PERTURBATION OF BISEMIGROUPS AND BOUNDARY VALUE PROBLEMS FOR ABSTRACT KINETIC EQUATIONS

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ABSTRACT

The question of unique solvability of an abstact linear kinetic equation in a half space is settled. The methods utilized include perturbation of analytic bisemigroups and an analysis of the Krein space structure of certain finite dimensional subspaces.

I. INTRODUCTION

Numerous stationary kinetic processes in neutron transport, radiative transfer and rarefied gas dynamics are described by linear kinetic equations, such as

$$\mathbf{v}\cdot\nabla_{\mathbf{r}} f(\mathbf{v},\mathbf{r}) = -\nu(\mathbf{v}) f(\mathbf{v},\mathbf{r}) + (Jf)(\mathbf{v},\mathbf{r})$$

with appropriate boundary conditions. Such boundary value problems in plane parallel half space geometry can be modeled by the abstract boundary value problem (BVP) (e.g. [16], [3], [18])

$$T\psi'(x) = -A\psi(x), \quad \pm x>0, \tag{1.1}$$

$$Q_{\pm}\psi(0)=\varphi_{\pm}, \quad \|\psi(x)\|=0(x^{n}) \text{ as } x\to\pm\infty, \tag{1.2}$$

for some $n \in \mathbb{N}$, where the $+/-$ sign stands for the right/left half space problem, $\varphi_{\pm}$ is the incoming flux, $\psi(x)$ is a vector in a Hilbert space H, T is a self adjoint injective operator on H, and the collision operator A typically has the form "identity plus a compact" or "Sturm–Liouville plus a compact". In studying this BVP one proceeds in two steps: (i) prove that $T^{-1}A$ generates a holomorphic bisemigroup $E(x,T^{-1}A)$ with separating projectors $P_{\pm}$ (see below), (ii) prove that $H=Q_{\mp}H \oplus P_{\pm}H$. Having (i) it is easy to check that each solution of (1) (at times we will talk about the right half space problem only to make the notation clearer) has the form $\psi(x)=\exp(-xT^{-1}A)h$, $x>0$, for $h \in P_{+}H$ such that $Q_{+}h=\varphi_{+}$, the solvability of which is equivalent to (ii). This is in line with the general procedure of analyzing a BVP on a manifold X, where one reduces it to a problem on $X \cup_{\partial X} X$ and a problem on ∂X. In our case the spatial dependence is very simple but the difficulties arise from the coefficients which are operators.

A recurrent theme in the study of BVP's is the Wiener–Hopf (WH) problem (e.g., [9]). The original scalar WH-equation was derived to model radiative transfer in a stellar atmosphere. The study of the matrix valued WH-equation is connected with the work of Gohberg, Krein, Feldman, Maslennikov, etc. (see [12],[24],[11],[10]). The WH-equation equivalent to the BVP (1) has an operator valued integral kernel. Set $\mathcal{H}(x)=T^{-1}E(x,T^{-1})$ and $(\mathcal{L}\psi)(x)=\int_{-\infty}^{\infty}\mathcal{H}(x-y)B\psi(y)dy$, where $\psi \in C(\mathbb{R},H)$ and $B=I-A$. Write $\mathcal{P}_{\pm}$ for the natural projectors $C(\mathbb{R},H)\to C(\mathbb{R}_{\pm},H)$ and $\mathcal{L}_{\pm}=\mathcal{P}_{\pm}\mathcal{L}\mathcal{P}_{\pm}$. Then (1) is equivalent to the WH-equation, $(I-\mathcal{L}_{\pm})\psi=\omega$, where $\omega(x)=E(x,T^{-1})(\varphi_{+}+\varphi_{-})$. The steps analogous to (i–ii) above are: (i) invert the full line convolution operator $(I-\mathcal{L})$, (ii) obtain a canonical WH-factorization of the symbol $W(\lambda) = (T^{-1}-\lambda)^{-1}(T^{-1}A-\lambda)^{-1}$, $\mathrm{Re}\lambda=0$, of $(I-\mathcal{L})$.

