



A Matrix Schrödinger Approach to Focusing Nonlinear Schrödinger Equations with Nonvanishing Boundary Conditions

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Abstract

We relate the scattering theory of the focusing AKNS system with equally sized nonvanishing boundary conditions to that of the matrix Schrödinger equation. This (shifted) Miura transformation converts the focusing matrix nonlinear Schrödinger (NLS) equation into a new nonlocal integrable equation. We apply the matrix triplet method of solving the Marchenko integral equations by separation of variables to derive the multisoliton solutions of this nonlocal equation, thus proposing a method to solve the reflectionless matrix NLS equation.

Keywords Miura transformation · Matrix KdV equation · Matrix NLS equation · Matrix triplet method · Integrable nonlocal equation

Mathematics Subject Classification 35Q53 · 35Q55 · 35P25 · 35R30 · 35C08

1 Introduction

The nonlinear Schrödinger (NLS) equations have served as the basic models for surface waves on deep waters (Ablowitz 2011; Ablowitz and Segur 1981; Zakharov and Shabat 1972), signals along optical fibres (Hasegawa and Tappert 1973; Hasegawa 2002; Shaw 2004), plasma oscillations (Zakharov 1971), magnetic spin waves (Chen et al. 1994; Zakharov and Popkov 1983), and particle states in Bose–Einstein condensates (Pethick and Smith 2002; Pitaevskii and Stringari 2016; Kevrekidis et al. 2008). The NLS equations with solutions decaying at infinity have been studied in detail (Ablowitz

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et al. 1974; Ablowitz and Segur 1981; Calogero and Degasperis 1982; Faddeev and Takhtajan 1987; Ablowitz et al. 2004). After finding the Peregrine solutions (Peregrine 1983), various solutions of the NLS equations with nonvanishing boundary conditions have been presented in Ma (1979), Akhmediev et al. (1985, 2009), Akhmediev and Korneev (1986), Its et al. (1988), Mihalache et al. (1993), Tajiri and Watanabe (1998), Zakharov and Gelash (2013).

In 1972 in their seminal paper Zakharov and Shabat (1972), Zakharov and Shabat showed that the NLS equation can be solved by means of the Inverse Scattering Transform (IST) technique. To this aim, they introduced a scattering problem now known as the Zakharov–Shabat (ZS) system. The ZS system was used to solve the scalar NLS system with zero and nonzero boundary conditions (Zakharov and Shabat 1972, 1973). In particular, in Zakharov and Shabat (1973), Zakharov and Shabat considered the case of nonzero boundary conditions in the defocusing regime, introducing a spectral parameter belonging to a suitable two-sheeted Riemann surface and studying the analyticity properties of the scattering data on this surface. Moreover, in Ma (1979), it was proven that, in order to develop the IST for the focusing NLS equation with nonvanishing boundary conditions, the associated ZS system leads to introducing a spectral parameter λ which belongs again to a suitable two-sheeted Riemann surface. The introduction of a two-sheeted Riemann surface evidently makes the study of the NLS equation with nonvanishing boundary conditions via the IST much more complicated with respect to the vanishing case. Furthermore, in 1974 Ablowitz, Kaup, Newell and Segur proposed an alternative but equivalent way to develop the IST for the NLS equation consisting of associating to this equation the so-called AKNS system (Ablowitz et al. 1974). In the AKNS system, one (matrix) equation represents the spectral equation, whereas a second (matrix) equation describes the time evolution of the scattering data. Similarly to what happens with the ZS system, developing the IST from the AKNS pairs is significantly more complicated in the nonvanishing cases than in the vanishing case.

Systematic studies of the inverse scattering transform theory (IST) of the (scalar and matrix) NLS equation with nonvanishing boundary conditions have been carried out in the defocusing case in Kawata and Inoue (1977, 1978), Asano and Kato (1981, 1984), Faddeev and Takhtajan (1987), Prinari et al. (2006), Demontis et al. (2013) and in the focusing case in Biondini and Kovačić (2014), Demontis et al. (2014), Ortiz and Prinari (2020), Biondini et al. (2021). In Bilman and Miller (2019), the IST with full account of the spectral singularities leads to rogue wave solutions of the focusing NLS with nonvanishing boundary conditions.

In all the papers cited above, a ZS system or an AKNS system is associated to the NLS equation. If one considers the focusing NLS with nonvanishing boundary conditions, it is customary, as we have remarked above, to introduce a new spectral complex parameter, say λ , defined as $\lambda = \sqrt{k^2 + \mu^2}$ (it should be noted that λ is defined through a multivalued function). The study of the analyticity properties of the scattering data with respect to the parameter λ is quite difficult and requires special care. In this article, we show how to associate a Schrödinger equation with a vanishing potential as a spectral problem for the NLS equation with nonzero boundary conditions. In this way, to the best of our knowledge, for the first time we develop the IST for

the focusing NLS system with nonzero boundary conditions without associating to it the AKNS system (or the Zakharov–Shabat system). The advantage of associating the Schrödinger equation with vanishing boundary conditions instead of the AKNS system is immediate because the construction of the scattering data for the Schrödinger equation with zero boundary conditions does not require the introduction of a new spectral parameter. Consequently, the study of the analyticity properties of these coefficients can be done in a more transparent way with respect to the analogous study while using the AKNS system. However, we have achieved the important task of establishing a connection between the scattering data of the AKNS system and the scattering data of the Schrödinger equation.

In other words, a major obstacle encountered in the above-cited studies of the IST for the nonvanishing NLS systems is the change of variable from the initial spectral parameter k to a new spectral parameter $\lambda = \sqrt{k^2 + \mu^2}$ which complicates analyticity issues for Jost solutions and scattering coefficients considerably, especially if such change of variable is considered in the entire complex plane. The main purpose of this article is to greatly simplify these issues by relating the focusing NLS equation to a suitable matrix Schrödinger equation, where the spectral parameter (in this case, λ) is typically chosen in the closed upper half complex half-plane $\mathbb{C}^+ \cup \mathbb{R}$. Here we can rely on a substantial body of knowledge on the direct and inverse scattering theory of the scalar Schrödinger equation on the line (Faddeev 1964; Deift and Trubowitz 1979; Calogero and Degasperis 1982; Chadan and Sabatier 1989) and the matrix Schrödinger equation on the half-line (Aktosun and Weder 2018, 2020) and the full-line (Wadati and Kamijo 1974; Aktosun et al. 2001). In particular, the small λ asymptotics of the scattering data, which is crucial to a rigorous matrix Schrödinger scattering theory, has been developed in detail in Aktosun et al. (2001).

In this article, we study the focusing $m + m$ AKNS system

$$v_x = (-ik\sigma_3 + \mathcal{Q})v, \quad (1.1)$$

where $v = v(x, k)$ is a vector function with $n = 2m$ components, I_m is the identity matrix of order m , $\sigma_3 = I_m \oplus (-I_m)$, the potential \mathcal{Q} anticommutes with σ_3 , and the complex conjugate transpose $\mathcal{Q}^\dagger = -\mathcal{Q}$. The potential \mathcal{Q} is to satisfy the integrability condition

$$\int_0^\infty dy (1 + |y|) (\|\mathcal{Q}(-y) - \mathcal{Q}_l\| + \|\mathcal{Q}(y) - \mathcal{Q}_r\| + \|\mathcal{Q}_y(y)\| + \|\mathcal{Q}_y(-y)\|) < +\infty, \quad (1.2)$$

where \mathcal{Q}_y is the y -derivative of \mathcal{Q} and $[\mathcal{Q}_{r,l}]^2 = -\mu^2 I_n$ for some $\mu > 0$.

We pursue an approach that is quite different from the one expounded in Biondini and Kovačić (2014), Demontis et al. (2014), Biondini et al. (2021). Letting $L = i\sigma_3[\partial_x I_n - \mathcal{Q}]$ stand for the AKNS Hamiltonian, we easily verify that $\mathcal{L} = L^2 + \mu^2 \mathbf{1}$ is the matrix Schrödinger Hamiltonian given by

$$\begin{aligned} \mathcal{L}v &= (L^2 + \mu^2 \mathbf{1})v = -\sigma_3[\partial_x I_n - \mathcal{Q}]\sigma_3[\partial_x I_n - \mathcal{Q}]v + \mu^2 v \\ &= -[\partial_x I_n + \mathcal{Q}][\partial_x I_n - \mathcal{Q}]v + \mu^2 v \end{aligned}$$

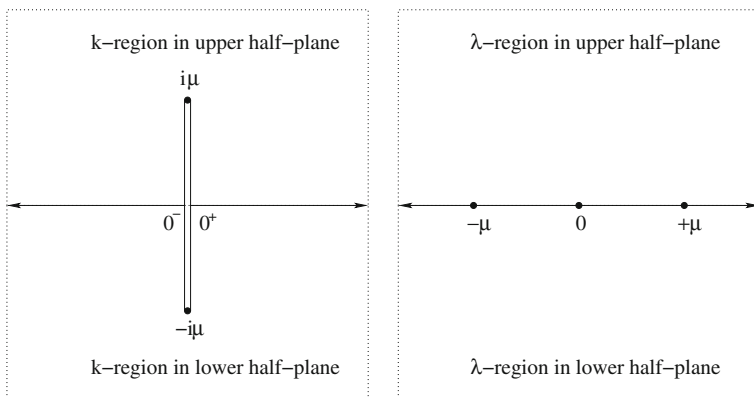


Fig. 1 The regions $k \in \mathbb{K}^\pm$ and $\lambda \in \mathbb{C}^\pm$ with manifold boundary

$$= -v_{xx} + Q^2v - Qv_x + (Qv)_x + \mu^2v = -v_{xx} + Qv,$$

where $\mathbf{1}$ stands for the identity operator on a suitable function space and

$$Q = Q^2 + Q_x + \mu^2 I_n \tag{1.3}$$

is a matrix Faddeev class Schrödinger potential obtained from Q by the (shifted) Miura transform (Ablowitz and Segur 1981). In other words, $\|Q(\cdot)\| \in L^1(\mathbb{R}; (1 + |x|)dx)$. Then any solution v of the AKNS system (1.1) is also a solution of the matrix Schrödinger equation

$$\mathcal{L}v = (-\partial_x^2 I_n + Q)v = \lambda^2 v, \tag{1.4}$$

where

$$\lambda = \sqrt{k^2 + \mu^2} \tag{1.5}$$

is the conformal transformation from the complex k -plane \mathbb{K} cut along the segment $[-i\mu, i\mu]$ onto the complex λ -plane satisfying $\lambda \sim k$ at infinity. This transformation provides a 1, 1-correspondence between the open upper/lower half k -plane \mathbb{K}^\pm cut along $[-i\mu, i\mu]$ onto the open upper/lower half λ -plane \mathbb{C}^\pm as well as a 1, 1-correspondence between their boundaries $\partial\mathbb{K}^\pm$ and \mathbb{R} and their closures $\mathbb{K}^\pm \cup \partial\mathbb{K}^\pm$ and $\mathbb{C}^\pm \cup \mathbb{R}$ (Fig. 1).

In this article, we wish to take advantage of the well-developed direct and inverse scattering theory of the matrix Schrödinger equation with selfadjoint potential [Agronovich and Marchenko (1963), Aktosun and Weder (2018, 2020) on the half-line, Wadati and Kamijo (1974), Aktosun et al. (2001) on the full line], especially the established custom of choosing its spectral variable λ in $\mathbb{C}^+ \cup \mathbb{R}$, in deriving the focusing NLS solutions with nonvanishing boundary conditions. In a previous paper, Demontis and van der Mee (2021), such full-line theory has been made to fit potentials satisfying

$$Q^\dagger = \sigma_3 Q \sigma_3. \tag{1.6}$$

The traditional applications of the matrix Schrödinger equation to quantum graphs, quantum wires, and quantum mechanical scattering of particles with internal structure (Berkolaiko 2017; Berkolaiko et al. 2006; Berkolaiko and Kuchment 2013; Berkolaiko and Liu 2017; Boman and Kurasov 2005; Exner et al. 2008; Gerasimenko 1988; Gerasimenko and Pavlov 1988; Gutkin and Smilansky 2001; Harmer 2002, 2004, 2005; Kostykin and Schrader 1999, 2000; Kuchment 2004, 2005; Kurasov and Nowaczyk 2010, 2005; Kurasov and Stenberg 2002) have led to the almost exclusive development of matrix Schrödinger scattering theory for selfadjoint potentials satisfying $Q^\dagger = Q$ [see Agranovich and Marchenko (1963), Aktosun and Weder (2018, 2020) for the half-line theory and Wadati and Kamijo (1974), Aktosun et al. (2001) for the full-line theory]. Energy losses in such systems naturally lead to potentials whose imaginary part $[\mathcal{Q} - \mathcal{Q}^\dagger]/2i$ has constant sign. In the present context where \mathcal{Q} satisfies (1.6), we thus require the modified matrix Schrödinger scattering theory given in Demontis and van der Mee (2021) when solving the focusing matrix NLS equation.

Let us discuss the contents of the various sections. In Sect. 2, we introduce the Lax pair $\{\mathcal{L}, A\}$ and the AKNS pair $\{X, T\}$ whose compatibility conditions lead to an integrable nonlocal equation for \mathcal{Q} . We also relate the solutions of this integrable equation to those of a modified matrix NLS equation which is converted into the usual matrix NLS equation by a trivial gauge transformation. Next, in Sects. 3–4 we state the direct and inverse scattering theory of the matrix Schrödinger equation (1.4) with Faddeev class potentials \mathcal{Q} satisfying (1.6), disregarding any time dependence. In particular, we introduce the Jost solutions and the scattering coefficients, write them as Fourier transforms of L^1 -functions, and state the Marchenko integral equations to solve the inverse scattering problem. We then go on to derive the time evolution of the scattering data [Sect. 5]. In Sect. 6, we apply the so-called matrix triplet method to derive the multisoliton solutions of the nonlocal integrable equation and the focusing matrix NLS equation by separation of variables in the Marchenko integral equations.

