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WHITHER EXISTENCE THEORY?

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ABSTRACT

It is suggested that in certain cases the Fredholm alternative theorem can be used to simplify existence proofs in time-independent linear transport theory. Existing results in abstract kinetic equations theory are generalized as to encompass polarized light transfer as well as certain inhomogeneous finite slab media.

1. Introduction

The proof that a transport equation (or any mathematical equation) is well-posed includes a demonstration that a unique solution exists. In many instances, it is relatively simple that a solution, if it exists, is unique, whereas the existence question is often considerably more delicate. This situation occurs for the generalized transport equation originally studied by Beals[1] and generalized by a number of authors[2,3]. Here it turns out that simple positivity arguments[4] modeled after an idea introduced originally by Case and Zweifel[5] can be used to prove uniqueness. The existence proof is rather involved, requiring the introduction of various auxiliary inner products and Hilbert spaces with associated projection operators, and a number of technical arguments which tend to obscure the subject to the physicist or engineer. Another dis-

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advantage of this approach is that it seems to defy generalization to inhomogeneous media problems and non-plane-parallel geometry, and requires selfadjointness assumptions of the two basic operators appearing in the abstract equation.

The purpose of this article is to point out that in certain cases, at least, the separate existence proof is entirely redundant, i.e., existence is already implied by the demonstration of uniqueness. These cases are those in which the Fredholm alternative can be brought into action. By way of review, we recall Halmos' particularly elegant statement of the alternative,[6] which we paraphrase as follows. Let C be a compact operator from a Banach space H into itself, and consider the operator equation

$$f = Cf + g;\ f,\ g \in H, \tag{1}$$

where g is known. Then if the solution to Eq. (1) is unique, it exists. We refer the reader to Reference 6 for a proof of the alternative theorem.[7]

A class of generalized kinetic equations for which the above Fredholm alternative can be applied in a most elegant fashion is the class of boundary value problems

$$(Tf)'(x) + (I-B)f(x) = g(x), \quad x \in (0,\tau) \tag{2}$$

$$Q_+f(0) = Q_+\phi,\ Q_-f(\tau) = Q_-\phi. \tag{3}$$

Here T and B are operators on an abstract Hilbert space H, T is (bounded or unbounded) self-adjoint and injective and B is a compact selfadjoint operator satisfying

$$\exists 0<\alpha<1: \quad \text{Ran } B \subset \text{Ran}|T|^\alpha, \tag{4}$$

while A = I-B is positive self-adjoint. By the closed graph theorem, this condition implies the existence of a bounded operator D such that

$$B = |T|^\alpha D. \tag{5}$$

The operators Q_+ and Q_- appearing in the boundary conditions (3) are the orthogonal projections of H onto the maximal positive and negative T-invariant subspaces, respectively. This class of abstract boundary value problems which model a large variety of kinetic equations in neutron transport, radiative transfer, rarefied gas dynamics and even phonon transport (see References 3, 9 for many references) has been discussed at great length by Van der Mee[2,10] (T bounded) and Greenberg, Van der Mee and Walus[9] (T unbounded). For these models uniqueness can be proved by a generalization of the argument used in Reference 4. Furthermore, Van der Mee proved (for T bounded in Reference 2, for T unbounded in Reference 10) that the operator

$$(L_\tau f)(x) = \int_0^\tau H(x-y)Bf(y)dy, \quad 0 < x < \tau(<\infty), \tag{6}$$

is compact on $L_p((0,\tau);H)$, $1 \le p \le \infty$.[11] (Here $H(\cdot)$ is the "propagator function" of T, i.e., the integral kernel of the operator $(I+T\frac{d}{dx})^{-1}$.) He then goes on to prove that the boundary value problem (2) - (3) is equivalent to the vector equation

$$f = L_\tau f + \omega \tag{7}$$

on $L_\infty((0,\tau);H)$, where ω is a known function.[12] At this point, existence follows from the Fredholm alternative theorem; no further argument is needed.

