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Stationary Solutions of the Non-linear Boltzmann Equation in a Bounded Spatial Domain

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A contraction mapping (or, alternatively, an implicit function theory) argument is applied in combination with the Fredholm alternative to prove the existence of a unique stationary solution of the non-linear Boltzmann equation on a bounded spatial domain under a rather general reflection law at the piecewise C^1 boundary. The boundary data are to be small in a weighted L_∞-norm.

1. Introduction

By far most existence and uniqueness theory for the solution of the non-linear Boltzmann equation has been developed for the time-dependent equation. Not many results are available on the well-posedness of the stationary problem. The best results so far are the ones published by Ukai and Asano[28–30] who have proved the unique solvability of the stationary non-linear Boltzmann equation for a gas confined to the exterior of a bounded convex body with piecewise C^1 boundary, under a variety of boundary conditions, as well as the convergence of the solution of the time-dependent problem to the corresponding steady state as time goes to infinity. Another important source of results is provided by Maslova,[21] who gives results for bounded domains and exterior regions of bounded domains and for general boundary conditions, but without detailed proofs. The oldest results on the subject have been obtained for finite-slab domains and vacuum boundary conditions by Young-Ping Pao.[26] His method has recently been analyzed thoroughly by Cercignani and Palczewski[17] who have applied the Fredholm alternative to retrieve the well-posedness result for the linearized problem used in Pao's paper. A simple one-dimensional model problem was studied by Cercignani[6] in an L_1 space. In all these papers the boundary data have to be small.

Let us give a short outline of the prevailing method. We consider the stationary non-linear Boltzmann equation

$$\xi \cdot \frac{\partial f}{\partial \mathbf{x}}(x, \xi) = (Lf)(x, \xi) + \nu(\xi)[\Gamma(f, f)](x, \xi), \quad \mathbf{x} \in \Omega, \ \xi \in \mathbb{R}^3, \tag{1}$$

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with boundary condition

$$f(x,\xi)=[\mathbb{B}f](x,\xi)+g(x,\xi),\quad (x,\xi)\in\partial\Omega\times\mathbb{R}^3,\ \mathbf{n}(\mathbf{x})\cdot\xi<0. \tag{2}$$

Here Ω is a bounded open set in $\mathbb{R}^3$ with the same interior as its closure, $\partial\Omega=\bar{\Omega}\setminus\Omega$ its boundary and $\mathbf{n}(\mathbf{x})$ the unit outer normal at $\mathbf{x}\in\partial\Omega$. This set Ω represents the spatial region occupied by the gas and ξ denotes the velocity of a particle. Further, L is the linearized collision operator obtained when linearizing the non-linear Boltzmann equation about a global Maxwellian, $\nu(\xi)[\Gamma(\cdot,\cdot)](\mathbf{x},\xi)$ is the residual non-linear collision operator and $\nu(\xi)$ is the collision frequency. Condition (2) describes (partial) reflection at the spatial boundary. The boundary operator $\mathbb{B}$ maps functions on $\mathbb{D}_+=\{(\mathbf{x},\xi)\in\partial\Omega\times\mathbb{R}^3\colon\ \mathbf{n}(\mathbf{x})\cdot\xi>0\}$ into functions on $\mathbb{D}_-=\{(\mathbf{x},\xi)\in\partial\Omega\times\mathbb{R}^3\colon \mathbf{n}(\mathbf{x})\cdot\xi<0\}$. The solution $f(x,\xi)$ represents the deviation of the particle distribution from (Maxwellian) equilibrium.

The method used by Pao[26] and Cercignani and Palczewski[7] is to write equations (1) and (2) in integral form and to exploit the unique solvability of the corresponding problem with $\Gamma\equiv 0$ in combination with the implicit function theorem. Introducing two suitable linear operators U and U_0 and decomposing L as

$$L=-\nu+K, \tag{3}$$

where K is an integral operator, they obtained the vector equation

$$f-UKf=U_0g+U\nu\Gamma(f,f). \tag{4}$$

Writing this equation as $G(f,g)=0$, computing the Fréchet derivative $G_f(0,0)$ and applying the implicit function theorem they proved the existence of a unique solution of equation (4) for sufficiently small $\mathbf{g}$, using that $\mathbb{1}-UK$ is boundedly invertible. The latter amounts to the unique solvability of the corresponding stationary linearized problem. The technical implementation of the method requires an estimate of the type

$$\|\Gamma(f_1,f_2)\|_r\leqslant c_r\|f_1\|_r\|f_2\|_r \tag{5}$$

on a suitable Banach space of real functions on $\Omega\times\mathbb{R}^3$ and the invertibility of $\mathbb{1}-UK$ on the same space. The estimate (5) in the Banach space of measurable functions $f(\xi)$ such that $(1+|\xi|^2)^{r/2}f(\xi)$ with $r\geqslant 1$ is essentially bounded is due to Grad.[10] Alternatively, one may use the contraction mapping principle, whose application also relies on the invertibility of $\mathbb{1}-UK$. Thus, apart from a rather simple non-linear argument and an estimate which may be found in the literature, the boundary value problem (1), (2) is a linear problem.

