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Half- and Finite-Range Completeness for the Equation of Transfer of Polarized Light*)

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On neglecting reflection by the surface the existence and uniqueness are proved for the solution of the equation of transfer of polarized light in a homogeneous semi-infinite or finite plane-parallel medium. A general L_p-space formulation, where $1 \leqslant p < \infty$, is adopted. The analysis concerns a vector-valued convolution equation, which is an equivalent form of the equation of radiative transfer and is solved with the help of Wiener-Hopf factorization, Fredholm index and cone preservation methods. The results are also proved for the equations obtained from the full equation of transfer by means of Fourier expansion and symmetry relations.

1 Introduction

On neglecting vertical inhomogeneities and thermal emission, the equation of transfer of polarized light in a plane-parallel atmosphere of finite or semi-infinite *optical thickness b* is the vector-valued integro-differential equation

$$(1.1)\quad u\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{I}(\tau,u,\varphi) + \mathbf{I}(\tau,u,\varphi) = \frac{a}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi}\mathbf{Z}(u,u',\varphi-\varphi')\mathbf{I}(\tau,u',\varphi-\varphi')\,\mathrm{d}\varphi'\,\mathrm{d}u', \qquad 0<\tau<b.$$

Here $0 < a \leqslant 1$ is the *albedo of single scattering*, $\mathbf{Z}(u,u',\varphi-\varphi')$ the phase matrix and $\mathbf{I}(\tau,u,\varphi)$ a four-vector depending on optical depth τ, direction cosine of propagation u and azimuthal angle φ. The components I, Q, U and V of the vector $\mathbf{I}$ are the Stokes parameters, which describe the intensity and state of polarization of the beam. A consistent treatment of polarized light transfer based on the (equivalent) conventions for polarization parameters of Chandrasekhar [2] and Van de Hulst [15] is given in [14], on which we shall rely for notations.

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The *phase matrix* can be expressed as the product

(1.2) $$\mathbf{Z}(u,u',\varphi-\varphi') = \mathbf{L}(\pi-\sigma_2)\mathbf{F}(\theta)\mathbf{L}(-\sigma_1)$$

of two *rotation matrices* of the form

(1.3) $$\mathbf{L}(\alpha) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and the *scattering matrix*

(1.4) $$\mathbf{F}(\theta) = \begin{bmatrix} a_1(\theta) & b_1(\theta) & 0 & 0 \\ b_1(\theta) & a_2(\theta) & 0 & 0 \\ 0 & 0 & a_3(\theta) & b_2(\theta) \\ 0 & 0 & -b_2(\theta) & a_4(\theta) \end{bmatrix}.$$

The relationship between $u = -\cos\vartheta$, $u' = -\cos\vartheta'$ and $\theta (0 \leqslant \vartheta, \vartheta', \theta < \pi)$ on the one hand and $\varphi, \varphi', \sigma_1$ and σ_2 on the other hand is given by the formulas

(1.5) $$\cos\theta = \cos\vartheta\cos\vartheta' + \sin\vartheta\sin\vartheta'\cos(\varphi'-\varphi)$$

(1.6) $$\cos\sigma_1 = \frac{\cos\vartheta - \cos\vartheta'\cos\theta}{\sin\vartheta'\sin\theta}, \quad \cos\sigma_2 = \frac{\cos\vartheta' - \cos\vartheta\cos\theta}{\sin\vartheta\sin\theta},$$

where $\sin\sigma_1$ and $\sin\sigma_2$ have the same sign as $\sin(\varphi'-\varphi)$.
When the denominator of any of the equations (1.6) vanishes, the appropriate limits must be taken.

The present article offers a complete existence and uniqueness theory for the solution of the equation of polarized light transfer endowed with the boundary conditions

(1.7) $$\left.\begin{aligned} \mathbf{I}(0,u,\varphi) &= \mathbf{J}(u,\varphi) \text{ for } u > 0 \\ \mathbf{I}(b,u,\varphi) &= \mathbf{J}(u,\varphi) \text{ for } u < 0 \end{aligned}\right\} \textit{whenever } b \textit{ is finite,}$$

or

(1.8) $$\left.\begin{aligned} \mathbf{I}(0,u,\varphi) &= \mathbf{J}(u,\varphi) \text{ for } u > 0 \\ \mathbf{I}(\tau,u,\varphi) &= O(1)(\tau\to\infty) \end{aligned}\right\} \textit{whenever } b = \infty .$$

The results are transferred to various component equations associated with (1.1).

The existence and uniqueness problem requires a functional formulation of the above boundary value problems. Let H_p, $1 \leqslant p < \infty$, denote the direct sum of four copies of $L_p(\Omega)$, where Ω is the unit sphere in $\mathbf{R}^3$. The norm of a function $\mathbf{I}: \Omega \to \mathbf{C}^4$ is given by

$$\|\mathbf{I}\|_p = \left[\int_{-1}^{1}\int_{0}^{2\pi}\{|I(u,\varphi)|^p + |Q(u,\varphi)|^p + |U(u,\varphi)|^p + |V(u,\varphi)|^p\}\mathrm{d}\varphi\,\mathrm{d}u\right]^{1/p},$$

where $u = -\cos\vartheta$ and (ϑ, φ) are the polar coordinates of a point $\omega \in \Omega$. On H_p we define the bounded linear operators T, B, A, Q_+ and Q_- by

$$(1.9) \quad (T\mathbf{I})(u,\varphi) = u\mathbf{I}(u,\varphi), (A\mathbf{I})(u,\varphi) = \mathbf{I}(u,\varphi) - a(B\mathbf{I})(u,\varphi)$$

$$(1.10) \quad (B\mathbf{I})(u,\varphi) = (4\pi)^{-1} \int_{-1}^{1} \int_{0}^{2\pi} \mathbf{Z}(u,u',\varphi - \varphi')\mathbf{I}(u',\varphi')\mathrm{d}\varphi'\mathrm{d}u'$$

$$(1.11) \quad (Q_\pm \mathbf{I})(u,\varphi) = \begin{cases} \mathbf{I}(u,\varphi) & \text{for } u \gtrless 0 \\ 0 & \text{for } u \lesseqgtr 0 . \end{cases}$$

We define the H_p-vector $\mathbf{I}(\tau)(u,\varphi) = \mathbf{I}(\tau,u,\varphi)$. By a *solution* in H_p of the boundary value problem (1.1) and (1.7) for finite b we mean a vector-valued function $\mathbf{I}:(0,b) \to H_p$ such that $T\mathbf{I}$ is differentiable on $(0,b)$ in the strong sense and the following equations hold true:

$$(1.12) \quad (T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau) \qquad (0 < \tau < b)$$

$$(1.13) \quad \lim_{\tau\downarrow 0} \|Q_+\mathbf{I}(\tau) - Q_+\mathbf{J}\|_p = 0 , \quad \lim_{\tau\uparrow b} \|Q_-\mathbf{I}(\tau) - Q_-J\|_p = 0 .$$

A solution in H_p of the problem (1.1) and (1.8), where $b = \infty$, is defined as a vector-valued function $\mathbf{I}:(0,\infty) \to H_p$ such that $T\mathbf{I}$ is differentiable on $(0,\infty)$ in the strong sense and the following equations are satisfied:

$$(1.14) \quad (T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau) \qquad (0 < \tau < \infty)$$

$$(1.15) \quad \lim_{\tau\downarrow 0} \|Q_+\mathbf{I}(\tau) - Q_+\mathbf{J}\|_p = 0 , \quad \|\mathbf{I}(\tau)\|_p = O(1)(\tau \to \infty) .$$

Before explaining the mathematics in more detail, we notice that the Stokes vector $\mathbf{I} = (I, Q, U, V)$ must satisfy the inequalities

$$(1.16) \quad \mathbf{I} \geqslant \sqrt{Q^2 + U^2 + V^2} \geqslant 0 .$$

Rotation matrices leave invariant the set of vectors $\mathbf{I} = (I, Q, U, V)$ satisfying (1.16). The physics of the problem prescribes that the element $a_1(\theta)$, the so-called *phase function,* is nonnegative measurable with normalization

$$(1.17) \quad \int_{-1}^{1} a_1(\theta)\mathrm{d}(\cos\theta) = 2 .$$

Further, the entries of the scattering matrix are measurable functions and for almost every $\theta \in (0,\pi)$ the matrix $\mathbf{F}(\theta)$ maps four-vectors satisfying (1.16) into vectors of the same type. This implies ([14], (82) – (85))

$$(1.18) \quad |b_1(\theta)| \leqslant \frac{1}{2}\{a_1(\theta) + a_2(\theta)\} \leqslant a_1(\theta) ; \quad b_1(\theta)^2 + b_2(\theta)^2 + a_k(\theta)^2 \leqslant a_1(\theta)^2 (k = 3,4) .$$

Hence, all entries of $\mathbf{F}(\theta)$ are real L_1-functions of $\cos\theta$.

