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RELATIONS FOR THE ELEMENTS OF MATRICES APPEARING IN POLARIZED LIGHT SCATTERING

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ABSTRACT

Relations for the elements of the scattering and phase matrices relevant to the single and multiple scattering of polarized light are given in some detail. After a Fourier decomposition of the phase matrix with respect to azimuth has been made, some details of the expansion coefficients are presented.

1. INTRODUCTION

Studying the scattering of electromagnetic radiation by independently scattering particles is important in astrophysics. A great deal of knowledge may be obtained by analyzing the radiation scattered by particles in the atmospheres of planets, satellites and other objects, especially by taking polarization into account. Single scattering of a polarized beam of light by a particle is generally described by a 4×4 matrix, the scattering matrix, whose elements are 16 functions of wavelength and the directions of incidence and scattering. When writing down the equation of transfer of polarized light, however, one must pre- and postmultiply this matrix by suitable rotation matrices to get the integral kernel of the "collision term", which also is a 4×4 matrix, the so-called phase matrix. Finally, when doing an actual analysis of the equation of transfer one usually applies Fourier decomposition with respect to azimuth as a result of which the phase matrix is written as a sum of separated terms involving generalized spherical functions and the so-called Greek expansion coefficients. We are in fact engaged in a long-term project concerning various matrices relevant to the scattering of polarized light. The main purpose of this project is to unravel the internal structure of such matrices and to study relationships between them. The present contribution is an interim report primarily aimed at

(i) giving a summary of the main results obtained so far;
(ii) presenting some new results;
(iii) outlining some ideas for future research.

Let us outline some of the history of the present topic of research. After polarization parameters were introduced in different but equivalent ways by Chandrasekhar[1] and Van de Hulst[2], the latter posed the problem of investigating conditions satisfied by the elements of the scattering matrix. The problem was solved in a satisfactory way by Hovenier et al.[3] for the scattering matrix and by Hovenier and Van der Mee[4] for the phase matrix. After the observation by Kuščer and Ribarič[5] that the equation of transfer of polarized light displays certain symmetries of the rotation group, azimuth decompositions yielding complex component equations were frequently employed (see Ref. 6, for instance). A more economical decomposition in terms of real component equations was given by Siewert[7] and derived concisely, using symmetries and matrix algebra, by Hovenier and Van der Mee[8]. The Greek expansion coefficients were never studied systematically. The only available information was given by Germogenova and Konovalov[9] but requires a transformation from results in terms of complex polarization parameters. In this paper we will disclose some of our recent material on this topic[10].

In Sec. 2 we will introduce our major theoretical constructs. These will be used in Sec. 3 at the derivation of conditions for the elements of the scattering and the phase matrix. In Sec. 4 we will discuss the Fourier decomposition with respect to azimuth and finally, in Sec. 5, properties of the Greek expansion coefficients will be considered.

2. SOME IMPORTANT THEORETICAL CONSTRUCTS

The scattering of a simple monochromatic wave by an arbitrary particle may be described by means of a 2×2 amplitude matrix satisfying

$$\begin{bmatrix} E_\ell \\ E_r \end{bmatrix} = \begin{bmatrix} A_2 & A_3 \\ A_4 & A_1 \end{bmatrix} \begin{bmatrix} E_{\ell 0} \\ E_{r0} \end{bmatrix}. \tag{1}$$

Here the positive r-direction, the positive l-direction and the direction of propagation of the wave span a Cartesian coordinate system, E_ℓ and E_r represent the electric field components of the scattered wave parallel and perpendicular to the scattering plane, respectively, while $E_{\ell 0}$ and E_{r0} are the analogous quantities relating to the incident wave. In general, the elements of the amplitude matrix are complex functions of the directions of incidence and scattering[2]. For spherical particles we have

$$A_3 = A_4 = 0.$$

The Stokes parameters can now be defined as

$$I = E_l E_l^* + E_r E_r^*$$

$$Q = E_l E_l^* - E_r E_r^*$$

$$U = E_l E_r^* + E_r E_l^*$$

$$V = i(E_l E_r^* - E_r E_l^*)$$

for the scattered wave and analogously for the incident wave where an asterisk denotes the conjugate complex value. Thus we may construct a row vector $\mathbb{I}=\{I,Q,U,V\}$ satisfying $I\geq 0$ and $I=(Q^2+U^2+V^2)^{1/2}$. The relationship between $\mathbb{I}$ and $\mathbb{I}_0$, the Stokes vector of the incident beam, turns out to be linear. Writing the relationship as