II. PERTURBATIONS OF BISEMIGROUPS

To show that $T^{-1}A$ generates a bisemigroup, we proceed as follows. We view $W(\cdot)$ as an element of an appropriate algebra of Wiener type, whence the invertibility of $W(\lambda)$ for each λ on the imaginary axis implies, by a theorem of Bochner–Phillips ([5], also [1, [13]), that $(I-\mathcal{L})$ is invertible as an operator on $C(\mathbb{R},H)$. From this one shows that $T^{-1}A$ generates a bisemigroup. The development of the perturbation theory of bisemigroups with applications to matrix equations is due largely to Gohberg and his school (e.g., [2]), with applications in systems theory. Bisemigroup theory in the infinite dimensional setting is in a relatively embryonic state. Here we prove a theorem for perturbation of bisemigroups, which generalizes [2] and extends [26], [27].

A C_0 bisemigroup $E(t)$ on H is a function from $\mathbb{R}\setminus\{0\}$ to $L(H)$ with the properties:

(BS1) $E(t)E(s)=\pm E(t+s)$ if $\mathrm{sgn}(s)=\mathrm{sgn}(t)=\pm$ and $E(t)E(s)=0$ if $\mathrm{sgn}(s) = \mathrm{sgn}(t)$,

(BS2) $E(\cdot)$ is strongly continuous,

(BS3) $\Pi_+ + \Pi_- = I$,

where $\Pi_\pm = \text{s-lim}(\pm E(t))$ as $\pm t \downarrow 0$, are the separating projectors for the bisemigroup. An operator S is the generator of the bisemigroup $E(t,S)$ if $\Pi_\pm$ commute with S and $E(t,S) = \pm\exp(-tS)\Pi_\pm$, $\pm t>0$. The two-sided Laplace transform of $E(t,S)$, if it exists, is the resolvent of S along the imaginary axis. The bisemigroup will be called bounded, holomorphic, strongly decaying or exponentially decaying if the semigroups $\pm E(t)\Pi_\pm$, $\pm t>0$, have the respective property.

For an angle $0<\theta\le\pi/2$ we denote the two-sided sectors about the real axis of opening θ by Σ_θ. Assuming that S is a spectral operator of scalar type on H, $\sigma(S)\subset\Sigma_\theta$ for some $0<\theta\le\pi/2$ and that zero is either in the resolvent set or in the continuous spectrum, one has immediately that S generates a strongly decaying holomorphic bisemigroup with separating projectors $\Pi_\pm$ equal to the maximal positive/negative spectral projectors. If S^{-1} is bounded, then $E(t,S)$ is exponentially decaying. Our aim is to establish sufficient conditions so that a perturbation $S^{\times}=SA$ will again generate a bisemigroup. Assume one of the following (either (a) or (a$'$)):

(a) $B=I-A$ is compact and $\mathrm{Ran}B\subset D(|S|^{\alpha})\cap \mathrm{Ran}|S|^{\beta}$, for some $\alpha,\beta>0$

(a$'$) B is trace class

as well as

(b) the spectrum of $S^{\times}$ is contained is a sector around the real axis

(c) Ker $A = 0$.

Condition (c) is taken to simplify this exposition. Its removal, which leads to a more complicated existence and uniqueness theory, involves a reduction of the operator $T^{-1}A$ in terms of its zero root linear manifold and an appropriate complement. For details, see [14],[15]. We remark that the range condition in (a) assures the Bochner integrability of $\mathcal{H}(\cdot)B$. Also, $S^{\times}-S$ is S–relatively compact, so in checking condition (b) one need worry only about isolated eigenvalues of finite algebraic multiplicity, because of a Weyl type of argument. If we assume $\mathrm{Re}A\geq 0$ and Ker $A=\mathrm{Ker}(\mathrm{Re}A)=0$, where $\mathrm{Re}A=(A+A^{*})/2$, then it is easy to check that $T^{-1}A$ has no eigenvalues on the imaginary axis.

Theorem. With the above assumptions on S and B, $S^{\times}$ generates a holomorphic bisemigroup $E^{\times}(t)$ with separating projectors $\Pi_{\pm}^{\times}$. For any $t\in\mathbb{R}\setminus\{0\}$ we have that $E(t)-E^{\times}(t)$ and $\Pi_{\pm}-\Pi_{\pm}^{\times}$ are compact. The bisemigroup $E^{\times}(t)$ is strongly decaying. If $E(t)$ is exponentially decaying so is $E^{\times}(t)$.