Although for many applications the unboundedness of the operator A = I-B in Equation (2) causes the simplification suggested above to be non-applicable (for example, cf. References 13, 14, 15), we hope that our remarks have added a new dimension and have suggested a new direction for existence questions in transport theory. We recall that some classical existence proofs[16] have relied on the fact that the operator L_τ of Equation (7) obeys $\|L_\tau\| < 1$, so that the Neumann series solution to Equation (7) converges. Evidently, this gives only a sufficient condition for existence, which is not very sharp; the Fredholm alternative may well give substantially improved criteria.

In this article we shall also generalize the existing theory in two directions. First we shall assume that A = I-B has a non-negative real part, $\text{Re}A = \frac{1}{2}(A+A^*) \ge 0$, and satisfies

$$\mathrm{Ker}(\mathrm{Re}A) = \mathrm{Ker}\ A. \tag{8}$$

Beyond the models where A is positive selfadjoint, a new application is provided by polarized light transfer[17,18] leading to a novel proof of a recent result of Van der Mee[19]. The use of A with nonnegative real part was suggested by work of Beals[20]. A second generalization is to allow x-dependence of the compact operator B. In particular, B_x is a continuous function from $[0,\tau]$ into the compact operators on H, which satisfies the regularity condition (4) for all $x \in [0,\tau]$, while $A_x = I-B_x$ has nonnegative real part with $\mathrm{Ker}(\mathrm{Re}A_x) = \mathrm{Ker}\ A_x$ almost everywhere. For this general situation the Fredholm alternative argument still applies (although with some modifications) and the uniqueness argument goes through completely. As a result new existence and uniqueness results are obtained for inhomogeneous media problems.

We remark that the restriction $\tau < \infty$ in Equation (6) is necessary to guarantee the compactness of the operator L_τ and therefore the full machinery of the Fredholm alternative. For half-space problems the compactness of L_τ often (and always for homogeneous media) breaks down and the argument does not apply in the above way. However, for subcritical problems (ReA strictly positive) the operator $I-L_\tau$ is still a Fredholm operator and some modification of the Fredholm alternative argument might be sought for.

The article is organized as follows. In Section 2 we discuss the uniqueness problem for the case of a homogeneous medium ($\mathrm{Re}A \geq 0$, x-independent) as well as the precise form of the Fredholm alternative. In Section 3 we analyze the modifications required if B depends on x. In Section 4 we apply our results to the equation of transfer of polarized light.

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2. Homogeneous Media

Let T be injective selfadjoint, B compact satisfying (4), A have a nonnegative real part satisfying (8); let $\phi \in D(T)$ and let

g(x) meet the Hölder continuity condition of Reference 12. Then every continuous function $f:[0,\tau]\to H$ such that $f(x) \in D(T)$ for all $x \in (0,\tau)$, Tf is strongly differentiable on $(0,\tau)$ and Equations (2)-(3) hold true is a solution of the convolution equation (7) where $\omega(x)$ is given in Reference 12. Conversely, every solution f of Eq. (7) in $L_\infty((0,\tau);H)$ is continuous on $[0,\tau]$, has its values in D(T), has Tf strongly differentiable on $(0,\tau)$ and satisfies Equations (2)-(3)[?,10]. Hence, in view of the compactness of L_τ on $L_\infty((0,\tau);H)$, it suffices to prove that the boundary value problem

$$(Tf)'(x) + (I-B)f(x) \equiv 0, \quad 0 < x < \tau \tag{9}$$

$$Q_+f(0) = Q_-f(\tau) = 0 \tag{10}$$

has the trivial solution f = 0 only. Indeed, given a solution f of these equations we have

$$-2((\mathrm{Re}A)f(x),f(x)) = -(Af(x),f(x)) - (f(x),Af(x)) =$$
$$= ((Tf)'(x),f(x)) + (f(x),(Tf)'(x)) = \frac{d}{dx}(Tf(x),f(x))$$

