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STRONG SOLUTIONS OF STATIONARY EQUATIONS IN ABSTRACT KINETIC THEORY

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An abstract differential equation with "partial range" boundary conditions, modelling a variety of plane-symmetric stationary transport phenomena, is studied in Hilbert space. The collision operator is assumed to be a positive compact perturbation of the identity. A complete existence and uniqueness theory for the abstract equation is presented and two examples from rarefied gas dynamics are detailed.

1. INTRODUCTION

In recent years considerable effort has been devoted to the study of the boundary value problem

$$(T\psi)'(x) = -A\psi(x), \quad 0 < x < \infty, \tag{1.1}$$

$$Q_+\psi(0) = \varphi_+, \tag{1.2}$$

$$\limsup_{x\to\infty} \|\psi(x)\|_H < \infty. \tag{1.3}$$

Here T is an injective, self-adjoint operator defined on a complex Hilbert space H,

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Q_+ is the orthogonal projection onto the maximal T-positive, T-invariant subspace of H and A is a positive compact perturbation of the identity. In particular, we cite Hangelbroek [9,10], Lekkerkerker [17], Beals [2], Kaper [13], Van der Mee [18] and Greenberg et al. [8]. The theory so far developed either assumes T is bounded or seeks solutions which are H_T-valued for an "enlarged" space $H_T \supseteq D(T)$ (weak solutions). On the other hand, for certain specific one-dimensional models in gas dynamics of especially simple form, explicit representations of solutions of the boundary value problem posed on H are known. (See, for example, Kaper [13], which deals with A a concrete rank one perturbation of the identity.)

Equations (1.1)-(1.3) model a large variety of transport phenomena in semi-infinite media under steady state conditions. For most problems in radiative transfer and neutron transport in non-multiplying media, the operator T is bounded and A is a compact perturbation of the identity. In this case, the existence and uniqueness theory for H-valued functions ψ (strong solutions) has been developed by Hangelbroek and Lekkerkerker for degenerate neutron transport [9,17], by Hangelbroek [10] for $\mathbb{I} - A$ a concrete trace-class operator from neutron transport and by Van der Mee [18] for $\mathbb{I} - A$ an abstract compact operator with a "regularity" condition. In rarefied gas dynamics the operator T is unbounded. We shall study here the abstract boundary value problem for T unbounded and $\mathbb{I} - A$ a compact operator, again with a "regularity" condition. Our technique will parallel the arguments developed in [8] and [18] and, in some cases, the reader is referred to these sources for the proofs of preliminary propositions.

In Section 2 we will obtain existence and uniqueness results for A strictly positive. The first part of Section 3 contains decompositions needed to accomplish the reduction of the half-space problem with A non-strictly positive to the half-space problem with a strictly positive collision operator. The complete existence and (non) uniqueness theory for boundary value problems with T unbounded and A non-strictly positive is presented in the second part of Section 3. Finally, in the last section, two applications from rarefied gas dynamics are exhibited.

Throughout this article H will be a complex Hilbert space, T an injective self-adjoint operator on H and $Q_\pm$ the orthogonal projections of H onto the maximal T-positive/ negative T-invariant subspaces, while A is a positive operator such that $B = \mathbb{I} - A$ is compact. Positivity will always be understood in the sense of positive selfadjointness. The domain, the range, the kernel and the spectrum of a linear operator S will be denoted by D(S), Ran S, Ker S and $\sigma(S)$, respectively.

2. DISSIPATIVE MODELS

In this section we will assume that $A = \mathbb{I} - B$ is a strictly positive compact perturbation of the identity $\mathbb{I}$ on H. Then Ker $A = \{0\}$ and $A^{-1} = \mathbb{I} + C$, where $C = BA^{-1}$ is obviously compact. Let H_A denote the Hilbert space H with inner product

$$(f,g)_A = (Af,g). \tag{2.1}$$

This inner product is equivalent to the original inner product on H. Let $S = A^{-1}T$. Then $D(S) = D(T)$ and S is injective and self-adjoint with respect to the H_A-inner product (2.1). We define $P_\pm$ as the H_A-orthogonal projection of H onto the maximal S-positive/ negative S-invariant subspaces of H. These projections are complementary, as are the projections $Q_\pm$ associated with T. Moreover, they leave D(T) invariant and are bounded on the complete inner product space D(T) with the T-graph norm defined by

$$(f,g)_{GT} = (f,g) + (Tf,Tg). \tag{2.2}$$

The selfadjointness of S with respect to the inner product (2.1) allows the machinery of the Spectral Theorem to be introduced. If $F(\cdot)$ is the resolution of the identity associated with S, we can define the operator functions

$$e^{\mp xT^{-1}A}P_\pm h = \pm\int_0^{\pm\infty} e^{\mp x/t}dF(t)h \tag{2.3}$$

for Re $x > 0$. Then the restrictions of $\exp(\mp xT^{-1}A)P_\pm$ to Ran $P_\pm$ are bounded analytic semigroups on Ran $P_\pm$ whose infinitesimal generators are the inverses of the restrictions of $\mp A^{-1}T$ to Ran $P_\pm$. From the injectivity of A and the dominated convergence theorem, we have $\lim_{x\to\infty} \|\exp(\mp xT^{-1}A)P_\pm h\|_H = 0$ for all $h \in H$. Moreover, the strong (and even uniform) derivative of the expression (2.3) exists for $x \in (0,\infty)$, belongs to D(T) and satisfies the differential equation (1.1). Using the invariance of D(T) under these semigroups, we may also prove that these semigroups are bounded and analytic on Ran $P_\pm \cap D(T)$ relative to the topology generated by the graph norm (2.2).

We define a solution of the boundary value problem (1.1)-(1.3) for given $\varphi_+ \in Q_+[D(T)]$ to be a strongly continuous function ψ: $[0,\infty) \to H$ such that $\psi(x) \in D(T)$

for all $x \in (0,\infty)$, $T\psi$ is strongly differentiable on $(0,\infty)$ and Eqs. (1.1)-(1.3) hold true.