Let us give an outline of the history of the linear theory, skipping publications primarily devoted to BGK models. Well-posedness results on the linearized Boltzmann equation for bounded regions and conservative boundaries first appeared in the early but often overlooked work of Guiraud.[14–16] His results were improved considerably by Ukai and Asano[27] for exterior domains of bounded convex bodies and by Maslova[20, 23] for unbounded regions. For plane-parallel domains there are more refined results. We mention the results presented by Maslova[22] and Bardos *et al.*[2] which may, in fact, be obtained as a direct consequence of the abstract kinetic theory of Beals,[3] Greenberg *et al.*[11, 12, 25] and van der Mee.[24] All these results have been obtained in an L_2 space, although some of them can also be formulated in a different L_p setting using a Fredholm argument (cf. Reference 11, Section VI.6). It should be observed that all these results have been given for Maxwellian or hard interactions with angular cut-off. Abstract kinetic theory may also be applied to obtain results on plane-parallel domains for Maxwellian or hard interactions with radial potential cut-off, though the existence and uniqueness theory for linearized Boltzmann equations with soft interactions remains open.

In Sections 2 and 3 we will study the well-posedness of the linearized equation on a number of function spaces, including those where the estimate (5) is known to be valid. Contrary to the approach of Reference 7, where a direct proof of the power compactness of UK is given, we will first exploit a well-known Grad estimate[9] to prove K to be power compact and then pass to the power compactness of UK. Here we will generalize compactness results of Grad[9] and Drange[8] for K on an L_2 space to arbitrary (weighted and unweighted) L_p spaces. At the same time we will generalize the existence and uniqueness theory for the stationary linearized Boltzmann equation to an arbitrary L_p setting with rather general reflective boundary conditions. In Section 4 we will solve the non-linear problem (1), (2) for small data g, again under reflective boundary conditions. However, since the estimate (5) is only known to be true in a (weighted) L_∞ space, our non-linear results will be restricted to this L_∞ setting. Throughout the paper we will assume that the collision frequency is bounded below by a positive constant, which amounts to assuming a Maxwellian or hard intermolecular interaction.[9, 5]

We remark that, under suitable constraints, all our results remain valid if the bounded stationary domain Ω is replaced by the finite slab $\{(x, y, z) \in \mathbb{R}^3 \colon 0 \leqslant z \leqslant L)$ with the boundary data g and the solution f independent of the transverse coordinates y and z. At the end of this paper we will indicate how to derive this result also.

2. Auxiliary linear theory: compactness properties of K

As shown by Grad,[9] the operator K is an integral operator with kernel $k(\xi, \xi_1)$ satisfying

$$|k(\xi, \xi_1)| \leqslant \frac{c}{|\xi - \xi_1|} \exp\left(-\alpha|\xi - \xi_1|^2 - \alpha \frac{(\xi^2 - \xi_1^2)^2}{|\xi - \xi_1|^2} \right) \text{ for some } \alpha > 0. \tag{6}$$

This estimate is obtained under the assumption that the intermolecular potential satisfies $0 \leqslant B(\theta, V) \leqslant b_1 |\sin\theta\cos\theta| \{V + V^{\varepsilon - 1}\}$ for some $\varepsilon < 1$. The collision kernel can then be written as the difference of two integral kernels $k_2(\xi, \xi_1)$ and $k_1(\xi, \xi_1)$ which satisfy $0 \leqslant k_1(\xi, \xi_1) \leqslant a_1 \{|\mathbf{\eta} - \xi| + |\mathbf{\eta} - \xi|^{\varepsilon - 1}\}\ \exp\{-\tfrac{1}{4}(\xi^2 + \mathbf{\eta}^2)\}$ and $0 \leqslant k_2(\xi, \xi_1) \leqslant a_2 |\mathbf{\eta} - \xi|^{-1} \exp\{-\tfrac{1}{8}|\mathbf{\eta} - \xi|^2 - \tfrac{1}{8}[(\mathbf{\eta}^2 - \xi^2)^2/|\mathbf{\eta} - \xi|^2]\}$ [cf. Reference 13, equations (55)–(59)], so that $k(\xi, \xi_1)$ itself satisfies (6). In this section we will prove that all integral operators satisfying the estimate (6) are compact on $L_p(\mathbb{R}^3)$ for $1 < p < \infty$ and have a compact square on $L_p(\mathbb{R}^3)$ for $p = 1$ and $p = \infty$. To start with the elementary offshoots of equation (6), we note that the estimates

$$|k(\xi, \xi_1)| \leqslant l(\xi - \xi_1) \overset{\text{def.}}{=} \frac{c}{|\xi - \xi_1|} \exp\{-\alpha|\xi - \xi_1|^2\} \text{ for some } \alpha > 0 \tag{7}$$

and

$$\int_{\mathbb{R}^3} l(\mathbf{t})\,\mathrm{d}^3\mathbf{t} = \int_{\mathbb{R}^3} \frac{1}{|\mathbf{t}|} \mathrm{e}^{-\alpha t^2}\,\mathrm{d}^3\mathbf{t} = 4\pi \int_0^\infty r\mathrm{e}^{-\alpha r^2}\,\mathrm{d}r = \frac{2\pi}{\alpha} < +\infty \tag{8}$$

imply that K is bounded on the function spaces $L_p(\mathbb{R}^3)$ $(1 \leqslant p \leqslant \infty)$ with $2\pi/\alpha$ as an upper bound for the norm of K.

We will now prove the following result.