Let us explain the mathematics to be used. The boundary value problem (1.12) – (1.13) for finite b, or (1.14) – (1.15) for infinite b can be shown to have the same bounded solutions as the vector-valued convolution equation

$$(1.19)\quad \mathbf{I}(\tau) - a\int_0^b H(\tau-\tau')B\mathbf{I}(\tau')\mathrm{d}\tau' = \omega(\tau) \qquad (0<\tau<b)\,,$$

where

$$(1.20)\quad (H(\sigma)\mathbf{I})(u,\varphi) = \begin{cases} |u|^{-1}\mathrm{e}^{-\sigma/u}\mathbf{I}(u,\varphi) & \text{for } \sigma u > 0 \\ 0 & \text{for } \sigma u < 0\,, \end{cases}$$

defines the *propagator function* and $\omega(\tau)$ is a suitable right-hand side. For unpolarized light transfer the analogous result is known to be true (see [25]). For $b = \infty$ we have a Wiener-Hopf operator integral equation with *symbol*

$$W(\lambda) = \mathbb{I} - a\int_{-\infty}^{\infty} \mathrm{e}^{\sigma/\lambda}H(\sigma)B\mathrm{d}\sigma = \mathbb{I} - a\lambda(\lambda - T)^{-1}B\,, \qquad \mathrm{Re}\lambda = 0\,,$$

where T is noted to have real spectrum only. We consider (1.19) on the Banach space $L_q(H_p)_b$ of strongly measurable functions $\mathbf{I}:(0,b)\to H_p$ which are bounded with respect to the L_q-norm. Using a technique modeled on Feldman [4] one shows that

$$\int_{-\infty}^{\infty} \|H(\sigma)B\|_{H_p}\mathrm{d}\sigma < \infty\,,$$

whenever, as functions of $\cos\theta$, the entries of $\mathbf{F}(\theta)$ belong to $L_r[-1,1]$ for some $r > 1$. For such scattering matrices the theory of convolution equations of [9,5,8] (plus infinite-dimensional analogues) will apply. Invoking a norm equality of Germogenova, Konovalov and Kuzmina ([7,23]; also [22]), namely $\|B\| = 1$ on H_p, we obtain the useful estimate

$$\|W(\lambda) - \mathbb{I}\|_{H_p} \leqslant a\|\lambda(\lambda - T)^{-1}\|_{H_p}\|B\|_{H_p} \leqslant a.1.1. = a\,, \qquad \mathrm{Re}\lambda = 0\,.$$

A straightforward application of a factorization result of Gohberg and Leiterer [10] for Hilbert space operator functions close to the identity yields the unique solvability for $b = \infty$ and $0 < a < 1$ of the Wiener-Hopf equation (1.19) on the spaces $L_q(H_2)_\infty$. With the help of Fredholm techniques (as in [27]) one extends the result to all spaces $L_q(H_p)_\infty$.

As observed by Germogenova and Konovalov [7], the physical requirement (1.16) on the solutions $\mathbf{I}(\tau,u,\varphi)$ strongly suggests the use of the cone

$$K_p = \{\mathbf{I} = (I,Q,U,V)/I \geqslant \sqrt{Q^2+U^2+V^2} \geqslant 0\}$$

on the real Banach space H_p. In [7,22,23] the cone preservation techniques of Krein and Rutman [20] and Krasnoselskii [19] have been applied to K_p to derive information on the position and multiplicity of the zeros of the characteristic equation (which are the eigenvalues of $T^{-1}A$) and the structure of the corresponding eigenfunctions, thereby generalizing the results of Maslennikov [24] for unpolarized light transfer. We shall exploit cone preservation methods to deal with (1.19) (rather than with (1.12) – (1.13) or (1.14) – (1.15)). On the real space $L_q(H_p)_b$ we define the cone

$$L_q(K_p)_b = \{\mathbf{I} \in L_q(H_p)_b / \mathbf{I}(\tau) \in K_p \text{ almost everywhere}\}$$

and write (1.19) as the vector equation

$$(1.21) \quad (\mathbb{I} - aL_b)\mathbf{I} = \omega ,$$

where

$$(1.22) \quad (L_b\mathbf{I})(\tau) = \int_0^b H(\tau - \tau')B\mathbf{I}(\tau')\mathrm{d}\tau' , \qquad 0 < \tau < b .$$

Observe that L_b is a bounded (and for finite b a compact) operator on $L_q(H_p)_b$, which leaves invariant the cone $L_q(K_p)_b$. A monotonicity argument with respect to the cone K_p, modeled on methods in [28], implies the existence and uniqueness of the solution of (1.19) on H_p for finite b and $0 < a \leqslant 1$. It requires the temporary use of an auxiliary space $H_{1,2}$ instead of H_p. As an ancillary result we find that the solution $\mathbf{I}(\tau) \in K_p$ almost everywhere. At the same time, writing for $0 < a \leqslant 1$ the solution to (1.21) as a Neumann series, a mathematical justification is obtained for the method of solution of the equation of radiative transfer by expansion with respect to successive orders of multiple scattering (see [13]).

For practical purposes the full equation of transfer is reduced by Fourier decomposition and symmetry relations. As indicated by Kuščer and Ribarič [21], the equation is first written in terms of complex polarization parameters. Expanding the elements of the transformed scattering matrix into the generalized spherical functions of Gelfand and Shapiro [6] and using the addition formula for these functions one arrives at complex Fourier component equations. Exploiting symmetry relations for the phase matrix ([12], also [3]) one may accomplish a further reduction to real component equations. This was first done by Kuščer and Ribarič [21] for the azimuth-independent part and by Siewert [30] for the azimuth-dependent parts. An alternative route to these results can be found in [14]. We shall also prove the existence and uniqueness of the solution for the various component equations.

Sofar we have exempted the case $b = \infty$ and $a = 1$, because this case is not amenable to Wiener-Hopf factorization and Fredholm techniques. However, one of the Fourier components equations has the form (1.14) – (1.15) for $H_p = L_p[-1,1] \oplus L_p[-1,1]$, where for $p = 2$ the operator A is bounded positive self-adjoint with one-dimensional kernel. This component problem can be solved uniquely along the route of [25]. The information thus obtained allows us to solve (1.14) – (1.15) for the full equation of transfer, where for the remaining component equations we use the factorization result from [10]. Using the method of [26] we also prove that the solution of the full equation of transfer in the conservative $a = 1$ case can be approximated from the non-conservative case by taking the limit as $a \uparrow 1$.

In Section 2 we store preliminaries and state the equivalence theorem for boundary value problem and convolution equation. In Section 3 we treat the semi-infinite medium for $0 < a < 1$. Section 4 deals with media of finite optical thickness. In Section 5 we discuss the various component equations, after which we return to the case $a = 1$ and $b = \infty$ in Section 6. We conclude with a short discussion.

2 Preliminaries and Equivalence Theorems

Let us first list some properties of the integral operator B.

Proposition 2.1 *For* $1 \leqslant p < \infty$ *the operator*

$$(B\mathbf{I})(u,\varphi) = \frac{1}{4\pi}\int_{-1}^{1}\int_{0}^{2\pi} \mathbf{Z}(u,u',\varphi-\varphi')\mathbf{I}(u',\varphi')\,d\varphi'\,du'$$

is compact and has unit norm on H_p. *If* $a_1 \in L_r[-1,1]$ *for some* $r > 1$, *then* B *is a bounded operator from* H_p *into* H_{pr} *and*

$$\int_{-\infty}^{\infty} \|H(\sigma)B\|_{H_p}\,d\sigma < \infty\,. \tag{2.1}$$

Moreover, in this case B *acts as a compact operator from* $H_{r/(r-1)}$ *into the space* $C^{(4)}(\Omega)$ *of continuous functions* $\mathbf{h}:\Omega \to \mathbf{C}^4$ *with supremum norm.*

Proof. The compactness of B on H_p and from $H_{r/(r-1)}$ into $C^{(4)}(\Omega)$ (if $a_1 \in L_r[-1,1]$) are parts of the statement of Theorem 1 of [7]. The third part of Theorem 3 of [7] implies that B has unit norm on H_p. Using the specific form (1.2) of the phase matrix, the equations (1.18) (which imply that $\{a_1,a_2,a_3,a_4,b_1,b_2\} \subseteq L_r[-1,1]$ as functions of $\cos\theta$, if $a_1 \in L_r[-1,1]$) and Theorem 1 (2.X) of [17] one finds that B maps H_p into H_{pr} as a bounded operator. For $0 < \alpha < (r-1)/pr$ the operator

$$(S_\alpha\mathbf{I})(u,\varphi) = |u|^{-\alpha}\mathbf{I}(u,\varphi)$$

is bounded from $L_{pr}[-1,1]$ into $L_p[-1,1]$, while

$$\|\,|T|^\alpha H(\sigma)\| = \sup_{0<u\leqslant 1} |u|^{\alpha-1}e^{-|\sigma|/u} = O(|\sigma|^{\alpha-1})(\sigma \to 0)\,. \tag{2.2}$$

Equation (2.1) now follows from the boundedness of S_α and the estimate (2.2). ■

As in the introduction, let $L_q(H_p)_b$ be the (real or complex) Banach space of strongly measurable functions $\mathbf{I}:(0,b) \to H_p$, which are finite with respect to the norm

$$\|\mathbf{I}\|_{L_q(H_p)_b} = \begin{cases} \left[\int_0^b \|\mathbf{I}(\tau)\|_{H_p}^q\,d\tau\right]^{1/q}, & 1 \leqslant q < \infty \\ \operatorname{ess\,sup}_{0<\tau<b} \|\mathbf{I}(\tau)\|_{H_p}, & q = \infty\,. \end{cases}$$

Strong measurability is defined in the sense of Section 31 of [32]. If $a_1 \in L_r[-1,1]$ for some $r > 1$, then (2.1) guarantees that

$$(L_b \mathbf{I})(\tau) = \int_0^b H(\tau - \tau') B \mathbf{I}(\tau') \mathrm{d}\tau' , \qquad 0 < \tau < b ,$$

is a bounded operator on $L_q(H_p)_b$, where $1 \leqslant q \leqslant \infty$. As B is uniformly approximable on $H_p (1 \leqslant p < \infty)$ by operators of finite rank, the usual theory of Wiener-Hopf operator integral equations, as stated in [5] as an infinite-dimensional generalization of [9], applies as also does the infinite-dimensional generalization of [8]. As a result we find that L_b, with b finite, is a compact operator (cf. [8], Lemma 1.1), where we assume that $a_1 \in L_r[-1,1]$ for some $r > 1$.

We now state the equivalence theorems mentioned in the introduction. Their proof can be given along the lines of [25] (also [27]) and will be omitted.

