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# SINGULAR INTEGRAL EQUATIONS ON CLOSED CONTOURS

*(Conferenza tenuta dal prof. P. F. Zweifel il 12 febbraio 1988)*

SOMMARIO. — Vengono costruite soluzioni per equazioni integrali singolari su contorni chiusi utilizzando una generalizzazione delle relazioni di ortogonalità della teoria del trasporto. Si ottengono semplificazioni notevoli rispetto alla teoria classica, che fa uso della trasformazione di Hilbert.

## 1. - INTRODUCTION.

Recently [1], the authors proposed to solve singular integral equations of Cauchy type on intervals, such as commonly arise in transport theory [2], by an orthogonality method. Similar methods were introduced in transport theory long ago [3] but were restricted specifically to the transport equation. The point of Ref. 1 was that a similar method could be used for quite general equations. This approach has a number of advantages. Among them are elegance and simplicity (the Hilbert transform used in the standard method of solving singular integral equations [4, 5] need not be introduced); familiarity (the method of solution becomes closely analogous to classical techniques for solving partial differential equations); and, perhaps most important, insight (for example, the so-called endpoint conditions [6] usually introduced in a completely *ad hoc* manner are seen to arise naturally, as a condition that certain contour integrals exist).

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In the present work the analysis of Ref. 1 is extended to the case that the contour of integration is a closed rectifiable Jordan curve in the complex plane which we assume to be the unit circle $S^1$. Of course, endpoint conditions are no longer involved, but we find advantages in this case also. In particular, for certain problems (the so-called « non-normal » problems of Ref. 5) the gain in simplicity is quite spectacular. Also, certain problems not previously solved can be treated by a limiting procedure. These points are elaborated in more detail in Section 3.

## 2. - The Basic Method.

We consider the equation

$$f(t) = \lambda(t) A(t) + P \oint \frac{\eta(\nu) A(\nu)}{\nu - t} d\nu , \tag{1}$$

where the symbol $P$ mean Cauchy principal value and the contour integration is, as mentioned previously, the unit circle $S^1$. We seek complex-valued uniformly Hölder continuous solutions $A(t)$ for $|t| = 1$, under the assumption that the given functions $f$, $\lambda$ and $\eta$ are uniformly Hölder continuous on $S^1$.

The more general problem

$$g(t) = \lambda(t) B(t) + \eta_1(t) P \oint \frac{\eta_2(\nu) B(\nu)}{\nu - t} d\nu$$

can be reduced to Eq. (1) by the substitutions

$$B(\nu) = \eta_1(\nu) A(\nu),$$
$$g(t) = \eta_1(t) f(t),$$
$$\eta_1(t) \eta_2(t) = \eta(t),$$

so we shall consider only Eq. (1).

In solving Eq. (1), it is necessary, as in Ref. 1, to introduce the solution $X(z)$ to a homogeneous Riemann-Hilbert problem, i.e. we seek a solution holomorphic on $\mathbb{C} \setminus S^1$ whose boundary values $X^{\pm}$ satify

$$\frac{X^+(t)}{X^-(t)} = \frac{\lambda(t) + \pi i \eta(t)}{\lambda(t) - \pi i \eta(t)}, \quad t \in S^1, \tag{2}$$

with

$$X^{\pm}(t) = \lim_{\varepsilon \downarrow 0} X((1 \pm \varepsilon) t) .$$

We observe the important fact that if $X(z)$ is a solution of Eq. (2), so is

$$\frac{P(z)\,X(z)}{Q(z)}$$

for any polynomials $P(z)$ and $Q(z)$.

For simplicity, we introduce the abbreviations

$$\Lambda^{\pm}(t) = \lambda(t) \pm \pi i \eta(t)$$

without any implication that there exists an analytic function of which $\Lambda^{\pm}$ are the boundary values. Actually, in transport theory such a function $\Lambda$ often does exist which clearly simplifies the solution to the Riemann-Hilbert problem (2)). It is called the *symbol* and the function $X$ is referred to as the *factorization* of $\Lambda$. This Wiener-Hopf approach of reducing Eq. (1) to a Riemann-Hilbert problem which is solved by factorization techniques, has been exploited extensively by I. Gohberg and co-workers [7] and, it must be admitted, is the method of choice for dealing with vector problems. This is true because of the inherent difficulty posed by the matrix Riemann-Hilbert problem [8]. However, in the present paper we deal with the scalar problem and do not adopt the latter approach. For the scalar case standard algorithms exist for computing $X(z)$, and, as we shall see, in many cases $X$ can actually be written down by inspection. In any case, let us assume $X$ is known.