Theorem 1. *The operator K is compact on $L_p(\mathbb{R}^3)$ if $1 < p < \infty$, and has a compact square if $p = 1$ or $p = \infty$.*

Proof. Consider the convolution operator

$$(L_n f)(\xi) = \int_{\{|\xi_1| \leqslant n\}} \frac{c}{|\xi - \xi_1|} \exp\{-\alpha|\xi - \xi_1|^2\} f(\xi_1)\, d\xi_1.$$

Observing that the convolution kernel belongs to $L_1(\mathbb{R}^3)$,[8] we approximate it in the L_1 norm by measurable step functions of compact support. As each such step function is a linear combination of characteristic functions on a compact Borel set E of $\mathbb{R}^3$, it is sufficient to prove the compactness of the operator

$$(L_E f)(\xi) = \int_{\{|\xi_1| \leqslant n\}} \chi_E(\xi - \xi_1) f(\xi_1)\, d\xi_1.$$

However, since $L_E \mathbf{f}$ has compact support, we may again apply Theorem 3 (2.X) of Reference 17 to derive the compactness of L_E on $L_p(\mathbb{R}^3)$ $(1 \leqslant p < \infty)$. Its compactness on $L_\infty(\mathbb{R}^3)$ is obtained by 'dualizing' the corresponding L_1 result, which proves the compactness of L_n. Using $0 \leqslant |K_n| \leqslant L_n$, we find that K_n is weakly compact on $L_1(\mathbb{R}^3)$ [cf. Reference 13, Proposition 2.1(b)] and hence that its square is compact on $L_1(\mathbb{R}^3)$. By taking the dual we find the compactness of K_n^2 on $L_\infty(\mathbb{R}^3)$.

If $1 < p < \infty$ and $k_n(\xi, \xi_1) = k(\xi, \xi_1)\chi_n(\xi_1)$ where $\chi_n(\xi) = 1$ for $|\xi| \leqslant n$ and $\chi_n(\xi) = 0$ for $|\xi| > n$, we use the square integrability of $k_n(\xi, \xi_1)$ and compact interpolation (i.e. Reference 18, Theorem 3.10 or 3.11) to prove the compactness of K_n on $L_p(\mathbb{R}^3)$ where $1 < p < \infty$.

So far we have based our boundedness and compactness results for K on the estimate (7). In combination with the principle of dominated convergence, this estimate allows one to prove that K_n converges to K in the strong topology of each $L_p(\mathbb{R}^3)$ $(1 \leqslant p \leqslant \infty)$, which is clearly insufficient for proving the (power) compactness of K. We will need the stronger estimate (6) to prove the convergence of K_n to K in the operator norm. In order to do so, we introduce

$$B(w) = 2\pi \int_0^\infty \int_{-1}^1 r \exp\{-\alpha r^2 - \alpha(r + 2w\cos\theta)^2\}\, d(\cos\theta)\, dr, \tag{9}$$

which is obtained by integrating the right-hand side of the inequality (6), with $c = 1$, with respect to $\xi \in \mathbb{R}^3$ and putting $r = |\xi - \xi_1|$, $\cos\theta = \{(\xi - \xi_1)\cdot\xi_1\}/r|\xi_1|$ and $w = |\xi_1|$. Then $\sup\{B(\mathbf{w})\colon w \geqslant n\}$ is an upper bound for the norm of $K - K_n$ on both $L_1(\mathbb{R}^3)$ and $L_\infty(\mathbb{R}^3)$ and hence, by interpolation, also for the norm of $K - K_n$ on the intermediate spaces $L_p(\mathbb{R}^3)$. By making the change of variables $(r, \cos\theta) \to (r, \tau)$ in the double integral (9) where $\tau = r + 2w\cos\theta$, and using $\operatorname{erf}(z) = 2\pi^{-1/2}\int_0^z \exp(-t^2)\, dt$, we obtain

$$B(w) = \frac{2}{w}\left(\frac{\pi}{\alpha}\right)^{1/2} \int_0^\infty r e^{-\alpha r^2}(\operatorname{erf}(r + 2w)\sqrt{\alpha}) - \operatorname{sgn}(r - 2w)(\operatorname{erf}(|r - 2w|\sqrt{\alpha}))\, dr.$$

From this equation we trivially derive

$$\lim_{w \to \infty} B(w) = 0,$$

implying

$$\lim_{n \to \infty} \| K - K_n \|_{L_p(\mathbb{R}^3)} = 0, \ 1 \leqslant p \leqslant \infty,$$

which completes the proof. □

It is an easy exercise to prove that for every real r the function $k^r(\xi, \xi_1) = (1 + |\xi|^2)^{r/2} k(\xi, \xi_1)(1 + |\xi_1|^2)^{-r/2}$ satisfies the estimate (6) whenever $k(\xi, \xi_1)$ does. Consequently, K is a power compact operator (compact if $1 < p < \infty$ and with a compact square if $p = 1$ or $p = \infty$) on the space $L_{p,r}(\mathbb{R}^3)$ of those Lebesgue measurable functions $f(\xi)$ on $\mathbb{R}^3$ for which $(1 + |\xi|^2)^{r/2} f(\mathbf{x}, \xi)$ belongs to $L_p(\mathbb{R}^3)$. Here the norm of $L_{p,r}(\mathbb{R}^3)$ is defined as the norm of $(1 + |\xi|^2)^{r/2} f(\mathbf{x}, \xi)$ on $L_p(\mathbb{R}^3)$.