(see Appendix for the last equality) and, as $\mathrm{Re}A \geq 0$,

$$0 \geq -2\int_0^\tau ((\mathrm{Re}A)f(x),f(x))dx = (Tf(\tau),f(\tau)) - (Tf(0),f(0)) \geq 0,$$

because $f(0) = Q_-f(0)$ and $f(\tau) = Q_+f(\tau)$. Hence, $((\mathrm{Re}A)f(x), f(x)) \equiv 0$ and thus

$$(\mathrm{Re}A)f(x) \equiv 0, \quad 0 \leq x \leq \tau.$$

Using (8), we have $Af(x) \equiv 0$ and therefore [cf. (9)] $(Tf)'(x) \equiv 0$. Hence, $f(x) \equiv h$ with $h \in \mathrm{Ker}A$. Finally, we have [cf. (10)]

$$h = Q_+h + Q_-h = Q_+f(0) + Q_-f(\tau) = 0,$$

whence f = 0. Uniqueness, and thus existence, of the solution of Equations (2)-(3) is clear.

3. Inhomogeneous Media

Let T, ϕ and $q(x)$ be as before, and let $x \mapsto B_x$ be a continuous function (with respect to the norm topology) from $[0,\tau]$ into the compact operators on H. Suppose that for fixed $\alpha \in (0,1)$

$$\text{Ran}\, B_x \subset \text{Ran}|T|^{\alpha}, \quad 0 < x < \tau, \tag{11}$$

where the family of operators $\{|T|^{-\alpha}B_x | x \in [0,\tau]\}$ is bounded. Also suppose that $\text{Re}A_x \geq 0$ and $\text{Ker Re}A_x = \text{Ker}A_x$ for almost every $x \in [0,\tau]$. Then every continuous function $f:[0,\tau] \to H$ such that $f(x) \in D(T)$ for all $x \in (0,\tau)$, Tf is strongly differentiable on $(0,\tau)$ and Equations (2)-(3) hold true (with B replaced by B_x) is a solution of the integral equation (7), where $\omega(x)$ is given in Reference 12 and

$$(L_\tau f)(x) = \int_0^\tau H(x-y)B_y f(y)dy, \quad x \in (0,\tau). \tag{12}$$

Conversely, every solution f of Equation (7), with L_τ defined by (12), in $L_\infty((0,\tau);H)$ is continuous on $[0,\tau]$, has its values in $D(T)$, has Tf strongly differentiable on $(0,\tau)$ and satisfies Equations (2)-(3) (with B replaced by B_x). Indeed, for fixed $0 < \beta < \alpha$ one may prove $x \mapsto |T|^{-\beta} B_x$ continuous on $[0,\tau]$. Exactly repeating the equivalence proof of either Reference 2 (T bounded) or Reference 10 (T unbounded) and considering the compact operator-valued function $x \mapsto D_x = |T|^{-\beta} B_x$ on $[0,\tau]$, one easily proves L_τ compact on $L_p((0,\tau);H)$, $1 \leq p \leq \infty$. The uniqueness proof of the previous section goes through completely, with only the nominal change $B \mapsto B_x$. Hence, under the above conditions Equations (2)-(3) (with B replaced by B_x) are uniquely solvable.