The proof of the next lemma is straightforward.

LEMMA 2.1. *The vector function* $\psi(x)$ *is a solution of the boundary value problem* (1.1)-(1.3) *if and only if*

$$\psi(x) = e^{-xT^{-1}A}h, \quad 0 < x < \infty, \tag{2.4}$$

for some $h \in \operatorname{Ran} P_+ \cap D(T)$ with $Q_+h = \varphi_+$. *Such solutions are strongly differentiable on* $(0,\infty)$ *and vanish at infinity with respect to the original norm on* H *as well as the graph norm on* D(T).

Hence the problem reduces to finding the vector h. Let $V = Q_+P_+ + Q_-P_-$. Then an obvious candidate for h will be $E\varphi_+ = V^{-1}\varphi_+$ if we show that V is boundedly invertible and V maps D(T) onto D(T). In order to prove this we shall establish the injectivity of V on H and the compactness of $\mathbb{I} - V$ on H and on D(T) equipped with the norm (2.2). Once these are proved, the Fredholm alternative gives the boundedness of V^{-1} and shows that $V[D(T)] = D(T)$. The operator V was first introduced in [11].

We present three technical lemmas. The first is a consequence of the norm closedness of the algebra of compact operators, and the second one is a moment inequality which follows easily from the Spectral Theorem and Hölder's inequality. The third was proved by Krein and Sobolevskii [15]; an accessible proof can be found in Krasnoselskii et al. [14].

LEMMA 2.2. *The integral of a (norm) continuous compact operator-valued function with integrable norm is a compact operator.*

LEMMA 2.3. *Let* A *be a positive self-adjoint operator. Then for any* $\tau \in (0,1)$ *and any* $x \in D(A)$ *we have* $\|A^\tau x\|_H \le k\, \|Ax\|_H^{\tau}\|x\|_H^{1-\tau}$.

LEMMA 2.4. *Let* A *be a strictly positive self-adjoint operator and* B *a closed operator satisfying* $\|Bx\|_H \le k\, \|Ax\|_H^{\tau}\|x\|_H^{1-\tau}$ *for any* D(A) *and some* $\tau \in (0,1)$. *Then, for all* $\delta > \tau$, $D(A^\delta) \subseteq D(B)$ *and, for all* $x \in D(A^\delta)$, $\|Bx\|_H \le k_0\|A^\delta\|_H$.

In addition to the compactness of $\mathbb{I} - A$, we shall assume throughout the regularity condition

$$\exists\ \alpha \in (0,1) \text{ and } \omega > \max\left\{\frac{1+\alpha}{2},\frac{2-\alpha}{2}\right\}: \ \text{Ran}\ (\mathbb{I} - A) \subseteq \text{Ran}\,(|T|^{\alpha}) \cap D(|T|^{1+\omega}) \quad (2.5)$$

LEMMA 2.5. *The operator* $P_+ - Q_+$ *is compact on* H *and the restriction of* $P_+ - Q_+$ *to* D(T) *is compact on* D(T) *(endowed with the graph inner product* (2.2)*). Moreover,* $(P_+ - Q_+)[H] \subseteq D(T)$.

Proof: We will prove first that $P_+ - Q_+$ is compact on H and $(P_+ - Q_+)[H] \subseteq D(T)$. Let $\Delta_1 = \Delta(\epsilon,M)$ denote the oriented curve composed of the straight line segments from $-i\epsilon$ to $-i$, from $-i$ to $M-i$, from $M+i$ to i, and from i to $i+\epsilon$. Let $\Delta_2 = \Delta(M)$ denote the oriented curve composed of the straight line segments from $M-i$ to $+\infty-i$ and from $+\infty+i$ to $M+i$. Denote $\Delta = \Delta_1 \cup \Delta_2$ with the orientation inherited from Δ_1 and Δ_2. We recall that the projections P_+ and Q_+ are bounded on H and on D(T) endowed with the graph inner product (2.2). We have the integral representations

$$P_+ = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Delta} (\lambda - S)^{-1}d\lambda, \quad Q_+ = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Delta} (\lambda - T)^{-1}d\lambda,$$

where the limits are taken in the strong topology. Let $P_+^{(1)}$ and $P_+^{(2)}$ be defined as P_+ but with Δ replaced by Δ_1 and Δ_2 and $Q_+^{(1)}$ and $Q_+^{(2)}$ as Q_+ with the same change of integration curve. Then $P_+ - Q_+ = \{P_+^{(1)} - Q_+^{(1)}\} + \{P_+^{(2)} - Q_+^{(2)}\}$. We will show that $P_+^{(1)} - Q_+^{(1)}$ and $P_+^{(2)} - Q_+^{(2)}$ are compact on H, and $(P_+^{(1)} - Q_+^{(1)})[H] \subseteq D(T)$ as well as $(P_+^{(2)} - Q_+^{(2)})[H] \subseteq D(T)$.