3. Auxiliary linear theory: compactness properties of $U_0 K$

Let us first consider the simple boundary value problem

$$\xi \cdot \frac{\partial f}{\partial \mathbf{x}} + v(\mathbf{x}, \xi) f(\mathbf{x}, \xi) = F(\mathbf{x}, \xi), \tag{10}$$

$$f(\mathbf{x}, \xi) = g(\mathbf{x}, \xi), \ \mathbf{n}(\mathbf{x}) \cdot \xi < 0. \tag{11}$$

Then it is well-known that for every $F \in L_p(\Omega \times \mathbb{R}^3)$ and $g \in L_p(\mathbb{D}_-; d\tilde{\sigma})$, where $d\tilde{\sigma} = |\mathbf{n}(\mathbf{x}) \cdot \xi| \, d\sigma \, d\xi$ with $d\sigma$ the surface Lebesgue measure on $\partial\Omega$, there exists a unique solution f of equations (10) and (11) in $L_p(\Omega \times \mathbb{R}^3)$ which is non-negative whenever F and g are non-negative and whose 'restrictions' $f_\pm$ to $\mathbb{D}_\pm$ belong to the spaces $L_p(\mathbb{D}_\pm; d\tilde{\sigma})$. Here some remarks are appropriate.

1. We must assume $1 \leqslant p \leqslant \infty$, and $v(\mathbf{x}, \xi)$ must be a non-negative Borel function on $\Omega \times \mathbb{R}^3$ which is integrable on every bounded Borel subset of $\Omega \times \mathbb{R}^3$ and essentially bounded away from zero; we will write v_0 for the essential infimum of $v(\mathbf{x}, \xi)$, so that v_0 is positive.
2. The 'restrictions' $f_\pm$ are defined via a so-called trace theorem, i.e. for every bounded Borel function φ on $\Omega \times \mathbb{R}^3$ that is continuously differentiable along the integral curves of the vector field $\xi \cdot (\partial / \partial \mathbf{x})$ with bounded 'directional' derivative and such that the total length of the integral curves meeting its compact support is bounded away from zero we have the Green's identity

$$\int_\Omega \int_{\mathbb{R}^3} \left(\xi \cdot \frac{\partial f}{\partial \mathbf{x}} \varphi + \xi \cdot \frac{\partial \varphi}{\partial \mathbf{x}} f \right) d\xi \, d\mathbf{x} = \int_{\mathbb{D}_+} f_+ \varphi \, d\tilde{\sigma} - \int_{\mathbb{D}_-} f_- \varphi \, d\tilde{\sigma}.$$

3. We have the equality

$$\int_{\mathbb{D}_+} |f_+|^p \, d\tilde{\sigma} + p \int_\Omega \int_{\mathbb{R}^3} v |f|^p \, d\xi \, d\mathbf{x} = \int_{\mathbb{D}_-} |g|^p \, d\tilde{\sigma} + p \int_\Omega \int_{\mathbb{R}^3} \operatorname{sgn}(f) |f|^{p-1} F \, d\xi \, d\mathbf{x}. \tag{12}$$

4. We will write the solution as

$$f = L_0 g + U_0 F$$

where L_0 and U_0 are suitable linear operators. In fact, the norms of L_0 and U_0 are bounded above by $(p\nu_0)^{-1/p}$ and $(\nu_0)^{-1}$, respectively. Moreover, the operator $L_0^+ g = [L_0 g]_+$ is a contraction and $U_0^+ F = [U_0 F]_+$ is bounded with norm $\leqslant p^{1/p}(\nu_0)^{-1/q}$ where $q = p/(p-1)$. These norm estimates are trivial consequences of equation (12) and Hölder's inequality.

5. If we replace F by νF, put $g=0$ in equation (12) and use Hölder's inequality we easily obtain

$$\max\{p^{-1/p}\|U_0^+\nu F\|_p, \|U_0 \nu F\|_p\} \leqslant \|F\|_{L_p(\Omega \times \mathbb{R}^3;\, d\xi\, dx)},$$

irrespective of the lower bound on ν.

Since the velocity ξ is constant on each integral curve of the above vector field, the functions $L_0 g$ and $U_0 F$ have their support on $\Omega \times \mathscr{V}$, where $\mathscr{V} \subseteq \mathbb{R}^3$, if g and F have their support on $\Omega \times \mathscr{V}$. Since $\mathbb{R}^3$ can be obtained as a union of countably many mutually disjoint bounded Borel sets $\mathscr{V}$ and Ω is bounded, we may solve equations (10) and (11) for $g \in L_\infty(\mathbb{D}_-; d\tilde{\sigma})$ and $F \in L_\infty(\Omega \times \mathbb{R}^3)$ by decomposing the problem into countably many analogous problems on a bounded phase space $\Omega \times \mathscr{V}$ and using the corresponding L_p results. However, because $\nu(\mathbf{x}, \xi)$ is bounded away from zero, the norm upper bounds obtained for L_p spaces have a finite upper limit as $p \to \infty$. Consequently, equations (10) and (11) are also well-posed in an L_∞ setting. In particular, L_0, $\nu_0 U_0$, L_0^+ and $\nu_0 U_0^+$ are now contraction operators and so are $U_0 \nu$ and $U_0^+ \nu$. In the latter case $L_\infty(\Omega \times \mathbb{R}^3; \nu\, d\xi\, dx)$ is one of the spaces involved.