We adopt boldface symbols for many of the quantities pertaining to the matrix Schrödinger equation and calligraphic symbols for many of the quantities pertaining to the AKNS system. We deviate from the praxis of Ablowitz et al. (1974), Ablowitz et al. (2004) in allowing right and left to correspond to the real line endpoints involved in defining the Jost solutions, both in the (matrix) Schrödinger and the AKNS cases. Hence, we prioritize traditional notations regarding (matrix) Schrödinger equations (Faddeev 1964; Deift and Trubowitz 1979; Chadan and Sabatier 1989) over those regarding AKNS systems (Ablowitz et al. 1974, 2004).

2 Lax Pair for the New Integrable Model

It is well-known that the matrix NLS system is governed by a Lax pair $\{L, A\}$ of linear operators (Lax 1968; Ablowitz and Segur 1981; Eckhaus and van Harten 1981)

$$L = i\sigma_3(\partial_x I_n - \mathcal{Q}), \quad (2.1a)$$

$$A = i\sigma_3 \left(2\partial_x^2 I_n - 2\mathcal{Q}\partial_x - \mathcal{Q} \right), \quad (2.1b)$$

where Q is given by (1.3), $Lv = kv$ is the AKNS eigenvalue problem, and $v_t = Av$ describes the time evolution. Then the zero curvature condition

$$L_t + LA - AL = \mathbf{0},$$

where $\mathbf{0}$ denotes the zero operator on a suitable function space, leads to the integrable PDE

$$i\sigma_3 Q_t + Q_{xx} - 2Q^3 - 2\mu^2 Q = 0_{n \times n} \tag{2.2}$$

which coincides with the usual matrix NLS equation, studied in Ablowitz et al. (2004), Ablowitz et al. (1974), apart from the extra term $-2\mu^2 Q$.

Putting $\mathcal{L} = L^2 + \mu^2 \mathbf{1} = -\partial_x^2 + Q$, we now compute

$$\begin{aligned} i\sigma_3[\mathcal{L}_t + \mathcal{L}A - A\mathcal{L}] &= i\sigma_3 Q_t \\ &\quad - (-\partial_x^2 + \sigma_3 Q \sigma_3) \left[2\partial_x^2 - 2Q\partial_x - Q \right] + \left[2\partial_x^2 - 2Q\partial_x - Q \right] (-\partial_x^2 + Q) \\ &= i\sigma_3 Q_t + 4(-Q_x + \frac{1}{2}[Q - \sigma_3 Q \sigma_3])\partial_x^2 \\ &\quad + 2(-Q_{xx} + Q_x + \sigma_3 Q \sigma_3 Q - Q Q)\partial_x \\ &\quad + Q_{xx} + \sigma_3 Q \sigma_3 Q - 2Q Q_x - Q^2. \end{aligned}$$

Then the ∂_x^2 term vanishes iff $Q = D + Q_x$ for some D commuting with σ_3 and vanishing as $x \rightarrow \pm\infty$. Hence, the coefficient of the ∂_x term equals $2(D - Q^2)_x + 2[D, Q] = 0_{n \times n}$. Putting $E = D - Q^2 - \mu^2 I_n$ so that E vanishes as $x \rightarrow \pm\infty$, we obtain $E_x + [E, Q] = 0_{n \times n}$. Writing the latter as

$$\left(e^{-x Q_r} E e^{x Q_r} \right)_x = -e^{-x Q_r} [Q(x) - Q_r] e^{x Q_r}$$

and using that $e^{\pm x Q_r} = \cos(\mu x) I_n \pm \frac{\sin(\mu x)}{\mu} Q_r$ to arrive at the estimate $\|e^{\pm x Q_r}\| \leq \frac{\sqrt{\mu^2 + \|Q_r\|^2}}{\mu}$, we can apply Gronwall’s inequality to the estimate

$$\|E(x)\| \leq \frac{\mu^2 + \|Q_r\|^2}{\mu^2} \int_x^\infty dy \|E(y)\| \|Q(y) - Q_r\|,$$

to see that E vanishes identically and therefore $D = Q^2 + \mu^2 I_n$. Thus, for this particular choice of D we arrive at the nonlinear evolution equation

$$i\sigma_3 Q_t + Q_{xx} - Q^2 + \sigma_3 Q \sigma_3 Q - 2Q Q_x = 0_{n \times n}, \tag{2.3}$$

where

$$Q(x; t) = Q_r - \int_x^\infty dy \frac{1}{2} (Q - \sigma_3 Q \sigma_3), \tag{2.4a}$$

$$Q(x; t) = Q_l + \int_{-\infty}^x dy \frac{1}{2} (Q - \sigma_3 Q \sigma_3). \tag{2.4b}$$

for time invariant matrices $Q_{r,l}$ satisfying $[Q_{r,l}]^2 = -\mu^2 I_n$ for every $t \in \mathbb{R}$.

Conversely, substituting

$$Q = D + Q_x,$$

where D commutes with σ_3 , Q_x anticommutes with σ_3 , and D vanishes as $x \rightarrow \pm\infty$, into (2.3), we obtain

$$0_{n \times n} = i\sigma_3 D_t + (D_x - 2Q Q_x)_x + (i\sigma_3 Q_t + Q_{xx} - 2QD)_x.$$

Separating the block off-diagonal and block diagonal components, we get

$$\begin{aligned} i\sigma_3 D_t + (D_x - 2Q Q_x)_x &= 0_{n \times n}, \\ i\sigma_3 Q_t + Q_{xx} - 2QD &= 0_{n \times n}, \end{aligned}$$

where Q_t , Q_{xx} , and D vanish as $x \rightarrow \pm\infty$. If there exists a solution Q of the differential Riccati equation $Q^2 + Q_x = Q - \mu^2 I_n$ which anticommutes with σ_3 and satisfies $Q \rightarrow Q_{r,l}$ as $x \rightarrow \pm\infty$, then $D = Q^2 + \mu^2 I_n$ and

$$[i\sigma_3 Q_t + Q_{xx} - 2Q^3 - 2\mu^2 Q, Q] = 0_{n \times n}, \tag{2.5a}$$

$$i\sigma_3 Q_t + Q_{xx} - 2Q^3 - 2\mu^2 Q = 0_{n \times n}, \tag{2.5b}$$

where a matrix commutator appears. The gauge transformation

$$Q(x; t) = e^{-i\mu^2 t \sigma_3} \mathcal{R}(x; t) e^{i\mu^2 t \sigma_3} \tag{2.6}$$

then converts (2.5b) into the usual matrix NLS equation

$$i\sigma_3 \mathcal{R}_t + \mathcal{R}_{xx} - 2\mathcal{R}^3 = 0_{n \times n},$$

where the limits $\mathcal{R}_{l,r}(t)$ of $\mathcal{R}(x; t)$ as $x \rightarrow \pm\infty$ satisfy $[\mathcal{R}_{r,l}]_t = -2i\mu^2 \sigma_3 \mathcal{R}_{r,l}$. This is in agreement with Q_t vanishing as $x \rightarrow \pm\infty$ and with the well-known time evolution [see Demontis et al. (2014) for $m = 1$]

$$\mathcal{R}_{r,l}(t) = i\mu e^{2i\mu^2 t \sigma_3} e^{i\theta_{r,l} \sigma_3} (\sigma_2 \otimes I_m),$$

where $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ denotes the second Pauli matrix, $\sigma_2 \otimes I_m = \begin{pmatrix} 0_{m \times m} & -iI_m \\ iI_m & 0_{m \times m} \end{pmatrix}$ is a Kronecker product (cf. Horn and Johnson 1994), and $\theta_{r,l} \in \mathbb{R}$ are phases. Furthermore, (2.5b) and (1.3) imply the nonlinear equation (2.3). In fact,

$$\begin{aligned} i\sigma_3 Q_t + Q_{xx} - (Q - \sigma_3 Q \sigma_3) Q - 2Q Q_x &= Q[Q_{xx} - 2Q^3 - 2\mu^2 Q] \\ &\quad - [Q_{xx} - 2Q^3 - 2\mu^2 Q] Q - [Q_{xxx} - 2(Q^3)_x - 2\mu^2 Q_x] \\ &\quad + [Q Q_{xx} + Q_{xx} Q + 2Q_x^2 + Q_{xxx}] - [2Q_x Q^2 + 2Q_x^2 + 2\mu^2 Q_x] \end{aligned}$$

$$- 2[\mathcal{Q}^2 \mathcal{Q}_x + \mathcal{Q} \mathcal{Q}_x \mathcal{Q} + \mathcal{Q} \mathcal{Q}_{xx}] = 0_{n \times n}.$$

Recall that the Lax pair $\{\mathcal{L}, A\}$ for the modified nonlinear matrix Schrödinger equation (2.3) is given by (2.1). Let us now derive an AKNS pair $\{X, T\}$ for the same equation. Indeed, (2.3) is compatible with the linear system

$$\mathcal{L}v = \lambda^2 v, \quad v_t = Av,$$

where $\mathcal{L} = -\partial_x^2 + \mathcal{Q}$. We may therefore write

$$\begin{aligned} v_t = Av &= 2i\sigma_3 v_{xx} - 2i\sigma_3 \mathcal{Q}v_x - i\sigma_3 \mathcal{Q}v \\ &= 2i\sigma_3(\mathcal{Q} - \lambda^2 \mathbf{1})v - 2i\sigma_3 \mathcal{Q}v_x - i\sigma_3 \mathcal{Q}v \\ &= i\sigma_3 \left\{ (\mathcal{Q} - 2\lambda^2 \mathbf{1})v - 2\mathcal{Q}v_x \right\}. \end{aligned}$$

Let us compute

$$\begin{aligned} (v_x)_t = (Av)_x &= i\sigma_3 \left((\mathcal{Q} - 2\lambda^2 \mathbf{1})v_x + \mathcal{Q}_x v - 2\mathcal{Q}_x v_x - 2\mathcal{Q}(\mathcal{Q} - \lambda^2 \mathbf{1})v \right) \\ &= i\sigma_3 \left(\mathcal{Q}_x - 2\mathcal{Q}\mathcal{Q} + 2\lambda^2 \mathcal{Q} \right) v + i\sigma_3 (\mathcal{Q} - 2\lambda^2 \mathbf{1} - 2\mathcal{Q}_x)v_x. \end{aligned}$$

Hence, putting $V = \begin{pmatrix} v \\ v_x \end{pmatrix}$ we get the linear system

$$V_x = X(x, \lambda; t)V, \quad V_t = T(x, \lambda; t)V,$$

where $\{X, T\}$ is the AKNS pair given by

$$X(x, \lambda; t) = \begin{pmatrix} 0_{n \times n} & I_n \\ \mathcal{Q}(x; t) - \lambda^2 I_n & 0_{n \times n} \end{pmatrix}, \tag{2.7a}$$

$$T(x, \lambda; t) = \begin{pmatrix} i\sigma_3(\mathcal{Q} - 2\lambda^2 I_n) & -2i\sigma_3 \mathcal{Q} \\ i\sigma_3(\mathcal{Q}_x - 2\mathcal{Q}\mathcal{Q} + 2\lambda^2 \mathcal{Q}) & i\sigma_3(\mathcal{Q} - 2\lambda^2 I_n - 2\mathcal{Q}_x) \end{pmatrix}. \tag{2.7b}$$

Then we easily compute

$$i(\sigma_3 \oplus \sigma_3) (X_t - T_x + XT - TX) = \begin{pmatrix} 0_{n \times n} & 0_{n \times n} \\ E_{21} & 0_{n \times n} \end{pmatrix},$$

where

$$\begin{aligned} E_{21} &= i\sigma_3 \mathcal{Q}_t + \mathcal{Q}_{xx} - 2(\mathcal{Q}\mathcal{Q})_x + 2\lambda^2 \mathcal{Q}_x \\ &\quad - \sigma_3(\mathcal{Q} - \lambda^2 I_n)\sigma_3(\mathcal{Q} - 2\lambda^2 I_n) + (\mathcal{Q} - 2\lambda^2 I_n - 2\mathcal{Q}_x)(\mathcal{Q} - \lambda^2 I_n) \\ &= i\sigma_3 \mathcal{Q}_t + \mathcal{Q}_{xx} + \mathcal{Q}^2 - \sigma_3 \mathcal{Q}\sigma_3 \mathcal{Q} - 2\mathcal{Q}\mathcal{Q}_x - 4\mathcal{Q}_x \mathcal{Q} \\ &\quad + \lambda^2 (2\mathcal{Q}_x + 2\sigma_3 \mathcal{Q}\sigma_3 + \mathcal{Q} - \mathcal{Q} - 2\mathcal{Q} + 2\mathcal{Q}_x) \end{aligned}$$

$$\begin{aligned}
 &= i\sigma_3 Q_t + Q_{xx} + Q^2 - \sigma_3 Q\sigma_3 Q - 2Q Q_x - 2(Q^2 - \sigma_3 Q\sigma_3 Q) \\
 &= i\sigma_3 Q_t + Q_{xx} - Q^2 + \sigma_3 Q\sigma_3 Q - 2Q Q_x,
 \end{aligned}$$

as claimed. Thus, the zero curvature condition for the AKNS pair $\{X, T\}$ is equivalent to the nonlinear evolution equation (2.3).