Consider first

$$P_+^{(1)} - Q_+^{(1)} = \lim_{\epsilon\to 0} \frac{1}{2\pi i}\int_{\Delta_1} [(\lambda - S)^{-1} - (\lambda - T)^{-1}]d\lambda. \quad (2.6)$$

We shall see that this limit can be taken in the norm topology. We exploit the regularity condition (2.5) and obtain from the Closed Graph Theorem the existence of a bounded operator D such that $B = |T|^{\alpha}D$. Then, for non-real λ, $(\lambda-S)^{-1} - (\lambda-T)^{-1} = (\lambda-T)^{-1}(S-T)(\lambda-S)^{-1} = (\lambda-T)^{-1}BS(\lambda-S)^{-1}$, which shows that $(\lambda-S)^{-1} - (\lambda-T)^{-1}$ is a compact operator on H. Next, since S is self-adjoint on H with respect to the inner product (2.1), we may use the Spectral Theorem to prove

$$\|S(\iota\mu - S)^{-1}\|_{L(H_A)} \leq \operatorname*{Sup}_{t\in\mathbb{R}} \left|\frac{t}{i\mu-t}\right| \leq 1.$$

But the inner products on H and H_A are equivalent and thus also are the L(H)- and $L(H_A)$-norms, so there is a constant c_0 such that $\|S(i\mu-S)^{-1}\|_{L(H)} \leq c_0$. Also, from the Spectral Theorem, $\||T|^\alpha(i\mu-T)^{-1}\|_{L(H)} \leq \operatorname*{Sup}_{t\in\mathbb{R}} \left|\frac{t^\alpha}{i\mu-t}\right| \leq c_\alpha|\mu|^{\alpha-1}$. Thus

$$\left\|\left(\int_{\Delta(\epsilon,M)} - \int_{\Delta(\gamma,M)}\right)[(\lambda - S)^{-1} - (\lambda - T)^{-1}]d\lambda\right\|_{L(H)} \leq 2\|D\|_{L(H)}c_0c_\alpha\int_\epsilon^\gamma \mu^{\alpha-1}d\mu,$$

which shows that the limit (2.6) exists in the operator norm topology and consequently proves the compactness of $P_+^{(1)} - Q_+^{(1)}$.

Let us take $x \in H$. Since $[(\lambda-S)^{-1} - (\lambda-T)^{-1}]x$ and $T[(\lambda-S)^{-1} - (\lambda-T)^{-1}]x = T(\lambda-T)^{-1}BS(\lambda-S)^{-1}x$ are bounded and continuous functions on Δ_1, we find that the vector $\frac{1}{2\pi i}\int_{\Delta_1} [(\lambda-S)^{-1} - (\lambda-T)^{-1}]xd\lambda \in D(T)$ and

$$T\left(\frac{1}{2\pi i}\int_{\Delta_1} [(\lambda-S)^{-1} - (\lambda-T)^{-1}]xd\lambda\right) = \frac{1}{2\pi i}\int_{\Delta_1} T[(\lambda - S)^{-1} - (\lambda - T)^{-1}]xd\lambda.$$

Now, note that

$$\left\|\left(\int_{\Delta(\epsilon,M)} - \int_{\Delta(\gamma,M)}\right)T[(\lambda - S)^{-1} - (\lambda - T)^{-1}]d\lambda\right\|_{L(H)} \leq 2c_0\|B\|_{L(H)}|\epsilon - \gamma|$$

implies the existence of the limit

$$\lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{\Delta(\epsilon,M)} T[(\lambda - S)^{-1} - (\lambda - T)^{-1}]xd\lambda$$

in the operator norm topology. Therefore, by the closedness of T,

$$(P_+^{(1)} - Q_+^{(1)})x = \lim_{\epsilon\to 0} \frac{1}{2\pi i} \int_{\Delta_1} [(\lambda - S)^{-1} - (\lambda - T)^{-1}]xd\lambda \in D(T),$$

which proves the inclusion $(P_+^{(1)} - Q_+^{(1)})[H] \subseteq D(T)$.

Next let us consider

$$P_+^{(2)} - Q_+^{(2)} = \frac{1}{2\pi i} \int_{\Delta_2} [(\lambda - S)^{-1} - (\lambda - T)^{-1}]d\lambda.$$

Since, for non-real λ, $(\lambda-S)^{-1} - (\lambda-T)^{-1} = (\lambda-T)^{-1}BS(\lambda-S)^{-1}$ is compact, it is sufficient to show the integrability of this operator. We rewrite this operator as

$$(\lambda - S)^{-1} - (\lambda - T)^{-1} = (\lambda - T)^{-1}CT(\lambda - T)^{-1}[(\lambda - T)(\lambda - S)^{-1}].$$

Obviously, $\operatorname{Ran}(\lambda-S)^{-1} = D(\lambda-T)$ and, by the Closed Graph Theorem, $(\lambda-T)(\lambda-S)^{-1}$ is a bounded operator on H. In fact, we will show that the norm of this operator is uniformly bounded for $\lambda \in \Delta_2$. We have the identity $(\lambda-T)(\lambda-S)^{-1} = \mathbb{I} + CT(\lambda-S)^{-1}$. By the estimate $\|(\lambda-S)^{-1}\|_{L(H)} \le c_0$ for $\lambda \in \Delta_2$, it is sufficient to show that CT is bounded on $D(T) = D(S)$. But, by the condition (2.5), $\operatorname{Ran} C = \operatorname{Ran} B \subseteq D(|T|^{1+\omega}) \subseteq D(T)$ and thus, by the Closed Graph Theorem, the operator TC is bounded on H, so that $CT \subseteq (TC)^*$ is bounded on D(T). Finally, for $\lambda \in \Delta_2$ we have $\|(\lambda-T)(\lambda-S)^{-1}\|_{L(H)} \le 1 + \|(TC)^*\|_{L(H)}c_0$, providing a λ-uniform bound, as claimed.