If we replace $L_p(\Omega \times \mathbb{R}^3)$ with $L_{p,r}(\Omega \times \mathbb{R}^3)$ for some $r > 0$ and $1 \leqslant p < \infty$, we merely add a weight $(1+|\xi|^2)^{pr/2}$ to the Lebesgue measure $dx\, d\xi$ which is constant on each integral curve of the vector field $\xi \cdot (\partial/\partial \mathbf{x})$. The corresponding boundary measure $d\tilde{\sigma}_r$ (in the sense of the trace theorem) is given by $d\tilde{\sigma}_r = (1+|\xi|^2)^{pr/2}\, d\tilde{\sigma}$, whereas the Green's identity and the equality (12) remain the same, except for these changes in the measures. Thus none of the previous results is affected when replacing $L_p(\Omega \times \mathbb{R}^3)$ with $L_{p,r}(\Omega \times \mathbb{R}^3)$, not even the upper bounds given for the operators L_0, U_0, L_0^+, U_0^+, $U_0 \nu$ and $U_0^+ \nu$. Similarly, if we consider $L_{\infty,r}(\Omega \times \mathbb{R}^3)$ and write $\mathbb{R}^3$ as an increasing sequence of bounded Borel sets $\mathscr{V}_n$, the reasoning of the preceding paragraph may be used to extend the previous L_∞ results.

A thorough study of trace theorems for equations (10) and (11) with $\nu(\mathbf{x}, \xi) \equiv 0$ was made by Voigt,[31] mainly (but not exclusively) in an L_1 setting and in the context of neutron transport theory. We have followed here the presentation of Beals and Protopopescu[4] (see also Reference 11, Chapters XI–XII) who have constructed a theory of abstract time-dependent kinetic equations. We will rely on these sources in the subsequent discussion of equation (10) with boundary condition (2). First we will derive an important compactness property.

Proposition 2. *The operator $U_0 K$ is compact on $L_p(\Omega \times \mathbb{R}^3)$ if $1 < p < \infty$ and has a compact square on $L_p(\Omega \times \mathbb{R}^3)$ if $p = 1$ or $p = \infty$. Similarly, the operator $U_0^+ K$ is compact (resp. has a compact square) from $L_p(\Omega \times \mathbb{R}^3)$ into $L_p(\mathbb{D}_+; d\tilde{\sigma})$ if $1 < p < \infty$ (resp. if $p = 1$ or $p = \infty$).*

Proof. It suffices to prove the compactness of $U_0 K$ where $\nu(\mathbf{x}, \xi)$ is replaced with a positive and constant lower bound ν_0. Putting tildes on top of the corresponding operators to indicate this change in the collision frequency and putting

$t(\mathbf{x}, \xi) = \inf\{\tau > 0: \mathbf{x} - \tau\xi \notin \Omega\}$ and $l(\mathbf{x}, \xi) = t(\mathbf{x}, \xi) + t(\mathbf{x}, -\xi)$, we obtain

$$(\tilde{U}_0 KF)(\mathbf{x}, \xi) = \int_{\mathbf{x}(\text{in})}^{\mathbf{x}} \frac{1}{|\xi|} e^{-\nu_0(\mathbf{x}-\mathbf{y})\cdot\xi/|\xi|^2}(KF)(\mathbf{y}, \xi)\, d\mathbf{y}, \qquad (\mathbf{x}, \xi) \in \Omega \times \mathbb{R}^3,$$

$$(\tilde{U}_0^+ KF)(\mathbf{x}, \xi) = \int_{\mathbf{x}(\text{in})}^{\mathbf{x}(\text{out})} \frac{1}{|\xi|} e^{-\nu_0(\mathbf{x}-\mathbf{y})\cdot\xi/|\xi|^2}(KF)(\mathbf{y}, \xi)\, d\mathbf{y}, \qquad (\mathbf{x}, \xi) \in \mathbb{D}_+,$$

Here $\mathbf{x}(\text{in}) = \mathbf{x} - \xi t(\mathbf{x}, \xi)$ and $\mathbf{x}(\text{out}) = \mathbf{x} + \xi t(\mathbf{x}, -\xi)$.

Let us apply $\tilde{U}_0 K$ and $\tilde{U}_0^+ \mathrm{K}$ to functions F of the type $F(\mathbf{x}, \xi) = \sum_{i=1}^{n} H_i(\mathbf{x}) f_i(\xi)$ where $H_i \in L_p(\Omega)$ and $f_i \in L_p(\mathbb{R}^3)$. We obtain

$$(\tilde{U}_0 KF)(\mathbf{x}, \xi) = \sum_{i=1}^{n} (Kf_i)(\xi)(L_0 H_i)(\mathbf{x}, \xi),$$

$$(\tilde{U}_0^+ KF)(\mathbf{x}, \xi) = \sum_{i=1}^{n} (Kf_i)(\xi)(L_0^+ H_i)(\mathbf{x}, \xi),$$

where L_0 and L_0^+ and L_0^+ are convolution integral operators along integral curves. It now follows from the density of the finite sums $\sum_{i=1}^{n} H_i(\mathbf{x}) f_i(\xi)$ and the (power) compactness of K in $L_p(\Omega \times \mathbb{R}^3)$ that $\tilde{U}_0 K$ and $\tilde{U}_0^+ K$ are (power) compact. Using that $0 \leqslant |U_0 K| \leqslant \tilde{U}_0 K$ and $0 \leqslant |U_0 K| \leqslant \tilde{U}_0 K$, we easily obtain the (power) compactness of UK and $U_0^+ K$. □