4. Application to Polarized Light Transfer

The equation of transfer of polarized light for a plane-parallel atmosphere of finite optical thickness τ reads[17]

$$\mu\frac{\partial}{\partial x}\underset{\sim}{f}(x,\mu,\phi) + \underset{\sim}{f}(x,\mu,\phi) = \frac{c}{4\pi}\int_{-1}^{1}\int_0^{2\pi} \underset{\sim}{Z}(\mu,\mu',\phi-\phi')\underset{\sim}{f}(x,\mu',\phi')d\phi'd\mu', \quad 0 < x < \tau \tag{13}$$

$$\underset{\sim}{f}(0,\mu,\phi) = \underset{\sim}{\phi}(0,\mu,\phi) \text{ for } \mu > 0, \quad \underset{\sim}{f}(\tau,\mu,\phi) = 0 \text{ for } \mu < 0. \tag{14}$$

Here $\underset{\sim}{f}(x,\mu,\phi)$ is the four-vector of polarization parameters I,Q,U,V with I the intensity, $\underset{\sim}{Z}(\mu,\mu',\phi-\phi')$ is the phase matrix describing single scattering, total absorption by the planetary surface is assumed and $c \in (0,1]$. Introducing the Hilbert space H of measurable functions $\underset{\sim}{h}:[-1,1]\times[0,2\pi] \to \mathbb{C}^4$ which are bounded with respect to the L_2-norm, and defining the operators

$$(T\underset{\sim}{h})(\mu,\phi) = \mu\underset{\sim}{h}(\mu,\phi),$$
$$(B\underset{\sim}{h})(\mu,\phi) = \frac{c}{4\pi}\int_{-1}^{1}\int_0^{2\pi} \underset{\sim}{Z}(\mu,\mu',\phi-\phi')\underset{\sim}{f}(x,\mu',\phi')d\phi'd\mu' \tag{15}$$

$$(Q_+\underset{\sim}{h})(\mu,\phi) = \begin{cases} \underset{\sim}{h}(\mu,\phi), & \mu > 0; \\ 0, & \mu < 0 \end{cases} \quad (Q_-\underset{\sim}{h})(\mu,\phi) = \begin{cases} 0, & \mu > 0 \\ \underset{\sim}{h}(\mu,\phi), & \mu < 0, \end{cases} \tag{16}$$

an example of Equations (2)-(3) arises. Inhomogeneous media problems arise for x-dependent phase functions. The phase function allows the factorization[17]

$$\underset{\sim}{Z}(\mu,\mu',\phi-\phi') = \underset{\sim}{L}(\pi-\sigma_2)\underset{\sim}{F}(\theta)\underset{\sim}{L}(-\sigma_1)$$

for suitable angles θ,σ_1,σ_2 depending on μ, μ' and $(\phi-\phi')$, where

$$\underset{\sim}{L}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \underset{\sim}{F}(\theta) = \begin{bmatrix} a_1(\theta) & b_1(\theta) & 0 & 0 \\ b_1(\theta) & a_2(\theta) & 0 & 0 \\ 0 & 0 & a_3(\theta) & b_2(\theta) \\ 0 & 0 & -b_2(\theta) & a_4(\theta) \end{bmatrix},$$

and where $\underset{\sim}{F}(\theta)$ leaves invariant the positive cone of vectors (I,Q,U,V) satisfying $I \geq (Q^2+U^2+V^2)^{1/2} \geq 0$ and $\int_{-1}^{1} a_1(\theta)d(\cos\theta) = 2$. On the basis of these properties it can be shown[18,21] that B is a compact operator on H whose eigenvalues are situated in the half plane $\{\lambda|\text{Re}\lambda < 1\}$ if $c \in (0,1)$, and in the set $\{\lambda|\text{Re}\lambda < 1\} \cup \{1\}$ if $c = 1$. Since $\text{Re}B = \frac{1}{2}(B+B^*)$ has the form (15) with $b_2(\theta) \equiv 0$, we also have the eigenvalues of $\text{Re}B$ in the set $\{\lambda|\text{Re}\lambda < 1\}$ if $c \in (0,1)$ and $\{\lambda|\text{Re}\lambda < 1\} \cup$