Proof. First we deal with the case (a), where one easily checks that $\mathscr{K}(t)=SE(t)B$ is Bochner integrable. Then L, the operator representing convolution by $\mathscr{K}$, is a bounded operator on $C(\mathbb{R},H)$ and on $L^{p}(\mathbb{R},H)$, the space of H–valued Bochner L^{p}–integrable functions. Also, $W(\cdot)$ belongs to the algebra of operator valued Bochner integrable functions, which is a Banach algebra of Wiener type. Thus, by the Bochner–Phillips theorem, the inverse of $(I-\mathcal{L})$ is in the form $(I+\mathcal{L}^{x})$ where $\mathcal{L}^{x}$ is again a convolution operator. We claim that $E^{x}(t)h=(I+\mathcal{L}^{x})E(t)h$ for $h\in H$, is the bisemigroup generated by S^{x}. Because $\mathcal{L}$ is bounded on $C(\mathbb{R},H)$, $E^{x}(t)$ is strongly continuous. To check that $\Pi_{+}^{x}+\Pi_{-}^{x}=I$ one uses the smoothing property of convolutions, so the jump of E^{x} at zero is equal to the jump of E at zero, hence $(\Pi_{+}^{x}+\Pi_{-}^{x})h=(\Pi_{+}+\Pi_{-})h=h$. Taking a two-sided Laplace transform of E^{x}, we immediately get $(S^{x}-\lambda)^{-1}$, thus S^{x} is the generator of E^{x}. If we take an angle φ inside the sector assumed in (b), we can apply the same reasoning to the operators $e^{i\varphi}S$ and $e^{i\varphi}S^{x}$, i.e., $e^{i\varphi}S^{x}$ generates a bisemigroup. Therefore, by a theorem from [20], S^{x} generates a holomorphic bisemigroup. Because $E^{x}(t)-E(t)=\int_{-\infty}^{\infty}\mathcal{K}(t-s)E^{x}(s)ds$, the compactness of this difference will follow from the compactness of B, since the Bochner integral of a compact operator valued function is compact. Finally, $D((S^{x})^{-1})=D(S^{-1})$ dense implies that zero is either in the resolvent set or in the continuous spectrum of S^{x}, so E^{x} is strongly decaying.

When we abandon assumption (a), the difficulty consists of showing that $\mathcal{L}$ is bounded on $C(\mathbb{R},H)$, since $\mathcal{H}(\cdot)$ by itself is only Gelfand integrable. That is to say, $\mathcal{H}\in L^{1}(\mathbb{R})\otimes_{\epsilon}L(H)$, the injective tensor product of $L^{1}(R)$ and the bounded operators on H (see [8]). Note that it is not clear whether $L^{1}(\mathbb{R})\otimes_{\epsilon}L(H)$ is a Banach algebra. For a fixed trace class operator B it is easy to check that $L^{1}(\mathbb{R})\otimes_{\epsilon}L(H)B$ is a Banach algebra and $\mathcal{L}$ is a bounded operator that is smoothing [14]. Hence, everything follows as before, with the exception of the compactness of $E(t)-E^{x}(t)$. To obtain compactness we approximate B by a sequence B_n so that the range condition of (i) is satisfied. Then $E(t)-E_{n}^{x}(t)$ are compact and converge in norm to $E(t)-E^{x}(t)$, so the latter is compact. ∎

III. BOUNDARY VALUE PROBLEM

Applying the above theorem with $S=T^{-1}$, we get a bisemigroup $E(x,T^{-1}A)$ with separating projectors $P_{\pm}$. Set $V=Q_+P_++Q_-P_-$ (this operator was first introduced in [19],[23]). It is immediate that step (ii) for the BVP is equivalent to the invertibility of V. There are two ways of showing the invertibility of V.