Theorem 2.2 *Let b be finite, $1 \leqslant p \leqslant \infty$ and $a_1 \in L_r[-1,1]$ for some $r > 1$. Let $\omega : [0,b] \to H_p$ be a continuous function such that $T\omega$ is strongly differentiable on $(0,b)$. Then a function $\mathbf{I} \in L_\infty(H_p)_b$ is a solution of the boundary value problem*

$$(2.3) \quad (T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau) + (T\omega)'(\tau) + \omega(\tau) \qquad (0 < \tau < b)$$

$$(2.4) \quad \lim_{\tau \downarrow 0} \| Q_+ \mathbf{I}(\tau) - Q_+ \omega(0) \|_{H_p} = 0 , \qquad \lim_{\tau \uparrow b} \| Q_- \mathbf{I}(\tau) - Q_- \omega(b) \|_{H_p} = 0 ,$$

if and only if $\mathbf{I}$ is a solution of the convolution equation

$$(2.5) \quad \mathbf{I}(\tau) - a \int_0^b H(\tau - \tau') B \mathbf{I}(\tau') \mathrm{d}\tau' = \omega(\tau) , \qquad 0 < \tau < b .$$

Any such solution is continuous on $[0,b]$.

The equivalence between the boundary value problem (1.12)–(1.13) and the convolution equation (2.5) is effectuated by choosing

$$(2.6) \quad \omega(\tau)(u,\varphi) = \mathrm{e}^{-\tau/u} \mathbf{J}(u,\varphi) \qquad \text{for } u > 0 ,$$
$$\omega(\tau)(u,\varphi) = \mathrm{e}^{(b-\tau)/u} \mathbf{J}(u,\varphi) \quad \text{for } u < 0 .$$

If $\mathbf{f} : [0,b] \to H_p$ is a uniformly Hölder continuous function, then the reasoning of [28] can be applied to prove the equivalence of (2.5) with right-hand side

$$\omega(\tau)(u,\varphi) = \begin{cases} [1 - \mathrm{e}^{-\tau/u}] \mathbf{f}(\tau,u) + \int_0^\tau u^{-1} \mathrm{e}^{-(\tau-\tau')/u} [\mathbf{f}(\tau',u) - \mathbf{f}(\tau',u)] \mathrm{d}\tau' , \\ u > 0 \\ [1 - \mathrm{e}^{(b-\tau)/u}] \mathbf{f}(\tau,u) - \int_\tau^b u^{-1} \mathrm{e}^{-(\tau-\tau')/u} [\mathbf{f}(\tau',u) - \mathbf{f}(\tau,u)] \mathrm{d}\tau' , \\ u < 0 \end{cases}$$

to the boundary value problem

$$(2.7)\quad (T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau) + \mathbf{f}(\tau) \qquad (0 < \tau < b)$$

$$(2.8)\quad \lim_{\tau\downarrow 0} \|Q_+\mathbf{I}(\tau)\|_{H_p} = 0\,, \qquad \lim_{\tau\uparrow b} \|Q_-\mathbf{I}(\tau)\|_{H_p} = 0\,.$$

In this way a thermal emission term $\mathbf{f}(\tau, u)$ may be added to (1.1).

Theorem 2.3 *Let* $1 \leqslant p < \infty$, *and let* $a_1 \in L_r[-1,1]$ *for some* $r > 1$. *Let* $\omega : [0,\infty) \to H_p$ *be a bounded continuous function such that* $T\omega$ *is strongly differentiable on* $(0,\infty)$. *Then a function* $\mathbf{I} \in L_\infty(H_p)_\infty$ *is a solution of the boundary value problem*

$$(2.9)\quad (T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau) + (T\omega)'(\tau) + \omega(\tau) \qquad (0 < \tau < \infty)$$

$$(2.10)\quad \lim_{\tau\downarrow 0} \|Q_+\mathbf{I}(\tau) - Q_+\omega(0)\|_{H_p} = 0\,, \qquad \|\mathbf{I}(\tau)\|_{H_p} = O(1) \qquad (\tau \to \infty)\,,$$

if and only if $\mathbf{I}$ *is a solution of the Wiener-Hopf operator integral equation*

$$(2.11)\quad \mathbf{I}(\tau) - a\int_0^\infty H(\tau - \tau')B\mathbf{I}(\tau')\,\mathrm{d}\tau' = \omega(\tau) \qquad (0 < \tau < \infty)\,.$$

Any such solution is bounded and continuous on $[0,\infty)$.

The equivalence between the boundary value problem (1.14) – (1.15) and the convolution equation (2.11) is obtained, for instance, by choosing

$$(2.12)\quad \omega(\tau)(u,\varphi) = \mathrm{e}^{-\tau/u}\mathbf{J}(u,\varphi) \quad \text{for } u > 0\,, \qquad \omega(\tau)(u,\varphi) = 0 \quad \text{for } u < 0\,.$$

If $\mathbf{f}: [0,\infty) \to H_p$ is a bounded uniformly Hölder continuous function, then (2.11) with right-hand side

$$\omega(\tau)(u,\varphi) = \begin{cases} [1 - \mathrm{e}^{-\tau/u}]\,\mathbf{f}(\tau,u) + \int_0^\tau u^{-1}\mathrm{e}^{-(\tau-\tau')/u}[\mathbf{f}(\tau',u) - \mathbf{f}(\tau',u)]\,\mathrm{d}\tau'\,, \\ u > 0 \\ \mathbf{f}(\tau,u) - \int_\tau^\infty u^{-1}\mathrm{e}^{-(\tau-\tau')/u}[\mathbf{f}(\tau',u) - \mathbf{f}(\tau,u)]\,\mathrm{d}\tau'\,, \\ u < 0 \end{cases}$$

is equivalent to the boundary value problem

$$(2.13)\quad (T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau) + \mathbf{f}(\tau) \qquad (0 < \tau < \infty)$$

$$(2.14)\quad \lim_{\tau\downarrow 0} \|Q_+\mathbf{I}(\tau)\|_{H_p} = 0\,, \qquad \|\mathbf{I}(\tau)\|_{H_p} = O(1)(\tau \to \infty)\,,$$

which again pertains to the addition of the thermal emission term $\mathbf{f}(\tau,u)$ to (1.1).

3 Semi-infinite Non-conservative Media

In this section we prove the following

Theorem 3.1 *Let* $1 \leqslant p < \infty$, *and let* $a_1 \in L_r[-1,1]$ *for some* $r > 1$. *Then for every* $1 \leqslant q \leqslant \infty$ *the Wiener-Hopf operator integral equation* (2.11), *with albedo* *of single scattering* $0 < a < 1$, *has a unique solution in* $L_q(H_p)_\infty$ *for every right-hand side* $\omega \in L_q(H_p)_\infty$. *In particular, if* $\omega : [0,\infty) \to H_p$ *is a bounded continuous function such that* $T\omega$ *is strongly differentiable on* $(0,\infty)$, *then the boundary value problem* (2.13) – (2.14), *with* $0 < a < 1$, *has a unique bounded solution and this solution is continuous on* $[0,\infty)$.

Proof. Up to a trivial change of variable ($i\lambda \to \lambda^{-1}$) the symbol of (2.11) is given by

$$W(\lambda) = \mathbb{I} - a \int_{-\infty}^{\infty} e^{\sigma/\lambda} H(\sigma) B \mathrm{d}\sigma = \mathbb{I} - a\lambda(\lambda - T)^{-1}B, \qquad \mathrm{Re}\lambda = 0.$$

The Fredholm characteristics of the operator $\mathbb{I} - aL_b$ on $L_q(H_p)_\infty$ are completely characterized by the Wiener-Hopf factorization properties of the symbol (see [9, 5]). Using Proposition 1 we find

$$\sup_{\mathrm{Re}\lambda = 0} \|\mathbb{I} - W(\lambda)\|_{H_p} \leqslant a\|B\|_{H_p} \sup_{\mathrm{Re}\lambda = 0} \|\lambda(\lambda - T)^{-1}\|$$

$$= a \cdot \|B\|_{H_p} \cdot 1 = a < 1.$$

Theorem 4.1 of [10] implies that for $p = 2$ there exist continuous functions W_+ and W_- from the extended imaginary line into the group of invertible operators on H_2 such that $W_\pm$ and $W_\pm^{-1}$ have analytic continuations to the left/right half-plane with continuous limiting values up to the extended imaginary axis, and such that the canonical factorization

$$W(\lambda) = W_-(\lambda)W_+(\lambda), \quad \mathrm{Re}\lambda = 0, \tag{3.1}$$

holds true. For $1 \leqslant p < 2$ and $2 < p < \infty$ we may not draw this conclusion from [10], as it applies exclusively to Hilbert space operator functions. Nevertheless we shall draw this conclusion below for general p but use a different method.

