t$) has a continuous extension to the endpoints of $\nu^{\pm}$- (resp. $\nu_{\pm,t}$)-almost every integral curve. We then define $\Phi_T^0$ (resp. $\Psi_t^0$) as the subspace of those $w$ that vanish at the endpoints. For every $u\in L_p(\Sigma_T, d\mu_T)$ we then define the distributional derivative $Yu$ by

$$\int_{\Sigma_T}(Yu)v\,d\mu_T=-\int_{\Sigma_T}u(Yv)\,d\mu_T\,,\qquad v\in\Phi_T^0\,.$$

Similarly we define $Xu$ for all $u\in L_p(\Lambda_t, d\varrho_t)$. Then one may prove every function $u\in L_p(\Sigma_T, d\mu_T)$, $1<p<\infty$, to have unique «restrictions» or «traces» $u^{\pm}\in L_{p,\mathrm{loc}}(D^{\pm}, d\nu^{\pm})$ such that

$$\int_{\Sigma_T}\{(Yu)v+u(Yv)\}\,d\mu_T=\int_{D^+}u^+v\,d\nu^+-\int_{D^-}u^-v\,d\nu^-\,,\qquad v\in\Phi_T\,.$$

Similarly every $u\in L_p(\Lambda_t, d\varrho_t)$, $1<p<\infty$, has unique «traces» $u_{\pm,t}\in L_{p,\mathrm{loc}}\cdot(D_{\pm,t}, d\nu_{\pm,t})$ such that

$$\int_{\Lambda_t}\{(Xu)v+u(Xv)\}\,d\varrho_t=\int_{D_{+,t}}u_{+,t}v\,d\nu_{+,t}-\int_{D_{-,t}}u_{-,t}v\,d\nu_{-,t}\,,\qquad v\in\Psi_t\,.$$

Here $L_{p,\mathrm{loc}}(D^{\pm}, d\nu^{\pm})$ is the space of all $\nu^{\pm}$-measurable functions $u$ such that $u\chi_E\in L_p(D^{\pm}, d\nu^{\pm})$ for all sets $E$ of bounded $\nu^{\pm}$-measure on which $l(\mathbf{z})^{-1}$ is bounded. Here $\chi_E$ is the characteristic function of $E$. Similarly one defines $L_{p,\mathrm{loc}}(D_{\pm,t}, d\nu_{\pm,t})$. One may then prove that $u^{\pm}$ can be decomposed in temporal and spatial-velocity pieces as

$$u^-=(u_0, u_-)\,,\qquad u^+=(u_T, u_+)\,,\tag{2.3}$$

in accordance with (2.1) and (2.2).

THEOREM 2.1. Suppose $h$ is a nonnegative function on $\Sigma_T$ that is Lebesgue integrable on every bounded Borel set of $\Sigma_T$. Then for every $f\in L_p(\Sigma_T, d\mu_T)$ and $g\in L_p(D^-, d\nu^-)$, $1<p<\infty$, there exists a unique function $u\in L_p(\Sigma_T, d\mu_T)$ such that $(Y+h)u\in L_p(\Sigma_T, d\mu_T)$, $u^{\pm}\in L_p(D^{\pm}, d\nu^{\pm})$ and the system of equations

$$(Y+h)u=f\,,$$

$$u^-=g\,,$$

is satisfied. This function is given by

$$u(\mathbf{z}, s)=\exp\left[-\int_0^s h(\mathbf{z}, r)\,dr\right]g(\mathbf{z})+\int_0^s\exp\left[-\int_r^s h(\mathbf{z}, t)\,dt\right]f(\mathbf{z}, r)\,dr\,,\tag{2.4}$$

is nonnegative if $f$ and $g$ are nonnegative, and satisfies the bound

$$|u(\mathbf{z}, s)|\leq|g(\mathbf{z})|+\int_0^s|f(\mathbf{z}, r)|\,dr\,.$$

The expression (2.4) proves also that $u$ has a natural «restriction» or trace on each slice $\Lambda_t$ which belongs to $L_p(\Lambda_t, d\varrho_t)$ and in turn has traces on each set $D_{\pm}E_t$ belonging to $L_p(D_{\pm,t}, d\varrho_{\pm,t})$.

Finally, we observe that the solution $u$ increases if either $f$ or $g$ increases or $h$ decreases. This observation will play a crucial role in the approximation scheme for the local in time solution of the nonlinear problem.

## 3. - Local in time solutions.

In this section we prove the existence of a local in time solution of the nonlinear Boltzmann type equation

$$\frac{\partial u}{\partial t}+Xu=J_t[u, u]\,,\tag{3.1}$$

$$u(t=0)=g_0\,,\tag{3.2}$$

$$u_-=K_t(u_+)+g_-\,,\tag{3.3}$$

where $J_t$ is an abstract bilinear form and $K_t$ is an abstract linear or nonlinear operator. Rather than to study eqs. (3.1)-(3.3) in an $L_p$-setting directly, we shall stipulate the existence of a nonnegative function $\varphi\in(D^-, d\nu^-)$, $1<p<\infty$, with respect to which $J_t$ and $K_t$ have certain continuity properties, and solve the above system for small time intervals in the space of all functions $u$ such that $|u| \leq C\varphi$ for some finite constant $C$.

Let us adopt a functional formulation in which $u \in L_p(\Sigma_T, d\mu_T)$ for some fixed $T > 0$. Putting $g = (g_0, g_-)$, $u^- = (u_0, u_-)$, $u_+ = (u_T, u_+)$ and $\mathcal{K} = (0, K)$ we may reformulate eqs. (3.1)-(3.3) as

$$Yu = J[u, u], \tag{3.4}$$

$$u^- = \mathcal{K}(u^+) + g. \tag{3.5}$$

For some fixed nonnegative function $\varphi \in L_p(D^-, d\nu^-)$ such that $\varphi(\mathbf{z}) > 0$ $\nu^-$-almost everywhere, let $\mathcal{N}$ be the space of all $\mu_T$-measurable functions $u$ on $\Sigma_T$ such that

$$|u(\mathbf{z}, s)| \leq C\varphi(\mathbf{z})$$

for some finite constant $C$. Then $\mathcal{N}$ is a Banach space with respect to the ideal norm

$$\|u\|_{\mathcal{N}} = \operatorname{Inf}\{C > 0 : |u(\mathbf{z}, s)| \leq C\varphi(\mathbf{z}) \text{ on } \Sigma_T\}.$$

In the same way $\mathcal{N}^\pm$ will denote the Banach space of all $\nu^\pm$-measurable functions $u^\pm$ on $D^\pm$ such that $|u^-(\mathbf{z})| \leq C\varphi(\mathbf{z})$ (resp. $|u^+(\mathbf{z}, l(\mathbf{z}))| \leq C\varphi(\mathbf{z})$) for some finite constant $C$, where the ideal norm $\|u^\pm\|_{\mathcal{N}}$ is the infimum of all such constants $C$. Then $\mathcal{N}$ and $\mathcal{N}^\pm$ are continuously and densely imbedded in $L_p(\Sigma_T, d\mu_T)$ and $L_p(\Lambda_t, d\varrho_t)$, respectively. For the traces $u^\pm$ of $u \in \mathcal{N}$ we have

$$\|u^\pm\|_{\mathcal{N}} \leq \|u\|_{\mathcal{N}},$$

so that $u^\pm \in L_p(D^\pm, d\nu^\pm)$ for all $u \in \mathcal{N}$.