Next we consider the boundary value problem

$$\xi \cdot \frac{\partial f}{\partial \mathbf{x}} + \nu(\mathbf{x}, \xi) f(\mathbf{x}, \xi) = F(\mathbf{x}, \xi), \tag{13}$$

$$f(\mathbf{x}, \xi) = (\mathbb{B}f)(\mathbf{x}, \xi) + g(\mathbf{x}, \xi), \quad \mathbf{n}(\mathbf{x}) \cdot \xi < 0, \tag{14}$$

where we assume the operator $\mathbb{B}$ to be a positive contraction from $L_p(\mathbb{D}_+; d\tilde{\sigma})$ into $L_p(\mathbb{D}_-; d\tilde{\sigma})$. Then every solution f of equations (13) and (14) satisfies

$$f = L_0(\mathbb{B}f_+ + g) + U_0 F,$$

where

$$f_+ - L_0^+ \mathbb{B} f_+ = L_0^+ g + U_0^+ F.$$

Thus if $\mathbb{1} - \mathbb{B}L_0^+$ is invertible (which is equivalent to the invertibility of $\mathbb{1} - L_0^+ \mathbb{B}$ and which is the case if $\|\mathbb{B}\| < 1$), we have

$$f = L_{\mathbb{B}} g + U_{\mathbb{B}} F = L_0(\mathbb{1} - \mathbb{B}L_0^+)^{-1} g + [U_0 + L_0(\mathbb{1} - \mathbb{B}L_0^+)^{-1} \mathbb{B} U_0^+] F. \tag{15}$$

Clearly, the operators $L_{\mathbb{B}}$, $U_{\mathbb{B}}$, $L_{\mathbb{B}}^+ g = [L_{\mathbb{B}} g]_+$ and $U_{\mathbb{B}}^+ F = [U_{\mathbb{B}} F]_+$ are bounded and positive whereas $U_{\mathbb{B}} \nu$ and $U_{\mathbb{B}}^+ \nu$ are bounded from $L_p(\Omega \times \mathbb{R}^3;\ \nu\, d\xi\, d\mathbf{x})$ into $L_p(\Omega \times \mathbb{R}^3)$ and $L_p(D_+; d\tilde{\sigma})$, respectively. Moreover, it is immediate from Proposition 2 and equation (15) that $U_{\mathbb{B}} K$ is compact on $L_p(\Omega \times \mathbb{R}^3)$ if $1 < p < \infty$ and has a compact square if $p = 1$ or $p = \infty$. Here we recall that the collisional invariants correspond to those f for which $(\nu - K) f = 0$. We have

Theorem 3. *Suppose $\mathbb{1} - \mathbb{B}L_0^+$ is boundedly invertible on $L_p(\Omega \times \mathbb{R}^3)$ where $1 \leqslant p \leqslant \infty$, and that no collisional invariant φ satisfies $\varphi_- = \mathbb{B}\varphi_+$. Then for every*

$g \in L_p(D_-; d\tilde{\sigma})$ and $F \in L_p(\Omega \times \mathbb{R}^3; \nu\, d\xi\, dx)$ the boundary value problem

$$\xi \cdot \frac{\partial f}{\partial \mathbf{x}} + \nu(\mathbf{x}, \xi) f(\mathbf{x}, \xi) = (Kf)(\mathbf{x}, \xi) + F(\mathbf{x}, \xi), \tag{16}$$

$$f(\mathbf{x}, \xi) = (\mathbb{B}f)(\mathbf{x}, \xi) + g(\mathbf{x}, \xi), \quad \mathbf{n}(\mathbf{x}) \cdot \xi < 0, \tag{17}$$

has a unique solution f in $L_p(\Omega \times \mathbb{R}^3)$ which satisfies $f_\pm \in L_p(\mathbb{D}_\pm; d\tilde{\sigma})$.

Proof. If f is a solution, we may write

$$f = L_{\mathbb{B}} g + U_{\mathbb{B}}(Kf + F).$$

As $U_{\mathbb{B}}K$ is power compact, it suffices to prove the injectivity of $\mathbb{1} - U_{\mathbb{B}}K$, i.e. the occurrence of only the zero solution if $g = 0$ and $F = 0$. Because $U_{\mathbb{B}}K$ is power compact on each one of the spaces $L_p(\Omega \times \mathbb{R}^3)$, it suffices to prove the uniqueness property for $p = 2$. Using the identity (12) for $p = 2$, $g \to (\mathbb{B}f_-)$ and $F \to Kf$, we obtain

$$\| f_+ \|_2^2 + 2 \| \nu f^2 \|_1 = \| \mathbb{B}f_- \|_2^2 + 2 \| f(Kf) \|_1.$$

As $\nu - K$ is a positive self-adjoint operator on $L_2(\Omega \times \mathbb{R}^3)$, we obtain $f = 0$ if $\| \mathbb{B} \| < 1$. Otherwise we find $(\nu - K)f = 0$ and $\mathbb{B}f_- = f_+$, turning f into a collisional invariant satisfying $\mathbb{B}f_- = \mathbb{B}_+$, a situation which we have excluded. □

Note that Theorem 3 remains valid in $L_{p,r}(\Omega \times \mathbb{R}^3)$ if $1 \leqslant p \leqslant \infty$ and $r > 0$.