{1} if c = 1. Hence, $\mathrm{Re}A \geq 0$ and $\mathrm{Ker}(\mathrm{Re}A) = \mathrm{Ker}A = \mathrm{span}\{(1,0,0,0),(0,0,0,1)\}$ if $a_1(\theta) \equiv a_4(\theta)$ and $\mathrm{Ker}(\mathrm{Re}A) = \mathrm{Ker}A = \mathrm{span}\{(1,0,0,0)\}$ otherwise. On assuming that

$$\exists r > 1: \int_{-1}^{1} a_1(\theta)^r d(\cos\theta) < \infty, \tag{17}$$

we may obtain the regularity assumption (4)[19]. Hence, if (17) is satisfied and the scattering matrix $\underset{\sim}{F}(\theta)$ leaves invariant the positive cone of vectors (I,Q,U,V) with $I \geq (Q^2+U^2+V^2)^{1/2} \geq 0$, the transport problem (13)-(14) is uniquely solvable. Hence, we have derived in a different way a result of Van der Mee[19]. In order to have an application for inhomogeneous atmospheres, we have to assume that $\underset{\sim}{F}(\theta)$ depends on x: $\underset{\sim}{F}(\theta) = \underset{\sim}{F}(\theta;x)$. Also the functions a_1,a_2,a_3,a_4,b_1 and b_2 must satisfy the continuity assumptions

$$\forall\varepsilon:\exists\delta_c:[\int_{-1}^{1}|c(\theta;x)-c(\theta;y)|^r\, d(\cos\theta)]^{1/r} < \varepsilon \text{ if } |x-y| < \delta_c$$

with $x,y \in [0,\tau]$ and fixed $r > 1$, as well as the property that all matrices $\underset{\sim}{F}(\theta;x)$ leave invariant the vectors (I,Q,U,V) satisfying $I \geq (Q^2+U^2+V^2)^{1/2} \geq 0$. Hence, the transport problem (13)-(14) with $\underset{\sim}{Z}(\mu,\mu',\phi-\phi')$ replaced by $\underset{\sim}{Z}(\mu,\mu',\phi-\phi';x)$ is uniquely solvable also.

5. Concluding Remarks

In an almost trivial way we have derived existence and uniqueness results which were previously known to be deducible rigorously only using heavy functional analysis, including an extensive apparatus of inner products, projections and scattering operators. We have extended these results to nonnegative real parts for A = I-B, while such an extension is far from obvious if one applies the usual arguments in abstract kinetic equations theory. The present approach also seems promising since it appears to render results on the Achilles heel of abstract kinetic equations theory: inhomogeneous media.

Appendix

Let $f: [0,\tau] \to H$ be continuous, $f(x) \in D(T)$ for $0 < x < \tau$, and Tf strongly differentiable on $(0,\tau)$. Then $x \mapsto (Tf(x),f(x))$ is differentiable on $(0,\tau)$ and

$$\frac{d}{dx}(Tf(x),f(x)) = ((Tf)'(x),f(x))+(f(x),(Tf)'(x)),\ x \in (0,\tau). \tag{18}$$

Since Tf is strongly differentiable rather than f, the identity is not completely trivial and a proof, however straightforward, is required. Indeed, writing

$$\frac{1}{\varepsilon}\{(Tf(x+\varepsilon),f(x+\varepsilon)) - (Tf(x),f(x))\} =$$

$$= \frac{1}{\varepsilon}(T[f(x+\varepsilon)-f(x)],f(x+\varepsilon)) + \frac{1}{\varepsilon}(f(x),T[f(x+\varepsilon)-f(x)]),$$

using the strong differentiability of Tf and the (local) boundedness of f, Eq. (18) is easily seen to be fulfilled.

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12. In fact, $\omega(x) = e^{-xT^{-1}} Q_+\phi + e^{(\tau-x)T^{-1}} Q_-\phi + \int_0^\tau H(x-y)g(y)dy$, where we have to assume $\| g(x)-g(y) \|_H \leq M|x-y|^\gamma$ for some $M<\infty$ and $\gamma \in (0,1)$ with $x,y \in [0,\tau]$.

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