Let us stipulate a positive linear operator $R$ on $\mathcal{N}$ such that $Ru$ is Lebesgue integrable on every bounded Borel set of $\Sigma_T$ whenever $u \in \mathcal{N}$. We assume that $R$ is local in time in the sense that $R(ru) = r(Ru)$ for all $u \in \mathcal{N}$ and all bounded continuous functions $r = r(t)$ of time alone. Later, in the statement of Theorem 3.2, we will impose a third condition on $R$. Next let us stipulate a positive symmetric bilinear form $Q$ on $\mathcal{N}$ such that

$$\left\| \int_0^s Q[u_1, u_2](\mathbf{z}, r)\, dr \right\|_{\mathcal{N}} \leq \gamma \|u_1\|_{\mathcal{N}} \|u_2\|_{\mathcal{N}}, \qquad \{u_1, u_2\} \subset \mathcal{N}, \tag{3.6}$$

for some finite constant $\gamma$. Fixing $T > 0$ the minimal such $\gamma$ will be denoted



by $\gamma(T)$. We assume that $Q$ is local in time in the sense that

$$Q[r_1 u_1, r_2 u_2] = r_1 r_2 Q[u_1, u_2]$$

for all $u_1, u_2 \in \mathcal{N}$ and all bounded continuous functions $r_1 = r_1(t)$ and $r_2 = r_2(t)$ of time alone. Finally, we stipulate a positive linear or nonlinear operator $\mathcal{K}: \mathcal{N}^+ \to \mathcal{N}^-$ such that

$$\|\mathcal{K}(u_1^+) - \mathcal{K}(u_2^+)\|_{\mathcal{N}} \leq k\|u_1^+ - u_2^+\|_{\mathcal{N}}, \qquad \{u_1^+, u_2^+\} \subset \mathcal{N}^+, \tag{3.7}$$

for some finite constant $k$. Fixing $T > 0$ the minimal such $k$ is denoted as $k(T)$. We assume that $\mathcal{K}$ is local in time in the sense that $\mathcal{K}(ru^+)$ has its support on $\{(\mathbf{x}, \mathbf{v}, t): t \in \bar{S}\}$ for all $u^+ \in \mathcal{N}^+$ and all bounded continuous functions $r = r(t)$ of time alone with support on $\bar{S} \subset (0, T)$. Using the decomposition (2.3) of the traces of $u \in L_p(\Sigma_T, d\mu_T)$ in temporal and spatial-velocity pieces, we require that $\mathcal{K} = (0, K)$, i.e. $\mathcal{K}(u_T, u_+) = (0, K(u_+))$. Decomposing the nonlinear term $J[u, u]$ in eq. (3.1) as the difference of the gain term $Q[u, u]$ and the loss term $uRu$, we finally obtain the system of equations

$$(Y + Ru)u = Q[u, u], \tag{3.8}$$

$$u^- = \mathcal{K}(u^+) + g. \tag{3.9}$$

In order to solve eqs. (3.8)-(3.9) for $g \geq 0$ in $\mathcal{N}$ and $\mathcal{K} = 0$ (vacuum boundary conditions) we fix two initial functions $l_0$, $u_0 \in \mathcal{N}$ such that $0 \leq l_0 \leq u_0$, and consider two recursively defined sequences of lower solutions $\{l_k\}_{k=0}^\infty$ and upper solutions $\{u_k\}_{k=0}^\infty$ satisfying the system of equations

$$(Y + Ru_k)l_{k+1} = Q[l_k, l_k], \tag{3.10a}$$

$$(l_{k+1})^- = g, \tag{3.10b}$$

and

$$(Y + Rl_k)u_{k+1} = Q[u_k, u_k], \tag{3.11a}$$

$$(u_{k+1})^- = g, \tag{3.11b}$$

and try to obtain a solution to eqs. (3.8)-(3.9) by approximation from below and above by those two sequences. We then obtain the Kaniel-Shinbrot approximation scheme (cf. [15]).

LEMMA 3.1. Suppose $0 \leqslant l_0 \leqslant l_1 \leqslant u_1 \leqslant u_0$ in $\mathcal{N}$. Then

$$0 \leqslant l_0 \leqslant l_1 \leqslant \ldots \leqslant l_k \leqslant \ldots \leqslant u_k \leqslant \ldots \leqslant u_1 \leqslant u_0 , \tag{3.12}$$

while $l = \operatorname{Sup} l_k$ and $u = \operatorname{Inf} u_k$ belong to $\mathcal{N}$ and satisfy the equations

$$(Y + Ru)\, l = Q[l, l] , \tag{3.13a}$$

$$(Y + Ru)\, u = Q[u, u] , \tag{3.13b}$$

$$l^- = u^- = g . \tag{3.13c}$$

PROOF. Suppose (3.12) is true for certain $k$. Then $Ru_k \geqslant Rl_k \geqslant 0$ and $0 \leqslant Q[l_k, l_k] \leqslant Q[u_k, u_k]$ imply that $0 \leqslant l_{k+1} \leqslant u_{k+1}$ with $l_{k+1}, u_{k+1} \in \mathcal{N}$. The latter follows immediately from the estimates

$$0 \leqslant \int_0^s Q[l_k, l_k](\mathbf{z}, r)\, dr \leqslant \int_0^s Q[u_k, u_k](\mathbf{z}, r)\, dr \leqslant \gamma(T) \|u_k\|^2_{\mathcal{N}} \varphi(\mathbf{z}) .$$

Next, since $0 \leqslant Ru_k \leqslant \ldots \leqslant Ru_1 \leqslant Ru_0$ and $0 \leqslant Q[l_0, l_0] \leqslant Q[l_1, l_1] \leqslant \ldots \leqslant Q[l_k, l_k]$, we have $0 \leqslant l_1 \leqslant l_2 \leqslant \ldots \leqslant l_{k+1}$ in $\mathcal{N}$. Similarly, since $0 \leqslant Rl_0 \leqslant Rl_1 \leqslant \ldots \leqslant Rl_k$ and $Q[u_0, u_0] \geqslant Q[u_1, u_1] \geqslant \ldots \geqslant Q[u_k, u_k]$, we have $u_1 \geqslant u_2 \geqslant \ldots \geqslant u_{k+1} \geqslant 0$. By induction we easily finish the proof. ■

In the sequel we assume $R$ to satisfy the condition

$$\left\| \int_0^s u_1(\mathbf{z}, r)(Ru_2)(\mathbf{z}, r)\, dr \right\|_{\mathcal{N}} \leqslant \delta(T) \|u_1\|_{\mathcal{N}} \|u_2\|_{\mathcal{N}} , \qquad \{u_1, u_2\} \subset \mathcal{N} , \tag{3.14}$$

for some $\delta(T) > 0$. For later convenience we introduce the notation

$$\beta(T) = \gamma(T) + \tfrac{1}{2}\delta(T) .$$

THEOREM 3.2. Suppose $g \geqslant 0$ and $\|g\|_{\mathcal{N}} \leqslant [4\beta(T)]^{-1}$. Then eqs. (3.8)-(3.9) with vacuum boundary conditions ($K = 0$) have a nonnegative solution $u$ on $[0, T]$ such that $(u/\varphi)$ is bounded. Any two such solutions $u^1$, $u^2$ that satisfy $\|u^1\|_{\mathcal{N}} \leqslant 2\|g\|_{\mathcal{N}}$ and $\|u^2\|_{\mathcal{N}} \leqslant 2\|g\|_{\mathcal{N}}$, coincide.