$$\mathbb{I} = \mathbb{F}(\theta)\mathbb{I}_0,$$

where $\mathbb{F}(\theta)$ is the scattering matrix, we find[2]

$$\mathbb{F}(\theta) = \begin{bmatrix} a_1 & b_1 & b_3 & b_5 \\ c_1 & a_2 & b_4 & b_6 \\ c_3 & c_4 & a_3 & b_2 \\ c_5 & c_6 & c_2 & a_4 \end{bmatrix} \qquad (2)$$

whose elements can be expressed directly in the real quantities $M_k=A_kA_k^*$, $S_{kj}=\frac{1}{2}(A_kA_j^*+A_jA_k^*)$ and $D_{kj}=\frac{1}{2}i(A_kA_j^*-A_jA_k^*)$. For a spherical particle the diagonality of the amplitude matrix gives immediately

$$\mathbb{F}(\theta) = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_4 & b_2 \\ 0 & 0 & -b_2 & a_4 \end{bmatrix} \qquad (3)$$

where

$$a_1 = \tfrac{1}{2}(M_1 + M_2), \quad b_1 = \tfrac{1}{2}(-M_1 + M_2), \quad a_4 = S_{21}, \quad b_2 = -D_{21}$$

and therefore

$$a_1 = (b_1^2 + b_2^2 + a_4^2)^{1/2} \geq 0. \qquad (4)$$

From the scattering matrix we now define the phase matrix. Let us consider a plane-parallel atmosphere illuminated at the top and let us specify directions by means of $-1 \leq u \leq 1$ (cosine of the angle with the downward normal) and $0 \leq \varphi \leq 2\pi$ (azimuth measured clockwise when viewing upward). Using the meridian plane as the plane of reference for the Stokes parameters of a beam, the light scattered from a direction (u', φ') into a direction (u, φ) is decribed by the phase matrix $\mathbb{Z}(u, u', \varphi-\varphi')$ defined by[8]

$$\mathbb{Z}(u, u', \varphi-\varphi') = \mathbb{L}(\pi-\sigma_2)\mathbb{F}(\theta)\mathbb{L}(-\sigma_1) \tag{5}$$

where the rotation matrix $\mathbb{L}(\alpha)$ has the form

$$\mathbb{L}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and σ_1, σ_2 and θ are the angles and ϑ_1, ϑ_2 and $(\varphi'-\varphi)$ are the opposite sides of a spherical triangle if $0<\varphi'-\varphi<\pi$. Here $u=-\cos\vartheta_1$ and $u'=-\cos\vartheta_2$.[11]

In reality light consists of many simple waves in rapid succession. In this case[1,2] the Stokes parameters I, Q, U and V of the beam may be obtained from those of the constituent simple waves by addition. As a result,

$$I \geq (Q^2 + U^2 + V^2)^{1/2} \geq 0. \tag{6}$$

We call $p=(Q^2+U^2+V^2)^{1/2}/I$ the degree of polarization, with $p=0$ for completely unpolarized light and $p=1$ for completely polarized light. Further, $p_l=(Q^2+U^2)^{1/2}/I$ is the degree of linear polarization and $p_c=V/I$ is the degree of circular polarization. In an atmosphere with many independent scatterers every infinitesimal volume-element has a scattering matrix obtained by summing up the scattering matrices of its constituent particles. Using symmetries[2], such as having randomly oriented optically inactive particles each with a plane of symmetry or having particles and their mirror particles in equal numbers and with random orientation, we may simplify the scattering matrix and obtain one of the form

$$\mathbb{F}(\theta) = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_4 \end{bmatrix}. \tag{7}$$