From the existence of the canonical factorization (3.1) on H_2 we find that on H_2 and for $0 < a < 1$ the equation (2.11) has a unique solution in $L_q(H_2)_\infty$ for every $\omega \in L_q(H_2)_\infty$. We may prove this along the usual procedure for solving Wiener-Hopf operator integral equations by means of factorization (see [9]; also [5]). The operator $\mathbb{I} - aL_\infty$ is invertible on $L_q(H_2)_\infty$. On the other hand, Theorem 4.3 (or 4.4) of [11] implies that on H_p $(1 \leqslant p < \infty)$ the symbol W has a (left and right) Wiener-Hopf factorization within the Wiener algebra of functions $c\mathbb{I} + \int_{-\infty}^{\infty} e^{\sigma/\lambda} k(\sigma) \mathrm{d}\sigma$ on the extended imaginary line, where k is assumed Bochner integrable with values in the algebra of bounded operators on H_p and the norm

$$\|c\mathbb{I} + \int_{-\infty}^{\infty} e^{\sigma/\lambda} k(\sigma)\mathrm{d}\sigma\|_{\mathrm{algebra}} \overset{\mathrm{def.}}{=\!=} |c| + \int_{-\infty}^{\infty} \|k(\sigma)\|_{H_p} \mathrm{d}\sigma$$

is used. This factorization property implies that $\mathbb{I} - aL_\infty$ is a Fredholm operator on $L_q(H_p)_\infty$, where $1 \leqslant p < \infty$. For $2 < p < \infty$ $L_q(H_p)_\infty$ is densely embedded in $L_q(H_2)_\infty$, while $\mathbb{I} - aL_\infty$ on $L_q(H_p)_\infty$ is the restriction to $L_q(H_p)_\infty$ of this operator on $L_q(H_2)_\infty$. Thus $\mathbb{I} - aL_\infty$ is invertible on $L_q(H_p)_\infty$ for $2 \leqslant p < \infty$. For $1 \leqslant p \leqslant 2$ one employs instead that $L_q(H_2)_\infty$ is densely embedded in $L_q(H_p)_\infty$. ■

We define the cone $L_q(K_p)_\infty$ by the expression

$$L_q(K_p)_\infty = \{\mathbf{I} \in L_q(H_p)_\infty / \mathbf{I}(\tau) \in K_p \text{ almost everywhere on } (0, \infty)\}.$$

Putting $\mathbf{I}(\tau) = (I(\tau), Q(\tau), U(\tau), V(\tau))$, we may write every real $\mathbf{I} \in L_q(H_p)_\infty$ as the difference

$$\mathbf{I}(\tau) = \mathbf{I}_+(\tau) - \mathbf{I}_-(\tau)$$

of two vectors in $L_q(K_p)_\infty$, where

$$\mathbf{I}_+(\tau) = \begin{cases} \mathbf{I}(\tau) \text{ if } I(\tau) \geqslant \sqrt{Q(\tau)^2 + U(\tau)^2 + V(\tau)^2} \\ (\sqrt{Q(\tau)^2 + U(\tau)^2 + V(\tau)^2}, \quad Q(\tau), U(\tau), V(\tau)) \text{ otherwise}, \end{cases}$$

and

$$\mathbf{I}_-(\tau) = \begin{cases} (0,0,0,0) \text{ if } I(\tau) \geqslant \sqrt{Q(\tau)^2 + U(\tau)^2 + V(\tau)^2} \\ (\sqrt{Q(\tau)^2 + U(\tau)^2 + V(\tau)^2} - I(\tau), 0, 0, 0) \text{ otherwise}. \end{cases}$$

Using the terminology of [19] we find that $L_q(K_p)_\infty$ is a reproducing cone in $L_q(H_p)_\infty$ (cf. [7], part (4) of Lemma 2), which has an empty interior.

Now observe that $\mathbf{I} = \mathbf{I}_2 - \mathbf{I}_1$ is a vector satisfying the inequalities (1.16), if and only if

$$I_2 - I_1 \geqslant \sqrt{(Q_1 - Q_2)^2 + (U_1 - U_2)^2 + (V_1 - V_2)^2}.$$

Adding $I_1 + \sqrt{Q_2^2 + U_2^2 + V_2^2}$ to both sides and using Minkowski's inequality we find

$$\text{(3.2)} \quad I_2 + \sqrt{Q_2^2 + U_2^2 + V_2^2} \geqslant I_1 + \sqrt{Q_1^2 + U_1^2 + V_1^2}.$$

Lifting both sides to the p-th power, applying $(a + b)^r \geqslant a^r + b^r$ for $a, b \geqslant 0$ to the right-hand side thus obtained twice $\left(\text{for } r = p \text{ and } \frac{1}{2}p, \text{ respectively}\right)$ and applying Minkowski's inequality to the left-hand side, we obtain

$$\sqrt{3}\,(|I_2|^p + |Q_2|^p + |U_2|^p + |V_2|^p)^{1/p}$$
$$\geqslant (|I_1|^p + |Q_1|^p + |U_1|^p + |V_1|^p)^{1/p}.$$

Hence, if $\mathbf{I}_2 - \mathbf{I}_1 \in K_p$, then

$$\text{(3.3)} \quad \sqrt{3}\,\|\mathbf{I}_2\|_{H_p} \geqslant \|\mathbf{I}_1\|_{H_p},$$

which proves the normality of the cone K_p (in the sense of [19]). If one now chooses, for $1 \leqslant q \leqslant \infty$, $\mathbf{I}_2 - \mathbf{I}_1 \in L_q(K_p)_\infty$, then (3.3) implies

$$\sqrt{3}\,\|\mathbf{I}_2\|_{L_q(K_p)_\infty} \geqslant \|\mathbf{I}_1\|_{L_q(K_p)_\infty},$$

which establishes the normality of the cone $L_q(K_p)_\infty$.

Let us now consider the auxiliary Banach space $H_{1.2}$ of measurable functions $\mathbf{I} = (I, Q, U, V): \Omega \to \mathbf{C}^4$, which are bounded with respect to the norm

$$\|\mathbf{I}\|_{1.2} \stackrel{\text{def.}}{=} \int_{-1}^{1} \int_{0}^{2\pi} \{|I(u,\varphi)| + \sqrt{|Q(u,\varphi)|^2 + |U(u,\varphi)|^2 + |V(u,\varphi)|^2}\} \, d\varphi \, du.$$

The space $H_{1.2}$ contains H_2 as a dense subspace and is densely embedded in H_1. We now introduce the cone $K_{1.2}$, the spaces $L_q(H_{1.2})_\infty$, and the cones $L_q(K_{1.2})_\infty$, in a way analogous to the introduction of K_p, $L_q(H_p)_\infty$ and $L_q(K_p)_\infty$. As a result we find all these cones to be reproducing and normal and we recover Theorems 2.2 to 3.1 with $H_{1.2}$ instead of H_p. On inspecting (3.2) one finds that $H_{1.2}$ has the property

(3.4) $\|\mathbf{I}_2\|_{H_{1.2}} > \|\mathbf{I}_1\|_{H_{1.2}}$ *whenever* $\mathbf{I}_2 - \mathbf{I}_1 \in K_{1.2} \setminus \{0\}$,

which carries over to the spaces $L_q(H_{1.2})_\infty$. So one also finds

$$\|\mathbf{I}_2\|_{L_q(H_{1.2})_\infty} > \|\mathbf{I}_1\|_{L_q(H_{1.2})_\infty} \text{ whenever } \mathbf{I}_2 - \mathbf{I}_1 \in L_q(K_{1.2})_\infty \setminus \{0\}.$$

Corollary 3.2 *Let* $1 \leqslant p < \infty$, *and let* $a_1 \in L_r[-1,1]$ *for some* $r > 1$. *Then for every* $1 \leqslant q \leqslant \infty$, $0 < a < 1$ *and* $\omega \in L_q(H_p)_\infty$ *the unique solution to* (2.11) *is given by the series*

(3.5) $\mathbf{I} = \sum_{n=0}^{\infty} a^n L_\infty^n \omega$,

which is absolutely convergent in the norm of $L_q(H_p)_\infty$. *In particular, if* $\omega \in L_q(K_p)_\infty$, *so does the solution* (3.5). *Hence, to every boundary value function* $\mathbf{J} \in K_p$ *the unique solution to* (1.14)–(1.15) *takes its values in the cone* K_p.

Proof. Because for $a = 1$ the symbol W of the Wiener-Hopf operator integral equation (2.11) fails to be invertible at infinity (note that $W(\infty) = A$; cf. [7], Th. 5, part (1)), the operator $\mathbb{I} - L_\infty$ cannot possibly be invertible on any one of the spaces $L_q(H_p)_\infty$ (cf. [9,5]). Thus $a = 1$ belongs to the spectrum of L_∞, while $(1, \infty)$ is contained in the resolvent set of L_∞. Consequently, the spectral radius $r(L_\infty)$ of L_∞ is not less than unity.

We notice that L_∞ is a bounded operator on $L_q(H_p)_\infty$, which leaves invariant $L_q(K_p)_\infty$. This cone is reproducing and normal. Using [1] (part (2) of Th. 1) we find that the adjoint cone of bounded linear functionals on $L_q(H_p)_\infty$ leaving invariant $L_q(K_p)_\infty$ is reproducing. According to Theorem 4 of [18] the spectral radius of L_∞ must belong to the spectrum of L_∞. Hence, $r(L_\infty) = 1$. The series (3.5) now turns out to be a Neumann series for L_∞ and the remaining part of the proof is straightforward. ■

The representation of solutions to (2.11) by means of the series (3.5) is known in radiative transfer theory as the method of expansion with respect to orders of multiple scattering [13] (for unpolarized light transfer, also [16]). Corollary 3.2 is the mathematical justification of this method for $b = \infty$ and $0 < a < 1$.

4 Media With Finite Optical Thickness

Let us first prove the main result of this section.

Theorem 4.1 *Let* $1 \leqslant p < \infty$, $0 < b < \infty$ *and* $0 < a \leqslant 1$, *and let* $a_1 \in L_r[-1,1]$ *for some* $r > 1$. *Then for every* $1 \leqslant q \leqslant \infty$ *the convolution equation* (2.5) *has a unique solution in* $L_q(H_p)_b$ *for every right-hand side* $\omega \in L_q(H_p)_b$. *In particular, if* $\omega : [0,b] \to H_p$ *is a continuous function such that* $T\omega$ *is strongly differentiable on* $(0,b)$, *then the boundary value problem* (2.3)–(2.4) *has a unique bounded solution and this solution is continuous on* $[0,b]$.