PROOF. Put

$$F(x) = \frac{2}{1 + (1 - 4x)^{\frac{1}{2}}} , \qquad 0 \leqslant x \leqslant \tfrac{1}{4} .$$



Then $F(x)$ increases monotonically from $F(0) = 1$ to $F(\frac{1}{4}) = 2$. Now set $C = \|g\|_{\mathcal{N}} F(\gamma(T)\|g\|_{\mathcal{N}})$ where $\gamma(T)\|g\|_{\mathcal{N}} \leqslant \frac{1}{4}$. Then $C$ is the lower root of the quadratic equation

$$\|g\|_{\mathcal{N}} + \gamma(T)\, C^2 = C . \tag{3.15}$$

Now put $l_0 = 0$ and $u_0 = C\varphi$ as the initial conditions of the recursion scheme (3.10)-(3.11). Then obviously $0 = l_0 \leqslant l_1 \leqslant u_1$, while $u_1$ is the unique solution of the linear problem

$$Yu_1 = Q[u_0, u_0] , \qquad (u_1)^- = g ,$$

which has the form

$$u_1(\mathbf{z}, s) = g(\mathbf{z}) + C^2 \int_0^s Q[\varphi, \varphi](\mathbf{z}, r)\, dr \leqslant$$
$$\leqslant \|g\|_{\mathcal{N}} \varphi(\mathbf{z}) + \gamma(T)\, C^2 \varphi(\mathbf{z}) = C\varphi(\mathbf{z}) = u_0(\mathbf{z}, s) .$$

Thus the initial string of inequalities $0 \leqslant l_0 \leqslant l_1 \leqslant u_1 \leqslant u_0$ is satisfied and therefore also (3.12). Now let $l = \operatorname{Sup} l_k$ and $u = \operatorname{Inf} u_k$. Then $0 \leqslant l \leqslant u \leqslant C\varphi$.

In order to prove that $l$ and $u$ coincide, we subtract eq. (3.13a) from eq. (3.13b), use (3.13c) and $0 \leqslant l \leqslant u$ and find

$$Y(u - l) + (u - l) Ru = Q[u - l, u + l) + uR(u - l) , \tag{3.16a}$$

$$(u - l)^- = 0 , \tag{3.16b}$$

so that

$$\|u - l\|_{\mathcal{N}} \leqslant \gamma(T) \|u - l\|_{\mathcal{N}} \|u + l\|_{\mathcal{N}} + \delta(T) \|u\|_{\mathcal{N}} \|u - l\|_{\mathcal{N}} \leqslant$$
$$\leqslant 2\beta(T) \|u\|_{\mathcal{N}} \|u - l\|_{\mathcal{N}} \leqslant 4\beta(T) \|g\|_{\mathcal{N}} \|u - l\|_{\mathcal{N}} .$$

Thus if $\|g\|_{\mathcal{N}} < [4\beta(T)]^{-1}$, we have $u = l$, as required.

Suppose now that $\|g\|_{\mathcal{N}} = [4\beta(T)]^{-1}$. Then $\|g\|_{\mathcal{N}} \leqslant [4\gamma(T)]^{-1}$ guarantees that the lower solution $l$ and the upper solution $u$ exist. If we consider a sequence $\{\alpha_k\}_{k=1}^{\infty}$ with $\alpha_k \uparrow 1$, $g_{[k]} = \alpha_k g$ as initial data, $l_{[k]}$ as lower solution and $u_{[k]}$ as upper solution, we have on the one hand $l_{[k]} = u_{[k]}$ and on the other hand $l_{[k]} \uparrow l$ and $u_{[k]} \uparrow u$ (provided the corresponding $l_0 = 0$ and $u_0 = C_{[k]}\varphi$ with $C_{[k]}$ satisfying eq. (3.15) with $\|g\|_{\mathcal{N}}$ replaced by $\alpha_k \|g\|_{\mathcal{N}}$), so that $l = u$ also for $\|g\|_{\mathcal{N}} = [4\beta(T)]^{-1}$.

In order to prove the uniqueness of any two solutions $u^1$ and $u^2$ for which $\|u^i\|_{\mathcal{N}}<2\|g\|_{\mathcal{N}}$ $(i=1,2)$, we first note that it is sufficient that the solution $u$ obtained from the recursion scheme of eqs. (3.10)-(3.11) does not depend on $l_0$ and $u_0$ if $0<l_0<u_0<D\varphi$ with $D$ the lower root of the quadratic equation

$$\|g\|_{\mathcal{N}}+\beta(T)D^2=D \tag{3.17}$$

(which exists and is positive if $\|g\|_{\mathcal{N}}<[4\beta(T)]^{-1}$). This will be apparent if we start the scheme for $l_0=u_0=u$ the solution and observe that the corresponding approximate solutions satisfy $l_k=u_k=u$.

Let us prove that the solution $u$ obtained from eqs. (3.10)-(3.11) under the above conditions does not depend on $l_0$ and $u_0$. Indeed, let us begin with two pairs $(l_0,u_0)$ and $(\tilde l_0,\tilde u_0)$ satisfying $0<l_0<u_0<D\varphi$ and $0<\tilde l_0<\tilde u_0<D\varphi$. If $l_0=\tilde l_0$ and $u_0<\tilde u_0$, we have $\tilde l_k<l_k<u_k<\tilde u_k$ and hence $u=\tilde u$ for the corresponding solutions. Similarly, we find $u=\tilde u$ for the solutions if $l_0>\tilde l_0$ and $u_0=\tilde u_0$. Next, if $l_0=\tilde l_0$ and $u_0$ and $\tilde u_0$ do not satisfy either $u_0<\tilde u_0$ or $u_0>\tilde u_0$, we consider $\hat l_0=l_0=\tilde l_0$ and $\hat u_0=\max\{u_0,\tilde u_0\}$, which satisfy $0<\hat l_0<\hat u_0<D\varphi$, and conclude that $u=\tilde u$ for the corresponding solutions. All other cases of different pairs $(l_0,u_0)$ and $(\tilde l_0,\tilde u_0)$ satisfying the above conditions can be treated in an analogous way. ■

If $\gamma(T)\downarrow 0$ and $\delta(T)\downarrow 0$, there is at most one nonnegative solution of eqs. (3.1)-(3.3). Indeed, if there are two different nonnegative solutions, they must differ from a certain $t=t_1$ on. Considering eqs. (3.1)-(3.3) on $t\in[t_1,t_2]\subset(0,T)$ with initial condition $u(t_1)$, we may apply the previous theorem to prove these solutions to coincide on some interval $[t_1,t_2]$ for $(t_2-t_1)$ small anough, which leads to a contradiction.

COROLLARY 3.3. Suppose $\beta(T)$ is a monotonically nondecreasing function of $T$ with $\beta(\infty)<\infty$. Then for all nonnegative $g\in\mathcal{N}$ with $\|g\|_{\mathcal{N}}<[4\beta(\infty)]^{-1}$ there exists a unique nonnegative global solution $u$ of eqs. (3.8)-(3.9) with $(u/\varphi)$ bounded and satisfying $\|u\|_{\mathcal{N}}<2\|g\|_{\mathcal{N}}$.