*We shall assume throughout that $\Lambda^{\pm}$ are non-vanishing, except when explicitly stated otherwise.* This assumption is necessary to turn Eq. (1) into a Fredholm problem.

Let us now introduce a suggestive notation by rewriting Eq. (1) in the form

$$f(t) = \oint A(\nu)\,\varphi_\nu(t)\,d\nu, \tag{3}$$

where the $\varphi_\nu$ are distributions of the type

$$\varphi_\nu(t) = \lambda(t)\,\delta(\nu - t) + P\,\frac{\eta(\nu)}{\nu - t}\,.$$

We can now state.

PROPOSITION 1. (Orthogonality) Let $X(z)$ obey Eq. (2), be analytic on $\mathbb{C} \setminus S^1$ and bounded at infinity. Then

$$\oint W(t)\,\varphi_\nu(t)\,\varphi_{\nu'}(t)\,dt = N(\nu)\,\delta(\nu - \nu'), \tag{4a}$$

where the weight function $W$ is given by

$$W(t) = \eta(t) \frac{X^+(t)}{\Lambda^+(t)} \tag{4b}$$

and the normalization constant $N(\nu)$ by

$$N(\nu) = \eta(\nu) X^+(\nu) \Lambda^-(\nu) . \tag{4c}$$

We shall not present proofs in this paper, since they can be found elsewhere [9]. For example, Proposition 1 is proved by substituting $W$ and $\varphi_\nu$ into the left-hand side of (4a) and performing the integration.

COROLLARY. A solution to Eq. (1) (or Eq. (3)), *if it exist,* is given by

$$A_X(\nu) = \frac{1}{N(\nu)} P \oint W(t) f(t) \varphi_\nu(t) \, dt . \tag{5}$$

This follows by multiplying (3) by $W(t)\varphi_\nu(t)$ and integrating over $S^1$, taking into account (4a). The subscript $X$ on $A_X$ serves to remind us that $A$ depends in an explicit way on $X$.

The Corollary clearly exemplifies some of the advantages claimed in the introduction, namely simplicity, elegance and familiarity. However, we still need

PROPOSITION 2. Let $X(z)$ obey Eq. (2), and let $1/X(z)$ be analytic on $\mathbb{C} \setminus S^1$ and bounded at infinity. Then a solution to Eq. (1) (or Eq. (3)) is given by $A_X$ in Eq. (5).

The proof of this proposition is actually identical to the proof of the corresponding proposition in Ref. 1. The idea is to substitute the putative solution (5) into (3) and to perform the integrations, arriving at the tautology $f(t) = f(t)$.

Note that Proposition 2 requires that $1/X(z)$ be bounded at infinity. Uniqueness depends on the actual asymptotic behavior of $1/X$.

PROPOSITION 3. Let $X(z)$ be as in Proposition 2, and let $P_N(z)$ be a polynomial of degree less than or equal to $N$ such that

$P_N(z)/X(z) \to 0$ as $|z| \to \infty$. Then

$$\frac{P_N(t)}{X^+(t)\,\Lambda^-(t)}$$

is a solution to the homogeneous Eq. (1) (or (3)).

Finally, if $1/X(z)$ is not analytic, we can state

PROPOSITION 4. Let $X(z)$ obey Eq. (2), and let $1/X(z)$ be bounded at infinity and analytic on $\mathbb{C} \setminus S^1$ except for a pole of order $m$ at $z = 0$. Then there exists a solution to Eq. (1) (or (3)) if and only if the following $m$ constraints are satisfied:

$$\oint W(t) f(t)\, t^l\, dt = 0\,, \quad l = -1, 0, 1, \ldots, m-2\,. \tag{6}$$

and this solution is given by $A_X$ in Eq. (7).

We note that if $1/X(z)$ has a pole of order $m$ at $z = z_0$, the pole can always be shifted to $z = 0$ by choosing

$$X_1(z) = \left(\frac{z}{z - z_0}\right)^m X(z)\,,$$

while $X_1$ still obeys Eq. (2).