Thus in putting the individual particles together to get an infinitesimal volume of the assembly the quantities b_3, b_4, b_5, b_6, c_3, c_4, c_5 and c_6 cancel out while the combinations (b_1-c_1) and (b_2+c_2) vanish. For such a special scattering matrix valid in many cases of astrophysical interest we find

$$\mathbb{Z}(u,u',\varphi-\varphi') =$$

$$\begin{pmatrix} a_1 & b_1C_1 & -b_1S_1 & 0 \\ b_1C_2 & C_2a_2C_1-S_2a_3S_1 & -C_2a_2S_1-S_2a_3C_1 & -b_2S_2 \\ b_1S_2 & S_2a_2C_1+C_2a_3S_1 & -S_2a_2S_1+C_2a_3C_1 & b_2C_2 \\ 0 & -b_2S_1 & -b_2C_1 & a_4 \end{pmatrix} \tag{8}$$

where $C_k=\cos 2\sigma_k$ and $S_k=\sin 2\sigma_k$ $(k=1,2)$ and the dependence of a_1, a_2, a_3, a_4, b_1 and b_2 on θ has not been written. For a plane-parallel atmosphere we thus get the equation of transfer[8]

$$u\frac{\partial}{\partial\tau}\mathbb{I}(\tau,u,\varphi) + \mathbb{I}(\tau,u,\varphi) =$$

$$= \frac{a}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} \mathbb{Z}(u,u',\varphi-\varphi')\mathbb{I}(\tau,u',\varphi')d\varphi' du' \tag{9}$$

where $a\in[0,1]$ is the <u>albedo of single scattering</u>, τ is the optical depth (measured down with $\tau=0$ at the top) and $a_1(\theta)$, the <u>phase function</u>, is normalized by

$$\int_{-1}^{1} a_1(\theta)d(\cos\theta) = 2. \tag{10}$$

3. CONDITIONS FOR ELEMENTS OF SCATTERING AND PHASE MATRICES

Using the quantities $M_k = A_k A_k^*$, $S_{kj} = \frac{1}{2}(A_k A_j^* + A_j A_k^*)$ and $D_{kj} = \frac{1}{2}i(A_k A_j^* - A_j A_k^*)$ we easily derive the identities

$$S_{kj}^2 + D_{kj}^2 = M_k M_j \tag{11}$$

$$S_{kt}S_{jt} + D_{kt}D_{jt} = S_{kj}M_t \tag{12}$$

$$D_{kt}S_{jt} - S_{kt}D_{jt} = D_{kj}M_t. \tag{13}$$

These identities may be obtained to derive a large number of identities for the elements of the scattering matrix for a single particle. From the identities of type (11) we obtain

$$(a_3 \pm a_4)^2 + (b_2 \mp c_2)^2 + (b_1 \pm c_1)^2 = (a_1 \pm a_2)^2 \tag{14}$$

$$(b_3 \pm b_4)^2 + (b_5 \pm b_6)^2 + (a_2 \pm b_1)^2 = (a_1 \pm c_1)^2 \tag{15}$$

$$(c_3 \pm c_4)^2 + (c_5 \pm c_6)^2 + (a_2 \pm c_1)^2 = (a_1 \pm b_1)^2. \tag{16}$$

From the various identities of type (12) and (13) we find many other identities for the elements of the scattering matrix of a single particle.

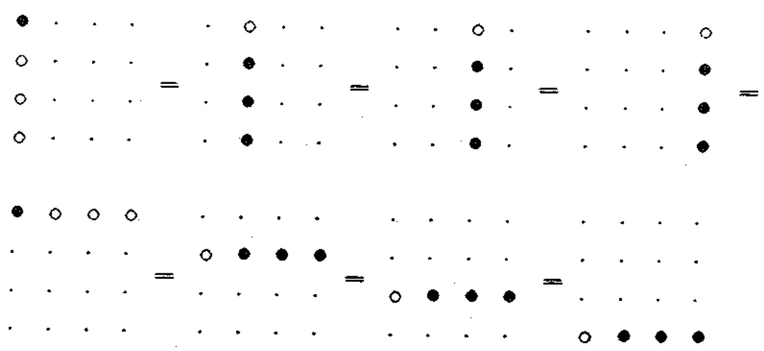
A different way of deriving such equalities is based on the observation that the scattering matrix of a single particle transforms the Stokes vector of a fully polarized beam of light into the Stokes vector of a fully polarized beam. Collecting the coefficients in the identity

$$I^2 - Q^2 - U^2 - V^2 = 0$$

when $\mathbf{I} = \{I,Q,U,V\}$ is given by $\mathbf{I} = F(\theta)\mathbf{I}_0$ and taking for $\mathbf{I}_0$ one of the nine vectors $\{1,\pm1,0,0\}$, $\{1,0,\pm1,0\}$, $\{1,0,0,\pm1\}$, $\{1,1/\sqrt{2},1/\sqrt{2},0\}$, $\{1,1/\sqrt{2},0,1/\sqrt{2}\}$ and