Proof. Let us take $0 < a < 1$. Given $m \in \mathbf{N}$ and infinite b, let us choose $\mathbf{I} \in L_q(K_{1.2})_b$ such that for some $\varepsilon > 0$ we have

$$\|\mathbf{I}\|_{L_q(H_{1.2})_b} = 1\,, \qquad \|L_b^m \mathbf{I}\|_{L_q(H_{1.2})_b} \geqslant (1-\varepsilon)\|L_b^m\|_{L_q(H_{1.2})_b}\,.$$

Extending $\mathbf{I}$ to $L_q(H_{1.2})_\infty$ by defining $\mathbf{I}(\tau) \equiv 0$ for $\tau > b$, we must certainly have

$$(L_\infty^m \mathbf{I})(\tau) - (L_b^m \mathbf{I})(\tau) \in K_{1.2}\,, \qquad 0 < \tau < b\,.$$

Using the monotonicity property (3.4) we find

$$\|L_\infty^m \mathbf{I}\|_{L_q(H_{1.2})_\infty} \geqslant (1-\varepsilon)\|L_b^m\|_{L_q(H_{1.2})_b}\,,$$

where the extended vector $\mathbf{I}$ has unit norm in $L_q(H_{1.2})_\infty$. Thus

$$\|L_\infty^m\|_{L_q(H_{1.2})_\infty} \geqslant \|L_b^m\|_{L_q(H_{1.2})_b}\,, \qquad m \in \mathbf{N}\,.$$

Taking m-th roots at both sides and computing the limits as $m \to \infty$ we obtain

$$1 = r(L_\infty) \geqslant r(L_b)\,.$$

Hence, for $0 < a < 1$ Theorem 4.1 holds true with H_p instead of $H_{1.2}$. Using the compactness of L_b for all finite optical thicknesses b and all spaces $L_q(H_p)_b$ and $L_q(H_{1.2})_b$ (see Section 2), we obtain Theorem 4.1 for $0 < a < 1$ in general.

The case $a = 1$ requires more effort. First we consider the case $p = r/(r-1)$ and let C denote the (real or complex) Banach space of functions $\mathbf{I} : \Omega \to \mathbf{C}^4$ with supremum norm

$$\|\mathbf{I}\|_C = \max\{\|I\|_\infty,\ \|Q\|_\infty,\ \|U\|_\infty,\ \|V\|_\infty\}\,.$$

According to Proposition 2.1, B is a bounded operator from H_p into C. We shall prove that L_b is $\mathbf{u}_0$-bounded above (in the sense of [19]) when defined on $L_q(H_p)_b$, where $\mathbf{u}_0(u,\varphi) \equiv (1,0,0,0)$. Let us take $\mathbf{I} \in L_\infty(K_p)_b$. Then $L_b\mathbf{I}$ is a continuous function from $[0,b]$ into H_p and therefore $(BL_b\mathbf{I})(\tau)$ depends continuously on τ as a function from $[0,b]$ into C. So we can find a positive constant c such that $(BL_b\mathbf{I})(\tau) \leqslant c(1,0,0,0)$ for $0 \leqslant \tau \leqslant b$, which implies that

$$(L_b^2\mathbf{I})(\tau) = \int_0^b H(\tau-\tau')(BL_b\mathbf{I})(\tau')\,d\tau' \leqslant c(1,0,0,0)\,.$$

Hence on $L_\infty(H_p)_b$ the operator L_b as $\mathbf{u}_0$-bounded above.

We also intend to prove that on $L_\infty(H_p)_b$ the operator L_b is $\mathbf{u}_0$-bounded below in the sense of [19]. For this reason we first show that

$$\{\mathbf{I} \in K_p / B\mathbf{I} = 0\} = \{0\}. \tag{4.1}$$

Let us consider a vector $\mathbf{I} \in K_p$ such that $B\mathbf{I} = 0$. Then

$$\mathbf{Z}(u, u', \varphi - \varphi')\mathbf{I}(u', \varphi') = 0$$

for almost all $u, u' \in [-1,1]$ and $\varphi, \varphi' \in [0, 2\pi]$. Define the well-defined measurable function $\sigma_1 : \Omega \to [0, \pi)$ by requiring that

$$(\cos 2\sigma_1(u', \varphi'))\,Q(u', \varphi') - (\sin 2\sigma_1(u', \varphi'))\,U(u', \varphi') = 0\,. \tag{4.2}$$

The normalization condition (1.17) on the phase function a_1 implies the existence of $\varepsilon > 0$ and a subset E of $[-1,1]$ of positive Borel measure such that $a_1(\theta) \geqslant \varepsilon > 0$ for almost every $\cos\theta \in E$. For almost all $u, u' \in [-1,1]$ (with $u = -\cos\vartheta$, $u' = -\cos\vartheta'$ and $0 \leqslant \vartheta, \vartheta' < \pi$) and $\varphi, \varphi' \in [0, 2\pi)$ one may choose unique angles $0 \leqslant \sigma_1, \sigma_2, \theta < \pi$ such that (1.5) – (1.6) hold true. Now remark that the set of points

$$\Phi = \{(\vartheta, \varphi) \in \Omega / \text{Eqs. } (1.5), (1.6) \text{ and } (4.2) \text{ are fulfilled and } \cos\theta \in E\} \subsetneq \Omega$$

has positive Borel measure. But for $(\vartheta, \varphi) \in \Phi$ we have for the first component of $\mathbf{Z}(u, u', \varphi - \varphi')\mathbf{I}(u', \varphi')$:

$$0 = [\mathbf{Z}(u, u', \varphi - \varphi')\mathbf{I}(u', \varphi')]_1 = a_1(\theta) I(u', \varphi') \geqslant \varepsilon I(u', \varphi')\,,$$

where u' and φ' run over their ranges freely. Thus $I(u', \varphi')$ vanishes almost identically. On using (1.16) one finds $\mathbf{I}(u', \varphi') \equiv 0$, which establishes (4.1).

In order to prove that L_b is $\mathbf{u}_0$-bounded below, take $0 \neq \mathbf{I} \in L_\infty(K_p)_b$, where $p = r/(r-1)$. Take a subset F of $(0, b)$ of positive measure and a constant $\varepsilon > 0$ such that $\mathbf{I}(\tau) \neq 0$ almost everywhere on F. Then in the partial order of the real space H_p induced by the cone K_p one must have

$$(L_b \mathbf{I})(\tau) \geqslant \int_F H(\tau - \tau') B\mathbf{I}(\tau')\,\mathrm{d}\tau' \neq 0\,, \qquad 0 \leqslant \tau \leqslant b\,;$$

otherwise $B\mathbf{I}(\tau') = 0$ for almost all $\tau' \in F$, which implies $\mathbf{I}(\tau') = 0$ almost everywhere on F. One gets

$$(BL_b \mathbf{I})(\tau) \geqslant \int_F BH(\tau - \tau') B\mathbf{I}(\tau')\,\mathrm{d}\tau' \neq 0\,, \qquad 0 \leqslant \tau \leqslant b\,,$$

where both sides represent continuous functions from $[0, b]$ into C. We now easily show that, for some positive constant d,

$$(L_b^2 \mathbf{I})(\tau) = \int_0^b H(\tau - \tau')(BL_b \mathbf{I})(\tau')\,\mathrm{d}\tau' \geqslant d(1, 0, 0, 0)\,,$$

which proves that L_b is $\mathbf{u}_0$-bounded below in the sense of [19].

We now prove that the equation

$$(\mathbb{I} - aL_b)\mathbf{I} = \omega$$

is uniquely solvable on the Banach space $L_\infty(H_p)_b$, where $0 < a \leqslant 1$, b is finite and $p = r/(r-1)$. Because L_b is $\mathbf{u}_0$-positive in the sense of [19], there exist a unique positive number $c(b)$ and (up to multiplication by a positive constant) a unique $0 \neq \mathbf{I}_b \in L_\infty(K_p)_b$ such that

$$(4.3) \quad (\mathbb{I} - c(b)L_b)\mathbf{I}_b = 0 .$$

Now choose $0 < b < b' < \infty$. Then (4.3) with b replaced by b' can be written as the equation

$$(\mathbb{I} - c(b')L_{b'})\mathbf{I}_{b'} = \omega$$

on $L_\infty(H_p)_b$, where $\mathbf{I}_{b'}$ has been restricted from $(0,b')$ to $(0,b)$ and the right-hand side $\omega \in L_\infty(K_p)_b$. In fact,

$$\omega(\tau) = c(b') \int_b^{b'} H(\tau - \tau')B\mathbf{I}_{b'}(\tau')\mathrm{d}\tau' \neq 0 , \qquad 0 \leqslant \tau \leqslant b .$$

Using Theorem 2.16 of [19] we must conclude that

$$0 < c(b') < c(b) < \infty .$$

If we would assume that $c(b) \leqslant 1$, then $c(b') < 1$ and a contradiction arises. Hence, the theorem holds true for $0 < a \leqslant 1$, $p = r/(r-1)$ and $q = \infty$.

On exploiting the compactness of L_b on the spaces $L_q(H_p)_b$ we extend Theorem 4.1 to more general p and q. ■

Let us define the cone

$$L_q(K_p)_b = \{\mathbf{I} \in L_q(H_p)_b / \mathbf{I}(\tau) \in K_p \text{ almost everywhere on } (0,b)\}$$

on $L_q(H_p)_b$. One easily establishes that $L_q(K_p)_b$ is a reproducing and normal cone. Repeating the proof of Corollary 3.2 for finite b, one finds

Corollary 4.2 *Let* $1 \leqslant p < \infty$, $0 < b < \infty$ *and let* $a_1 \in L_r[-1,1]$ *for some* $r > 1$. *Then for every* $1 \leqslant q \leqslant \infty$ *and* $\omega \in L_q(H_p)_b$ *the unique solution to* (2.5) *is given by the series*

$$(4.4) \quad \mathbf{I} = \sum_{n=0}^{\infty} a^n L_b^n \omega ,$$

which is absolutely convergent in the norm of $L_q(H_p)_b$. *In particular, if* $\omega \in L_q(K_p)_b$, *so does the solution* (4.4). *Hence, to every boundary value function* $\mathbf{J} \in K_p$ *the unique solution to* (1.12) – (1.13) *takes its values in the cone* K_p.