Incidentally, suppose that $X(z)$ obeys Eq. (2), that it is analytic on $\mathbb{C} \setminus S_1$, that it has no zeros (except for a possible zero of order $m$ at the origin), and that $X(z) \sim c\, z^N$ as $|z| \to \infty\, (c \neq 0)$; then its index, i.e. the winding number of $X$ with respect to $z = 0$, is defined by

$$k = m - N\,.$$

with $m = 0$ if $X(0) \neq 0$.

It is now clear that $X(z)$ can always be chosen so that $1/X$ will never simultaneously vanish at infinity and have a finite pole. For instance, suppose $X(z)^{-1} \sim z^{-1}$ at infinity and also has a first order pole at zero. Then the function $X_1(z) = X(z)/z$ can be defined as the solution to he Riemann-Hilbert problem (2), and $1/X_1$ approaches a nonzero constant at infinity and is analytic at zero. Keeping this important fact in mind, we can now collect all our propositions as

THEOREM 1. Let $X(z)$ obey Eq. (2), and let $k$ be the index of $X$. Then

(a) If $k = 0$, the unique solution to Eq. (1) is given by $A_X(t)$ in Eq. (5).

(b) If $k < 0$, the general solution to Eq. (1) is given by

$$A_X(t) + \frac{P_{-k-1}(t)}{X^+(t)\,\Lambda^-(t)}$$

where $P_{-k-1}$ is an arbitrary polynomial of indicated degree.

(c) If $k > 0$, there exists a unique solution to Eq. (1) if and only if the $k$ constraints in Eq. (6) are satisfied, and this solution is given by Eq. (5).

Theorem 1 incorporates all of the classical results [4, 5], obtained here by what we believe to be more elegant and familiar methods. We now turn to applications.

## 3. - APPLICATIONS.

We consider first some cases in which $X$ can be constructed by inspection.

THEOREM 2. Let

$$\frac{\Lambda^+(t)}{\Lambda^-(t)} = \frac{\Omega^+(t)}{\Omega^-(t)} \tag{7}$$

where $\Omega^\pm(t)$ are functions that can be continued analytically into $S^\pm = \{z \in \mathbb{C} : |z| \lessgtr 1\}$, respectively. We sall denote the analytic continuation by $\Omega^\pm(z)$. Define

$$X_0(z) = \frac{1}{\Omega^+(z)}, \; z \in S^+ ; \quad X_0(z) = \frac{1}{\Omega^-(z)}, \; z \in S^- .$$

Suppose $z^l X_0(z) \to \text{cont} \neq 0$ as $|z| \to \infty$. Then.

(a) $l$ is the index, $k$, of the singular integral equation (1).

(b) If $l < 0$, the solutions are given by

$$A_{X_0}(t) + \frac{P_{-l-1}(t)}{X_0^+(t)\,\Lambda^-(t)} .$$

(c) If $l > 0$, define $X_1^+(z) = z^l X_0(z)$. Then the solution is given by $A_{X_1}(t)$, provided $l$ constraints, as in Theorem 1, are satisfied.

(Note that the weight function $W(t)$ in Eqs. (5) and (6) is defined by (4b) with, in this case, $X = X_1$).

EXAMPLE 1.

$$\frac{A^+(t)}{A^-(t)} = \frac{\prod_{i=1,\,|c_i|>1}^{N_1} (c_i - t) \prod_{k=1,\,|c_k|<1}^{N_2} (c_k - t)}{\prod_{l=1,\,|d_l|>1}^{M_1} (d_l - t) \prod_{n=1,\,|d_n|<1}^{M_2} (d_n - t)}$$

Then

$$\Omega_- = \frac{\prod_{k=1,\,|c_k|<1}^{N_2} (c_k - t)}{\prod_{n=1,\,|d_n|<1}^{M_2} (d_n - t)}$$

$$\Omega_+ = \frac{\prod_{l=1,\,|d_l|>1}^{M_1} (d_l - t)}{\prod_{i=1,\,|c_i|>1}^{N_1} (c_i - t)}$$

and $l = k = N_2 - M_2$.

EXAMPLE 2. Consider

$$f(t) = tA(t) + \frac{c}{\pi i} P \oint \frac{A(\nu)}{\nu - t} d\nu, \tag{8}$$

where $c$ is a complex number such that $|c| \neq 1$.