Before we consider the case of arbitrary boundary conditions, we obtain some bounds on the solution for $\mathcal{K}=0$. Fix $T>0$, and let $g\in\mathcal{N}$ be nonnegative with $\beta(T)\|g\|_{\mathcal{N}}<\frac14$. Then there exists a unique nonnegative local solution $u$ of eqs. (3.8)-(3.9) on $[0,T]$ with $(u/\varphi)$ bounded and satisfying $\|u\|_{\mathcal{N}}<2\|g\|_{\mathcal{N}}$, to be denoted as $u=S(g)$. Then $\|u\|_{\mathcal{N}}\le D$, whence

$$\|S(g)\|_{\mathcal{N}}<F(\beta(T)\|g\|_{\mathcal{N}})\|g\|_{\mathcal{N}}<2\|g\|_{\mathcal{N}}\,. \tag{3.17}$$



On the other hand, suppose $0<g_1<g$ in $\mathcal{N}$ and consider the corresponding solutions $S(g_1)$ and $S(g)$, assuming $\|g\|_{\mathcal{N}}<[4\beta(T)]^{-1}$. Then

$$S(g_1)(\mathbf{z},s)=\exp\left[-\int_0^s S(g_1)(\mathbf{z},r)\,dr\right]g_1(\mathbf{z})+\int_0^s\exp\left[-\int_r^s S(g_1)(\mathbf{z},t)\,dt\right]Q[S(g_1),S(g_1)](\mathbf{z},r)\,dr\,,$$

$$S(g)(\mathbf{z},s)=\exp\left[-\int_0^s S(g)(\mathbf{z},r)\,dr\right]g(\mathbf{z})+\int_0^s\exp\left[-\int_0^s S(g)(\mathbf{z},t)\,dt\right]Q[S(g),S(g)](\mathbf{z},r)\,dr\,.$$

Then

$$S(g)(\mathbf{z},s)-S(g_1)(\mathbf{z},s)\le g(\mathbf{z})-g_1(\mathbf{z})+\int_0^s Q[S(g)-S(g_1),S(g)+S(g_1)](\mathbf{z},r)\,dr\,.$$

In the same way we obtain for $0<g_2<g$ in $\mathcal{N}$ with $\|g\|_{\mathcal{N}}<[4\beta(T)]^{-1}$

$$S(g)(\mathbf{z},s)-S(g_2)(\mathbf{z},s)\le g(\mathbf{z})-g_2(\mathbf{z})+\int_0^s Q[S(g)-S(g_2),S(g)+S(g_2)](\mathbf{z},r)\,dr\,.$$

Hence, for $0<g_1,g_2$ in $\mathcal{N}$ with $g=\max\{g_1,g_2\}$ and $\|g\|_{\mathcal{N}}<[4\beta(T)]^{-1}$ we obtain

$$|S(g_1)(\mathbf{z},s)-S(g_2)(\mathbf{z},s)|\le|g_1(\mathbf{z})-g_2(\mathbf{z})|+\int_0^s|Q[S(g)-S(g_1),S(g)+S(g_1)](\mathbf{z},r)+Q[S(g_2),S(g)+S(g_2)](\mathbf{z},r)|\,dr\,,$$

and hence

$$\|S(g_1)-S(g_2)\|_{\mathcal{N}}\le\|g_1-g_2\|_{\mathcal{N}}+\gamma(T)\{\|S(g)-S(g_1)\|_{\mathcal{N}}\|S(g)+S(g_1)\|_{\mathcal{N}}+\|S(g)-S(g_2)\|_{\mathcal{N}}\|S(g)+S(g_2)\|_{\mathcal{N}}\}\le$$
$$\le\|g_1-g_2\|_{\mathcal{N}}+2\gamma(T)\|S(g)\|_{\mathcal{N}}\{\|S(g)-S(g_1)\|_{\mathcal{N}}+\|S(g)-S(g_2)\|_{\mathcal{N}}\}\,.$$

To obtain an estimate for $\|S(g)-S(g_i)\|_{\mathcal{N}}$, $i=1,2$, we approximate $S(g)$ and $S(g_i)$ by the sequences of lower solutions $\{l_k\}_{k=0}^\infty$ and $\{l_k^{(i)}\}_{k=0}^\infty$ and the sequences of upper solutions $\{u_k\}_{k=0}^{\infty}$ and $\{u_k^{(i)}\}_{k=0}^{\infty}$, where $l_0 = l_0^{(1)} = l_0^{(2)} = 0$ and $u_0 = u_0^{(1)} = u_0^{(2)}$. We then obtain the estimates

$$\|l_{k+1} - l_{k+1}^{(i)}\|_{\mathcal{N}} \leq \|g - g_i\|_{\mathcal{N}} + \left\| \int_0^s \{Q[l_k, l_k] - Q[l_k^{(i)}, l_k^{(i)}]\}(\mathbf{x}, r)\, dr \right\|_{\mathcal{N}} \leq$$

$$\leq \|g - g_i\|_{\mathcal{N}} + \gamma(T)\|l_k - l_k^{(i)}\|_{\mathcal{N}} \|l_k + l_k^{(i)}\|_{\mathcal{N}} \leq$$

$$\leq \|g - g_i\|_{\mathcal{N}} + 2\gamma(T)\|g\|_{\mathcal{N}} F(\beta(T)\|g\|_{\mathcal{N}})\|l_k - l_k^{(i)}\|_{\mathcal{N}},$$

so that for $l = \operatorname{Sup} l_k$ and $l^{(i)} = \operatorname{Sup} l_k^{(i)}$

$$\|l - l^{(i)}\|_{\mathcal{N}} \leq \|g - g_i\|_{\mathcal{N}} \sum_{j=0}^{\infty} [2k(T)\|g\|_{\mathcal{N}} F(\beta(T)\|g\|_{\mathcal{N}})]^j =$$

$$= \frac{\|g - g_i\|}{1 - 2\gamma(T)\|g\|_{\mathcal{N}} F(\beta(T)\|g\|_{\mathcal{N}})}.$$

Now recall that $l = S(g)$ and $l^{(i)} = S(g_i)$, $i = 1, 2$. Also notice that $g = \max\{g_1, g_2\}$ implies

$$\|g - g_1\|_{\mathcal{N}} + \|g - g_2\|_{\mathcal{N}} = \|g_1 - g_2\|_{\mathcal{N}}.$$

We then finally obtain

$$\|S(g_1) - S(g_2)\|_{\mathcal{N}} \leq \frac{\|g_1 - g_2\|_{\mathcal{N}}}{1 - 2\gamma(T)\|g\|_{\mathcal{N}} F(\beta(T)\|g\|_{\mathcal{N}})} \tag{3.18}$$

where $g = \max\{g_1, g_2\}$.

Let us now consider eqs. (3.8)-(3.9) with arbitrary boundary operator $\mathcal{K}$. Then for $\|\mathcal{K}(u^+) + g\|_{\mathcal{N}} \leq [4\beta(T)]^{-1}$ we may write

$$u = S(\mathcal{K}(u^+) + g). \tag{3.19}$$

On defining $\mathcal{L}(u^+) = [S(\mathcal{K}(u^+) + g)]^+$ on the set $\mathcal{M}(T, \mathcal{K})$ of all nonnegative $u^+ \in \mathcal{N}^+$ such that $\|\mathcal{K}(u^+) + g\|_{\mathcal{N}} \leq [4\beta(T)]^{-1}$, we must solve the nonlinear equation

$$u^+ = \mathcal{L}(u^+) \tag{3.20}$$

on $\mathcal{M}(T, \mathcal{K})$, after which (3.19) yields a local solution $u$ of eqs. (3.8)-(3.9) on $[0, T]$ with $(u/\varphi)$ bounded.