If there exist collisional invariants φ with $\varphi_- = \mathbb{B}\varphi_+$, then $f(\mathbf{x}, \xi) \equiv \varphi(\xi)$ is a solution of equations (16) and (17) for $g = 0$ and $F = 0$ which belongs to all spaces $L_p(\Omega \times \mathbb{R}^3)$. In this case equations (16) and (17) are non-uniquely solvable. This situation occurs for specular, reverse specular and diffuse reflection with accommodation coefficient 1.

Another important issue is the invertibility of $\mathbb{1} - \mathbb{B}L_0^+$ for $\| \mathbb{B} \| = 1$, i.e. whether equations (13) and (14) only have the trivial solution if $g = 0$ and $F = 0$. If $1 \leqslant p < \infty$, we use the identity (12) for $g \to \mathbb{B}f_-$ and $F = 0$ and find

$$\| f_+ \|_p^p + p \| \nu |f|^p \|_1 = \| \mathbb{B}f_- \|_p^p$$

while $\mathbb{B}$ is a contraction, so that $f = 0$. Thus $\mathbb{1} - \mathbb{B}L_0^+$ is one-to-one. If $p = \infty$, however, a more subtle argument has to be used. We have to assume that $\mathbb{R}^3$ is the union of an increasing sequence of bounded Borel sets $\mathscr{V}_n$ such that $\mathbb{B}$ maps functions with support on $\mathbb{D}_+ \cap [\Omega \times \mathscr{V}_n]$ into functions with support on $\mathbb{D}_- \cap [\Omega \times \mathscr{V}_n]$. (Examples of such $\mathbb{B}$ are specular and reverse specular reflections which preserve the speed of a particle upon collision with the boundary.) By restricting the velocity variable to $\mathscr{V}_n$, we may repeat the above argument in an L_p, setting with finite p and conclude that $\mathbb{1} - \mathbb{B}L_0^+$ is one-to-one on L_∞ space also. Nevertheless, if $\| \mathbb{B} \| = 1$, this operator is one-to-one but need not have a bounded inverse.

4. Solution of the non-linear boundary value problem

In this section we will solve equations (1) and (2) using the implicit function theorem and the contraction mapping principle. Introducing the Banach space $L_{\infty,r}(\Omega \times \mathbb{R}^3)$ of all Lebesgue measurable functions $f(\mathbf{x}, \xi)$ on $\Omega \times \mathbb{R}^3$ such that $(1 + |\xi|^2)^{r/2} f(\mathbf{x}, \xi)$ is essentially bounded, and endowing this space with the corresponding L_∞ norm $\|\cdot\|_{\infty,r}$, we have the estimate

$$\|\Gamma(f_1, f_2)\|_{\infty,r} \leqslant c_r \|f_1\|_{\infty,r} \|f_2\|_{\infty,r}; \quad f_1, f_2 \in L_{\infty,r}(\Omega \times \mathbb{R}^3), \tag{18}$$

where $r \geqslant 1$. This estimate is valid for hard or Maxwellian interactions as stipulated in the beginning of Section 2 (cf. Reference 10, Appendix). Then any solution f of equations (1) and (2) in $L_{\infty,r}(\Omega \times \mathbb{R}^3)$ satisfies the equality

$$\mathscr{G}(f, g) = f - U_{\mathbb{B}}\{Kf + \nu\Gamma(f, f)\} - L_{\mathbb{B}}g = 0, \tag{19}$$

or the equivalent equality

$$f = \mathscr{H}(f) = (\mathbb{1} - U_{\mathbb{B}}K)^{-1}\{L_{\mathbb{B}}g + U_{\mathbb{B}}\nu\Gamma(f, f)\}. \tag{20}$$

Obviously, there are two strategies to deal with equations (1) and (2). We may apply the implicit function theorem to equation (19) or we may apply the contraction mapping principle to equation (20). In either case we will obtain a unique solution for small data g.

To apply the implicit function theorem we compute the Fréchet derivative (cf. Reference 19, Section II.4) of $\mathscr{G}(f, g)$. Using the symmetry of the bilinear form Γ we obtain

$$\begin{aligned}\|\mathscr{G}(f+h, g) - \mathscr{G}(f, g) - (\mathbb{1} - U_{\mathbb{B}}K)h\|_{\infty,r} &\leqslant \|-U_{\mathbb{B}}\nu\Gamma(h, f+h)\|_{\infty,r} \\ &\leqslant c_r \|U_{\mathbb{B}}\nu\| \, \|h\|_{\infty,r} \|f+h\|_{\infty,r},\end{aligned}$$

so that $\mathscr{G}(f, g)$ has $\mathbb{1} - U_{\mathbb{B}}K$ as its partial Fréchet derivative with respect to the first variable. Under the conditions of Theorem 3, this operator is boundedly invertible on $L_{\infty,r}(\Omega \times \mathbb{R}^3)$. Since equation (19) is satisfied for $f = 0$ and $g = 0$, we may apply the implicit function theorem (cf. Reference 17, Theorem 1 (4.XVII)) and prove that equation (19) has a solution f in $L_{\infty,r}(\Omega \times \mathbb{R}^3)$ satisfying $f_\pm \in E_r^\pm$, provided that $\|g\|_{\infty,r}$ is small enough. This solution depends continuously on g in the norm of $L_{\infty,r}(\Omega \times \mathbb{R}^3)$.