Thus it is sufficient to show the integrability of $F(\lambda) = (\lambda-T)^{-1}CT(\lambda-T)^{-1}$. Let Q_0 be a spectral projection belonging to a spectral decomposition of T such that the resolvent set of the restriction of T to the range of $Q_1 = \mathbb{I} - Q_0$ contains a real neighbourhood of zero. We can decompose $F(\lambda)$ as follows:

$$F(\lambda) = (\lambda-T)^{-1}|T|^{-\omega}Q_1|T|^{\omega}C|T|^{1+\nu}|T|^{-1-\nu}T(\lambda-T)^{-1}Q_1 + (\lambda-T)^{-1}|T|^{-\omega}Q_1|T|^{\omega}CT(\lambda-T)^{-1}Q_0$$

$$+ (\lambda - T)^{-1}Q_0C|T|^{1+\nu}|T|^{-1-\nu}T(\lambda - T)^{-1}Q_1 + (\lambda - T)^{-1}Q_0CT(\lambda - T)^{-1}Q_0,$$

where $\nu = \frac{1}{2}\alpha$ and $2\omega > \max\{1+\alpha,2-\alpha\}$, and we may choose $2\omega < 2 + \alpha$. Note that $\nu+\omega > 1$. For $\lambda \in [M\pm i,\infty\pm i)$ we have the following estimates:

$$\|(\lambda-T)^{-1}|T|^{-\omega}Q_1\|_{L(H)} \le c_1(\operatorname{Re}\lambda)^{-\omega}, \quad \||T|^{-1-\nu}T(\lambda-T)^{-1}Q_1\|_{L(H)} \le c_2(\operatorname{Re}\lambda)^{-\nu},$$

$$\|(\lambda-T)^{-1}Q_0\|_{L(H)} \le c_3(\operatorname{Re}\lambda)^{-1}, \quad \|T(\lambda-T)^{-1}Q_0\|_{L(H)} \le c_4(\operatorname{Re}\lambda)^{-1}.$$

Moreover, as $\operatorname{Ran} C = \operatorname{Ran} B \subseteq D(|T|^{1+\omega}) \subseteq D(|T|^{1+\nu}) \subseteq D(|T|^{\omega})$, $|T|^{\omega}C$ and $(C|T|^{1+\nu})^* = |T|^{1+\nu}C$ are bounded, thus also $C|T|^{1+\nu}$ (on $D(|T|^{1+\nu})$). Let us consider $|T|^{\omega}C|T|^{1+\nu}$.

Evidently, it is sufficient to consider this product on $\operatorname{Ran} Q_1$. Choose $\sigma \in (0,1)$. As $C|T|^{1+\omega}$ is bounded on $D(|T|^{1+\omega})$, we have $\|Ch\| \le k\,\||T|^{-1-\omega}h\|$ for all $h \in D(|T|^{-1-\omega}) = \operatorname{Ran}(|T|^{1+\omega})$. Then, by Lemma 2.3, for $h \in D(|T|^{-1-\omega})$ we have $\||C|^{\sigma}h\| \le k^{\sigma}\||T|^{-1-\omega}h\|^{\sigma}\|h\|^{1-\sigma}$. Hence, by Lemma 2.4, since $|T|^{1+\omega}$ is strictly positive on $\operatorname{Ran} Q_1$, $\||C|^{\sigma}h\| \le k_0\||T|^{-\delta(1+\omega)}h\|$ for all $h \in D(|T|^{-\delta(1+\omega)})$ and any $\delta > \sigma$. Therefore, $|T|^{\delta(1+\omega)}|C|^{\sigma}$ and $|C|^{\sigma}|T|^{\delta(1+\omega)}$ are bounded. For $\delta = \frac{\omega}{1+\omega}$ and $\delta = \frac{1+\nu}{1+\omega}$, respectively, and $\sigma = \frac{1}{2}$ we recover $|T|^{\omega}|C|^{1/2}$ and $|C|^{1/2}|T|^{1+\nu}$ as bounded operators. Then, using the polar decomposition $B = U|B|$, we can represent $|T|^{\omega}C|T|^{1+\nu}$ as a composition of bounded operators; one has $|T|^{\omega}C|T|^{1+\nu} = |T|^{\omega}|C|^{1/2}U|C|^{1/2}|T|^{1+\nu}$. Now we estimate

$$\|F(\lambda)\|_{L(H)} \le c\{(\mathrm{Re}\,\lambda)^{-\nu-\omega} + (\mathrm{Re}\,\lambda)^{-1-\nu} + (\mathrm{Re}\,\lambda)^{-1-\nu} + (\mathrm{Re}\,\lambda)^{-2}\} \le c(\mathrm{Re}\,\lambda)^{-s},$$

where $s = \min\{\nu+\omega, 1+\omega, 1+\nu, 2\}$ and c is a constant. This estimate, along with the uniform boundedness of $(\lambda-T)(\lambda-S)^{-1}$ for $\lambda \in \Delta_2$, proves the integrability of $(\lambda-S)^{-1} - (\lambda-T)^{-1}$ on Δ_2 and completes the proof of the compactness of $P_+^{(2)} - Q_+^{(2)}$.

Let $x \in H$. Note that $[(\lambda-S)^{-1} - (\lambda-T)^{-1}]x \in D(T)$ for any $\lambda \in \Delta_2$. To prove $(P_+^{(2)} - Q_+^{(2)})x \in D(T)$, it is sufficient to show that $T[(\lambda-S)^{-1} - (\lambda-T)^{-1}]x$ is Bochner integrable on Δ_2 (see [12], Theorem 3.6.12). Since

$$\|T[(\lambda-S)^{-1} - (\lambda-T)^{-1}]x\| \le \left\{1 + \|(TC)^*\|_{L(H)}\right\}\|T(\lambda - T)^{-1}CT(\lambda - T)^{-1}\|_{L(H)}\|x\|,$$

it is sufficient to prove the integrability of $\|T(\lambda - T)^{-1}CT(\lambda - T)^{-1}\|_{L(H)}$ on Δ_2. But this can be done in the same way as in the case of $\|F(\lambda)\|_{L(H)}$, the only change being that one must use $|T|^{-1-\kappa}|T|^{1+\kappa}$ instead of $|T|^{-\omega}|T|^{\omega}$ in the decomposition of $TF(\lambda)$, for some $\kappa < \omega$ satisfying (2.5). Thus $P_+ - Q_+$ is compact on H and maps H into D(T).