Corollary 4.2 is the mathematical justification, for finite b and $0 < a \leqslant 1$, of the method of expansion with respect to orders of multiple scattering. This method is explained in [13], where one also finds explicit expressions for the lower order terms.

5 Transition to Component Equations

In this section we state how, in two steps, Fourier decomposition and symmetry relations may be applied to write (1.1) as a set of component equations without azimuthal dependence. The kernels of these equations can be determined explicitly. Following [24] we indicate how this decomposition can be made rigorous and how the solvability of the full equation is related to the solvability of the component equations. The same relationship we derive for the multiple scattering expansion.

Consider the full equation (1.1) and write

$$\text{(5.1)}\quad \mathbf{Z}(u,u',\varphi-\varphi') = \mathbf{Z}^{c0}(u,u') + 2\sum_{j=1}^{\infty}[\mathbf{Z}^{cj}(u,u')\cos j(\varphi-\varphi') + \mathbf{Z}^{sj}(u,u')\sin j(\varphi-\varphi')]$$

$$\text{(5.2)}\quad \mathbf{I}(\tau,u,\varphi) = \mathbf{I}^{c0}(\tau,u) + 2\sum_{j=1}^{\infty}[\mathbf{I}^{cj}(\tau,u)\cos j\varphi + \mathbf{I}^{sj}(\tau,u)\sin j\varphi]\,.$$

For the functions appearing in the boundary conditions and for thermal emission terms added to Eq. (1.1) one applies an analogous Fourier decomposition. From (1.1) one arrives at the set of equations

$$\text{(5.3)}\quad u\frac{\mathrm{d}\mathbf{I}^{c0}(\tau,u)}{\mathrm{d}\tau} = -\mathbf{I}^{c0}(\tau,u) + \frac{1}{2}a\int_{-1}^{1}\mathbf{Z}^{c0}(u,u')\mathbf{I}^{c0}(\tau,u')\,\mathrm{d}u'$$

$$\text{(5.4)}\quad u\frac{\mathrm{d}\mathbf{I}^{cj}(\tau,u)}{\mathrm{d}\tau} = -\mathbf{I}^{cj}(\tau,u) + \frac{1}{2}a\int_{-1}^{1}[\mathbf{Z}^{cj}(u,u')\mathbf{I}^{cj}(\tau,u') - \mathbf{Z}^{sj}(u,u')\mathbf{I}^{sj}(\tau,u')]\,\mathrm{d}u'$$

$$\text{(5.5)}\quad u\frac{\mathrm{d}\mathbf{I}^{sj}(\tau,u)}{\mathrm{d}\tau} = -\mathbf{I}^{sj}(\tau,u) + \frac{1}{2}a\int_{-1}^{1}[\mathbf{Z}^{sj}(u,u')\mathbf{I}^{cj}(\tau,u') + \mathbf{Z}^{cj}(u,u')\mathbf{I}^{sj}(\tau,u')]\,\mathrm{d}u'$$

where $j = 1,2,3,\dots$. For the kernels of these equations one finds the symmetry relations (see [14], Eqs. (114))

$$\mathbf{Z}^{c0}(u,u') = \mathbf{D}\mathbf{Z}^{c0}(u,u')\mathbf{D},\ \mathbf{Z}^{cj}(u,u') = \mathbf{D}\mathbf{Z}^{cj}(u,u')\mathbf{D},\ \mathbf{Z}^{sj}(u,u') = -\mathbf{D}\mathbf{Z}^{sj}(u,u')\mathbf{D}\,,$$

where $\mathbf{D} = \operatorname{diag}(1,1,-1,-1)$. We now write ([14], Eqs. (115), (124) and (125))

$$\mathbf{W}^{j}(u,u') = \mathbf{Z}^{cj}(u,u') - \mathbf{D}\mathbf{Z}^{sj}(u,u') = \mathbf{Z}^{cj}(u,u') + \mathbf{Z}^{sj}(u,u')\mathbf{D}$$

$$\mathbf{Y}^{j}(\tau,u) = \frac{1}{2}(\mathbb{I}+\mathbf{D})\mathbf{I}^{cj}(\tau,u) + \frac{1}{2}(\mathbb{I}-\mathbf{D})\mathbf{I}^{sj}(\tau,u)$$

$$\mathbf{X}^j(\tau,u) = -\frac{1}{2}(\mathbb{I}-\mathbf{D})\mathbf{I}^{cj}(\tau,u) + \frac{1}{2}(\mathbb{I}+\mathbf{D})\mathbf{I}^{sj}(\tau,u),$$

and arrive at the transformed equations

$$(5.6)\quad u\frac{d\mathbf{Y}^j(\tau,u)}{d\tau} = -\mathbf{Y}^j(\tau,u) + \frac{1}{2}a\int_{-1}^{1}\mathbf{W}^j(u,u')\mathbf{Y}^j(\tau,u')du'$$

$$(5.7)\quad u\frac{d\mathbf{X}^j(\tau,u)}{d\tau} = -\mathbf{X}^j(\tau,u) + \frac{1}{2}a\int_{-1}^{1}\mathbf{W}^j(u,u')\mathbf{X}^j(\tau,u')du' .$$

These equations have the same form, but in general the boundary conditions and the thermal emission terms to be added are different. Conversely, one finds

$$\mathbf{I}^{cj}(\tau,u) = \frac{1}{2}(\mathbb{I}+\mathbf{D})\mathbf{Y}^j(\tau,u) - \frac{1}{2}(\mathbb{I}-\mathbf{D})\mathbf{X}^j(\tau,u)$$

$$\mathbf{I}^{sj}(\tau,u) = \frac{1}{2}(\mathbb{I}-\mathbf{D})\mathbf{Y}^j(\tau,u) + \frac{1}{2}(\mathbb{I}+\mathbf{D})\mathbf{X}^j(\tau,u)$$

$$\mathbf{Z}^{cj}(u,u') = \frac{1}{2}\{\mathbf{W}^j(u,u') + \mathbf{D}\mathbf{W}^j(u,u')\mathbf{D}\}$$

$$\mathbf{Z}^{sj}(u,u') = \frac{1}{2}\{\mathbf{W}^j(u,u')\mathbf{D} - \mathbf{D}\mathbf{W}^j(u,u')\}$$

and similar identities for boundary conditions and thermal emission terms.

For $j = 1, 2, 3, \ldots$ we have obtained component equations of the form (5.6)–(5.7), which are four-group equations. For $j = 0$ one has $\mathbf{W}^0(u,u') = \mathbf{Z}^{c0}(u,u')$ (using the convention $\mathbf{Z}^{s0}(u,u') \equiv 0$) and a further decoupling into two two-group equations follows. Boundary conditions and thermal emission terms decouple similarly. For $j = 0$ these results were first obtained by Kuščer and Ribarič [21]. They were extended to the $j \geqslant 1$ case by Siewert [30]. Complex Fourier decomposition was applied to (1.1) by Domke [3], but if one counts real and imaginary parts as separate components Domke's equations are four-group equations for $j = 0$ and eight-group equations for $j \neq 0$. Here we have followed the treatment of [14].

In [24] the equation of transfer without polarization taken into account was Fourier decomposed in a mathematically rigorous way. This procedure was repeated in [7] (also [22, 23]) for the complex Fourier components of the equation of transfer of polarized light. Essentially the same thing can be done for the real Fourier decomposition through (5.1) and (5.2). One defines the subspaces

$$H_p^{c0} = \{\mathbf{I} \in L_p(\Omega)/\mathbf{I}(u,\varphi) \text{ does not depend on } \varphi\}$$

$$H_p^{cj} = \{\mathbf{I} \in L_p(\Omega)/\mathbf{I}(u,\varphi) = \mathbf{J}(u)\cos j\varphi,\ \mathbf{J} \in H_p^{c0}\}$$

$$H_p^{sj} = \{\mathbf{I} \in L_p(\Omega)/\mathbf{I}(u,\varphi) = \mathbf{J}(u)\sin j\varphi,\ \mathbf{J} \in H_p^{c0}\} .$$

All these spaces can be identified with the direct sum of four copies of $L_p[-1,1]$, while

(5.8) $$H_p = H_p^{c0} \oplus \bigoplus_{j=1}^{\infty} [H_p^{cj} \oplus H_p^{sj}] .$$

The projection of H_p onto any finite direct sum of the spaces H_p^{c0}, H_p^{cj} and H_p^{sj} along the infinite direct sum of the remaining spaces has unit norm. The direct sum decomposition (5.8) reduces the operators T, Q_+ and Q_-, but the operators A and B are reduced by the decomposition

(5.9) $$H_p = H_p^{c0} \oplus \bigoplus_{j=1}^{\infty} H_p^{csj} ,$$

where $H_p^{csj} \overset{\text{def.}}{=\!=} H_p^{cj} \oplus H_p^{sj}$.

The restrictions of (1.1) to the subspaces H_p^{c0} and H_p^{csj} coincide with (5.3) and the coupled set (5.4)–(5.5), respectively. The restrictions of the operators T, Q_+, Q_-, A and B to H_p^{csj}, however, are reduced by the following two operators of unit norm:

$$\mathsf{P}_j^{\pm} \begin{bmatrix} \mathbf{c} \\ \mathbf{s} \end{bmatrix} = \pm\frac{1}{2}(\mathbb{I} \pm \mathbf{D})\mathbf{c} + \frac{1}{2}(\mathbb{I} \mp \mathbf{D})\mathbf{s} .$$

These operators have complementary ranges in H_p^{csj} and the restrictions of the coupled set of equations (5.4) and (5.5) to these ranges coincide with (5.6) and (5.7), respectively.