Let us obtain a similar, though slightly more explicit, result by applying the contraction mapping principle (cf. Reference 19, Section IV.1) to equation (20). We first estimate

$$\begin{aligned}\|\mathscr{H}(f_1) - \mathscr{H}(f_2)\|_{\infty,r} &\leqslant \|(\mathbb{1} - U_{\mathbb{B}}K)^{-1}\| \, \|U_{\mathbb{B}}\nu\| \, \|\Gamma(f_1 - f_2, f_1 + f_2)\|_{\infty,r} \\ &\leqslant c_r \|(\mathbb{1} - U_{\mathbb{B}}K)^{-1}\| \, \|U_{\mathbb{B}}\nu\| \, \|f_1 - f_2\|_{\infty,r} \|f_1 + f_2\|_{\infty,r},\end{aligned}$$

where we have used the symmetry of the bilinear form Γ. Consider the iteration scheme

$$f_0 = 0, \quad f_{k+1} = \mathscr{H}(f_k).$$

Then

$$\begin{aligned}\|f_{k+1}\|_{\infty,r} \leqslant\; & \|(\mathbb{1} - U_{\mathbb{B}}K)^{-1}\| \, \|L_{\mathbb{B}}\| \, \|g\|_{\infty,r} \\ & + c_r \|(\mathbb{1} - U_{\mathbb{B}}K)^{-1}\| \, \|U_{\mathbb{B}}\nu\| \, \|f_k\|^2_{\infty,r}.\end{aligned}$$

Now put

$$C = 2c_r \|(\mathbb{1} - U_{\mathbb{B}}K)^{-1}\|^2 \|U_{\mathbb{B}}\nu\| \, \|L_{\mathbb{B}}\|, \; \delta = 1 - \sqrt{(1 - 2C\|g\|_{\infty,r})}, \tag{21}$$

and assume that $\|g\|_{\infty,r} < (1/2C)$. Then $\|f_k\|_{\infty,r} \leqslant \delta/\{2c_r \|(\mathbb{1} - U_{\mathbb{B}}K)^{-1})\| \, \|U_{\mathbb{B}}\nu\|\}$, so that

$$\|f_{k+m} - f_k\|_{\infty,r} \leqslant \frac{\delta^k - \delta^{k+m}}{1-\delta} \|f_1 - f_0\|_{\infty,r},$$

whence $\|f\|_{\infty,r} \leqslant (1-\delta)^{-1} \|f_1\|_{\infty,r} \leqslant 1/\{4(1-\delta)c_r \|(\mathbb{1} - U_{\mathbb{B}}K)^{-1}\| \, \|U_{\mathbb{B}}\nu\|\}$. Then f is the only solution of equation (20) satisfying this bound.

We have

Theorem 4. *Suppose $\mathbb{1} - \mathbb{B}L_0^+$ is boundedly invertible on $L_{\infty,r}(\Omega \times \mathbb{R}^3)$, and that no collisional invariant φ satisfies $\varphi_- = \mathbb{B}\varphi_+$. Let C be the constant defined in equation (21). Then for $\|g\|_{\infty,r} < (1/2C)$ there exists a solution f of equations (1) and (2) in $L_{\infty,r}(\Omega \times \mathbb{R}^3)$ satisfying $f_\pm \in L_{\infty,r}(D_\pm;\ \mathrm{d}\tilde{\sigma}_r)$ and this function is the only solution of equations (1) and (2) satisfying the bound $\|f\|_{\infty,r} \leqslant 1/\{4(1-\delta)c_r \|(\mathbb{1} - U_{\mathbb{B}}K)^{-1}\| \, \|U_{\mathbb{B}}\nu\|\}$.*

If $\|\mathbb{B}\| < 1$, the conditions of the theorem are fulfilled. The theorem does not apply to specular, reverse specular and diffuse reflection with accommodation coefficient 1.

5. Concluding remark

If $\Omega = \mathbb{R}^2 \times [-a, a]$, a plane-parallel layer of finite thickness $2a$, and the function $g(\mathbf{x}, \boldsymbol{\xi})$ and the operator $\mathbb{B}$ appearing in the boundary condition (2) are independent of the transverse co-ordinates x and y, the operator U_0K becomes a convolution operator on $[-a, a]$ whose kernel is a (power) compact operator on $L_p(\mathbb{R}^3)$. It is well known that such an operator is (power) compact on $L_p([-a, a] \times \mathbb{R}^3)$ (cf., for instance, Reference 11, Proposition VIII.3.1; though this result restricts itself to the compact case, its proof can be easily modified to the weakly compact case). The (power) compactness of U_0^+K may be established in the same way. As a result we find existence results for the solution of the stationary non-linear Boltzmann equation in plane-parallel geometry under conditions which are analogous to those of Section 4. In this way we may reproduce the results in References 26 and 7, and generalize them to reflective boundary conditions of norm less than 1.

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