3 Direct Scattering Theory

In this section, we introduce the Jost solutions and scattering coefficients for the matrix Schrödinger equation (1.4) with Faddeev class potential Q satisfying (1.6). For the scalar Schrödinger equation with real Faddeev class potential, the direct scattering theory is well documented (Faddeev 1964; Deift and Trubowitz 1979; Calogero and Degasperis 1982; Novikov et al. 1984; Chadan and Sabatier 1989). The matrix theory is discussed at great length in Aktosun and Weder (2018, 2020) for the half-line and in Wadati and Kamijo (1974), Aktosun et al. (2001) for the full line. Here Aktosun et al. (2001) contains the essential small λ asymptotics of scattering coefficients that is lacking in Wadati and Kamijo (1974). The adjoint symmetry Q requires some modifications of existing theory (cf. Demontis and van der Mee 2021).

3.1 Jost Solutions of the Matrix Schrödinger Equation

3.1.1 $n \times n$ Jost Solutions

Let us define the *Jost solution from the left* $F_l(x, \lambda)$ and the *Jost solution from the right* $F_r(x, \lambda)$ as those solutions of the matrix Schrödinger equation (1.4) which satisfy the asymptotic conditions

$$F_l(x, \lambda) = e^{i\lambda x} [I_n + o(1)], \quad x \rightarrow +\infty, \tag{3.1a}$$

$$F_r(x, \lambda) = e^{-i\lambda x} [I_n + o(1)], \quad x \rightarrow -\infty, \tag{3.1b}$$

where $n = 2m$. Calling $m_l(x, \lambda) = e^{-i\lambda x} F_l(x, \lambda)$ and $m_r(x, \lambda) = e^{i\lambda x} F_r(x, \lambda)$ *Faddeev functions*, we easily define them as the unique solutions of the Volterra integral equations

$$m_l(x, \lambda) = I_n + \int_x^\infty dy \frac{e^{2i\lambda(y-x)} - 1}{2i\lambda} Q(y) m_l(y, \lambda), \tag{3.2a}$$

$$m_r(x, \lambda) = I_n + \int_{-\infty}^x dy \frac{e^{2i\lambda(x-y)} - 1}{2i\lambda} Q(y) m_r(y, \lambda). \tag{3.2b}$$

Then, for each $x \in \mathbb{R}$, $m_l(x, \lambda)$ and $m_r(x, \lambda)$ are continuous in $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$, are analytic in $\lambda \in \mathbb{C}^+$, and tend to I_n as $\lambda \rightarrow \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$. For $0 \neq \lambda \in \mathbb{R}$,

we can reshuffle (3.2) and arrive at the asymptotic relations

$$F_l(x, \lambda) = e^{i\lambda x} A_l(\lambda) + e^{-i\lambda x} B_l(\lambda) + o(1), \quad x \rightarrow -\infty, \quad (3.3a)$$

$$F_r(x, \lambda) = e^{-i\lambda x} A_r(\lambda) + e^{i\lambda x} B_r(\lambda) + o(1), \quad x \rightarrow +\infty, \quad (3.3b)$$

where

$$A_{r,l}(\lambda) = I_n - \frac{1}{2i\lambda} \int_{-\infty}^{\infty} dy \mathcal{Q}(y) m_{r,l}(y, \lambda), \quad (3.4a)$$

$$B_{r,l}(\lambda) = \frac{1}{2i\lambda} \int_{-\infty}^{\infty} dy e^{\mp 2i\lambda y} \mathcal{Q}(y) m_{r,l}(y, \lambda). \quad (3.4b)$$

Then $A_{r,l}(\lambda)$ is continuous in $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$, is analytic in $\lambda \in \mathbb{C}^+$ and tends to I_n as $\lambda \rightarrow \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$, while $2i\lambda[I_n - A_{r,l}(\lambda)]$ has the finite limit $-\Delta_{r,l} = \int_{-\infty}^{\infty} dy \mathcal{Q}(y) m_{r,l}(y, \lambda)$ as $\lambda \rightarrow 0$ from within $\mathbb{C}^+ \cup \mathbb{R}$. By the same token, $B_{r,l}(\lambda)$ is continuous in $0 \neq \lambda \in \mathbb{R}$, vanishes as $\lambda \rightarrow \pm\infty$, and satisfies $2i\lambda B_{r,l}(\lambda) \rightarrow -\Delta_{r,l}$ as $\lambda \rightarrow 0$ along the real λ -axis.

3.1.2 $2n \times 2n$ Jost Solutions

Putting

$$\mathbf{F}_l(x, \lambda) = \begin{pmatrix} F_l(x, -\lambda) & F_l(x, \lambda) \\ F'_l(x, -\lambda) & F'_l(x, \lambda) \end{pmatrix}, \quad \mathbf{F}_r(x, \lambda) = \begin{pmatrix} F_r(x, \lambda) & F_r(x, -\lambda) \\ F'_r(x, \lambda) & F'_r(x, -\lambda) \end{pmatrix}, \quad (3.5)$$

where the prime denotes differentiation with respect to x , we obtain

$$\mathbf{F}_r(x, \lambda) = \mathbf{F}_l(x, \lambda) \begin{pmatrix} A_r(\lambda) & B_r(-\lambda) \\ B_r(\lambda) & A_r(-\lambda) \end{pmatrix}, \quad (3.6a)$$

$$\mathbf{F}_l(x, \lambda) = \mathbf{F}_r(x, \lambda) \begin{pmatrix} A_l(-\lambda) & B_l(\lambda) \\ B_l(-\lambda) & A_l(\lambda) \end{pmatrix}, \quad (3.6b)$$

where $0 \neq \lambda \in \mathbb{R}$. Using that $\mathbf{F}_{r,l}(x, \lambda)$ satisfies the linear first-order system

$$\begin{pmatrix} V \\ V' \end{pmatrix}' = \begin{pmatrix} 0_{n \times n} & I_n \\ \mathcal{Q}(x) - \lambda^2 I_n & 0_{n \times n} \end{pmatrix} \begin{pmatrix} V \\ V' \end{pmatrix} \quad (3.7)$$

with traceless system matrix, we see that, for $0 \neq \lambda \in \mathbb{R}$, $\mathbf{F}_{r,l}(x, \lambda)$ has a determinant not depending on $x \in \mathbb{R}$. Using (3.1), we easily verify that $\det \mathbf{F}_{r,l}(x, \lambda) = (2i\lambda)^n$ for $0 \neq \lambda \in \mathbb{R}$.

Let us now apply the x -independence (A proof of this property will be given in Appendix A) of $\mathbf{W}(x, \lambda)^\dagger (\sigma_2 \otimes \sigma_3) \mathbf{V}(x, \lambda)$, where $\sigma_2 \otimes \sigma_3 = \begin{pmatrix} 0_{n \times n} & -i\sigma_3 \\ i\sigma_3 & 0_{n \times n} \end{pmatrix}$, for any two square matrix solutions \mathbf{V} and \mathbf{W} of (3.7) to derive identities for the A and B coefficients by equating the asymptotics as $x \rightarrow +\infty$ to the asymptotics as $x \rightarrow -\infty$.

Using $V = W = \Phi = F_r e_1 + F_l e_2$, where $e_1 = I_n \oplus 0_{n \times n}$ and $e_2 = 0_{n \times n} \oplus I_n$, we get

$$A_{r,l}(\lambda)^\dagger \sigma_3 A_{r,l}(\lambda) - B_{r,l}(\lambda)^\dagger \sigma_3 B_{r,l}(\lambda) = \sigma_3, \tag{3.8a}$$

$$B_{r,l}(\lambda)^\dagger = -\sigma_3 B_{l,r}(\lambda) \sigma_3, \tag{3.8b}$$

where $0 \neq \lambda \in \mathbb{R}$. Using $V = W = F_{r,l}$, we get

$$A_{r,l}(\lambda)^\dagger \sigma_3 B_{r,l}(-\lambda) = B_{r,l}(\lambda)^\dagger \sigma_3 A_{r,l}(-\lambda), \tag{3.9}$$

where $0 \neq \lambda \in \mathbb{R}$. Using the x -independence of $W(x, -\lambda^*)^\dagger (\sigma_2 \otimes \sigma_3) V(x, \lambda)$ for $V = F_l$ and $W = F_r$, we obtain

$$A_r(\lambda)^\dagger = \sigma_3 A_l(-\lambda) \sigma_3, \quad B_r(\lambda)^\dagger = -\sigma_3 B_l(\lambda) \sigma_3, \tag{3.10}$$

where $0 \neq \lambda \in \mathbb{R}$. Finally, for $V = W = \Phi$ we get

$$A_{r,l}(-\lambda^*)^\dagger = \sigma_3 A_{l,r}(\lambda) \sigma_3, \tag{3.11}$$

where $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$.

3.1.3 Reflection Coefficients

Introducing the *reflection coefficients*

$$R_{r,l}(\lambda) = B_{r,l}(\lambda) A_{r,l}(\lambda)^{-1} = -A_{l,r}(\lambda)^{-1} B_{l,r}(-\lambda) \tag{3.12}$$

and the *transmission coefficients* $A_{r,l}(\lambda)^{-1}$, we obtain the *Riemann–Hilbert problem*

$$\begin{pmatrix} F_l(x, -\lambda) & F_r(x, -\lambda) \end{pmatrix} = \begin{pmatrix} F_r(x, \lambda) & F_l(x, \lambda) \end{pmatrix} \begin{pmatrix} A_r(\lambda)^{-1} & -R_l(\lambda) \\ -R_r(\lambda) & A_l(\lambda)^{-1} \end{pmatrix}, \tag{3.13}$$

where the matrix $S(\lambda)$ containing the A and R quantities is called the *scattering matrix* and a discussion of the nonsingularity of $A_{r,l}(\lambda)$ will be presented shortly. Then it is easily verified that

$$R_{r,l}(\lambda)^\dagger = \sigma_3 R_{r,l}(-\lambda) \sigma_3, \tag{3.14a}$$

and

$$S(\lambda)^\dagger (\sigma_3 \oplus \sigma_3) S(\lambda) = \sigma_3 \oplus \sigma_3, \tag{3.14b}$$

provided $0 \neq \lambda \in \mathbb{R}$ and $\det A_{r,l}(\lambda) \neq 0$.

Above we have defined $\Delta_{r,l}$ as follows:

$$\Delta_{r,l} = \lim_{\lambda \rightarrow 0} 2i\lambda A_{r,l}(\lambda) = - \lim_{\lambda \rightarrow 0^\pm} 2i\lambda B_{r,l}(\lambda),$$

where the first limit may be taken from the closed upper half-plane. Then the matrices $\mathbf{\Delta}_{r,l}$ have the same determinant. If $\mathbf{\Delta}_{r,l}$ is nonsingular, we are said to be in the *generic case*; if instead $\mathbf{\Delta}_{r,l}$ is singular, we are said to be in the *exceptional case* (Aktosun et al. 2001). We are said to be in the *superexceptional case* if $\mathbf{\Delta}_{r,l} = 0_{n \times n}$ and $A_{r,l}(\lambda)$ tends to a nonsingular matrix, $A_{r,l}(0)$ say, as $\lambda \rightarrow 0$ from within $\mathbb{C}^+ \cup \mathbb{R}$.

Throughout this article, we assume the absence of *spectral singularities*, i.e., the absence of nonzero real λ for which $\det A_{r,l}(\lambda) = 0$. Under this condition, the reflection coefficients $R_{r,l}(\lambda)$ are continuous in $0 \neq \lambda \in \mathbb{R}$.

3.1.4 Triangular Representations

The Jost solutions allow the triangular representations

$$F_l(x, \lambda) = e^{i\lambda x} I_n + \int_x^\infty dy e^{i\lambda y} K(x, y), \quad (3.15a)$$

$$F_r(x, \lambda) = e^{-i\lambda x} I_n + \int_{-\infty}^x dy e^{-i\lambda y} J(x, y), \quad (3.15b)$$

where for every $x \in \mathbb{R}$

$$\int_x^\infty dy \|K(x, y)\| + \int_{-\infty}^x dy \|J(x, y)\| < +\infty. \quad (3.16)$$

The integral equations satisfied by the auxiliary matrix functions $K(x, y)$ and $J(x, y)$ derived in Demontis and van der Mee (2021) imply that

$$K(x, x) = \frac{1}{2} \int_x^\infty dy \mathbf{Q}(y), \quad J(x, x) = \frac{1}{2} \int_{-\infty}^x dy \mathbf{Q}(y). \quad (3.17)$$

3.1.5 Wiener Algebras

For convenience, we introduce the well-known Wiener algebra (Gelfand et al. 1964). By the (continuous) Wiener algebra \mathcal{W} , we mean the complex vector space of constants plus Fourier transforms of L^1 -functions

$$\mathcal{W} = \{c + \hat{h} : c \in \mathbb{C}, h \in L^1(\mathbb{R})\}$$

endowed with the norm $|c| + \|h\|_1$. Here we define the Fourier transform as follows: $\hat{h}(k) = \int_{-\infty}^\infty dy e^{iky} h(y)$. The invertible elements of the commutative Banach algebra \mathcal{W} with unit element are exactly those $c + \hat{h} \in \mathcal{W}$ for which $c \neq 0$ and $c + \hat{h}(k) \neq 0$ for each $k \in \mathbb{R}$ (Gelfand et al. 1964).