Now $A^\pm(t) = t \pm c$, so

$$\frac{A^+(t)}{A^-(t)} = \frac{t + c}{t - c}.$$

CASE 1. $|c| > 1$. We have from Theorem 2

$$X_0(z) = \frac{z+c}{z-c}, \quad z \in S+, \qquad X_0(z) = 1, \quad z \in S- .$$

Then (Eqs. (4))

$$W(t) = \frac{\eta(t) X_0^+(t)}{\Lambda^+(t)} = \frac{c}{\pi i}\left(\frac{1}{t-c}\right)$$

and

$$N(\nu) = \eta(t) X_0^+(t) \Lambda^-(t) = \frac{c}{\pi i}(\nu + c) .$$

Computing $A_{X_0}$ from (5) gives immediately (noting that $k = 0$)

$$A(\nu) = \frac{\nu f(\nu)}{\nu^2 - c^2} + \frac{c}{\pi i(\nu + c)} \, P \oint \frac{f(t)}{(t-c)(\nu - t)} \, dt .$$

CASE 2. $|c| < 1$. Now Theorem 2 tells us that

$$X_0(z) = 1, \quad z \in S+; \quad \text{and} \quad X_0(z) = \frac{z-c}{z+c}, \; z \in S- .$$

Again one finds easily from Eq. (5) (again $k = 0$)

$$A(\nu) = \frac{\nu f(\nu)}{\nu^2 - c^2} + \frac{c}{\pi i(\nu + c)} \, P \oint \frac{f(t)}{(t+c)(\nu - t)} \, dt .$$

The condition $|c| \neq 1$ is enforced so that $\Lambda^{\pm}(t)$ will be non-zero on $S^1$. This, incidentally, is Example 3, Sec. 4 of Ref. 5.

If we now relax the requirement that $\Lambda^{\pm}$ will be non-zero, we can treat the case $|c| = 1$ by a limiting procedure. We rewrite Eq. (8) as

$$f(t) = t A(t) + \frac{c(1+\varepsilon)}{\pi i} \, P \oint \frac{A(\nu)}{\nu - t} \, d\nu, \quad \varepsilon > 0, \quad |c| = 1, \tag{9}$$

and plan to take the limit as $\varepsilon \downarrow 0$ at the end of the calculation. As in Case 1, the solution to Eq. (9) is given by

$$A(\nu) = \frac{\nu f(\nu)}{\nu^2 - c^2(1+\varepsilon)^2} + \frac{c(1+\varepsilon)}{\pi i(\nu + c(1+\varepsilon))} \oint \frac{f(t)}{(t - c(1+\varepsilon))(\nu - t)} \, dt .$$

Taking the limit as $\varepsilon \downarrow 0$ (using the formulas of Plemelj [4]) gives

$$A(\nu) = \frac{\nu f(\nu) - c f(c)}{\nu^2 - c^2} + \frac{c}{\pi i (\nu + c)} \; P \oint \frac{f(t)}{(t - c)(\nu - t)} \, dt \, .$$

This agrees with the result obtained in Ref. 5 (Eq. (24a)) by a much more complicated method. (Sign disagreements between our results and those of Ref. 5 exist in certain cases, because the integrals there are taken in the negative sense with respect to $S^1$, while ours are taken in the positive sense). This last example is an illustration of a so-called « non-normal » equation, which simply means that either $A^+(t)$ or $A^-(t)$ or both vanish somewhere on the contour of integration (but not at the same point to the same order).

It appears that problems might arise if $A^{\pm}$ are irrational, but this is not necessarily the case, since only sectional analyticity is required (Theorem 2).

EXAMPLE 3. Consider

$$\frac{A^+(t)}{A^-(t)} = \sqrt{\frac{t - k_1}{t - k_2}} \, .$$

CASE 1. $|k_1| < 1$, $|k_2| < 1$. The function

$$X_0(z) = 1 \, , \quad z \in S^+ ; \quad X_0(z) = \sqrt{\frac{z - k_2}{z - k_1}} \, , \quad z \in S^- .$$

follow from Theorem 2 with $k = 0$.

CASE 2. $|k_1| > 1$, $|k_2| > 1$. Now

$$X_0(z) = \sqrt{\frac{z - k_1}{z - k_2}} \, , \quad z \in S^+ ; \quad X_0(z) = 1 \, , \quad z \in S^- ,$$

is calculated from Theorem 2, again with $k = 0$.