As a consequence one finds that for the case $0 < a < 1$ and $b = \infty$, and the case $0 < a \leqslant 1$ and $0 < b < \infty$ the boundary value problems and convolution equations associated with (5.3) to (5.5), or with (5.6)–(5.7) are uniquely solvable. The case $a = 1$ and $b = \infty$ can now be treated in the converse direction by first showing the unique solvability of the component problems, after which the unique solvability of the full problem follows. This will be done in the next section. Similar remarks hold true with regard to the multiple scattering expansion method for the various component equations, which now evidently are justified for the cases $0 < a < 1$ and $b = \infty$, and $0 < a \leqslant 1$ and $0 < b < \infty$.

6 Semi-infinite Conservative Media

In this section we prove as a main result.

Theorem 6.1 *Let $1 \leqslant p < \infty$, $a = 1$ and let $a_1 \in L_r[-1,1]$ for some $r > 1$. Then for every $Q_+\mathbf{J} \in Q_+[H_p]$ there exists a unique bounded solution of the boundary value problem*

(6.1) $$(T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau) \qquad (0 < \tau < \infty)$$

(6.2) $$\lim_{\tau \downarrow 0} \| Q_+\mathbf{I}(\tau) - Q_+\mathbf{J} \|_p = 0 , \qquad \|\mathbf{I}(\tau)\|_p = O(1)(\tau \to \infty) .$$

Furthermore, if $a_1(\theta) \not\equiv a_4(\theta)$, *there exists a unique solution of the boundary value problem*

$$(6.3)\quad (T\mathbf{I})'(\tau) = -A\mathbf{I}(\tau) \qquad (0<\tau<\infty)$$

$$(6.4)\quad \lim_{\tau\downarrow 0}\|Q_+\mathbf{I}(\tau)\|_p = 0\,,\quad \exists n\geqslant 0:\|\mathbf{I}(\tau)\| = O(\tau^n)(\tau\to\infty)\,.$$

$$(6.5)\quad \lim_{\tau\to\infty}\int_{-1}^{1} u\mathbf{I}(\tau,u)\,\mathrm{d}u = -\frac{1}{2}F(1,0,0,0)\,.$$

This solution has the form

$$(6.6)\quad \mathbf{I}(\tau,u) = \frac{3}{4}F(\alpha-u)\begin{bmatrix}1\\0\\0\\0\end{bmatrix} + \begin{bmatrix}I_0(\tau,u)\\Q_0(\tau,u)\\0\\0\end{bmatrix},$$

where $\lim_{\tau\to\infty}\|I_0(\tau)\|_p = \lim_{\tau\to\infty}\|Q_0(\tau)\|_p = 0$ *exponentially.*

Proof. Let us decompose the boundary value problems (6.1) – (6.2), and (6.3) to (6.5) according to the direct sum (5.9). This direct sum reduces the operators B, B^* and $(BB^*)^{1/2}$. These operators have $+1$ as a simple leading eigenvalue with corresponding constant eigenfunction (1,0,0,0) except for $a_1(\theta) \equiv a_4(\theta)$ (cf. [7]; this exceptional case was not observed there). By (1.18), we then have $b_1(\theta) \equiv b_2(\theta) \equiv 0$, so that some restriction of A is positive self-adjoint and B, B^* and $(BB^*)^{1/2}$ have (0,0,0,1) as another eigenfunction at the eigenvalue $+1$. Because these eigenfunctions belong to H_p^{c0}, the restrictions of $(BB^*)^{1/2}$ to H_p^{csj} all have their spectrum inside [0,1), which implies that

$$\|B|_{H_p^{csj}}\| < 1\,.$$

Using the arguments of the proof of Theorem 3.1 (i.e., the application of [10] followed by a Fredholm argument) we obtain the unique solvability of the restrictions of the problem (6.1) – (6.2) to the spaces H_p^{csj}. The corresponding restrictions of problem (6.3) to (6.5) have zero solutions.

Let us turn to the problems (6.1) – (6.2) and (6.3) to (6.5) restricted to H_p^{c0}. The equation can be written as

$$u\frac{\mathrm{d}}{\mathrm{d}\tau}\mathbf{I}^{c0}(\tau,u) + \mathbf{I}^{c0}(\tau,u) = \frac{1}{2}a\int_{-1}^{1}\mathbf{Z}^{c0}(u,u')\mathbf{I}^{c0}(\tau,u')\,\mathrm{d}u'\,,$$

where $\mathbf{Z}^{c0}(u,u')$ is the direct sum of two 2×2-matrix functions. This is clear from the symmetry relation

$$\mathbf{Z}^{c0}(u,u') = \mathbf{D}\mathbf{Z}^{c0}(u,u')\mathbf{D}\,,$$

where $\mathbf{D} = \mathrm{diag}\,(1,1,-1,-1)$. We accordingly decompose H_p^{c0} as the direct sum

$$H_p^{c0} = H_p^{c0s} \oplus H_p^{c0a}$$

(s = symmetric, a = antisymmetric) of the subspaces

$$H_p^{c0s} = \left\{ \begin{bmatrix} I^{c0} \\ Q^{c0} \end{bmatrix} \middle/ \mathbf{I}^{c0} \in H_p^{c0} \right\}, \qquad H_p^{c0a} = \left\{ \begin{bmatrix} U^{c0} \\ V^{c0} \end{bmatrix} \middle/ \mathbf{I}^{c0} \in H_p^{c0} \right\},$$

where the projection of H_p^{c0} onto H_p^{c0s} along H_p^{c0a} has unit norm. For $a_1(\theta) \not\equiv a_4(\theta)$ the reduction of the value problems (6.1) – (6.2), and (6.3) to (6.5) to H_p^{c0a} gives the same answer as for the reductions to the spaces H_p^{csj}, because $(1,0,0,0) \in H_p^{c0s}$ and therefore

$$\|B|_{H_p^{c0a}}\| < 1 .$$

It remains to consider the coordinate problem on H_p^{c0s} and for $a_1(\theta) \equiv a_4(\theta)$ on H_p^{c0a}.

We have the following symmetry relations (cf. [1z, (11) – (113)):

$$\mathbf{P}\tilde{\mathbf{Z}}^{c0}(u',u)\mathbf{P} = \mathbf{Z}^{c0}(u,u') , \qquad \mathbf{Z}^{c0}(-u,-u') = \mathbf{Z}^{c0}(u,u') ,$$

where $\mathbf{P} = \operatorname{diag}(1,1,-1,1)$ and tilde above a matrix denotes transposition. On restricting to H_p^{c0s} we see that the restriction B^s of B to H_p^{c0s} is self-adjoint and commutes with the signature operator

$$(J^s\mathbf{h})(u) = \mathbf{h}(-u) , \qquad \mathbf{h} \in H_p^{c0s} ,$$

while the restriction T^s of T to $H_{p_s}^{c0s}$ anticommutes with J^s. Furthermore, $\|B^s\| = 1$. Let us put $A^s = \mathbb{I} - B^s$. Then

$$\operatorname{Ker} A^s = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} .$$

On H_p^{c0s} the operator T^s is bounded, injective and self-adjoint, while A^s is positive self-adjoint with one-dimensional kernel. Furthermore, the condition $a_1 \in L_r[-1,1]$ with $r > 1$ implies $B^s = |T^s|^\alpha D^s$ for some $0 < \alpha < (r-1)/pr$ and some bounded operator D^s. In the terminology of [25] the pair (T^s, B^s) is a semi-definite admissible pair on H_2^{c0s}, which allows the inversion symmetry J^s. Because the zero root linear manifold

$$Z_0 \overset{\text{def.}}{=\!=} Z_0((T^s)^{-1}A^s) = \bigcap_{n=1}^{\infty} \operatorname{Ker}((T^s)^{-1}A^s)^n$$

must have an even dimension ([25], Sec. III. 7) and has dimension $\leqslant 2 \dim \operatorname{Ker} A^s$ ([25], Prop. III 3.2), it must have dimension 2, so that $(T^s)^{-1}A^s$ has one zero Jordan block only, which has order 2. According to Theorem IV 3.4 or [25] the boundary value problem (6.1) – (6.2) restricted to H_2^{c0s} has a unique solution, while Theorem IV 3.5 of [25] and $\dim Z_0 = 2$ imply the uniqueness of the solution of Eqs. (6.3) to (6.5) and its form (6.6).