The algebra \mathcal{W} has the two closed subalgebras \mathcal{W}^+ and \mathcal{W}^- consisting of those $c + \hat{h} \in \mathcal{W}$ for which h is supported on \mathbb{R}^+ and \mathbb{R}^- , respectively. The invertible elements of \mathcal{W}^\pm are exactly those $c + \hat{h} \in \mathcal{W}^\pm$ for which $c \neq 0$ and $c + \hat{h}(k) \neq 0$ for each $k \in \mathbb{C}^\pm \cup \mathbb{R}$ (Gelfand et al. 1964). Letting \mathcal{W}_0^\pm and \mathcal{W}_0 stand for the (nonunital)

closed subalgebras of \mathcal{W}^\pm and \mathcal{W} consisting of those $c + \hat{h}$ for which $c = 0$, we obtain the direct sum decompositions

$$\mathcal{W} = \mathbb{C} \oplus \mathcal{W}_0^+ \oplus \mathcal{W}_0^-, \quad \mathcal{W}_0 = \mathcal{W}_0^+ \oplus \mathcal{W}_0^-.$$

We denote the (bounded) projections of \mathcal{W} onto \mathcal{W}_0^\pm along \mathcal{W}^\mp by Π_\pm .

Throughout this article, we denote the vector spaces of $n \times m$ matrices with entries in \mathcal{W} , \mathcal{W}^\pm , and \mathcal{W}_0^\pm by $\mathcal{W}^{n \times m}$, $\mathcal{W}^{\pm n \times m}$, and $\mathcal{W}_0^{\pm n \times m}$, respectively. We write $L^1(\mathbb{R})^{n \times n}$ and $L^1(\mathbb{R}^\pm)^{n \times m}$ for the vector spaces of $n \times m$ matrices with entries in $L^1(\mathbb{R})$ and $L^1(\mathbb{R}^\pm)$, respectively. Using a submultiplicative matrix norm, we can turn all of these vector spaces into Banach spaces. It is then clear that $\mathcal{W}^{n \times n}$ and $\mathcal{W}^{\pm n \times n}$ are noncommutative Banach algebras with unit element and $\mathcal{W}_0^{\pm n \times n}$ are (nonunital) noncommutative Banach algebras. As above, we then define Π^\pm as the (bounded) projections of $\mathcal{W}^{n \times m}$ onto $\mathcal{W}_0^{\pm n \times m}$ along $\mathcal{W}^{\mp n \times m}$. The invertible elements of $\mathcal{W}^{n \times n}$ and $\mathcal{W}^{\pm n \times n}$ are exactly those elements whose determinants are invertible elements of \mathcal{W} and \mathcal{W}^\pm , respectively. Hence, according to (3.15) and (3.16), for each $x \in \mathbb{R}$ the Faddeev functions $m_{r,l}(x, \cdot) \in \mathcal{W}_+^{n \times n}$. We then easily prove with the help of (3.4) that $2i\lambda[I_n - A_{r,l}(\lambda)]$ belong to $\mathcal{W}_+^{n \times n}$ and $2i\lambda B_{r,l}(\lambda)$ belong to $\mathcal{W}^{n \times n}$.

Assuming the absence of spectral singularities and to be in the generic case, we proved in Demontis and van der Mee (2021) that the reflection coefficients $R_{r,l}(\lambda)$ belong to $\mathcal{W}_0^{n \times n}$ and the transmission coefficients $A_{r,l}(\lambda)^{-1}$ to $\mathcal{W}_+^{n \times n}$. In the superexceptional case, where $\Delta_{r,l} = 0_{n \times n}$, we proved in Demontis and van der Mee (2021) that $A_{r,l} \in \mathcal{W}_+^{n \times n}$, provided $\mathcal{Q} \in L^1(\mathbb{R}; (1 + |x|)^2 dx)$; assuming the absence of spectral singularities and using the nonsingularity of $A_{r,l}(0)$, we see that the reflection coefficients $R_{r,l}(\lambda)$ and the transmission coefficients $A_{r,l}(\lambda)^{-1}$ belong to $\mathcal{W}^{n \times n}$.

At present it is not known if, under the absence of spectral singularities, the reflection and transmission coefficients belong to $\mathcal{W}^{n \times n}$ in any other exceptional case and for general $\mathcal{Q} \in L^1(\mathbb{R}; (1 + |x|) dx)$. Under the condition $\mathcal{Q} \in L^1(\mathbb{R}; (1 + |x|) dx)$, the continuity of the reflection and transmission coefficients at $\lambda = 0$ is known for $n = 1$ (Klaus 1988) and for selfadjoint potentials (Aktosun et al. 2001). In neither case is it known if these continuous functions belong to \mathcal{W} .

4 Inverse Scattering Theory

In this section, we introduce the Marchenko integral equations for the matrix Schrödinger equation (1.4) with Faddeev class potential \mathcal{Q} satisfying (1.6). We make use of the hypothesis that the reflection coefficients $R_{r,l} \in \mathcal{W}_0^{n \times n}$, something proved in the generic case but not in the most general exceptional case. For the sake of simplicity, we assume that the poles of $A_{r,l}(\lambda)^{-1}$ in \mathbb{C}^+ are simple. The extension to multiple pole situations is rather technical but straightforward (Demontis and van der Mee 2008a). Inverse scattering theory is well documented in the scalar case (Faddeev 1964; Deift and Trubowitz 1979; Calogero and Degasperis 1982; Chadan and Sabatier 1989), in the matrix half-line case (Aktosun and Weder 2018, 2020), and in the matrix full-line

case (Wadati and Kamijo 1974; Aktosun et al. 2001). The adjoint symmetry (1.6) requires some modifications to existing theory (cf. Demontis and van der Mee 2021).

Let us write the *transmission coefficients* in the form

$$A_r(\lambda)^{-1} = A_{r0}(\lambda) + \sum_{s=1}^N \frac{\tau_{r;s}}{\lambda - \lambda_s}, \quad A_l(\lambda)^{-1} = A_{l0}(\lambda) + \sum_{s=1}^N \frac{\tau_{l;s}}{\lambda - \lambda_s}, \quad (4.1)$$

where $\lambda_1, \dots, \lambda_N$ are the distinct simple poles of $A_{r,l}(\lambda)^{-1}$ in \mathbb{C}^+ , $\tau_{r;s}$ and $\tau_{l;s}$ are the residues of $A_r(\lambda)^{-1}$ and $A_l(\lambda)^{-1}$ at $\lambda = \lambda_s$ ($s = 1, \dots, N$), and $A_{r0}(\lambda)$ and $A_{l0}(\lambda)$ are continuous in $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$, are analytic in $\lambda \in \mathbb{C}^+$, and tend to I_n as $\lambda \rightarrow \infty$ from within $\mathbb{C}^+ \cup \mathbb{R}$. Then it is easily proved that $\tau_{r;\bar{s}} = -\sigma_3 \tau_{l;s}^\dagger \sigma_3$ and $\tau_{l;\bar{s}} = -\sigma_3 \tau_{r;s}^\dagger \sigma_3$ whenever $\lambda_{\bar{s}} = -\lambda_s^*$ (cf. Demontis and van der Mee 2021).

Let us write

$$R_r(\lambda) = \int_{-\infty}^{\infty} d\alpha e^{-i\lambda\alpha} \hat{R}_r(\alpha), \quad R_l(\lambda) = \int_{-\infty}^{\infty} d\alpha e^{i\lambda\alpha} \hat{R}_l(\alpha), \quad (4.2)$$

where $\hat{R}_{r,l} \in L^1(\mathbb{R})^{n \times n}$. In fact, this has only been proved in the generic case and, under the condition that $\mathcal{Q} \in L^1(\mathbb{R}; (1 + |x|)^2 dx)$, in the superexceptional case. Using (3.14a), it follows that $\hat{R}_{r,l}(\alpha; t)$ are σ_3 -Hermitian matrices. Then the following Marchenko integral equations can be derived [see Demontis and van der Mee (2021) for details]:

$$K(x, y) + \Omega_r(x + y) + \int_x^\infty dz K(x, z)\Omega_r(z + y) = 0_{n \times n}, \quad (4.3a)$$

$$J(x, y) + \Omega_l(x + y) + \int_{-\infty}^x dz J(x, z)\Omega_l(z + y) = 0_{n \times n}, \quad (4.3b)$$

where the *Marchenko integral kernels* are given by

$$\Omega_r(w) = \hat{R}_r(w) + \sum_{s=1}^N e^{i\lambda_s w} N_{r;s}, \quad (4.4a)$$

$$\Omega_l(w) = \hat{R}_l(w) + \sum_{s=1}^N e^{-i\lambda_s w} N_{l;s}. \quad (4.4b)$$

Here $N_{r;s}$ and $N_{l;s}$ are the so-called norming constants defined by

$$F_r(x, \lambda_s)\tau_{r;s} = i F_l(x, \lambda_s)N_{r;s}, \quad (4.5a)$$

$$F_l(x, \lambda_s)\tau_{l;s} = i F_r(x, \lambda_s)N_{l;s}, \quad (4.5b)$$

where λ_s is a (simple) pole of $A_{r,l}(\lambda)^{-1}$ in \mathbb{C}^+ ($s = 1, 2, \dots, N$). Then $\tau_{r;s}$ and $N_{r;s}$ have the same rank and the same null space; the same thing is true for $\tau_{l;s}$ and $N_{l;s}$. As

in Demontis and van der Mee (2008a), we can prove the adjoint symmetry relations

$$\Omega_{r,l}(w) = \sigma_3 \Omega_{r,l}(w)^\dagger \sigma_3, \tag{4.6}$$

thus implying the following symmetry relations for the norming constants:

$$N_{r;s} = \sigma_3 N_{r;s}^\dagger \sigma_3, \quad N_{l;s} = \sigma_3 N_{l;s}^\dagger \sigma_3, \tag{4.7}$$

provided $\lambda_s = -\lambda_s^*$ is a simple pole of $A_{r,l}(\lambda)^{-1}$. For the rather tedious details, we refer to Appendix B of Demontis and van der Mee (2021).

Example. Let us now solve the Marchenko integral equations (4.3) in the one-soliton case, where $\Omega_r(w; t) = e^{-a_0 w} N_{r;0}(t)$ and $\Omega_l(w; t) = e^{a_0 w} N_{l;0}(t)$ for a suitable eigenvalue $\lambda_0 = ia_0 \in \mathbb{C}^+$. Then separation of variables yields

$$K(x, y; t) = -e^{-a_0(x+y)} \left[I_n + \frac{1}{2a_0} e^{-2a_0 x} N_{r;0}(t) \right]^{-1} N_{r;0}(t), \tag{4.8a}$$

$$J(x, y; t) = -e^{a_0(x+y)} \left[I_n + \frac{1}{2a_0} e^{2a_0 x} N_{l;0}(t) \right]^{-1} N_{l;0}(t). \tag{4.8b}$$

so that

$$\int_x^\infty dy \mathcal{Q}(y; t) = -2 \left[e^{2a_0 x} I_n + \frac{1}{2a_0} N_{r;0}(t) \right]^{-1} N_{r;0}(t), \tag{4.9a}$$

$$\int_{-\infty}^x dy \mathcal{Q}(y; t) = -2 \left[e^{-2a_0 x} I_n + \frac{1}{2a_0} N_{l;0}(t) \right]^{-1} N_{l;0}(t), \tag{4.9b}$$

where the σ_3 -Hermitian norming constants $N_{r;0}(t)$ and $N_{l;0}(t)$ will be expressed in their initial values shortly. The off-diagonal parts of these expressions yield explicit expressions for $\mathcal{Q}_r - \mathcal{Q}(x; t)$ and $\mathcal{Q}(x; t) - \mathcal{Q}_l$, respectively.

5 Time Evolution of the Scattering Data

In this section, we establish the time evolution of the scattering data of the matrix Schrödinger equation. We then go on to derive the Marchenko integral kernels as a function of time. These results allow us, in Sect. 6, to derive the reflectionless solutions of the integrable nonlocal equation (2.3) and hence of the focusing matrix NLS equation.