CASE 3. $|k_1| > 1$, $|k_2| < 1$. This case can be treated by a limiting procedure. We illustrate the idea with a somewhat simpler example.

EXAMPLE 4. Consider

$$f(t) = \frac{\sqrt{t+1}}{2} A(t) + \frac{1}{2\pi i} \oint (\sqrt{\nu - 1}) \frac{A(\nu)}{\nu - t} d\nu , \tag{10}$$

so

$$\Lambda^+(t) = \sqrt{t}, \qquad \Lambda^-(t) = 1 .$$

The winding number for this problem is $\frac{1}{2}$ (we avoid the term « index » here because of the possibility of confusion with the Fredholm index on some $L_p$-space, which is equal to the winding number of $\Lambda^+/\Lambda^-$ only when $\Lambda^+/\Lambda^-$ is continuous and non-zero on the unit circle [7]).

Since this equation cannot be solved as it stands, we introduce two complex numbers $\alpha$ $|\alpha_+| < 1$, $|\alpha_-| > 1$; at the end of the calculation we shall take $\alpha_\pm \to \alpha$, an arbitrary point on $S^1$. Then the equation

$$f(t) = \frac{1}{2}\left(\sqrt{t \frac{t-\alpha_-}{t-\alpha_+}} + 1\right) + \frac{1}{2\pi i} P \oint \left(\sqrt{\nu \frac{\nu-\alpha_-}{\nu-\alpha_+}} - 1\right) \frac{A(\nu)}{\nu - t} d\nu \tag{11}$$

is the same as (10) as $\alpha_\pm \to \alpha$. But (11) can be solved as in Example 3. In particular,

$$\Lambda^+(t) = \sqrt{t \frac{t-\alpha_-}{t-\alpha_+}}, \qquad \Lambda^-(t) = 1 .$$

So applying Theorem 2 once more we obtain

$$X_0(z) = \sqrt{z-\alpha_-}, \quad z \in S^+; \qquad X_0(z) = \sqrt{\frac{z-\alpha_+}{z}}, \quad z \in S^- ,$$

and again we have $k = 0$. The evaluation of the solution is somewhat complicated and it turns out that the final result (after taking the limit) is not unique, i.e. the homogeneous equation has non-trivial solutions. The details may be found in Ref. 9.

In conclusion, let us note that for many problems the function $X$ cannot be constructed by inspection, so that the standard algorithms [4, 5] must be used. For example, let

$$\frac{\Lambda^+(t)}{\Lambda^-(t)} = 2 - t^2, \quad 0 \le \arg t \le \pi; \qquad \frac{\Lambda^+(t)}{\Lambda^-(t)} = 1, \quad \pi \le \arg t \le 2\pi .$$

Then $\Lambda^+(t)/\Lambda^-(t)$ is continuous and non-vanishing on the circle, but $X(z)$ cannot be constructed by inspection. However, the standard formula [4, 5] which for $k=0$ takes the form

$$X(z)) = \exp \left\{ \frac{1}{2\pi} \oint ln \frac{\Lambda^+(t)/\Lambda^-(t)}{z-t} dt \right\},$$

can be used, after which the formalism described in Section 2 can be applied directly [10].

Hopefully, the reader will agree that the claims made in the introduction have been justified. (We also note that an equation not previously worked out in detail, Example 4 above, has been treated). Further details and more examples may be found in Ref. 9.

## 4. - Acknowledgements.

The authors wish to thank Professors Joseph Ball, Robert Bowden, Stefano Paveri-Fontana, Kenneth Case, Ivar Stakgold and Barton Willis for useful discussions. This research was supported by the U.S. Department of Energy Grant. No. DE-FG-05-87ER25033 and grant from the U.S. National Science Foundation, No. DMS-8701050 and DMS-8823102.

Summary. — Construction of solutions to singular integral equations on closed contours is carried out using a generalization of the orthogonality relations of transport theory. Considerable simplification is achieved over the classical (Hilbert transform) approach.

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[10] In this case $k$ is defined to be the winding number of $\Lambda^+/\Lambda^-$ around $S^1$ If $k \neq 0$, the formulas are somewhat more complicated. See, for example, Ref. 4.