In the first, one has the implications:

$$P_{\pm}-Q_{\pm}=\text{compact} \Rightarrow I-V=\text{compact} \Rightarrow V \text{ Fredholm of index zero.}$$

Next, assuming that A is positive (or accretive in the nonsymmetric case), one shows that $P_{\pm}H\cap Q_{\mp}H=0$, so $\mathrm{Ker}V=0$ and therefore V is invertible. If A has a nontrivial null space, one separates off the zero root linear manifold $Z_0(T^{-1}A)=\cup_{n=1}^{\infty}\mathrm{Ker}(T^{-1}A)^n$ and uses the above argument on its complement. Then, the unique solvability of the BVP depends on the structure of the finite dimensional root linear manifold $Z_0(T^{-1}A)$. This program was fully developed in [25] for T bounded and A positive self adjoint of the form "identity plus a compact". (We note that the existence of the bisemigroup $E(x,T^{-1}A)$ is trivial in this case.) It was expanded to unbounded T in [17]. The case of unbounded T and nonsymmetric A was developed in [15].

The second method seeks "weak" solutions in spaces larger than the initial Hilbert space. It was initiated in [3], and extended to unbounded positive self adjoint A in [18] and [4]. The indefinite Sturm–Liouville problem is investigated in [4], [21].

We propose a modification of the method of weak solutions, where we extensively use the Krein space structure of the enlarged space [14]. This not only allows us to deal with operators A of the type "Sturm–Liouville plus a nonsymmetric compact" (for an example of such a model see [28]) but also gives a new way of looking at the case of self adjoint A making it simpler and more transparent. For brevity we will limit ourselves to the case of bounded T and Ker A = Ker(ReA)=0.

Let H_T be the completion of H with respect to the scalar product $(|T|\cdot,\cdot)$. Endowed with the indefinite scalar product $(T\cdot,\cdot)$ this becomes a Krein space, with fundamental decomposition $H_T=Q_+H_T\oplus Q_-H_T$ and fundamental symmetry $Q=Q_+-Q_-$. One easily extends $T^{-1}A$ to a closed, densely defined operator in H_T which is positive in the T-scalar product (T-positive). Because T is bounded the only critical point of $T^{-1}A$ is at infinity. For a Sturm-Liouville operator A it follows from [4] and [7] that infinity is a regular critical point. Using the spectral theorem for definitizable operators [22] one immediately gets that $T^{-1}A$ generates a holomorphic bisemigroup (exponentially decaying, because T is bounded) and that the separating projectors $P_\pm$ give maximal T-positive/negative definite subspaces. Step (ii), which is equivalent to $Q_\pm$ mapping $\mathrm{Ran}P_\pm$ bijectively onto $\mathrm{Ran}Q_\pm$, will follow from the next proposition (see [6]).

Proposition. A T-positive subspace M_+ of H_T is maximal T-positive iff $Q_+M_+=Q_+H_T$.

Now assume that $A=A_1+A_2$ with A_1 Sturm-Liouville strictly positive and $A_1^{-1}A_2$ trace class. After extending everything to H_T we proceed as above to define the bisemigroup $E(x,T^{-1}A_1)$ and separating projectors $P_{1\pm}$ using the spectral theorem for definitizable operators. Next we use the theorem for perturbation of bisemigroups with $S=T^{-1}A_1$ and $B=A_1^{-1}A_2$ to get the bisemigroup $E(x,T^{-1}A)$ with separating projectors.

Lemma. If A is accretive then $P_\pm H_T$ are T-positive/negative definite.

Proof. If $K=T^{-1}A$ and $K^\#$ is the T-adjoint of K we get that $K+K^\#$ is a strictly T-positive operator. Take any $g\in P_+H_T$ and set $f(x) = \exp(-xK)g$, $x>0$. Because KP_+ generates a holomorphic semigroup $f(x)\in D(K)$ for $x>0$. So we have $0>-(T(K+K^\#)f(x),f(x))=\frac{d}{dx}(Tf(x),f(x))$. Integrating we obtain

$$0 > -\lim_{\tau\to\infty}\int_0^\tau (T(K+K^\#)f(x),f(x))dx = \lim_{\tau\to\infty}(Tf(x),f(x))-(Tg,g) = -(Tg,g).$$

Similarly for P_-H_T. This trick appears in another guise in [29]. ∎

From the fact that $P_++P_-=I$ and $P_\pm H_T$ are T-positive/negative definite, it follows that $P_\pm H_T$ are maximal T-positive/negative spaces. Putting the above together and using the proposition once more we have shown the unique solvability of BVP (1) in H_T.

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