Recall that the integrable equation (2.3) arises as the zero curvature condition of the AKNS pair $\{X, T\}$ given by (2.7). Thus, there exist nonsingular matrices $C_{F_r}(\lambda; t)$ and $C_{F_l}(\lambda; t)$ not depending on $x \in \mathbb{R}$ such that

$$F_r(x, \lambda; t) = V(x, \lambda; t) C_{F_r}(\lambda; t)^{-1}, \quad F_l(x, \lambda; t) = V(x, \lambda; t) C_{F_l}(\lambda; t)^{-1}.$$

Then a simple differentiation yields

$$[C_{F_r}(\lambda; t)]_t C_{F_r}(\lambda; t)^{-1} = F_r^{-1} T F_r - F_r^{-1} [F_r]_t, \tag{5.1a}$$

$$[C_{F_l}(\lambda; t)]_t C_{F_l}(\lambda; t)^{-1} = F_l^{-1} T F_l - F_l^{-1} [F_l]_t, \tag{5.1b}$$

where the two left-hand sides do not depend on $x \in \mathbb{R}$. Using (2.4), we now compute the $x \rightarrow \pm\infty$ limits of the two right-hand sides by evaluating the matrix product

$$\frac{1}{2i\lambda} \begin{pmatrix} i\lambda e^{i\lambda x} I_n & -e^{i\lambda x} I_n \\ i\lambda e^{-i\lambda x} I_n & e^{-i\lambda x} I_n \end{pmatrix} \begin{pmatrix} -2i\lambda^2 \sigma_3 & -2i\sigma_3 Q_{r,l} \\ 2i\lambda^2 \sigma_3 Q_{r,l} & -2i\lambda^2 \sigma_3 \end{pmatrix} \begin{pmatrix} e^{-i\lambda x} I_n & e^{i\lambda x} I_n \\ -i\lambda e^{-i\lambda x} I_n & i\lambda e^{i\lambda x} I_n \end{pmatrix}$$

and obtain

$$[C_{F_r}(\lambda; t)]_t C_{F_r}(\lambda; t)^{-1} = \begin{pmatrix} -\Lambda_r^{\text{up}}(\lambda) & 0_{n \times n} \\ 0_{n \times n} & -\Lambda_r^{\text{dn}}(\lambda) \end{pmatrix}, \tag{5.2a}$$

$$[C_{F_l}(\lambda; t)]_t C_{F_l}(\lambda; t)^{-1} = \begin{pmatrix} -\Lambda_l^{\text{up}}(\lambda) & 0_{n \times n} \\ 0_{n \times n} & -\Lambda_l^{\text{dn}}(\lambda) \end{pmatrix}, \tag{5.2b}$$

where

$$\Lambda_{r,l}^{\text{up}}(\lambda) = 2i\lambda^2 \sigma_3 + 2\lambda \sigma_3 Q_{r,l}, \tag{5.3a}$$

$$\Lambda_{r,l}^{\text{dn}}(\lambda) = 2i\lambda^2 \sigma_3 - 2\lambda \sigma_3 Q_{r,l}, \tag{5.3b}$$

are time invariant. Then, using that $Q^\dagger = -Q$ and $Q\sigma_3 = -\sigma_3 Q$, we arrive at the symmetry relations

$$\Lambda_{r,l}^{\text{up}}(\lambda) = \sigma_3 \Lambda_{r,l}^{\text{dn}}(\lambda) \sigma_3, \tag{5.4a}$$

$$\Lambda_{r,l}^{\text{up}}(-\lambda^*)^\dagger = -\Lambda_{r,l}^{\text{up}}(\lambda), \tag{5.4b}$$

$$\Lambda_{r,l}^{\text{dn}}(-\lambda^*)^\dagger = -\Lambda_{r,l}^{\text{dn}}(\lambda), \tag{5.4c}$$

where $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$. Relating $F_{r,l}(x, \lambda; t)$ by means of the equalities [cf. (3.6)]

$$F_r(x, \lambda; t) = F_l(x, \lambda; t) A_r(\lambda; t), \quad F_l(x, \lambda; t) = F_r(x, \lambda; t) A_l(\lambda; t),$$

where the factors $A_{r,l}(\lambda; t)$ are given by the matrices

$$A_r(\lambda; t) = \begin{pmatrix} A_r(\lambda; t) & B_r(-\lambda; t) \\ B_r(\lambda; t) & A_r(-\lambda; t) \end{pmatrix}, \quad A_l(\lambda; t) = \begin{pmatrix} A_l(-\lambda; t) & B_l(\lambda; t) \\ B_l(-\lambda; t) & A_l(\lambda; t) \end{pmatrix}, \tag{5.5}$$

for $0 \neq \lambda \in \mathbb{R}$ we compute

$$\begin{aligned} [A_r]_t &= -F_l^{-1} [F_l]_t F_l^{-1} F_r + F_l^{-1} [F_r]_t \\ &= -F_l^{-1} (T F_l - F_l [C_{F_l}(\lambda; t)]_t C_{F_l}(\lambda; t)^{-1}) A_r \end{aligned}$$

$$\begin{aligned}
 &+ F_l^{-1} \left(T F_r - F_r [C_{F_r}(\lambda; t)]_t C_{F_r}(\lambda; t)^{-1} \right) \\
 &= [C_{F_l}(\lambda; t)]_t C_{F_l}(\lambda; t)^{-1} A_r - A_r [C_{F_r}(\lambda; t)]_t C_{F_r}(\lambda; t)^{-1} \\
 &= A_r \begin{pmatrix} \Lambda_r^{\text{up}}(\lambda) & 0_{n \times n} \\ 0_{n \times n} & \Lambda_r^{\text{dn}}(\lambda) \end{pmatrix} - \begin{pmatrix} \Lambda_l^{\text{up}}(\lambda) & 0_{n \times n} \\ 0_{n \times n} & \Lambda_l^{\text{dn}}(\lambda) \end{pmatrix} A_r. \tag{5.6}
 \end{aligned}$$

Using that $A_l(\lambda; t) = A_r(\lambda; t)^{-1}$, we obtain from (5.6)

$$[A_l]_t = A_l \begin{pmatrix} \Lambda_l^{\text{up}}(\lambda) & 0_{n \times n} \\ 0_{n \times n} & \Lambda_l^{\text{dn}}(\lambda) \end{pmatrix} - \begin{pmatrix} \Lambda_r^{\text{up}}(\lambda) & 0_{n \times n} \\ 0_{n \times n} & \Lambda_r^{\text{dn}}(\lambda) \end{pmatrix} A_l. \tag{5.7}$$

Therefore, (5.5), (5.6), and (5.7) imply

$$[A_r]_t = A_r(\lambda; t) \Lambda_r^{\text{up}}(\lambda) - \Lambda_l^{\text{up}}(\lambda) A_r(\lambda; t), \tag{5.8a}$$

$$[A_l]_t = A_l(\lambda; t) \Lambda_l^{\text{dn}}(\lambda) - \Lambda_r^{\text{dn}}(\lambda) A_l(\lambda; t), \tag{5.8b}$$

where $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$, and

$$[B_r]_t = B_r(\lambda; t) \Lambda_r^{\text{up}}(\lambda) - \Lambda_l^{\text{dn}}(\lambda) B_r(\lambda; t), \tag{5.8c}$$

$$[B_l]_t = B_l(\lambda; t) \Lambda_l^{\text{dn}}(\lambda) - \Lambda_r^{\text{up}}(\lambda) B_l(\lambda; t), \tag{5.8d}$$

where $0 \neq \lambda \in \mathbb{R}$.

Proposition 5.1 *The reflection coefficients satisfy the following differential equations:*

$$[R_r]_t = R_r(\lambda; t) \Lambda_l^{\text{up}}(\lambda) - \Lambda_l^{\text{dn}}(\lambda) R_r(\lambda; t), \tag{5.9a}$$

$$[R_l]_t = R_l(\lambda; t) \Lambda_r^{\text{dn}}(\lambda) - \Lambda_r^{\text{up}}(\lambda) R_l(\lambda; t), \tag{5.9b}$$

where $0 \neq \lambda \in \mathbb{R}$. Moreover, for fixed λ the matrices $\sigma_3 R_{r,l}(\lambda; t)$ have time invariant traces.

Proof Using (3.12), we compute

$$\begin{aligned}
 [R_r]_t &= [B_r A_r^{-1}]_t = [B_r]_t A_r^{-1} - B_r A_r^{-1} [A_r]_t A_r^{-1} \\
 &= \left(B_r \Lambda_r^{\text{up}} - \Lambda_l^{\text{dn}} B_r \right) A_r^{-1} - B_r A_r^{-1} \left(A_r \Lambda_r^{\text{up}} - \Lambda_l^{\text{up}} A_r \right) A_r^{-1} \\
 &= B_r A_r^{-1} \Lambda_l^{\text{up}} - \Lambda_l^{\text{dn}} B_r A_r^{-1} = R_r \Lambda_l^{\text{up}} - \Lambda_l^{\text{dn}} R_r,
 \end{aligned}$$

where we have not written the dependence on $(\lambda; t)$. Similarly, we compute

$$\begin{aligned}
 [R_l]_t &= [B_l A_l^{-1}]_t = [B_l]_t A_l^{-1} - B_l A_l^{-1} [A_l]_t A_l^{-1} \\
 &= \left(B_l \Lambda_l^{\text{dn}} - \Lambda_r^{\text{up}} B_l \right) A_l^{-1} - B_l A_l^{-1} \left(A_l \Lambda_l^{\text{dn}} - \Lambda_r^{\text{dn}} A_l \right) A_l^{-1} \\
 &= B_l A_l^{-1} \Lambda_r^{\text{dn}} - \Lambda_r^{\text{up}} B_l A_l^{-1} = R_l \Lambda_r^{\text{dn}} - \Lambda_r^{\text{up}} R_l.
 \end{aligned}$$

Finally, since $\sigma_3 \Lambda_{r,l}^{\text{up}}(\lambda) \sigma_3 = \Lambda_{r,l}^{\text{dn}}(\lambda)$, we see that

$$[\sigma_3 R_r]_t = [\sigma_3 R_r(\lambda; t), \Lambda_l^{\text{up}}(\lambda)], \quad [\sigma_3 R_l]_t = [\sigma_3 R_l(\lambda; t), \Lambda_r^{\text{dn}}(\lambda)], \tag{5.10}$$

where the square brackets in the right-hand sides are matrix commutators. Consequently, $[\sigma_3 R_{r,l}]_t$ are traceless matrices.

Let us now derive the time evolution equations for the norming constants. First, writing (5.8) in the form

$$\begin{aligned} [A_r^{-1}]_t &= A_r(\lambda; t)^{-1} \Lambda_l^{\text{up}}(\lambda) - \Lambda_r^{\text{up}}(\lambda) A_r(\lambda; t)^{-1}, \\ [A_l^{-1}]_t &= A_l(\lambda; t)^{-1} \Lambda_r^{\text{dn}}(\lambda) - \Lambda_l^{\text{dn}}(\lambda) A_l(\lambda; t)^{-1}, \end{aligned}$$

and computing the residues at the simple poles λ_s , we get

$$[\tau_{r;s}]_t = \tau_{r;s}(t) \Lambda_l^{\text{up}}(\lambda_s) - \Lambda_r^{\text{up}}(\lambda_s) \tau_{r;s}(t), \tag{5.11a}$$

$$[\tau_{l;s}]_t = \tau_{l;s}(t) \Lambda_r^{\text{dn}}(\lambda_s) - \Lambda_l^{\text{dn}}(\lambda_s) \tau_{l;s}(t). \tag{5.11b}$$

Next, using (5.2) we write (5.1) in the form

$$[\mathbf{F}_{r,l}]_t = \mathbf{T}(x, \lambda; t) \mathbf{F}_{r,l}(x, \lambda; t) + \mathbf{F}_{r,l}(x, \lambda; t) \begin{pmatrix} \Lambda_{r,l}^{\text{up}}(\lambda) & 0_{n \times n} \\ 0_{n \times n} & \Lambda_{r,l}^{\text{dn}}(\lambda) \end{pmatrix}. \tag{5.12}$$

Using the standard block structure $\mathbf{T} = \begin{pmatrix} T_1 & T_2 \\ T_3 & T_4 \end{pmatrix}$ as a 2×2 matrix having $m \times m$ entries, from (5.12) we easily arrive at the identities

$$\begin{aligned} [F_l(x, \lambda_s; t)]_t &= \mathbf{T}_1(x, \lambda_s; t) F_l(x, \lambda_s; t) + \mathbf{T}_2(x, \lambda_s; t) F_l'(x, \lambda_s; t) \\ &\quad + F_l(x, \lambda_s; t) \Lambda_l^{\text{dn}}(\lambda_s), \end{aligned} \tag{5.13a}$$

$$\begin{aligned} [F_r(x, \lambda_s; t)]_t &= \mathbf{T}_1(x, \lambda_s; t) F_r(x, \lambda_s; t) + \mathbf{T}_2(x, \lambda_s; t) F_r'(x, \lambda_s; t) \\ &\quad + F_r(x, \lambda_s; t) \Lambda_r^{\text{up}}(\lambda_s). \end{aligned} \tag{5.13b}$$

Differentiating (4.5a) with respect to t , utilizing both of (5.13), and applying (4.5a) as well as its derivative with respect to x , we obtain

$$F_r(\Lambda_r^{\text{up}} \tau_{r;s} + [\tau_{r;s}]_t) = i F_l(\Lambda_l^{\text{dn}} N_{r;s} + [N_{r;s}]_t),$$

where we have omitted the arguments $(x, \lambda_s; t)$, λ_s , and t . With the help of (5.11a), we write the latter in the form

$$F_r \tau_{r;s} \Lambda_l^{\text{up}} = i F_l(\Lambda_l^{\text{dn}} N_{r;s} + [N_{r;s}]_t).$$

Using (4.5a) once again and considering the $x \rightarrow +\infty$ asymptotics of the resulting expression to lose the resulting common factors iF_l , we obtain

$$[N_{r;s}]_t = N_{r;s}(t)\Lambda_l^{\text{up}}(\lambda_s) - \Lambda_l^{\text{dn}}(\lambda_s)N_{r;s}(t). \tag{5.14}$$

Analogously, differentiating (4.5b) with respect to t , utilizing both of (5.13), and applying (4.5b) as well as its derivative with respect to x , we obtain

$$F_l \left(\Lambda_l^{\text{dn}} \tau_{l;s} + [\tau_{l;s}]_t \right) = iF_r \left(\Lambda_r^{\text{up}} N_{l;s} + [N_{l;s}]_t \right),$$

where we have omitted the arguments $(x, \lambda_s; t)$, λ_s , and t . With the help of (5.11b), we write the latter in the form

$$F_r \tau_{l;s} \Lambda_r^{\text{dn}} = iF_l \left(\Lambda_r^{\text{up}} N_{l;s} + [N_{l;s}]_t \right).$$

Using (4.5b) once again and considering the $x \rightarrow -\infty$ asymptotics of the resulting expression to lose the resulting common factors iF_r , we obtain

$$[N_{l;s}]_t = N_{l;s}(t)\Lambda_r^{\text{dn}}(\lambda_s) - \Lambda_r^{\text{up}}(\lambda_s)N_{l;s}(t). \tag{5.15}$$

As in the proof of Proposition 5.1, we can prove that for each λ the matrices $\sigma_3 N_{r;s}(t)$ are similar and the matrices $\sigma_3 N_{l;s}(t)$ are similar. Hence, the traces of $\sigma_3 N_{r;s}(t)$ and $\sigma_3 N_{l;s}(t)$ are time independent. Thus, the ranks of $N_{r;s}(t)$ and $N_{l;s}(t)$ are time independent. We recall [see (4.7)] that the norming constants corresponding to eigenvalues symmetrically located with respect to the imaginary axis are each other's σ_3 -adjoints.