$\{1,0,1/\sqrt{2},1/\sqrt{2}\}$, we obtain 9 identities for the elements of the scattering matrix which completely characterize the property of $F(\theta)$ of transforming the set of Stokes vectors of a fully polarized beam into itself. It can be shown, however, that this property by itself allows for 4×4 matrices of the type (2) which cannot be constructed from an amplitude matrix. Thus the scattering matrix of a single particle has more structure than the structure imposed by the invariance of the set of Stokes vectors of a fully polarized beam.

A comprehensive analysis[3] showed that it is possible to write down 9 equations for the elements of the scattering matrix of a single particle from which all others based on the existence of the amplitude matrix of Eq. (1) may be derived. By algebraic manipulations one may then obtain 7 equations involving only squares of elements and 30 equations involving only products of two different elements. As a matter of fact, all 120 possible products between two distinct elements appear and each such product appears only once. The arrangement of the elements appearing in a particular equation is not arbitrary but turns out to conform to a graphic code. A new example is provided by the figure below which illustrates the 7 equations only involving squares of elements. The sum of the (positive and negative) squares, indicated in each pictogram, is the same.



Interrelations for elements of the scattering matrix

● = (element)^2 ○ = −(element)^2

When we construct the scattering matrix of an infinitesimal volume-element by summing up the scattering matrices of the individual particles, almost all identities are lost. However, we often obtain a matrix of the type (7) whose elements satisfy certain inequalities. These inequalities are obtained by summing up each one of Eqs. (14)-(16) for the constituent particles of the assembly. The result is as follows:

$$|a_2 + b_1| \leq |a_1 + b_1| \tag{17}$$

$$|a_2 - b_1| \leq |a_1 - b_1| \tag{18}$$

$$|a_3 - a_4| \leq |a_1 - a_2| \tag{19}$$

$$(a_3 + a_4)^2 + 4b_2^2 \leq (a_1 + a_2)^2 - 4b_1^2. \tag{20}$$

Using that

$$\max(|p|,|q|) = \tfrac{1}{2}(|p+q| + |p-q|), \tag{21}$$

we find from Eqs. (19)-(20)

$$(a_k^2 + b_1^2 + b_2^2)^{1/2} \leq a_1, \quad k=3,4, \tag{22}$$

where we used [cf. Eq. (4)]

$$a_1 \geq 0. \tag{23}$$

Simple reasoning yields that $|a_2|$, $|b_1|$, $|b_2|$, $|a_3|$ and $|a_4|$ are all dominated by a_1 so that the absolute value bars at the right-hand sides of Eqs. (17)-(19) may be omitted.

The scattering matrix of an assembly of particles must also leave invariant the set of vectors **I**={I,Q,U,V} satisfying the Stokes vector criterion (6). Using this property one easily derives Eqs. (17), (18), (22) and (23). As shown by an example (a_1=8, a_2=6, a_3=4, a_4=0, b_1=$2\sqrt{6}$, b_2=0), this property does not imply (19). The inequality (19) is purely a consequence of the construction of $\mathbb{F}(\theta)$ from the amplitude matrix of the individual particles. Further interesting inequalities for the diagonal elements of $\mathbb{F}(\theta)$ of a volume-element may be derived as follows. Equation (20) gives $|a_3+a_4|\leq a_1+a_2$, which in combination with Eq. (19) yields the four inequalities

$$|a_2 \pm a_3| \leq a_1 \pm a_4$$

$$|a_2 \pm a_4| \leq a_1 \pm a_3,$$

which have not been published before.