For $u_1^+, u_2^+ \in \mathcal{M}(T, \mathcal{K})$ we obtain for $w^- = \max\{\mathcal{K}(u_1^+), \mathcal{K}(u_2^+)\}$,

$$\|\mathcal{L}(u_1^+) - \mathcal{L}(u_2^+)\|_{\mathcal{N}} \leq \|S(\mathcal{K}(u_1^+) + g) - S(\mathcal{K}(u_2^+) + g)\|_{\mathcal{N}} \leq$$

$$\leq [1 - 2\gamma(T)\|w^- + g\|_{\mathcal{N}} F(\beta(T)\|w^- + g\|_{\mathcal{N}}]^{-1} \|\mathcal{K}(u_1^+) - \mathcal{K}(u_2^+)\|_{\mathcal{N}} \leq$$

$$\leq [1 - 2\gamma(T)\|w^- + g\|_{\mathcal{N}} F(\beta(T)\|w^- + g\|_{\mathcal{N}})]^{-1} k(T)\|u_1^+ - u_2^+\|_{\mathcal{N}}.$$

For all $0 < \varepsilon < 1$ we define $\mathcal{M}_\varepsilon(T, \mathcal{K})$ as the set of those nonnegative $u^+ \in \mathcal{N}^+$ such that $\|\mathcal{K}(u^+) + g\|_{\mathcal{N}} \leq \varepsilon[4\beta(T)]^{-1}$. Then $\gamma(T)\|w^- + g\|_{\mathcal{N}} \leq \frac{1}{4}\varepsilon$, and hence

$$\|\mathcal{L}(u_1^+) - \mathcal{L}(u_2^+)\|_{\mathcal{N}} \leq \frac{k(T)}{1 - \frac{1}{2}\varepsilon F(\frac{1}{4}\varepsilon)} \|u_1^+ - u_2^+\|_{\mathcal{N}}.$$

If $k(T) < 1$, we choose $\varepsilon \in (0, 1)$ so small that $k(T)[1 - \frac{1}{2}\varepsilon F(\frac{1}{4}\varepsilon)]^{-1} < 1$. Then eq. (3.20) has a unique solution in $\mathcal{M}_\varepsilon(T, \mathcal{K})$. However, the set $\mathcal{M}_\varepsilon(T, \mathcal{K})$ is only non empty if $\|g\|_{\mathcal{N}} \leq \varepsilon[4\beta(T)]^{-1}$.

We have obtained

THEOREM 3.5. Choose $\varepsilon \in (0, 1)$ such that $k(T)[1 - \frac{1}{2}\varepsilon F(\frac{1}{4}\varepsilon)]^{-1} < 1$. Then for all nonnegative $g \in \mathcal{N}$ with $\|g\|_{\mathcal{N}} \leq \varepsilon[4\beta(T)]^{-1}$ there is a local solution $u$ of eqs. (3.8)-(3.9) on $[0, T]$ such that $(u/\varphi)$ is bounded.

COROLLARY 3.6. Choose $\varepsilon \in (0, 1)$ such that $k(T)[1 - \frac{1}{2}\varepsilon F(\frac{1}{4}\varepsilon)]^{-1} < 1$ and suppose $\beta(T)$ is monotonically nondecreasing with $\beta(\infty) < \infty$. Then for all nonnegative $g \in \mathcal{N}$ with $\|g\|_{\mathcal{N}} \leq \varepsilon[4\beta(\infty)]^{-1}$ there is a global solution $u$ of eqs. (3.8)-(3.9) such that $(u/\varphi)$ is bounded.

In order to obtain a useful estimate for the solution of eqs. (3.8)-(3.9) if $\mathcal{K}$ is nontrivial, we put $\delta = k(T)[1 - \frac{1}{2}\varepsilon F(\frac{1}{4}\varepsilon)]^{-1}$ where $\|g\|_{\mathcal{N}} \leq \varepsilon[4\beta(T)]^{-1}$. If $u_0^+$ is the initial vector in $\mathcal{M}_\varepsilon(T, \mathcal{K})$ for the iteration of eq. (3.20), we obtain for $u_{n+1}^+ = \mathcal{L}(u_n^+)$

$$\|u_{m+1}^+ - u_0^+\|_{\mathcal{N}} \leq \sum_{j=1}^{m} \|u_{j+1}^+ - u_j^+\|_{\mathcal{N}} \leq \frac{1 - \delta^{m+1}}{1 - \delta} \|u_1^+ - u_0^+\|.$$

For $u_0^+ = 0$, $u_1^+ = [S(g)]^+$ and $m \to \infty$ we find

$$\|u^+\|_{\mathcal{N}} \leq (1 - \delta)^{-1} \|[S(g)]^+\|_{\mathcal{N}} \leq (1 - \delta)^{-1} F(\beta(T)\|g\|_{\mathcal{N}})\|g\|_{\mathcal{N}},$$

so that

$$\|\mathcal{K}(u^+) + g\|_{\mathcal{N}} \leq \left(1 + \frac{k(T) F(\beta(T)\|g\|_{\mathcal{N}})}{1 - \delta}\right) \|g\|_{\mathcal{N}}.$$

On minimizing $\mathcal{E} \geq 4\beta(T)\|g\|_{\mathcal{N}}$ (and hence $(1-\delta)^{-1}$) we get

$$\delta = k(T)[1 - 2\beta(T)\|g\|_{\mathcal{N}} F(\beta(T)\|g\|_{\mathcal{N}})]^{-1},$$

and therefore

$$\|\mathcal{K}(u^+) + g\|_{\mathcal{N}} < E(T, g, k(T))\|g\|_{\mathcal{N}}$$

where

$$E(T, g, k(T)) = \frac{\big(1 - 2\beta(T)\|g\|_{\mathcal{N}} F(\beta(T)\|g\|_{\mathcal{N}})\big)\big(1 + k(T)F(\beta(T)\|g\|_{\mathcal{N}})\big) - k(T)}{1 - 2\beta(T)\|g\|_{\mathcal{N}} F(\beta(T)\|g\|_{\mathcal{N}}) - k(T)}.$$

Thus we find

$$E(T, g, k(T)) > \frac{1}{1 - k(T)} > 1.$$

For the solution we obtain

$$\|u\|_{\mathcal{N}} < F\big(\beta(T)E(T, g, k(T))\|g\|_{\mathcal{N}}\big)E(T, g, k(T))\|g\|_{\mathcal{N}} \tag{3.21}$$

where

$$F\big(\beta(T)E(T, g, k(T))\|g\|_{\mathcal{N}}\big)E(T, g, k(T)) > \frac{1}{1 - k(T)} \geq 1. \tag{3.22}$$

## 4. – Local solutions obtained by lining up time intervals.

Suppose $\beta(T)$ is a continuous function on $[0, \infty)$ satisfying $\beta(0^+) = 0$ and $\beta(\infty) = \beta_\infty$, such that $\beta(T)$ dominates the ideal norm of the bilinear form

$$(u_1, u_2) \to \int_\delta^{\delta+t} \{Q[u_1, u_2] + u_1 R u_2\}\, ds, \qquad t \in [0, T],$$

on every time interval of the type $[\delta, T+\delta]$ where $\delta \geq 0$. Then, given a nonnegative solution $u$ of eqs. (3.1)-(3.3) on $[0, \tau]$, there exists a solution on $[0, \tau+\delta]$ if $\beta(\delta) < (4\|u(\tau)\|_{\mathcal{N}})^{-1}$. Starting from the initial condition $g$, one then obtains the sequence $\{T_k\}_{k=1}^\infty$ such that

$$\beta(T_1) < (4\|g\|_{\mathcal{N}})^{-1} \quad \text{and} \quad \beta(T_{n+1}) < \left(4\left\|u\left(\sum_{k=1}^n T_k\right)\right\|_{\mathcal{N}}\right)^{-1}.$$



The question arises whether the combined time interval can be arbitrarily large. We will answer the question in the negative.

THEOREM 4.1. We have

$$\sum_{k=1}^\infty \beta(T_k) < \frac{1}{\|g\|_{\mathcal{N}}}.$$

Moreover, the sequence of time intervals $\{\beta(T_k)\}_{k=1}^\infty$ can be chosen in such a way that $\sum_{k=1}^\infty \beta(T_k)$ coincides with $1/\|g\|_{\mathcal{N}}$.