Let us now derive the differential equations for the Marchenko integral kernels. Using (4.2) and (5.3), we obtain the PDEs

$$[\hat{R}_r]_t = -2i \left([\hat{R}_r]_{\alpha\alpha} \sigma_3 - \sigma_3 [\hat{R}_r]_{\alpha\alpha} + [\hat{R}_r]_{\alpha} \sigma_3 \mathcal{Q}_l - \mathcal{Q}_l \sigma_3 [\hat{R}_r]_{\alpha} \right), \tag{5.16a}$$

$$[\hat{R}_l]_t = -2i \left([\hat{R}_l]_{\alpha\alpha} \sigma_3 - \sigma_3 [\hat{R}_l]_{\alpha\alpha} + [\hat{R}_l]_{\alpha} \sigma_3 \mathcal{Q}_r - \mathcal{Q}_r \sigma_3 [\hat{R}_l]_{\alpha} \right), \tag{5.16b}$$

provided $\int_{-\infty}^{\infty} d\alpha (1 + \alpha^2) \|\hat{R}_{r,l}(\alpha; t)\|$ converges for every $t \in \mathbb{R}$. Here we recall that $\hat{R}_{r,l}(\alpha; t)$ are σ_3 -Hermitian for all $(\alpha, t) \in \mathbb{R}^2$. Using (4.2) and Proposition 5.1, we see that the traces of $\sigma_3 \hat{R}_{r,l}(\alpha; t)$ are time independent. Using (5.16) and (4.4) to derive PDEs for the Marchenko integral kernels $\Omega_{r,l}(w; t)$, we obtain with the help of (5.14) and (5.15)

$$[\Omega_r]_t = -2i \left([\Omega_r]_{ww} \sigma_3 - \sigma_3 [\Omega_r]_{ww} + [\Omega_r]_w \sigma_3 \mathcal{Q}_l - \mathcal{Q}_l \sigma_3 [\Omega_r]_w \right), \tag{5.17a}$$

$$[\Omega_l]_t = -2i \left([\Omega_l]_{ww} \sigma_3 - \sigma_3 [\Omega_l]_{ww} + [\Omega_l]_w \sigma_3 \mathcal{Q}_r - \mathcal{Q}_r \sigma_3 [\Omega_l]_w \right), \tag{5.17b}$$

where $\Omega_{r,l}(w; t)$ are σ_3 -Hermitian for all $(w, t) \in \mathbb{R}^2$. Hence, the reflection kernels $\hat{R}_{r,l}(w; t)$ and the Marchenko integral kernels $\Omega_{r,l}(w; t)$ satisfy the same PDEs. Finally, the traces of $\sigma_3 \Omega_{r,l}(w; t)$ are time independent.

Recalling that $Q_{r,l}$ are time independent, we observe that the matrices $\Lambda_{r,l}^{up}(\lambda)$ and $\Lambda_{r,l}^{dn}(\lambda)$ are time independent as well. We easily compute

$$e^{t\Lambda_{r,l}^{up}(\lambda)} = \cos(2\lambda kt)I_n + \frac{\sin(2\lambda kt)}{2\lambda k} \left[2i\lambda^2\sigma_3 + 2\lambda Q_{r,l} \right], \tag{5.18a}$$

$$e^{t\Lambda_{r,l}^{dn}(\lambda)} = \cos(2\lambda kt)I_n + \frac{\sin(2\lambda kt)}{2\lambda k} \left[2i\lambda^2\sigma_3 - 2\lambda Q_{r,l} \right], \tag{5.18b}$$

where $k^2 = \lambda^2 - \mu^2$ and the expressions (5.18) are even functions of k for fixed λ [cf. Demontis et al. (2014) where these matrix groups also appear]. Using that the initial value problem for the matrix differential equation

$$F_t = B_1 F(t) - F(t) B_2$$

has the unique solution

$$F(t) = e^{tB_1} F(0) e^{-tB_2},$$

we obtain for the solutions of (5.9a) and (5.9b)

$$R_r(\lambda; t) = e^{-t\Lambda_l^{dn}(\lambda)} R_r(\lambda; 0) e^{t\Lambda_l^{up}(\lambda)}, \tag{5.19a}$$

$$R_l(\lambda; t) = e^{-t\Lambda_r^{up}(\lambda)} R_l(\lambda; 0) e^{t\Lambda_r^{dn}(\lambda)}. \tag{5.19b}$$

Because of (5.4a), the matrices $\sigma_3 R_r(\lambda; t)$ are similar and so are the matrices $\sigma_3 R_l(\lambda; t)$. In the same way, we get for the time evolution of the norming constants

$$N_{r;s}(t) = e^{-t\Lambda_l^{dn}(\lambda_s)} N_{r;s}(0) e^{t\Lambda_l^{up}(\lambda_s)}, \tag{5.20a}$$

$$N_{l;s}(t) = e^{-t\Lambda_r^{up}(\lambda_s)} N_{l;s}(0) e^{t\Lambda_r^{dn}(\lambda_s)}, \tag{5.20b}$$

where $k_s^2 = \lambda_s^2 - \mu^2$ and the expressions (5.20) are even functions of k_s for fixed λ_s . Because of (5.4a), the matrices $\sigma_3 N_{r;s}(t)$ are similar and so are the matrices $\sigma_3 N_{l;s}(t)$. In the same way, we derive from (5.8) the identities

$$A_r(\lambda; t) = e^{-t\Lambda_l^{up}(\lambda)} A_r(\lambda; 0) e^{t\Lambda_r^{up}(\lambda)}, \tag{5.21a}$$

$$A_l(\lambda; t) = e^{-t\Lambda_r^{dn}(\lambda)} A_l(\lambda; 0) e^{t\Lambda_l^{dn}(\lambda)}, \tag{5.21b}$$

where $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$, and

$$B_r(\lambda; t) = e^{-t\Lambda_l^{dn}(\lambda)} B_r(\lambda; 0) e^{t\Lambda_r^{up}(\lambda)}, \tag{5.21c}$$

$$B_l(\lambda; t) = e^{-t\Lambda_r^{up}(\lambda)} B_l(\lambda; 0) e^{t\Lambda_l^{dn}(\lambda)}, \tag{5.21d}$$

where $0 \neq \lambda \in \mathbb{R}$. Observe that (5.21) and (3.12) imply (5.19).

6 Multisoliton Solutions

In this section, we apply the matrix triplet method to write the reflectionless Marchenko integral kernels in separated form and solve the Marchenko equations by separation of variables. This method has been successfully applied to the Korteweg-de Vries (KdV) equation (Aden and Carl 1996; Aktosun and van der Mee 2006), the NLS equation (Aktosun et al. 2007; Demontis and van der Mee 2008b), the sine-Gordon equation (Schiebold 2002; Aktosun et al. 2010), the modified Korteweg-de Vries (mKdV) equation (Demontis 2011), the Toda lattice equation (Schiebold 1998), and the Heisenberg Ferromagnet equation (Demontis et al. 2018, 2019). An introduction to this method can be found in van der Mee (2013). In contrast to earlier work, we allow the time factors in these triplets to be absorbed by both the input and output matrices.

Before solving the Marchenko integral equations (4.4), we write the reflectionless Marchenko integral kernels in the form

$$\Omega_r(w; t) = \sum_{s=1}^N e^{-a_s w} N_{r;s}(t), \quad \Omega_l(w; t) = \sum_{s=1}^N e^{a_s w} N_{l;s}(t), \quad (6.1)$$

where $a_s = -i\lambda_s$ ($s = 1, \dots, N$). Then it is easily proved that, for $s = 1, \dots, N$, the norming constants $N_{r;s}(t)$ and $N_{l;s}(t)$ both have the same time-independent rank r_s . In fact, r_s coincides with the ranks of the residues $\tau_{r;s}$ and $\tau_{l;s}$ of $A_{r,l}(\lambda; t)^{-1}$ at $\lambda = \lambda_s$. Since $\sigma_3 N_{r;s}(t)$ and $\sigma_3 N_{l;s}(t)$ have $\sigma_3 N_{r;\bar{s}}(t)$ and $\sigma_3 N_{l;\bar{s}}(t)$ as their respective complex conjugate transposes whenever $\lambda_s = -\lambda_{\bar{s}}^*$, there exist $n \times r_s$ matrices $e_{r;s}(t)$ and $e_{l;s}(t)$ having $r_s = r_{\bar{s}}$ orthonormal columns and spanning the ranges of $\sigma_3 N_{r;s}(t)$ and $\sigma_3 N_{l;s}(t)$ and time-independent diagonal $r_s \times r_s$ matrices $d_{r;s} = d_{r;\bar{s}}^\dagger$ and $d_{l;s} = d_{l;\bar{s}}^\dagger$ having only nonzero diagonal entries such that

$$\sigma_3 N_{r;s}(t) = e_{r;s}(t) d_{r;s} e_{r;\bar{s}}(t)^\dagger, \quad \sigma_3 N_{l;s}(t) = e_{l;s}(t) d_{l;s} e_{l;\bar{s}}(t)^\dagger, \quad (6.2)$$

whenever $\lambda_s = -\lambda_{\bar{s}}^*$. Furthermore,

$$e_{r;s}(t) = e^{-t \Lambda_r^{\text{up}}(\lambda_s)} e_{r;s}(0), \quad e_{l;s}(t) = e^{-t \Lambda_r^{\text{dn}}(\lambda_s)} e_{l;s}(0). \quad (6.3)$$

If $\lambda_s = -\lambda_s^*$ is purely imaginary and therefore $\sigma_3 N_{r;s}(t)$ and $\sigma_3 N_{l;s}(t)$ are Hermitian matrices, the number of positive and negative diagonal entries of $d_{r;s}$ and $d_{l;s}$ corresponds to the (time-independent) number of positive and negative eigenvalues of $\sigma_3 N_{r;s}(t)$ and $\sigma_3 N_{l;s}(t)$.

Now define the matrix triplets as follows:

$$A_r = A_l = a_1 I_{r_1} \oplus \dots \oplus a_N I_{r_N}, \quad (6.4)$$

where $A_{r,l}$ are diagonal matrices of order $q = r_1 + \dots + r_N$ having r_s copies of $a_s = -i\lambda_s$ on the diagonal. Next, we define

$$B_r = \begin{pmatrix} d_{r;1} e_{r;\bar{1}}^\dagger \\ \vdots \\ d_{r;N} e_{r;\bar{N}}^\dagger \end{pmatrix}, \quad B_l = \begin{pmatrix} d_{l;1} e_{l;\bar{1}}^\dagger \\ \vdots \\ d_{l;N} e_{l;\bar{N}}^\dagger \end{pmatrix}, \tag{6.5a}$$

$$C_r = (\sigma_3 e_{r;1} \dots \sigma_3 e_{r;N}), \quad C_l = (\sigma_3 e_{l;1} \dots \sigma_3 e_{l;N}), \tag{6.5b}$$

where we have not written the time dependence. Then the Marchenko integral kernels in (6.1) are given by

$$\Omega_r(w; t) = C_r(t) e^{-wA_r} B_r(t), \tag{6.6a}$$

$$\Omega_l(w; t) = C_l(t) e^{wA_l} B_l(t), \tag{6.6b}$$

where the $q \times q$ matrices $A_{r,l}$ have only eigenvalues with positive real parts, $B_{r,l}(t)$ are $q \times n$ matrices, and $C_{r,l}(t)$ are $n \times q$ matrices.