When writing down the phase matrix with the help of a scattering matrix of block diagonal type [cf. Eq. (7)], one may again derive a multitude of identities between its elements, as is apparent from Eq. (8). Denoting the (i,j)-element of $\mathbf{Z}$ by Z_{ij} we obtain the identities

$$Z_{14} = 0$$

$$Z_{41} = 0$$

$$Z_{12}Z_{42} + Z_{13}Z_{43} = 0$$

$$Z_{12}^2 + Z_{13}^2 - Z_{21}^2 - Z_{31}^2 = 0$$

$$Z_{12}Z_{34} + Z_{21}Z_{43} = 0$$

$$Z_{12}Z_{24} - Z_{31}Z_{43} = 0$$

$$(Z_{12}Z_{31})Z_{22} + (Z_{13}Z_{31})Z_{23} - (Z_{12}Z_{21})Z_{32} - (Z_{13}Z_{21})Z_{33} = 0$$

$$-(Z_{13}Z_{21})Z_{22} + (Z_{12}Z_{21})Z_{23} - (Z_{13}Z_{31})Z_{32} + (Z_{12}Z_{31})Z_{33} = 0.$$

If $Z_{12}\neq 0$, every equation for the elements of $\mathbb{Z}$ based on Eq. (8) can be derived from the above eight equations only[4]. If $Z_{12}=0$ but one of the other quantities Z_{13}, Z_{21}, Z_{31}, Z_{42}, Z_{43}, Z_{24} or Z_{34} does not vanish, one must replace these eight equations by another set of eight[4].

A variety of methods may be used to obtain inequalities for the elements of the phase matrix. In the first place we may employ the invariance of the set of vectors $\mathbb{I}=\{I,Q,U,V\}$ satisfying (6) under the application of $\mathbb{Z}$ and find, among other things, that Z_{11} dominates the absolute values of all other elements of $\mathbb{Z}$. One may also use the inequalities[3] valid for the scattering matrix of Eq. (2) and replace each element of $\mathbb{F}$ by the corresponding element of $\mathbb{Z}$. This yields inequalities for the elements of $\mathbb{Z}$ which can be justified by performing rotations of the plane of reference on the amplitude matrix of each individual particle and then constructing the phase matrix per particle followed by summation over various particles. For more details we refer to Ref. 4.

4. AZIMUTH DECOMPOSITION OF THE PHASE MATRIX

If we write the phase matrix as

$$\mathbb{Z}(u,u',\varphi-\varphi') =$$

$$= \mathbb{Z}^{c0}(u,u') + 2\sum_{j=1}^{\infty}\left[\mathbb{Z}^{cj}(u,u')\cos\{j(\varphi-\varphi')\} + \mathbb{Z}^{sj}(u,u')\sin\{j(\varphi-\varphi')\}\right]$$

and the solution of the equation of transfer as

$$\mathbb{I}(\tau,u,\varphi) = \mathbb{I}^{c0}(\tau,u) + 2\sum_{j=1}^{\infty}\left[\mathbb{I}^{cj}(\tau,u)\cos j\varphi + \mathbb{I}^{sj}(\tau,u)\sin j\varphi\right],$$

we decompose the equation of transfer (9) into component equations in which $\mathbb{I}^{cj}(\tau,u)$ and $\mathbb{I}^{sj}(\tau,u)$ are coupled. A further reduction[7,8] may be accomplished by exploiting the symmetry relations

$$\mathbb{D}\mathbf{Z}^{c0}(u,u') = \mathbf{Z}^{c0}(u,u')\mathbb{D}$$

$$\mathbb{D}\mathbf{Z}^{cj}(u,u') = \mathbf{Z}^{cj}(u,u')\mathbb{D}$$

$$\mathbb{D}\mathbf{Z}^{sj}(u,u') = -\mathbf{Z}^{sj}(u,u')\mathbb{D}$$

where $\mathbb{D}$=diag(1,1,−1,−1). If we write alternatively

$$\mathbb{I}^{j}_{\pm}(\tau,u) = \pm\tfrac{1}{2}(1\pm\mathbb{D})\mathbb{I}^{cj}(\tau,u) + \tfrac{1}{2}(1\mp\mathbb{D})\mathbb{I}^{sj}(\tau,u)$$

we obtain for each $j\geq 1$ two copies of the same equation, viz.