It remains to prove that the restriction of $P_+ - Q_+$ to D(T) is compact with respect to the graph norm (2.2). Let $\hat{P}_+ = AP_+A^{-1}$. Then $\hat{P}_+ - Q_+ = P_+ - Q_+ + P_+C - BP_+ - BP_+C$ is compact on H. Moreover, for $h \in D(T)$ we have $(\hat{P}_+-Q_+)Th = T(P_+-Q_+)h$. Using the compactness of $\hat{P}_+ - Q_+$ and this intertwining property, one shows the compactness of the restriction of $P_+ - Q_+$ to D(T) with respect to the graph norm. This completes the proof of the lemma. □

COROLLARY 2.6. *In both topologies under consideration,* $\mathbb{I} - V$ *is compact.*

Here we note $\mathbb{I} - V = (Q_- - Q_+)(P_+ - Q_+)$. To prove Ker $V = \{0\}$ we have

LEMMA 2.7. *The operator* V *has zero null space.*

Proof: Let $Vh = 0$ for some $h \in H$. Then $Q_+P_+h = -Q_-P_-h$ yields that $P_+h = (Q_--P_-)P_+h = (P_+-Q_+)P_+h \in D(T)$ and $P_-h = -(P_+-Q_+)P_-h \in D(T)$, whence $h \in D(T)$ and Ker $V \subseteq D(T)$. Note that $P_+h \in \mathrm{Ran}\,Q_- \cap \mathrm{Ran}\,P_+$ (cf. [11]); thus $(TP_+h,P_+h) \le 0$, $(TP_+h,P_+h) = (A^{-1}TP_+h,P_+) \ge 0$ and hence $P_+h=0$. □

THEOREM 2.8. *The operator* V *is invertible and* $E = V^{-1}$ *is bounded (on* H, *and on* D(T) *with graph inner product* (2.2)). *The boundary value problem* (1.1)-(1.3) *is uniquely solvable for each* $\varphi_+ \in Q_+[D(T)]$, *and its solution is given by*

$$\psi(x) = e^{-xT^{-1}A}E\varphi_+, \quad 0 \leq x < \infty. \tag{2.7}$$

One may seek solutions of the boundary value problem for all $\varphi_+ \in Q_+[H]$ rather than just $\varphi_+ \in Q_+[D(T)]$. However, in this case it seems necessary to reformulate the problem by replaced (1.1) with $T\psi'(x) = -A\psi(x)$, $0 < x < \infty$, defining a solution to be a strongly continuous function ψ: $[0,\infty) \to H$ which is strongly continuously differentiable on $(0,\infty)$ such that $\psi'(x) \in D(T)$ for $x \in (0,\infty)$ and satisfies $T\psi'(x) = -A\psi(x)$ for $x \in (0,\infty)$, (1.2) and (1.3). We have

THEOREM 2.9. *The equation* $T\psi'(x) = -A\psi(x)$ *for* $x \in (0,\infty)$ *with boundary conditions* (1.2) *and* (1.3) *is uniquely solvable for each* $\varphi_+ \in Q_+[H]$ *and the solution is given by* (2.7).

3. CONSERVATIVE MODELS

In the previous section we have assumed that A is strictly positive. Requiring Ker A = {0} excludes from consideration many physically important problems, in particular linearized gas kinetics equations where conservation laws result in the collision operator A having a nontrivial kernel consisting of the collision invariants (cf. [5]). In this section we will generalize the results of Section 2 to the case where A is positive but has a nontrivial kernel. As before we will assume A to be a compact perturbation of the identity, but now it will satisfy the regularity condition

$$\exists\, \alpha \in (0,1) \text{ and } \omega > \max\{\tfrac{1+\alpha}{2},\tfrac{2-\alpha}{2}\}: \ \operatorname{Ran}(\mathbb{1} - A) \subseteq \operatorname{Ran}(|T|^{\alpha}) \cap D(|T|^{3+\omega}) \tag{3.1}$$

§ 3.1. Decompositions

First, note that $K = T^{-1}A$ and its H-adjoint K^* are closed and densely defined. Let us define the zero root linear manifold Z_0 of K by

$$Z_0 = \{f \in D(K):\ f \in D(K^n) \text{ and } K^n f = 0 \text{ for some } n \in \mathbb{N}\}.$$

In addition to condition (3.1) we will assume that $Z_0 \subseteq D(T)$. It then follows that

$$\{f \in Z_0: (Tf,g) = 0 \text{ for all } g \in Z_0\} = \{0\}. \tag{3.2}$$

In a similar way one can define $\hat{Z}_0$ as the zero root linear manifold of K^*. The following lemma characterizes Z_0 and $\hat{Z}_0$ and yields useful decompositions of H. For isotropic neutron transport, these results are due to Lekkerkerker [17] and for more general cases with T bounded to Van der Mee [18] and Greenberg et al. [8]. The proofs therein can be extended easily to unbounded T.

LEMMA 3.1. *If* $f \in Z_0$, *then there exists* $g \in Z_0$ *such that* $Kf = g$ *and* $Kg = 0$, *i.e.* $Z_0 = \text{Ker } K^2$. *Likewise* $\hat{Z}_0 = \text{Ker } (K^*)^2$. *One has*

$$T[Z_0] = \hat{Z}_0,$$

$$A[(\hat{Z}_0)^{\perp}] = (Z_0)^{\perp} = \overline{T[(\hat{Z}_0)^{\perp} \cap D(T)]},$$

and the following decompositions hold true:

$$Z_0 \oplus (\hat{Z}_0)^{\perp} = H, \tag{3.3a}$$

$$\hat{Z}_0 \oplus (Z_0)^{\perp} = H. \tag{3.3b}$$

The decompositions (3.3) will enable us to reduce a boundary value problem with given A (having nontrivial kernel) to one with a strictly positive collision operator. This reduction, in fact, follows immediately from the following proposition.