Let us now depart from arbitrary Marchenko integral kernels (6.6), where the $q \times q$ matrices $A_{r,l}$ have only eigenvalues with positive real parts, $B_{r,l}(t)$ are $q \times n$ matrices, $C_{r,l}(t)$ are $n \times q$ matrices, and the specific expressions (6.4) and (6.5) need not be applied. Solving the Marchenko integral equations (4.3), we get

$$K(x, y; t) = -W_r(x; t) e^{-yA_r} B_r(t), \tag{6.7a}$$

$$J(x, y; t) = -W_l(x; t) e^{yA_l} B_l(t), \tag{6.7b}$$

where

$$W_r(x; t) = C_r e^{-xA_r} + \int_x^\infty dz K(x, z; t) C_r(t) e^{-zA_r},$$

$$W_l(x; t) = C_l e^{xA_l} + \int_{-\infty}^x dz J(x, z; t) C_l(t) e^{zA_l}.$$

Substituting (6.7) into (4.3) and solving for $W_{r,l}(x; t)$ we get

$$W_r(x; t) = C_r(t) e^{-xA_r} \left[I_q + e^{-xA_r} P_r(t) e^{-xA_r} \right]^{-1},$$

$$W_l(x; t) = C_l(t) e^{xA_l} \left[I_q + e^{xA_l} P_l(t) e^{xA_l} \right]^{-1},$$

provided the inverse matrices exist. Here

$$P_{r,l}(t) = \int_0^\infty dz e^{-zA_{r,l}} B_{r,l}(t) C_{r,l}(t) e^{-zA_{r,l}}$$

are the unique solutions of the Sylvester equations

$$A_{r,l}P_{r,l}(t) + P_{r,l}(t)A_{r,l} = B_{r,l}(t)C_{r,l}(t).$$

More precisely, given $(x, t) \in \mathbb{R}^2$, the Marchenko integral equations (4.3) are uniquely solvable (in an L^1 -setting) iff the algebraic equations for $W_{r,l}(x; t)$ are uniquely solvable. Consequently,

$$\begin{aligned} K(x, y; t) &= -C_r(t)e^{-xA_r} \left[I_q + e^{-xA_r} P_r(t)e^{-xA_r} \right]^{-1} e^{-yA_r} B_r(t) \\ &= -C_r(t) \left[I_q + e^{-2xA_r} P_r(t) \right]^{-1} e^{-(x+y)A_r} B_r(t) \\ &= -C_r(t) \left[e^{2xA_r} + P_r(t) \right]^{-1} e^{-(y-x)A_r} B_r(t), \end{aligned} \tag{6.8a}$$

$$\begin{aligned} J(x, y; t) &= -C_l(t)e^{xA_l} \left[I_q + e^{xA_l} P_l(t)e^{xA_l} \right]^{-1} e^{yA_l} B_l(t) \\ &= -C_l(t) \left[I_q + e^{2xA_l} P_l(t) \right]^{-1} e^{(x+y)A_l} B_l(t) \\ &= -C_l(t) \left[e^{-2xA_l} + P_l(t) \right]^{-1} e^{-(x-y)A_l} B_l(t). \end{aligned} \tag{6.8b}$$

Using (3.17), we obtain

$$\int_x^\infty dy Q(y; t) = -2C_r(t) \left[e^{2xA_r} + P_r(t) \right]^{-1} B_r(t), \tag{6.9a}$$

$$\int_{-\infty}^x dy Q(y; t) = -2C_l(t) \left[e^{-2xA_l} + P_l(t) \right]^{-1} B_l(t). \tag{6.9b}$$

Consequently,

$$Q(x; t) = -4C_r(t) \left[e^{2xA_r} + P_r(t) \right]^{-1} A_r e^{2xA_r} \left[e^{2xA_r} + P_r(t) \right]^{-1} B_r(t), \tag{6.10a}$$

$$Q(x; t) = -4C_l(t) \left[e^{-2xA_l} + P_l(t) \right]^{-1} A_l e^{-2xA_l} \left[e^{-2xA_l} + P_l(t) \right]^{-1} B_l(t). \tag{6.10b}$$

Using the partitioning

$$C_{r,l}(t) = \begin{pmatrix} C_{r,l}^{\text{up}}(t) \\ C_{r,l}^{\text{dn}}(t) \end{pmatrix}, \quad B_{r,l}(t) = (B_{r,l}^{\text{lt}}(t) \ B_{r,l}^{\text{rt}}(t)),$$

and assuming the nonsingularity of $P_{r,l}(t)$, we obtain

$$Q(x; t) = Q_r + 2C_r^{\text{up}}(t)P_r(t)^{-1}B_r^{\text{rt}}(t) + 2C_r^{\text{dn}}(t)P_r(t)^{-1}B_r^{\text{lt}}(t)$$

$$\begin{aligned}
& -2C_r^{\text{up}}(t) \left[e^{2xA_r} + P_r(t) \right]^{-1} B_r^{\text{rt}}(t) \\
& -2C_r^{\text{dn}}(t) \left[e^{2xA_r} + P_r(t) \right]^{-1} B_r^{\text{lt}}(t), \tag{6.11a}
\end{aligned}$$

$$\begin{aligned}
Q(x; t) &= Q_l + 2C_l^{\text{up}}(t) P_l(t)^{-1} B_l^{\text{rt}}(t) + 2C_l^{\text{dn}}(t) P_l(t)^{-1} B_l^{\text{lt}}(t) \\
& -2C_l^{\text{up}}(t) \left[e^{-2xA_l} + P_l(t) \right]^{-1} B_l^{\text{rt}}(t) \\
& -2C_l^{\text{dn}}(t) \left[e^{-2xA_l} + P_l(t) \right]^{-1} B_l^{\text{lt}}(t). \tag{6.11b}
\end{aligned}$$

If (6.11) are solutions of the differential Riccati equation $Q^2 + Q_x = Q - \mu^2 I_n$, then they represent the multisoliton solutions of the focusing matrix NLS equation with extra term (2.2). Using the gauge transformation (2.6), we then get the multisoliton solutions of the usual focusing matrix NLS equation.

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A Wronskian Relations

In this appendix, we give the details of the proofs of the identities (3.8), (3.9), (3.10), (3.11). First of all, we prove the following

Proposition A.1 *For $\lambda \in \mathbb{R}$, let $V(x, \lambda)$ and $W(x, \lambda)$ be two $2n \times 2n$ matrix solutions of the first-order system (3.7). Then*

$$W(x, \lambda)^\dagger (\sigma_2 \otimes \sigma_3) V(x, \lambda)$$

is independent of $x \in \mathbb{R}$. In particular, its asymptotic forms as $x \rightarrow \pm\infty$ coincide.

Proof It is easily verified by using (1.6) that for $\lambda \in \mathbb{R}$ we have

$$(\sigma_2 \otimes \sigma_3) \begin{bmatrix} 0_{n \times n} & I_n \\ Q(x) - \lambda^2 I_n & 0_{n \times n} \end{bmatrix} (\sigma_2 \otimes \sigma_3) = - \begin{bmatrix} 0_{n \times n} & I_n \\ Q(x) - \lambda^2 I_n & 0_{n \times n} \end{bmatrix}^\dagger.$$

Then

$$\frac{\partial}{\partial x} \left[W(x, \lambda)^\dagger (\sigma_2 \otimes \sigma_3) V(x, \lambda) \right]$$

$$\begin{aligned}
 &= \mathbf{W}(x, \lambda)^\dagger \begin{bmatrix} 0_{n \times n} & I_n \\ \mathbf{Q}(x) - \lambda^2 I_n & 0_{n \times n} \end{bmatrix}^\dagger (\sigma_2 \otimes \sigma_3) \mathbf{V}(x, \lambda) \\
 &+ \mathbf{W}(x, \lambda)^\dagger (\sigma_2 \otimes \sigma_3) \begin{bmatrix} 0_{n \times n} & I_n \\ \mathbf{Q}(x) - \lambda^2 I_n & 0_{n \times n} \end{bmatrix} \mathbf{V}(x, \lambda) = 0_{2n \times 2n},
 \end{aligned}$$

as claimed.

Let us first apply Proposition A.1 to $\mathbf{V}(x, \lambda) = \mathbf{W}(x, \lambda) = \Phi(x, \lambda)$ and divide the resulting equation by 2λ . For $0 \neq \lambda \in \mathbb{R}$, we get by equating the $x \rightarrow +\infty$ asymptotics to the $x \rightarrow -\infty$ asymptotics and dividing the resulting equation by 2λ

$$\begin{bmatrix} -A_r^\dagger \sigma_3 A_r + B_r^\dagger \sigma_3 B_r & B_r^\dagger \sigma_3 \\ \sigma_3 B_r & \sigma_3 \end{bmatrix} = \begin{bmatrix} -\sigma_3 & -\sigma_3 B_l \\ -B_l^\dagger \sigma_3 & A_l^\dagger \sigma_3 A_l - B_l^\dagger \sigma_3 B_l \end{bmatrix},$$

where we have not written the λ -dependence. Consequently, for $0 \neq \lambda \in \mathbb{R}$ we have the equalities

$$A_r(\lambda)^\dagger \sigma_3 A_r(\lambda) - B_r(\lambda)^\dagger \sigma_3 B_r(\lambda) = \sigma_3, \tag{A.1a}$$

$$A_l(\lambda)^\dagger \sigma_3 A_l(\lambda) - B_l(\lambda)^\dagger \sigma_3 B_l(\lambda) = \sigma_3, \tag{A.1b}$$

$$B_r(\lambda)^\dagger = -\sigma_3 B_l(\lambda) \sigma_3, \quad B_l(\lambda)^\dagger = -\sigma_3 B_r(\lambda) \sigma_3. \tag{A.1c}$$

We observe that (A.1) coincide with (3.8).

Let us now apply Proposition A.1 to $\mathbf{V}(x, \lambda) = \mathbf{W}(x, \lambda) = F_{r,\lambda}(x, \lambda)$ and divide the resulting equation by 2λ . For $0 \neq \lambda \in \mathbb{R}$, we get by equating the $x \rightarrow +\infty$ asymptotics to the $x \rightarrow -\infty$ asymptotics and dividing by 2λ

$$\begin{aligned}
 &\begin{bmatrix} -A_r^\dagger \sigma_3 A_r + B_r^\dagger \sigma_3 B_r - A_r^\dagger \sigma_3 B_r^\# + B_r^\dagger \sigma_3 A_r^\# \\ A_r^{\#\dagger} \sigma_3 B_r - B_r^{\#\dagger} \sigma_3 A_r & A_r^{\#\dagger} \sigma_3 A_r^\# - B_r^{\#\dagger} \sigma_3 B_r^\# \end{bmatrix} = \begin{bmatrix} -\sigma_3 & 0_{n \times n} \\ 0_{n \times n} & \sigma_3 \end{bmatrix}, \\
 &\begin{bmatrix} -\sigma_3 & 0_{n \times n} \\ 0_{n \times n} & \sigma_3 \end{bmatrix} = \begin{bmatrix} -A_l^{\#\dagger} \sigma_3 A_l^\# + B_l^{\#\dagger} \sigma_3 B_l^\# & -A_l^{\#\dagger} \sigma_3 B_l + B_l^{\#\dagger} \sigma_3 A_l \\ A_l^{\#\dagger} \sigma_3 B_l^\# - B_l^{\#\dagger} \sigma_3 A_l^\# & A_l^{\#\dagger} \sigma_3 A_l - B_l^{\#\dagger} \sigma_3 B_l \end{bmatrix},
 \end{aligned}$$

respectively, where the short-hand notation $C^\#(k) = C(-k)$ is adopted. Equating the block diagonal entries implies (A.1a) and (A.1b). Equating the block off-diagonal entries implies

$$A_r(\lambda)^\dagger \sigma_3 B_r(-\lambda) = B_r(\lambda)^\dagger \sigma_3 A_r(-\lambda), \tag{A.2a}$$

$$A_l(\lambda)^\dagger \sigma_3 B_l(-\lambda) = B_l(\lambda)^\dagger \sigma_3 A_l(-\lambda), \tag{A.2b}$$

and these equalities coincide with (3.9)

Finally, let us now apply Proposition A.1 to $\mathbf{V}(x, \lambda) = F_l(x, \lambda)$ and $\mathbf{W}(x, \lambda) = F_r(x, \lambda)$ and divide the resulting equation by 2λ . For $0 \neq \lambda \in \mathbb{R}$, we get by equating the $x \rightarrow +\infty$ asymptotics to the $x \rightarrow -\infty$ asymptotics and dividing the

resulting equation by 2λ

$$\begin{bmatrix} -A_r(\lambda)^\dagger \sigma_3 & B_r(\lambda)^\dagger \sigma_3 \\ -B_r(-\lambda)^\dagger \sigma_3 & A_r(-\lambda)^\dagger \sigma_3 \end{bmatrix} = \begin{bmatrix} -\sigma_3 A_l(-\lambda) & -\sigma_3 B_l(\lambda) \\ \sigma_3 B_l(-\lambda) & \sigma_3 A_l(\lambda) \end{bmatrix}.$$

As a result, we arrive at the two identities

$$A_r(\lambda)^\dagger = \sigma_3 A_l(-\lambda) \sigma_3, \quad B_r(\lambda)^\dagger = -\sigma_3 B_l(\lambda) \sigma_3. \quad (\text{A.3})$$

Identities (A.3) coincide with (3.10).

Equation (3.11) can easily be derived from the x -independence of

$$\mathbf{W}(x, -\lambda^*)^\dagger (\sigma_2 \otimes \sigma_3) \mathbf{V}(x, \lambda)$$

for given solutions $\mathbf{V}(x, \lambda)$ and $\mathbf{W}(x, \lambda)$ of (3.7).