$$u\frac{\partial}{\partial\tau}\mathbb{I}^{j}_{\pm}(\tau,u) + \mathbb{I}^{j}_{\pm}(\tau,u) = \tfrac{1}{2}a\int_{-1}^{1} \mathbf{W}^{j}(u,u')\mathbb{I}^{j}_{\pm}(\tau,u')du' \qquad (24)$$

where

$$\mathbf{W}^{j}(u,u') = \mathbf{Z}^{cj}(u,u') - \mathbb{D}\mathbf{Z}^{sj}(u,u').$$

For j=0 (where $\mathbb{I}^{sj}\equiv 0$ and $\mathbf{Z}^{sj}\equiv 0$ by convention) we obtain only one equation of the type (24), which further decomposes in a trivial way as two decoupled equations for the vectors $\{I_0,Q_0\}$ and $\{U_0,V_0\}$. The relations between the elements of $\mathbf{Z}$ considered in the previous section may be used to obtain relations for the elements of $\mathbf{W}^{j}$, as shown in another paper[4].

The integral kernel $\mathbf{W}^{j}(u,u')$ of Eq. (24) can be expressed in the elements of the scattering matrix with the help of the generalized spherical functions $P^{l}_{mn}(x)$[12], as shown by Siewert[7]. (See Ref. 8 for a concise derivation.) In fact, we have

$$\mathbf{W}^{j}(u,u') = \sum_{l=j}^{\infty} \frac{(l-j)!}{(l+j)!}\Pi^{j}_{l}(u)\mathbf{B}_{l}\Pi^{j}_{l}(u')$$

where

$$\Pi_l^j(u) = \begin{bmatrix} P_l^j(u) & 0 & 0 & 0 \\ 0 & R_l^j(u) & -T_l^j(u) & 0 \\ 0 & -T_l^j(u) & R_l^j(u) & 0 \\ 0 & 0 & 0 & P_l^j(u) \end{bmatrix} \tag{25}$$

$$R_l^j(u) = -\tfrac{1}{2}(i)^j\left[\frac{(l+j)!}{(l-j)!}\right]^{1/2}\{P_{2j}^l(u) + P_{-2j}^l(u)\}$$

$$T_l^j(u) = -\tfrac{1}{2}(i)^j\left[\frac{(l+j)!}{(l-j)!}\right]^{1/2}\{P_{2j}^l(u) - P_{-2j}^l(u)\}$$

with $i=\sqrt{-1}$ and

$$\mathbb{B}_l = \begin{bmatrix} \beta_l & \gamma_l & 0 & 0 \\ \gamma_l & \alpha_l & 0 & 0 \\ 0 & 0 & \zeta_l & -\epsilon_l \\ 0 & 0 & \epsilon_l & \delta_l \end{bmatrix}. \tag{26}$$

Here the elements of the matrix defined by (25) are special functions which may be expressed in Jacobi polynomials (see Ref. 8, Appendix) and satisfy the orthogonality relation

$$\int_{-1}^{1} \Pi_l^j(u)\Pi_{l'}^j(u)du = \frac{2}{2l+1}\delta_{ll'}\frac{(l+j)!}{(l-j)!}\mathbf{E}_l^j \tag{27}$$

where $\mathbf{E}_l^j$ is the identity matrix for $l \geq \max(j,2)$ and $\mathbf{E}_l^j = \mathrm{diag}\{1,0,0,1\}$ if $j=0$ and $l=0,1$ or if $j=l=1$. In particular, $P_l^j(u)$ are associated Legendre polynomials. The elements of the matrix appearing in (26) are the expansion coefficients appearing in the equations

$$a_1(\theta) = \sum_{l=0}^{\infty} \beta_l P_l(\cos\theta) \tag{28}$$

$$a_2(\theta) = \sum_{l=2}^{\infty} \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2}\{\alpha_l R_l^2(\cos\theta) + \zeta_l T_l^2(\cos\theta)\} \tag{29}$$

$$a_3(\theta) = \sum_{l=2}^{\infty} \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} \{\zeta_l R_l^2(\cos\theta) + \alpha_l T_l^2(\cos\theta)\} \tag{30}$$

$$a_4(\theta) = \sum_{l=0}^{\infty} \delta_l P_l(\cos\theta) \tag{31}$$

$$b_1(\theta) = \sum_{l=2}^{\infty} \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} \gamma_l P_l^2(\cos\theta) \tag{32}$$

$$b_2(\theta) = -\sum_{l=2}^{\infty} \left[\frac{(l-2)!}{(l+2)!}\right]^{1/2} \epsilon_l P_l^2(\cos\theta) \tag{33}$$

where P_l is the l-th Legendre polynomial and $\alpha_k=\gamma_k=\epsilon_k=\zeta_k=0$ for k=0,1. Because of the normalization condition (10) we have $\beta_0=1$. Using the orthogonality property (27) we may write each of the expansion coefficients explicitly as integrals involving certain elements of $\Pi_l^j(u)$ and $\mathbb{F}(\theta)$[13].