PROPOSITION 3.2. *Let* β be an invertible operator on Z_0 *satisfying*

$$(T\beta h,h) \geq 0,\quad h \in Z_0. \tag{3.4}$$

Let P_0 *be the projection of* H *onto* $(\hat{Z}_0)^{\perp}$ *along* Z_0. *If* A *is a nonnegative, compact perturbation of the identity with nontrivial kernel and satisfies the regularity condition* (3.1), *then* A_β *defined by*

$$A_\beta h = T\beta^{-1}(\mathbb{I} - P_0)h + AP_0 h \tag{3.5}$$

is a strictly positive operator satisfying

$$A_\beta^{-1}T = \beta \oplus (T^{-1}A\Big|_{(\hat{Z}_0)^\perp})^{-1}. \tag{3.6}$$

The operator A_β *is a compact perturbation of the identity satisfying the condition*

$$\mathrm{Ran}\,(\mathbb{I} - A_\beta) \subseteq \mathrm{Ran}\,(|T|^\alpha) \cap D(|T|^{1+\omega}), \tag{3.7}$$

with α *and* ω *as in* (3.1).

<u>Proof:</u> The identity (3.6) follows immediately from the definition of A_β. Moreover, for $g \in H$ we have $(A_\beta g,g) = (AP_0 g,P_0 g) + (T\beta^{-1}(\mathbb{I} - P_0)g,(\mathbb{I} - P_0)g) \geq 0$, using (3.4) for $h = \beta^{-1}T(\mathbb{I}-P_0)g$. As $\sigma(A) \subseteq \{0\} \cup [\epsilon,\infty)$ for some $\epsilon > 0$ and Z_0 has finite dimension, we must have strict positivity for A_β from the triviality of its kernel.

Next, since $A_\beta - A = (A_\beta - A)(\mathbb{I}-P_0)$ has finite rank, A_β is a compact perturbation of the identity. Furthermore, $\mathbb{I} - A_\beta = (\mathbb{I}-A) + T(T^{-1}A_\beta - T^{-1}A)(\mathbb{I}-P_0)$, $Z_0 = \mathrm{Ker}\,(T^{-1}A)^2$ and (3.1) imply

$$\mathrm{Ran}\,(\mathbb{I} - A_\beta) \subseteq \mathrm{Ran}\,(|T|^\alpha) \cap D(|T|^{1+\omega}),$$

thus yielding (3.7). □

As in Section 2 one can construct the Hilbert space H_{A_β} with the H_{A_β}-norm equivalent to the original norm on H, so that the topology of H_{A_β} does not depend on the particular choice of β in Proposition 3.2. We may define $P_\pm$ as the H_{A_β}-orthogonal projections of H onto the maximal $A_\beta^{-1}T$-positive/ negative $A_\beta^{-1}T$-invariant subspaces. From the above proposition it follows that $P_{1,+} = P_0P_+$, $P_{1,-} = P_0P_-$ and P_0 form a family of complementary projections commuting with $T^{-1}A$ which do not depend on the particular choice of β.

The next proposition gives a decomposition of Z_0 into T-positive/ negative subspaces and a characterization of β compatible with the intended boundary value problems. A proof of this proposition can be found in [18] and [8] for bounded T; the unbounded T case introduces some technicalities connected with D(T).

PROPOSITION 3.3. *The subspaces*

$$\mathcal{M}_{\pm} = [\text{Ran } P_{1,\mp} \oplus \text{Ran } Q_{\pm}] \cap Z_0$$

satisfy the conditions

$$\pm(Tf,f) > 0, \quad 0 \neq f \in \mathcal{M}_{\pm},$$

$$\mathcal{M}_{+} \oplus \mathcal{M}_{-} = Z_0,$$

$$[\mathcal{M}_{+} \cap \text{Ker } A] \oplus [\mathcal{M}_{-} \cap \text{Ker } A] \oplus T^{-1}A[Z_0] = \text{Ker } A.$$

Moreover, it is possible to choose β such that

$$\text{Ran } P_{+} \subseteq \text{Ran } P_{1,+} \oplus \text{Ker } A.$$

§ 3.2. Existence and uniqueness theory for half-space problems

In this subsection we will analyze the boundary value problems

$$(T\psi)'(x) = -A\psi(x), \quad 0 < x < \infty, \tag{3.8}$$

$$Q_{+}\psi(0) = \varphi_{+}, \tag{3.9}$$

and

$$(T\psi)'(x) = -A\psi(x), \quad -\infty < x < 0, \tag{3.10}$$

$$Q_{-}\psi(0) = \varphi_{-}, \tag{3.11}$$

along with a condition at infinity, namely, one of

$$\lim_{x\to\pm\infty} \|\psi(x)\|_H = 0, \tag{3.12}$$

$$\|\psi(x)\|_H = O(1) \quad (x \to \pm\infty), \tag{3.13}$$

$$\|\psi(x)\|_H = O(x) \quad (x \to \pm\infty). \tag{3.14}$$

The upper and lower signs are to be taken with Eqs. (3.8)-(3.9) and Eqs. (3.10)-(3.11), respectively. As there is a complete symmetry between left and right half-space problems, we will consider the right half-space problem only. By a solution of the various boundary value problems for $\varphi_+ \in Q_+[D(T)]$ we shall mean a strongly continuous function ψ: $[0,\infty) \to D(T)$ such that $T\psi$ is strongly differentiable on $(0,\infty)$ and Eqs. (3.8), (3.9) and one of (3.12)-(3.14) are satisfied. Solutions of left half-space problems are defined analogously.