PROOF. Observe that

$$\frac{\left\|u\left(\sum_{j=1}^{k+1} T_j\right)\right\|_{\mathcal{N}}}{\left\|u\left(\sum_{j=1}^{k} T_j\right)\right\|_{\mathcal{N}}} < F\left(\beta(T_{k+1})\left\|u\left(\sum_{j=1}^k T_j\right)\right\|_{\mathcal{N}}\right).$$

Starting from $\beta(T_1) = \mathcal{E}_1/4\|g\|_{\mathcal{N}}$ for some $\mathcal{E}_1 \in (0,1)$ we define $\sigma_k = \beta(T_k)\cdot\|g\|_{\mathcal{N}}$ and $\delta_{k+1} = \sigma_{k+1}\|g\|_{\mathcal{N}} \prod_{j=1}^k F(\delta_j)$. We then obtain a sequence $\{\delta_k\}_{k=1}^\infty$ in $(0, \frac{1}{4})$ and a sequence $\{\sigma_k\}_{k=1}^\infty$ such that

$$\sigma_1 = \delta_1. \tag{4.1}$$

$$\sigma_{k+1} = \frac{\delta_{k+1}}{\prod_{j=1}^k F(\delta_j)}. \tag{4.2}$$

Here $F(x)$ is defined in the proof of Theorem 3.2.

At first, we observe that, for fixed $\delta \in (0, 1)$, the function

$$G(x) = x + \frac{\delta}{F(x)}$$

reaches its maximum $G_m$ at the point $\bar{x} = \frac{1}{4}(1 - \delta^2)$, while $G_m = \frac{1}{4}(1+\delta)^2$. We then obtain

$$\sum_{k=1}^n \sigma_k < \beta_n, \tag{4.3}$$

where $\beta_0 = 0$, $\beta_1 = \frac{1}{4}$ and $\beta_{k+1} = (1+\beta_k)^2/4$ and the upper bound at the right hand side can be approached arbitrarily closely by choosing $\delta_k = (1-\beta_{n-k}^2)/4$. Indeed, the result is ovbiously correct for $n = 1$. If the result is correct for $n$ replaced by $n-1$, we use the identity

$$\sum_{k=1}^{n}\sigma_k = \delta_1 + \frac{1}{F(\delta_1)}\left(\sum_{k=1}^{n-1}\sigma_k \text{ with } \delta_k \text{ replaced by } \delta_{k+1}\right).$$

The latter sum has $\beta_{n-1}$ as its supremum assumed for $\delta_k = (1-\beta_{n-k}^2)/4$ as $k = 2, \ldots, n$, while $x + \beta_{n-1}F(x)^{-1}$ has $(1+\beta_{n-1})^2/4$ as its supremum assumed for $\delta_1 = (1-\beta_{n-1}^2)/4$, which proves the assertion. Now, since $\beta_n \uparrow 1$ as $n \to \infty$, we obtain the desired result. Because $\mathcal{E}_1 = 1$ can be assumed, one can also assume the upper bound. ■

Let us assume that $\beta(t)$ is monotonically increasing from $\beta(0^+) = 0$ to $\beta(\infty) = \beta_\infty$ and continuously differentiable with monotonically nonincreasing derivative. Then the inequality $(\beta(t)/t) < \beta'(0)$ for $t > 0$ implies that

$$\sum_{k=1}^{\infty} T_k > \frac{1}{\beta'(0)} \sum_{k=1}^{\infty}\beta(T_k),$$

while the right-hand side can be chosen as to approach $(\beta'(0)\|g\|_{\mathcal{N}})^{-1}$ arbitrarily closely. Hence, we can solve the initial value problem at least on the time interval $0 < t < (\beta'(0)\|g\|_{\mathcal{N}})^{-1}$. On the other hand, the maximum size of the combination $\sum_{i=1}^{N} T_k$ of $N$ time intervals is given by $\operatorname{Sup}_{N\in\mathbb{N}} \sum_{i=1}^{N} \Gamma^{-1}(\sigma_i)$, where $\Gamma^{-1}$ is the inverse function of $\Gamma(t) = \beta(t)\|g\|_{\mathcal{N}}$ and $\sigma_i$ is defined in terms of the optimal sequence $\{\delta_k\}_{k=1}^{n}$. Hence, if infinitely many time intervals are taken, we find existence for $0 < t < T_{\max}$ where

$$T_{\max} = \sum_{i=0}^{\infty}\Gamma^{-1}(\zeta_i) > \frac{1}{\Gamma'(0)}\sum_{i=0}^{\infty}\zeta_i$$

with $\Gamma'(0)$ being the derivative of $\Gamma(t)$ at $t = 0$. For $\|g\|_{\mathcal{N}} < (1/4\beta_\infty)$ we thus find global in time existence. If $\beta'(t)$ is not monotonically decreasing throughout, we may essentially draw the same conclusions, though at the expense of an increased complexity of the arguments.

If we adopt reflective boundary conditions, we get a similar result. As apparent from Theorem 3.5, we would only have to replace $\beta(T)$ by $\beta(T)/\mathcal{E}(T)$ where $\mathcal{E} = \mathcal{E}(T) \in (0, 1)$ satisfies $k(T)[1 - \frac{1}{4}\mathcal{E}F(\frac{1}{4}\mathcal{E})]^{-1} < 1$.



## 5. – Examples.

In this section we will discuss a few applications of the abstract theory and show them to satisfy its assumptions. In particular, we shall indicate the ideal and the corresponding ideal norm and derive the bilinear estimate on the gain and loss terms of the collision operator with respect to the ideal norm.

### 5.1. *The plane Broadwell model.*

In the plane Broadwell model the particles can only have the four velocities

$$\boldsymbol{v}_1 = c\boldsymbol{i}, \qquad \boldsymbol{v}_2 = -c\boldsymbol{i}, \qquad \boldsymbol{v}_3 = c\boldsymbol{j}, \qquad \boldsymbol{v}_4 = -c\boldsymbol{j},$$

where $\boldsymbol{i}$ and $\boldsymbol{j}$ are the unit vectors in the positive $x$- and $y$-direction, respectively. If $S$ is some positive constant representing the effective cross-section, the kinetic equations read

$$\frac{\partial f_1}{\partial t} + c\frac{\partial f_1}{\partial x} = cS(f_3 f_4 - f_1 f_2), \tag{5.1a}$$

$$\frac{\partial f_2}{\partial t} - c\frac{\partial f_2}{\partial x} = cS(f_3 f_4 - f_1 f_2), \tag{5.1b}$$

$$\frac{\partial f_3}{\partial t} + c\frac{\partial f_3}{\partial y} = cS(f_1 f_2 - f_3 f_4), \tag{5.1c}$$

$$\frac{\partial f_4}{\partial t} - c\frac{\partial f_4}{\partial y} = cS(f_1 f_2 - f_3 f_4). \tag{5.1d}$$