5. PROPERTIES OF THE GREEK EXPANSION COEFFICIENTS

Evidently the Greek expansion coefficients play a fundamental role in the theory of polarized light transfer. However, almost nothing has been published about general bounds and relations for these coefficients. Some results were derived by Germogenova and Konovalov[9] on the basis of complex expansion results of Domke[6] and cone preservation techniques of functional analysis. Further, Benassi et al.[14] conjectured some inequalities which play a crucial role in a particular numerical method. These are

$$2l + 1 - a\beta_l > 0 \tag{34a}$$

$$(2l+1-a\beta_l)(2l+1-a\alpha_l) - a^2\gamma_l^2 > 0, \tag{34b}$$

both formulated for all $l \geq 0$ if $a \neq 1$ and for $l \geq 1$ if a=1, and

$$2l + 1 - a\delta_l > 0 \tag{35a}$$

$$2l + 1 - a\varsigma_l > 0, \tag{35b}$$

where the last two inequalities were formulated for $l=0,1,2,...$ and $0 \leq a \leq 1$.

We are currently involved in a study of inequalities for the Greek expansion coefficients. Some preliminary results are as follows. Obviously, we can use inequalities for the elements of the scattering matrix such as (17)-(20) and (22)-(23) to obtain inequalities for the Greek expansion coefficients. Therefore, without relying on the equation of transfer and functional analysis (as done by Germogenova and Konovalov[9]) we have derived a plethora of inequalities directly from Eqs. (17)-(20) and (22)-(23). In addition we employed Eqs. (6) and (27) which led to relations of the type

$$|\beta_l| \leq 2l+1$$

and bounds involving squares of expansion coefficients. Although our research in this area has not been completed yet, it has become clear that the inequalities (34) and (35) can be proven in an elementary way.

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REFERENCES

1. S. Chandrasekhar, "Radiative Transfer," Oxford Univ. Press, London (1950); Dover Publ., New York (1960).

2. H.C. van de Hulst, "Light Scattering by Small Particles," Wiley, New York (1957); Dover Publ., New York (1981).

3. J.W. Hovenier, H.C. van de Hulst and C.V.M. van der Mee, Astron. Astrophys., 157, 301 (1986).

4. J.W. Hovenier and C.V.M. van der Mee, "Scattering of Polarized Light: Properties of the Elements of the Phase Matrix," Astron. Astrophys., accepted for publication.

5. I. Kuščer and M. Ribarič, Optica Acta, 6, 42 (1959).

6. H. Domke, Astrophys. Space Sci., 29, 379 (1974).

7. C.E. Siewert, Astrophys. J., 245, 1080 (1981).

8. J.W. Hovenier and C.V.M. van der Mee, Astron. Astrophys., 128, 1 (1983).

9. T.A. Germogenova and N.V. Konovalov, "The Spectrum of the Characteristic Equation in the Theory of Transfer with Polarization Taken into Account," Inst. Appl. Math., Acad. Sci., Preprint No. 62, Moscow (1978) [Russian].

10. C.V.M. van der Mee and J.W. Hovenier, "Expansion Coefficients in Polarized Light Transfer," In Preparation.

11. If $0<\varphi-\varphi'<\pi$, the spherical triangle is formed by $-\sigma_1$, $-\sigma_2$, θ, ϑ_1, ϑ_2 and $(\varphi-\varphi')$.

12. I.M. Gelfand, R.A. Minlos and Z.Ya. Shapiro, "Representation of Rotation and Lorentz Groups and their Applications," Pergamon Press, Oxford (1963).

13. C.E. Siewert, Astron. Astrophys., 109, 195 (1982).

14. M. Benassi, R.D.M. Garcia and C.E. Siewert, Z. angew. Math. Phys., 35, 308 (1984).

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