First we outline the procedure which will be used to construct solutions to these boundary value problems. Let us reduce the half-space problem (3.8)-(3.9) to two subproblems. Writing $\psi_1 = (\mathbb{I} - P_0)\psi$ and $\psi_0 = P_0\psi$, Eq. (3.8) may be decomposed as follows:

$$(T\psi_1)'(x) = -A\psi_1(x), \quad 0 < x < \infty, \tag{3.15}$$

$$\psi_0'(x) = -T^{-1}A\psi_0(x), \quad 0 < x < \infty.$$

The second equation is an evolution equation on the finite-dimensional space Z_0 and therefore admits an elementary solution of the form

$$\psi_0(x) = e^{-xT^{-1}A}\psi_0(0) = (\mathbb{I} - xT^{-1}A)\psi_0(0),$$

using Lemma 3.1. Next consider Eq. (3.15). Add to it the dummy equation

$$(T\varphi_0)'(x) = -A_\beta\varphi_0(x), \; 0 < x < \infty, \tag{3.16}$$

on Z_0 where A_β is given by (3.5) for some β. The solution of Eq. (3.16) is easy to compute but does not concern us, as it will be projected out shortly. However, defining $\varphi = \varphi_0 + \psi_1$ we can combine Eqs. (3.15) and (3.16) to obtain

$$(T\varphi)'(x) = -A_\beta\varphi(x), \quad 0 < x < \infty. \tag{3.17}$$

Now since A_β is strictly positive (and a compact perturbation of the identity satisfying (3.7)), we apply the results of Section 2 to Eq. (3.17) and find its solution as

$\varphi(x) = \exp\{-xT^{-1}A_\beta\}Eg_+$, $x \in (0,\infty)$, where $g_+ \in Q_+[D(T)]$ and $E = (Q_+P_+ + Q_-P_-)^{-1}$. Then projecting $\varphi(x)$ onto $(\tilde{Z}_0)^\perp$ along Z_0 and adding $\psi_0(x)$ we obtain a solution of Eq. (3.8) in the form

$$\psi(x) = e^{-xT^{-1}A}(\mathbb{I} - P_0)Eg_+ + \psi_0(x), \quad 0 \le x < \infty.$$

Now we will fit the boundary condition (3.9). To do so we must indicate vectors $g_+ \in Q_+[D(T)]$ and $\psi_0(0) \in Z_0$ such that

$$Q_+[(\mathbb{I} - P_0)Eg_+ + \psi_0(0)] = \varphi_+.$$

Note that if $\psi_0(0) = 0$, $\psi_0(0) \in \operatorname{Ker} A$ or $\psi_0(0) \in Z_0$, then the respective right half-space condition (3.12), (3.13) or (3.14) is satisfied.

Let us define the measure of non-completeness for any of the boundary value problems to be the co-dimension in $\operatorname{Ran} Q_+$ of the subspace of boundary values $\varphi_+ \in \operatorname{Ran} Q_+$ for which the problem has at least one solution, and the measure of non-uniqueness to be the dimension of the solution space of the corresponding homogeneous boundary value problem. The principal results of this article are the following existence and uniqueness theorems.

THEOREM 3.4. *The boundary value problem* (3.8), (3.9) and (3.12) *has at most one solution for each* $\varphi_+ \in Q_+[D(T)]$ *and its measure of non-completeness coincides with the maximal number of linearly independent vectors* $g_1, \cdots, g_n \in \operatorname{Ker} A$ *satisfying*

$$(Tg_i,g_j) = 0, \quad 1 \le i,\ j \le n, \quad i \ne j,$$

$$(Tg_i,g_i) \ge 0, \quad 1 \le i \le n.$$

The solution, if it exists, is given by

$$\psi(x) = e^{-xT^{-1}A}(\mathbb{I} - P_0)Eg_+,$$

where g_+ *is the unique solution of the vector equation*

$$Q_+(\mathbb{I} - P_0)Eg_+ = \varphi_+.$$

THEOREM 3.5. *The boundary value problem* (3.8), (3.9) *and* (3.13) *has at least one solution for each* $\varphi_+ \in Q_+[D(T)]$ *and its measure of non-uniqueness coincides with the number of linearly independent vectors* $h_1, \cdots, h_p \in \text{Ker } A$ *satisfying*

$$(Th_i,h_j) = 0, \quad 1 \leq i, j \leq p, \quad i \neq j,$$

$$(Th_i,h_i) < 0, \quad 1 \leq i \leq p.$$

The solutions have the form

$$\psi(x) = e^{-xT^{-1}A}(\mathbb{I} - P_0)Eh_+ + h_0,$$

where $h_0 \in [\text{Ran } P_{1,+} \oplus \text{Ran } Q_-] \cap \text{Ker } A$ *and* h_+ *is the unique solution of*

$$Q_+(\mathbb{I} - P_0)Eh_+ + Q_+h_0 = \varphi_+.$$

THEOREM 3.6. *The boundary value problem* (3.8), (3.9) *and* (3.14) *has at least one solution for each* $\varphi_+ \in Q_+[D(T)]$ *and its measure of non-uniqueness coincides with the number of linearly independent vectors* $e_1, \cdots, e_m \in Z_0$ *satisfying*

$$(Te_i,e_j) = 0, \quad 1 \leq i, j \leq m, \quad i \neq j,$$

$$(Te_i,e_i) < 0, \quad 1 \leq i \leq m.$$

The solutions have the form

$$\psi(x) = e^{-xT^{-1}A}(\mathbb{I} - P_0)Ef_+ + (\mathbb{I} - xT^{-1}A)f_0,$$

where $f_0 \in [\text{Ran } P_{1,+} \oplus \text{Ran } Q_-] \cap Z_0$ *and* f_+ *is the unique solution of*

$$Q_+(\mathbb{I} - P_0)Ef_+ + Q_+f_0 = \varphi_+.$$

The proofs of Theorems 3.4-3.6 can be given in almost precise analogy with results derived in [18] (also [7,8]).