Let $\boldsymbol{n}$ and $\boldsymbol{m}$ denote the unit vectors in the directions $\boldsymbol{i} + \boldsymbol{j}$ and $-\boldsymbol{i} + \boldsymbol{j}$, respectively, and let

$$\varphi(\boldsymbol{x}) = \varphi_1(|\boldsymbol{x}\cdot\boldsymbol{n}|)\,\varphi_2(|\boldsymbol{x}\cdot\boldsymbol{m}|) \tag{5.2}$$

where $\varphi_1, \varphi_2 \in L_2(\mathbb{R}^+)$. Then if $0 < f_i(\boldsymbol{n} + \boldsymbol{v}_i t, t) < \varphi(\boldsymbol{x})$, $i = 1, 2, 3, 4$,

$$\int_0^t (f_1 f_2)(\boldsymbol{x} + \boldsymbol{v}_3 s, s)\, ds =$$

$$= \int_0^t f_1(\boldsymbol{x} + (\boldsymbol{v}_3 - \boldsymbol{v}_1)s + \boldsymbol{v}_1 s, s)\, f_2(\boldsymbol{x} + (\boldsymbol{v}_3 - \boldsymbol{v}_2)s + \boldsymbol{v}_2 s, s)\, ds <$$

$$\leq \int_0^t \varphi_1\big(|(\boldsymbol{x} + (\boldsymbol{v}_3 - \boldsymbol{v}_1)s)\cdot \boldsymbol{n}|\big)\cdot \varphi_2\big(|(\boldsymbol{x} + (\boldsymbol{v}_3 - \boldsymbol{v}_1)s)\cdot \boldsymbol{m}|\big)\cdot$$
$$\cdot \varphi_1\big(|(\boldsymbol{x} + (\boldsymbol{v}_3 - \boldsymbol{v}_2)s)\cdot \boldsymbol{n}|\big)\cdot \varphi_2\big(|(\boldsymbol{x} + (\boldsymbol{v}_3 - \boldsymbol{v}_2)s)\cdot \boldsymbol{m}|\big)\, ds =$$
$$= \int_0^t \varphi_1(|\boldsymbol{x}\cdot\boldsymbol{n}|)\,\varphi_2(|\boldsymbol{x}\cdot\boldsymbol{m} + c\sqrt{2}\,s|)\,\varphi_1(|\boldsymbol{x}\cdot\boldsymbol{n} + c\sqrt{2}\,s|)\,\varphi_2(|\boldsymbol{x}\cdot\boldsymbol{m}|)\, ds \leq$$
$$\leq \varphi_1(|\boldsymbol{x}\cdot\boldsymbol{n}|)\,\varphi_2(|\boldsymbol{x}\cdot\boldsymbol{m}|)\Big[\int_0^t \varphi_1(|\boldsymbol{x}\cdot\boldsymbol{n} + c\sqrt{2}\,s|)^2\, ds\Big]^{\frac12}\cdot$$
$$\cdot\Big[\int_0^t \varphi_2(|\boldsymbol{x}\cdot\boldsymbol{m} + c\sqrt{2}\,s|)^2\, ds\Big]^{\frac12} \leq \gamma(t)\,\varphi(\boldsymbol{x})$$

where

$$\gamma(t) = \sup_{a\in\mathbb{R}}\Big[\int_a^{a+t}\varphi_1(c\sqrt{2}\,s)^2\, ds\Big]^{\frac12} \sup_{a\in\mathbb{R}}\Big[\int_a^{a+t}\varphi_2(c\sqrt{2}\,s)^2\, ds\Big]^{\frac12}. \tag{5.3}$$

Owing to the absolute continuity of the measures

$$\lambda_i(E) = \int_E \varphi_i^2\, ds\,, \qquad i = 1, 2\,,$$

the function $\gamma(t)$ is nondecreasing and $\gamma(0^+) = 0$. Therefore, observing that analogous estimates can be derived for $f_3 f_4$, we obtain

$$\Big\|\int_0^t Q[u_1, u_2](\boldsymbol{z}, r)\, dr\Big\|_{\mathcal{N}} \leq \gamma(t)\,\|u_1\|_{\mathcal{N}}\,\|u_2\|_{\mathcal{N}}\,,$$

where $\varphi(\boldsymbol{z})$ is defined by (5.2). Precisely the same estimate can be found for the loss term. Moreover, since for $i = 1, 2$

$$\int_0^t \varphi_i(|b + c\sqrt{2}\,s|)^2\, ds \leq \int_{-\infty}^{\infty}\varphi_i^2(c\sqrt{2}\,s)\, ds = \frac{1}{c\sqrt{2}}\,\|\varphi_i\|_2^2\,,$$

we find

$$\gamma(t) = \delta(t) \leq \frac{1}{c\sqrt{2}}\,\|\varphi_1\|_2\,\|\varphi_2\|_2\,, \tag{5.4}$$



so that

$$\beta(t) \leq \frac{3}{4c\sqrt{2}}\,\|\varphi_1\|_2\,\|\varphi_2\|_2\,.$$

Thus when

$$\|g\|_{\mathcal{N}} \leq \frac{c\sqrt{2}}{6\|\varphi_1\|_2\,\|\varphi_2\|_2}\,,$$

global existence follows. Modifying (5.3), we also find global existence for small data if $\varphi_1 \in L_p(\mathbb{R}^+)$, $\varphi_2 \in L_q(\mathbb{R}^+)$ and $p^{-1} + q^{-1} = 1$. For arbitrary initial data we find local in time existence.

5.2. *The nonlinear Boltzmann equation in the full space.*

The nonlinear Boltzmann equation has the form

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{x}} + fRf = Q[f, f]\,, \tag{5.5}$$

where

$$(Q[f, f])(\boldsymbol{x}, \boldsymbol{v}, t) = \int_{\mathbb{R}^3\times[0,\pi/2]+\times[0,2\pi]} B(\theta, q)\, f(\boldsymbol{x}, \boldsymbol{v}', t)\, f(\boldsymbol{x}, \boldsymbol{v}_1', t)\, d\boldsymbol{v}_1\, d\theta\, d\varepsilon\,,$$

$$(Rf)(\boldsymbol{x}, \boldsymbol{v}, t) = \int_{\mathbb{R}^3\times[0,\pi/2]\times[0,2\pi]} B(\theta, q)\, f(\boldsymbol{x}, \boldsymbol{v}_1, t)\, d\boldsymbol{v}_1\, d\theta\, d\varepsilon\,,$$

and $|\boldsymbol{q}| = |\boldsymbol{v}_1 - \boldsymbol{v}| = |\boldsymbol{v}' - \boldsymbol{v}_1'|$. We shall assume the cut-off

$$0 \leq B(\theta, q) \leq Cq^\delta \sin\theta\cos\theta \tag{5.6}$$

where $\delta \in (-1, 1]$. Here $\boldsymbol{v}$ and $\boldsymbol{v}_1$ are the outgoing and $\boldsymbol{v}'$ and $\boldsymbol{v}_1'$ the incoming velocities of the particles in the binary collision, and $\theta$ and $\varepsilon$ are the polar and azimuthal angles of $\boldsymbol{v}'$ in a spherical coordinate system attached to $\boldsymbol{v}$ with the $z$-axis in the direction of $\boldsymbol{q}$. Moreover, $\delta \in (-1, 0)$ corresponds to soft interactions, $\delta = 0$ to Maxwell's molecules, $\delta \in (0, 1]$ to hald interactions and $B(\theta, q) = Cq \sin\theta\cos\theta$ to the hard sphere model. Suppose $h \in L_1(\mathbb{R}^+)$ is a continuous nonincreasing positive function satisfying $\sup_{s\geq 0}\{h(s/\sqrt{2})/h(s)\} < \infty$ and $0 \leq m \in L_1(\mathbb{R}^3)$. We consider the order ideal

$$\mathcal{N} = \big\{f = f(\boldsymbol{x}, \boldsymbol{v}, t)\colon |f| \leq c h(|\boldsymbol{x} - \boldsymbol{v}t|)\, m(\boldsymbol{v}) \text{ for some } c \in \mathbb{R}^+\big\}\,, \tag{5.7}$$

where

$$m(v) = \exp\{-\alpha|v|^2\} \tag{5.8}$$

with $\alpha > 0$ and $h$ is to be determined later. Then, using conservation of energy, we have for the ideal norm

$$\gamma(t) \le \operatorname*{Sup}_{x,v} \int_0^t \int_{\mathbf{R}^3\times[0,\pi/2]\times[0,2\pi]} B(\theta, q)\, h(|x - qs|)\, m(v_1)\, dv_1\, d\theta\, d\varepsilon\, ds\,.$$