Let us extend the result on H_2^{c0s} to the spaces H_p^{c0s}. Take invertible β on Z_0 without imaginary eigenvalues and put

(6.7) $\quad A_\beta^s = T^s\beta^{-1}(I - P^s) + A^sP^s ,$

where P^s is the $(T^s)^{-1}A^s$-reducing projection with kernel Z_0. This construction is along the lines of Prop. III 6.3 of [25] and yields an invertible A_β^s such that

$$(A^s_\beta)^{-1}T^s = \beta \oplus [(T^s)^{-1}A^s|_{\operatorname{Ran}P^s}]^{-1}$$

does not have imaginary eigenvalues. To the operator (6.7) we associate a Wiener-Hopf equation whose symbol admits a (left and right) Wiener-Hopf factorization. Using the operator (6.7) the boundary value problems (6.1) – (6.2) and (6.3) to (6.5) are decomposed to analogous problems on H_p^{c0s} with A^s_β instead of A^s and finite-dimensional evolution equations on Z_0. The evolution equations are identical for each $1 \leqslant p < \infty$ and the problems with A^s_β have the same solvability properties on all spaces H_p^{c0s}, because of the Fredholm argument in the proof of Theorem 3.1. Hence, the results on H_2^{c0s} may be transferred to all spaces H_p^{c0s}. If $a_1(\theta) \equiv a_4(\theta)$, we proceed analogously in H_p^{c0a}. ■

Using the explicit form for $\mathbf{Z}^{c0}(u, u')$ (see [14], (189)) one easily computes that the zero root linear manifold $Z_0(T^{-1}A)$ is two-dimensional and has Jordan basis

$$\left\{(1,0,0,0)\,,\quad \left[\frac{u}{1-\frac{1}{3}\beta_1},0,0,0\right]\right\},$$

where $\beta_1 = \frac{3}{2}\int_{-1}^{1}(\cos\theta)a_1(\theta)\,\mathrm{d}(\cos\theta)$. The nonnegativity of a_1 implies $\beta_1 \in (-3,3)$. In the exceptional case $a_1(\theta) \equiv a_4(\theta)$ the space $Z_0(T^{-1}A)$ is four-dimensional and a second linearly independent jordan chain is provided by

$$\left\{(0,0,0,1), (0,0,0,\frac{u}{1-\frac{1}{3}\beta_1}\right\}.$$

Equations (6.3) to (6.5) form a statement of the Milne problem, where F is the radiative flux coming the stellar interior (cf. [2]). In neutron physics the number α in (6.6) is usually called the extrapolation length. Equations (6.1) – (6.2) form a statement of the usual half-space problem. To (6.1) – (6.2) one may add a thermal emission term $\mathbf{f}\colon [0,\infty) \to H_p$, which is bounded and uniformly Hölder continuous, without affecting the unique solvability result.

Proposition 6.2 *Let* $1 \leqslant p < \infty$ *and let* $a_1 \in L_r[-1,1]$ *for some* $r > 1$. *Given* $Q_+\mathbf{J} \in Q_+[H_p]$, *the unique solution of the boundary value problem* (6.1) – (6.2) *with* $0 < a < 1$ *converges to the unique solution of* (6.1) – (6.2) *with* $a = 1$ *uniformly in* τ *on* H_p, *whenever* $a \uparrow 1$.

Proof. Let L_∞ be the operator defined by (1.22) (where $b = \infty$). For $1 \leqslant q \leqslant \infty$ this operator is reduced by the decomposition

$$L_q(H_p)_\infty = L_q(H_p^{c0s})_\infty \oplus L_q(H_p^{c0a})_\infty \oplus \bigoplus_{j=1}^{\infty} L_q(H_p^{csj})_\infty\,.$$

Let us denote by L_∞^{c0s}, L_∞^{c0a} and L_∞^{csj} the restrictions of L_∞ to $L_q(H_p^{c0s})_\infty$, $L_q(H_p^{c0a})_\infty$ and $L_q(H_p^{csj})_\infty$, respectively. By the proof of Theorem 6.1, the spectral radii coincide with those of the restrictions of B to H_p^{c0s}, H_p^{c0a} and H_p^{csj}, respectively, implying

$$r(L_\infty^{c0s}) = 1, \quad r(L_\infty^{c0a}) < 1, \quad r(L_\infty^{csj}) < 1 .$$

(For $a_1(\theta) \equiv a_4(\theta)$ we have $r(L_\infty^{c0a}) = 1$). First take $a_1(\theta) \equiv a_4(\theta)$. Then $\mathbb{I} - aL_\infty^{c0a}$ and $\mathbb{I} - aL_\infty^{c0j}$ are invertible for all $0 < a \leqslant 1$, whereas $\mathbb{I} - aL_\infty^{c0s}$ is invertible for $0 < a < 1$. Rewriting (6.1)–(6.2) as the equation

$$(\mathbb{I} - aL_\infty)\mathbf{I} = \omega$$

and decomposing this equation as the set of equations

$$(\mathbb{I} - aL_\infty^{c0s})\mathbf{I}_a^{c0s} = \omega^{c0s}, \qquad (\mathbb{I} - aL_\infty^{c0a})\mathbf{I}_a^{c0a} = \omega^{c0a},$$
$$(\mathbb{I} - aL_\infty^{csj})\mathbf{I}_a^{csj} = \omega^{csj},$$

where the albedo of single scattering a is added as a subscript to the solutions, we obtain

$$\lim_{a\uparrow 1} \|\mathbf{I}_a^{c0a} - \mathbf{I}_{a=1}^{c0a}\|_q = 0\,, \qquad \lim_{a\uparrow 1} \|\mathbf{I}_a^{csj} - \mathbf{I}_{a=1}^{csj}\|_q = 0\,.$$

So we have the proposition for all components except H_p^{c0s}.

For the component H_p^{c0s} we first consider the case $p = 2$ and exploit the positive self-adjointness of A^s on H_2^{c0s}. Because of the special Jordan structure of $(T^s)^{-1}A^s$ at zero (one Jordan block only, which has order 2), we may apply Theorem 2.4 of [26], and obtain

$$\lim_{a\uparrow 1} \|\mathbf{I}_a^{c0s} - \mathbf{I}_{a=1}^{c0s}\|_\infty = 0$$

for the special right-hand side $\omega^{c0s}(\tau) = \mathrm{e}^{-\tau(T^s)^{-1}} Q_s^+ \mathbf{J}^s$, where Q_s^+ and $\mathbf{J}^s$ are the H_p^{c0s}-components of Q_+ and $\mathbf{J}$. For the remaining $1 \leqslant p < \infty$ we employ a similar procedure as in the final paragraph of the proof of Theorem 6.1 in order to extend the H_2^{c0s}-result. The crux of the matter is that the zero root linear manifold Z_0 of $(T^s)^{-1}A^s$ is the same in all spaces H_p^{c0s} so that the part of the convergence within Z_0 does not depend on p.

Finally, we combine the convergence results on the component equations and finish the proof of Proposition 6.2 in this case. If $a_1(\theta) \equiv a_4(\theta)$, we repeat the $a = 1$ argument in H_p^{c0a}. ■

As a warning we mention that the proof of Theorem 2.4 of [26] is not completely correct. The flaw is contained in the paragraph at the top of page 588 of [26], where an incorrect (i.e., nonpositive) $\mathbb{I} - \tilde{B}_N$ was given. A generalization of Theorem 2.4 of [26] was found by Ran and Rodman [29].

We remark that the justification of the method of multiple scattering expansion is an open problem for $a = 1$ and $b = \infty$. Using a result of Littlewood ([31], Sec. 7.66) one only has to prove the boundedness of the sequence of norms

$$\text{(6.8)} \qquad \{n\|L_\infty^n\|_{L_q(H_p)_\infty}\}_{n=1}^\infty$$

in order to establish the weak convergence in $L_q(H_p)_\infty$ of the series (3.5) for $a = 1$ and ω as in (2.12). For $a = 1$ and $b = \infty$ the main consequence of Corollaries 3.2 and 4.2, namely that $\mathbf{I}(\tau) \in K_p$ whenever the boundary data are vectors in K_p, follows immediately from Proposition 6.2 and the last statement of Corollary 4.2.

7 Discussion

We have developed a complete existence and uniqueness theory for the equation of transfer of polarized light. Our main strategy has been the application of Wiener-Hopf factorization, Fredholm index and cone preservation methods. The phase matrix, however, satisfies the following symmetry relations (see [12], also [14]):

$$(7.1) \quad \mathbf{Z}(-u', -u, \varphi' - \varphi) = \mathbf{P}\tilde{\mathbf{Z}}(u, u', \varphi - \varphi')\mathbf{P}$$

$$(7.2) \quad \mathbf{Z}(-u, -u', \varphi' - \varphi) = \mathbf{Z}(u, u', \varphi - \varphi')$$

$$(7.3) \quad \mathbf{Z}(u, u', \varphi' - \varphi) = \mathbf{D}\mathbf{Z}(u, u', \varphi - \varphi')\mathbf{D},$$

where tilde above a matrix denotes transposition, $\mathbf{P} = \mathrm{diag}\,(1,1,-1,1)$ and $\mathbf{D} = \mathrm{diag}\,(1,1,-1,-1)$. From these we easily derive

$$(7.4) \quad \tilde{\mathbf{Z}}(u', u, \varphi' - \varphi) = \mathbf{Q}\mathbf{Z}(u, u', \varphi - \varphi')\mathbf{Q}$$

$$(7.5) \quad \mathbf{Z}(-u, -u', \varphi - \varphi') = \mathbf{D}\mathbf{Z}(u, u', \varphi - \varphi')\mathbf{D},$$

where $\mathbf{Q} = \mathrm{diag}\,(1,1,1,-1)$. Equations (7.2) and (7.5) imply that

$$(7.6) \quad JT = -TJ, \qquad JA = AJ, \qquad JB = BJ$$

for the signature operators (i.e., $J = J^* = J^{-1}$)

$$(J\mathbf{I})(u,\varphi) = \mathbf{I}(-u, -\varphi) \quad or \quad (J\mathbf{I})(u,\varphi) = \mathbf{D}\mathbf{I}(-u,\varphi).$$

Equation (7.4) implies the self-adjointness of the operators T, A and B with respect to the indefinite scalar product

$$\langle \mathbf{I}_1, \mathbf{I}_2 \rangle = (\mathbf{Q}\mathbf{I}_1, \mathbf{I}_2) = \int_{-1}^{1}\int_{0}^{2\pi} \{I_1(u,\varphi)\overline{I_2(u,\varphi)} + Q_1(u,\varphi)\overline{Q_2(u,\varphi)} + U_1(u,\varphi)\overline{U_2(u,\varphi)} - V_1(u,\varphi)\overline{V_2(u,\varphi)}\}\,d\varphi\,du$$

on H_2. The second signature operator and the indefinite scalar product can also be defined on the various components of H_2. Possibly this provides another method to deal with the equation of transfer of polarized light, which exploits (1.12) to (1.15) directly.

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