If the differential equation (3.8) is replaced by

$$T\psi'(x) = -A\psi(x), \quad 0 < x < \infty, \tag{3.18}$$

one can seek solutions of the boundary value problems with $\varphi_+ \in Q_+[H]$. Here by a solution we mean a strongly continuous function ψ: $[0,\infty) \to H$ which is continuously differentiable on $(0,\infty)$, $\psi'(x) \in D(T)$ for $x \in (0,\infty)$ and such that Eq. (3.18), the boundary condition $Q_+\psi(0) = \varphi_+$ and an appropriate condition at infinity are satisfied. Then one can prove the analogs of Theorems 3.4-3.6, where one has to substitute Eq. (3.18) for Eq. (3.8) and φ_+ belongs to $Q_+[H]$ rather than $Q_+[D(T)]$.

4. APPLICATIONS

This section contains two physical models leading to equations of the form (1.1) involving a time-independent one-dimensional transport problem in a semi-infinite medium with spatial variable $x \in [0,\infty)$. We will specify the Hilbert space H, the operators T and A, the kernel of A and the zero root linear manifold Z_0 and point out the impact of the existence and uniqueness theory of Sections 2 and 3.

§ 4.1. The one-dimensional BGK model equation (cf. [1,19])

$$v\frac{\partial f}{\partial x}(x,v) = -f(x,v) + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} [1 + 2vu + 2(v^2-\tfrac{1}{2})(u^2-\tfrac{1}{2})]f(x,u)e^{-u^2}du, \quad v\in\mathbb{R}.$$

The equation is posed in the space $H = L_2(\mathbb{R},\rho)$ where $d\rho = \pi^{-1/2}e^{-v^2}dv$. Put

$$(Tf)(v) = vf(v),$$

$$(Af)(v) = f(v) - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} [1 + 2vu + 2(v^2-\tfrac{1}{2})(u^2-\tfrac{1}{2})]f(u)e^{-u^2}du.$$

Then T is unbounded self-adjoint, A is bounded positive and $\mathbb{I} - A$ has finite rank. One can check that condition (3.1) is fulfilled, $\operatorname{Ker} A = \operatorname{span}\{1,v,v^2\}$ and $Z_0 = \operatorname{span}\{1,v,v^2,v^3\}$. Now we introduce the sesquilinear form

$$[h,k] = (Th,k) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} vh(v)\overline{k(v)}e^{-v^2}dv.$$

To apply Theorems 3.4 and 3.5 one has to represent this sesquilinear form as a diagonal matrix with respect to an appropriate (mutually $[\cdot,\cdot]$-orthogonal) basis of Ker A. The diagonalization of a symmetric bilinear form is a simple algebraic procedure and results in a matrix with 1, -1 and 0 on the diagonal. Then, by Theorem 3.4, a solution of Eqs. (1.1)-(1.2) vanishing as $x \to \infty$ may not exist and its measure of non-completeness is 2. Solutions of the boundary value problem which are bounded as $x \to \infty$ always exist by Theorem 3.5, and indeed have measure of non-uniqueness 1. Diagonalization of the symmetric bilinear form on Z_0 leads to a matrix with 1, -1, $\frac{3}{2}$ and $-\frac{3}{2}$ on the diagonal. Thus, solutions to the boundary value problem of order x as $x \to \infty$ have measure of non-uniqueness 2.

§ 4.2. BGK <u>equation</u> <u>for</u> <u>heat</u> <u>transfer</u> (cf. [4,5,6,16])

$$v\frac{\partial}{\partial x}\begin{bmatrix} f_1(x,v) \\ f_2(x,v) \end{bmatrix} = -\begin{bmatrix} f_1(x,v) \\ f_2(x,v) \end{bmatrix} + \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathbb{D}(v,u)\begin{bmatrix} f_1(x,u) \\ f_2(x,u) \end{bmatrix} e^{-u^2}du, \quad v \in \mathbb{R},$$

with

$$\mathbb{D}(v,u) = \begin{bmatrix} 1 + \frac{2}{3}(v^2-\frac{1}{2})(u^2-\frac{1}{2}) & \frac{2}{3}(v^2-\frac{1}{2}) \\ \frac{2}{3}(u^2-\frac{1}{2}) & \frac{2}{3} \end{bmatrix}.$$

This equation is posed in the space $H = L_2(\mathbb{R},\rho) \oplus L_2(\mathbb{R},\rho)$ with ρ as in the first example. Let f be the column vector with entries f_1 and f_2. We define T and A by

$$(Tf)(v) = vf(v),$$

$$(Af)(v) = f(v) - \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathbb{D}(v,u)f(u)e^{-u^2}du.$$

Then T is unbounded self-adjoint, A is bounded positive and $\mathbb{I} - A$ has finite rank. One easily checks that condition (3.1) is satisfied. Then

$$\text{Ker } A = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} v^2 \\ 1 \end{bmatrix}\right\}, \quad Z_0 = \text{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} v \\ 1 \end{bmatrix}, \begin{bmatrix} v^2 \\ 1 \end{bmatrix}, \begin{bmatrix} v^3 \\ v \end{bmatrix}\right\}.$$

Again we introduce the sesquilinear form

$$[h,k] = (Th,k) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} vh_1(v)\overline{k_1(v)}e^{-v^2}dv + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} vh_2(v)\overline{k_2(v)}e^{-v^2}dv.$$

Since Ker A is degenerate with respect to this sesquilinear form, Theorem 3.5 implies that the boundary value problem (1.1)-(1.2) has a unique bounded solution as $x \to \infty$. On the other hand, by Theorem 3.4, solutions vanishing as $x \to \infty$ may not exist, its measure of non-completeness being 2, which is a result of the conservation laws (of mass and energy). To apply Theorem 3.6 one has to represent the sesquilinear form on Z_0 as a diagonal matrix. One obtains a matrix with 1, -1, $\frac{5}{2}$ and $-\frac{5}{2}$ on the diagonal. Thus, solutions to the boundary value problem of order x as $x \to \infty$ have measure of non-uniqueness 2. The corresponding Kramers or slip-flow problems has a two-dimensional manifold of solutions.

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