Using (5.6), the monotonicity of $h$ on $\mathbf{R}^+$, $\cos\theta = \cos(v - v', q)$ and $\sin\theta = \cos(v - v_1', q)$, we obtain $\gamma(t) \le \max\{\gamma_1(t), \gamma_2(t)\}$ where

$$\gamma_1(t) \le \operatorname*{Sup}_{v} 2C \int_0^{t/2} \int_{\mathbf{R}^3\times[0,\pi/2]\times[0,2\pi]} q^\delta \sin\theta \cos\theta\, h\left(qs \begin{matrix}\cos\\ \sin\end{matrix}\theta\right) m(v_1)\, dv_1\, d\theta\, d\varepsilon\, ds\,.$$

However,

$$\int_0^{\pi/2} \cos\theta \sin\theta\, h\left(qs \begin{matrix}\cos\\ \sin\end{matrix}\theta\right) d\theta = \int_0^1 x h(qsx)\, dx\,.$$

As a result we get

$$\int_0^{t/2} \int_0^{\pi/2} \cos\theta \sin\theta\, h\left(qs \begin{matrix}\cos\\ \sin\end{matrix}\theta\right) d\theta = \int_0^1 \frac{1}{q} H\left(\frac{qtx}{2}\right) dx \le \frac{1}{q} H\left(\frac{qt}{2}\right),$$

where $H(z) = \int_0^z h(\sigma)\, d\sigma$. Hence, using $q = v_1 - v$ we get

$$\gamma(t) \le \operatorname*{Sup}_{v} 4\pi C \int_{\mathbf{R}^3} d^{\delta-1} H\left(\frac{qt}{2}\right) m(|q + v|)\, dq =$$

$$= \operatorname*{Sup}_{v} 8\pi^2 C \int_0^\infty \int_{-1}^1 q^{\delta+1} H\left(\frac{qt}{2}\right) m\left((q^2 + v^2 + 2qvx)^{\frac12}\right) dx\, dq\,.$$

Substituting (5.8) we get

$$\gamma(t) \le \operatorname*{Sup}_{v} \frac{8\pi^2 C}{\alpha} \int_0^\infty q^\delta \frac{\sinh(2\alpha q v)}{v} \exp[-\alpha v^2] H\left(\frac{qt}{2}\right) \exp[-\alpha q^2]\, dq\,. \tag{5.9}$$



Bounding $H(qt/2)$ above by $H(\infty)$, we use eq. 3.952(7) of [11] with $\mu = \delta + 1$ and the identity $\sin\gamma x = -i \sinh(i\gamma x)$ to compute the resulting integral yielding

$$k(t) \le \frac{8\pi^2 C}{\alpha^{1+\frac12\delta}} \|h\|_{L_1(\mathbf{R}^+)} \Gamma\left(1 + \frac{1}{2}\delta\right) {}_1F_1\left(\frac{1-\delta}{2}; \frac{3}{2}; -\alpha v^2\right),$$

where ${}_1F_1(\beta; \gamma; z)$ is a confluent hypergeometric function. With the help of eq. 9.211(1) of [11] we see that the right-hand side is monotonically decreasing in $v$. Substituting $v = 0$ we finally get for $\delta \in (-1, 1]$

$$\gamma(t) \le \frac{8\pi^2 C}{\alpha^{1+\frac12\delta}} \|h\|_{L_1(\mathbf{R}^+)} \Gamma\left(1 + \frac{1}{2}\delta\right) < \infty\,.$$

Precisely the same estimate is obtained for $\delta(t)$. We thus obtain global existence for small data. To get local existence for arbitrary data, we apply dominated convergence in conjunction with $\lim_{t\downarrow 0} H(qt/2) = 0$, which yields $\lim_{t\downarrow 0} \gamma(t) = \lim_{t\downarrow 0} \delta(t) = 0$.

If the initial condition $0 \le g \in \mathcal{N}$ with $\mathcal{N}$ as in (5.7) and (5.8) (for arbitrary $\alpha > 0$), there is local in time existence for arbitrarily large initial data in $\mathcal{N}$ and global in time existence for $\|g\|_{\mathcal{N}} < (1/4\beta(\infty))$. The global existence results include the case of a hard sphere interaction. Local existence results for the hard sphere model are also obtained for initial data bounded by a Maxwellian of the type (4.8). Observe that ILLNER and SHINBROT [14] proved only global existence for small data in a suitable order ideal in $L_1(\mathbf{R}^3\times\mathbf{R}^3)$ but failed to establish local existence for large initial data. We have thus generalized their result as well as some results of BELLOMO and TOSCANI [8, 19, 20, 21].

5.3. *The nonlinear Boltzmann equation with specularly reflective boundary conditions.*

The nonlinear Boltzmann equation of the form (5.5) is now restricted to the region $x \in V$ where $V$ is assumed to be a bounded convex body with piecewise $C^1$-boundary. On assuming (5.6) we may prove local existence for the problem with vacuum boundary conditions in a suitable ideal $\mathcal{N}$ by simply extending the intermolecular interaction and the initial condition as $B(x, \theta, q) \equiv 0$ and $g \equiv 0$ for $x$ outside $V$. Let $n(x)$ be the unit outer normal at almost every point $x \in \partial V$ and define the specularly reflecting boundary operator

$$(Kf)(x, v, t) = \alpha(x, v) f\big(x, v - 2(n(x)\cdot v) n(x), t\big)$$

with $\alpha$ being the essential supremum of $\alpha(\boldsymbol{x}, \boldsymbol{v})$. Here we assume $\alpha < 1$. If the ideal $\mathcal{N}$ is determined by $\|f\|_{\mathcal{N}} = \|f/\varphi\|_\infty$ for some almost everywhere positive $\varphi$, then

$$\|K(f-g)\|_{\mathcal{N}} < \alpha \left\| \frac{(f-g)\big(\boldsymbol{x}, \boldsymbol{v} - 2(n(\boldsymbol{x})\cdot \boldsymbol{v})\, n(\boldsymbol{x}), t\big)}{\varphi(\boldsymbol{x}, \boldsymbol{v}, t)} \right\|_\infty = \alpha \|f-g\|_\infty ,$$

provided

$$\varphi(\boldsymbol{x}, \boldsymbol{v}, t) = \varphi\big(\boldsymbol{x}, \boldsymbol{v} - 2(n(\boldsymbol{x})\cdot\boldsymbol{v})\, n(\boldsymbol{x}), t\big) \tag{5.10}$$

for almost $(\boldsymbol{x}, \boldsymbol{v}, t) \in D_{-,t}$. Then, in the notations of Sections 3, $k(T) < \alpha < 1$ so that local existence is guaranteed for initial data $g$ in this ideal. Condition (5.10) is satisfied if

$$\varphi(\boldsymbol{x}, \boldsymbol{v}, t) = \Phi(|\boldsymbol{v}|) , \tag{5.11}$$

where $\Phi \in L_1(\mathrm{R}^+)$ is monotonically increasing and continuous and satisfies the condition $\sup_{s \geq 0} \{\Phi(s/\sqrt{2}(/\Phi(s)\} < \infty$.

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SUMMARY

Using a general existence and uniqueness theory for linear time dependent kinetic equations, for general inhomogeneous multidimensional spatial and velocity domains and partially absorbing boundaries, we obtain local in time solutions of a class of nonlinear Boltzmann type equations. For small initial-boundary data we obtain global in time solutions. The ideal norm on certain ideals in the Banach space of $L_p$-functions on phase space is used to measure the « size » of initial-boundary data and solutions. Kaniel-Shinbrot type upper and lower approximation arguments are applied. The combined length of the time interval of existence when applying the method repeatedly is analyzed as a function of the size of the initial-boundary data. Specific applications to the nonlinear Boltzmann equation itself and to the plane Broadwell model are given.

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