References

- Ablowitz, M.J.: *Nonlinear Dispersive Waves. Asymptotic Analysis and Solitons*, Cambridge Texts in Applied Mathematics, vol. 47. Cambridge University Press, Cambridge (2011)
- Ablowitz, M.J., Segur, H.: *Solitons and Inverse Scattering Transforms*. SIAM, Philadelphia (1981)
- Ablowitz, M.J., Kaup, D.J., Newell, A.C., Segur, H.: The inverse scattering transform. *Fourier analysis for nonlinear problems*. *Stud. Appl. Math.* **53**, 249–315 (1974)
- Ablowitz, M.J., Prinari, B., Trubatch, A.D.: *Discrete and Continuous Nonlinear Schrödinger Systems*. Cambridge University Press, Cambridge (2004)
- Aden, H., Carl, B.: On realizations of solutions of the KdV equation by determinants on operator ideals. *J. Math. Phys.* **37**, 1833–1857 (1996)
- Agranovich, Z.S., Marchenko, V.A.: *The Inverse Problem of Scattering Theory*. Gordon and Breach, New York (1963)
- Akhmediev, N.N., Korneev, V.I.: Modulational instability and periodic solutions of the nonlinear Schrödinger equation. *Theor. Math. Phys.* **69**, 1089–1093 (1986)
- Akhmediev, N.N., Eleonskii, V.M., Kalagin, N.E.: Generation of a periodic sequence of picosecond pulses in an optical fiber. *Exact solutions*. *Sov. Phys. JETP* **89**, 1542–1551 (1985)
- Akhmediev, N.N., Ankiewicz, A., Soto-Crespo, J.M.: Rogue waves and rational solutions of nonlinear Schrödinger equation. *Phys. Rev. E* **80**, 026601 (2009)
- Aktosun, T., van der Mee, C.: Explicit solutions to the Korteweg-de Vries equation on the half-line. *Inverse Prob.* **22**, 2165–2174 (2006)
- Aktosun, T., Weder, R.: Inverse scattering on the half line for the matrix Schrödinger equation. *J. Math. Phys. Anal. Geom.* **14**, 237–269 (2018)
- Aktosun, T., Weder, R.: *Direct and Inverse Scattering for the Matrix Schrödinger Equation*, Applied Mathematical Sciences 203. Springer, New York (2020)
- Aktosun, T., Klaus, M., van der Mee, C.: Small-energy asymptotics of the scattering matrix for the matrix Schrödinger equation on the line. *J. Math. Phys.* **42**, 4627–4652 (2001)
- Aktosun, T., Demontis, F., van der Mee, C.: Exact solutions to the focusing nonlinear Schrödinger equation. *Inverse Prob.* **23**, 2171–2195 (2007)
- Aktosun, T., Demontis, F., van der Mee, C.: Exact solutions to the sine-Gordon equation. *J. Math. Phys.* **51**, 123521, 27 pp (2010)
- Asano, N., Kato, Y.: Non-self-adjoint Zakharov-Shabat operator with a potential of the finite asymptotic values, I. Direct and inverse scattering problems. *J. Math. Phys.* **22**, 2780–2793 (1981)
- Asano, N., Kato, Y.: Non-self-adjoint Zakharov-Shabat operator with a potential of the finite asymptotic values, II. Inverse problem. *J. Math. Phys.* **25**, 570–588 (1984)

- Berkolaiko, G.: An elementary introduction to quantum graphs. In: Geometric and Computational Spectral Theory, Contemporary Mathematics, vol. 700, pp. 41–72. American Mathematical Society, Providence RI (2017)
- Berkolaiko, G., Kuchment, P.: Introduction to Quantum Graphs, Mathematical Surveys and Monographs 186. American Mathematical Society, Providence RI (2013)
- Berkolaiko, G., Liu, W.: Simplicity of eigenvalues and non-vanishing of eigenfunctions of a quantum graph. *J. Math. Anal. Appl.* **445**, 803–818 (2017)
- Berkolaiko, G., Carlson, R., Fulling, S.A., Kuchment, P. (eds.): Quantum Graphs and their Applications, Contemporary Mathematics, vol. 415. American Mathematical Society, Providence (2006)
- Bilman, D., Miller, P.: A robust inverse scattering transform for the focusing nonlinear Schrödinger equation. *Commun. Pure. Appl. Math.* **72**, 1722–1805 (2019)
- Biondini, G., Kovačić, G.: Inverse scattering transform for the focusing nonlinear Schrödinger equation with nonzero boundary conditions. *J. Math. Phys.* **55**, 031506 (2014)
- Biondini, G., Lottes, J., Mantzavinos, D.: Inverse scattering transform for the focusing nonlinear Schrödinger equation with counterpropagating flows. *Stud. Appl. Math.* **46**, 371–439 (2021)
- Boman, J., Kurasov, P.: Symmetries of quantum graphs and the inverse scattering problem. *Adv. Appl. Math.* **35**, 58–70 (2005)
- Calogero, F., Degasperis, A.: Spectral Transforms and Solitons. North-Holland, Amsterdam (1982)
- Chadan, K., Sabatier, P.: Inverse Problems in Quantum Scattering Theory, 2nd edn. Springer, New York (1989)
- Chen, M., Tsankov, M.A., Nash, J.M., Patton, C.E.: Backward-volume-water microwave-envelope solitons in yttrium iron garnet films. *Phys. Rev. B* **49**, 12773–12790 (1994)
- Deift, P., Trubowitz, E.: Inverse scattering on the line. *Commun. Pure Appl. Math.* **32**, 121–251 (1979)
- Demontis, F.: Exact solutions to the modified Korteweg-de Vries equation. *Theor. Math. Phys.* **168**, 886–897 (2011)
- Demontis, F., van der Mee, C.: Marchenko equations and norming constants of the matrix Zakharov-Shabat system. *Oper. Matrices* **2**, 79–113 (2008a)
- Demontis, F., van der Mee, C.: Explicit solutions of the cubic matrix nonlinear Schrödinger equation. *Inverse Prob.* **24**, 02520, 16 pp (2008b)
- Demontis, F., van der Mee, C.: From the AKNS system to the matrix Schrödinger equation with vanishing potentials: Direct and inverse problems (2021). [arXiv:2109.11684](https://arxiv.org/abs/2109.11684) [nlin.SI]
- Demontis, F., Prinari, B., van der Mee, C., Vitale, F.: The inverse scattering transform for the defocusing nonlinear Schrödinger equation with nonzero boundary conditions. *Stud. Appl. Math.* **131**, 1–40 (2013)
- Demontis, F., Prinari, B., van der Mee, C., Vitale, F.: The inverse scattering transform for focusing nonlinear Schrödinger equation with asymmetric boundary conditions. *J. Math. Phys.* **55**, 101505, 40 pp (2014)
- Demontis, F., Lombardo, S., Sommacal, M., van der Mee, C., Vargiu, F.: Effective generation of closed-form soliton solutions of the continuous classical Heisenberg ferromagnet equation. *Commun. Nonlinear Sci. Numer. Simul.* **64**, 35–65 (2018)
- Demontis, F., Ortenzi, G., Sommacal, M., van der Mee, C.: The continuous classical Heisenberg ferromagnet equation with in-plane asymptotic conditions. II. IST and closed-form soliton solutions. *Ricerche mat.* **68**(1), 163–178 (2019)
- Eckhaus, W., van Harten, A.: The Inverse Scattering Transformation and the Theory of Solitons. North-Holland, Amsterdam (1981)
- Exner, P., Keating, J.P., Kuchment, P., Sunada, T., Teplyaev, A. (eds.) Analysis on Graphs and its Applications, Proceedings of Symposia in Pure Mathematics, vol. 77. Amer. Math. Soc., Providence (2008)
- Faddeev, L.D.: Properties of the S -matrix of the one-dimensional Schrödinger equation. *Am. Math. Soc. Transl.* **2**, 139–166 (1964)
- Faddeev, L.D., Takhtajan, L.A.: Hamiltonian Methods in the Theory of Solitons. Springer, Berlin (1987)
- Gelfand, I.M., Raikov, D.A., Shilov, G.E.: Commutative Normed Rings. Chelsea Publ, New York (1964)
- Gerasimenko, N.I.: The inverse scattering problem on a noncompact graph. *Theor. Math. Phys.* **75**, 460–470 (1988)
- Gerasimenko, I., Pavlov, B.S.: Scattering problems on noncompact graphs. *Theor. Math. Phys.* **74**, 230–240 (1988)
- Gutkin, B., Smilansky, U.: Can one hear the shape of a graph? *J. Phys. A Math. Gen.* **34**, 6061–6068 (2001)
- Harmer, M.S.: Inverse scattering for the matrix Schrödinger operator and Schrödinger operator on graphs with general self-adjoint boundary conditions. *ANZIAM J.* **44**, 161–168 (2002)

- Harmer, M.S.: The matrix Schrödinger operator and Schrödinger operator on graphs. Ph.D. thesis, University of Auckland, New Zealand (2004)
- Harmer, M.S.: Inverse scattering on matrices with boundary conditions. *J. Phys. A Math. Gen.* **38**, 4875–4885 (2005)
- Hasegawa, A.: *Optical Solitons in Fibers*, Springer Series in Photonics 9. Springer, New York (2002)
- Hasegawa, A., Tappert, F.: Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers. I. Anomalous dispersion, and II. Normal dispersion. *Appl. Phys. Lett.* **23**, 142–144 and 171–172 (1973)
- Horn, R.A., Johnson, C.J.: *Topics in Matrix Analysis*. Cambridge University Press, Cambridge (1994)
- Its, A.R., Rybin, A.V., Sall, M.A.: Exact integration of nonlinear Schrödinger equation. *Theor. Math. Phys.* **74**, 20–32 (1988)
- Kawata, T., Inoue, H.: Eigenvalue problem with nonvanishing potentials. *J. Phys. Soc. Jpn.* **43**, 361–362 (1977)
- Kawata, T., Inoue, H.: Inverse scattering method for the nonlinear evolution equations under nonvanishing conditions. *J. Phys. Soc. Jpn.* **44**, 1722–1729 (1978)
- Kevrekidis, P.G., Frantzeskakis, D.J., Carretero-González, R.: *Emergent Non-linear Phenomena in Bose-Einstein Condensates*. Springer, Berlin (2008)
- Klaus, M.: Low-energy behaviour of the scattering matrix for the Schrödinger equation on the line. *Inverse Prob.* **4**, 505–512 (1988)
- Kostykin, V., Schrader, R.: Kirchhoff's rule for quantum wires. *J. Phys. A Math. Gen.* **32**, 595–630 (1999)
- Kostykin, V., Schrader, R.: Kirchhoff's rule for quantum wires. II. The inverse problem with possible applications to quantum computers. *Fortschr. Phys.* **48**, 703–716 (2000)
- Kuchment, P.: Quantum graphs. I. Some basic structures. *Waves Random Media* **14**, S107–S128 (2004)
- Kuchment, P.: Quantum graphs. II. Some spectral properties of quantum and combinatorial graphs. *J. Phys. A Math. Gen.* **38**, 4887–4900 (2005)
- Kurasov, P., Nowaczyk, M.: Inverse spectral problem for quantum graphs. *J. Phys. A Math. Gen.* **38**, 4901–4915 (2005)
- Kurasov, P., Nowaczyk, M.: Geometric properties of quantum graphs and vertex scattering matrices. *Opusc. Math.* **30**, 295–309 (2010)
- Kurasov, P., Stenberg, F.: On the inverse scattering problem on branching graphs. *J. Phys. A Math. Gen.* **35**, 101–121 (2002)
- Lax, P.: Integrals of nonlinear equations of evolution and solitary waves. *Commun. Pure Appl. Math.* **21**, 467–490 (1968)
- Ma, Yan-chow: The perturbed plane-wave solutions of the cubic Schrödinger equation. *Stud. Appl. Math.* **60**, 43–58 (1979)
- Mihalache, D., Lederer, F., Baboiu, D.-M.: Two-parameter family of exact solutions of the nonlinear Schrödinger equation describing optical soliton propagation. *Phys. Rev. A* **47**, 3285–3290 (1993)
- Novikov, S.P., Manakov, S.V., Pitaevskii, L.B., Zakharov, V.E.: *Theory of Solitons. The Inverse Scattering Method*. Plenum Press, New York (1984)
- Ortiz, A.K., Prinari, B.: Inverse scattering transform and solitons for square matrix nonlinear Schrödinger equations with mixed sign reductions and nonzero boundary conditions. *J. Nonlinear Math. Phys.* **27**, 130–161 (2020)
- Peregrine, D.H.: Water waves, nonlinear Schrödinger equations and their solutions. *J. Aust. Math. Soc. B* **25**, 16–43 (1983)
- Pethick, C.J., Smith, H.: *Bose-Einstein Condensation in Dilute Gases*. Cambridge University Press, Cambridge (2002)
- Pitaevskii, L.P., Stringari, S.: *Bose-Einstein Condensation and Superconductivity*. Oxford University Press, Oxford (2016)
- Prinari, B., Ablowitz, M.J., Biondini, G.: Inverse scattering transform for the vector nonlinear Schrödinger equation with nonvanishing boundary conditions. *J. Math. Phys.* **47**, 063508, 33 pp (2006)
- Schiebold, C.: An operator theoretic approach to the Toda lattice equation. *Physica D* **122**, 37–61 (1998)
- Schiebold, C.: Solutions of the sine-Gordon equation coming in clusters. *Revista Matemática Complutense* **15**, 265–325 (2002)
- Shaw, J.K.: *Mathematical Principles of Optical Fiber Communications*, CBMS-NSF Regional Conference Series in Applied Mathematics 76. SIAM, Philadelphia (2004)
- Tajiri, M., Watanabe, Y.: Breather solutions to the focusing nonlinear Schrödinger equation. *Phys. Rev. E* **57**, 3510–3519 (1998)
- van der Mee, C.: *Nonlinear Evolution Models of Integrable Type*, SIMAI e-Lecture Notes 11 (2013)

- Wadati, M., Kamijo, T.: On the extension of inverse scattering method. *Prog. Theor. Phys.* **52**, 397–414 (1974)
- Zakharov, V.E.: Hamilton formalism for hydrodynamic plasma models. *Sov. Phys. JETP* **33**, 927–932 (1971)
- Zakharov, V.E., Gelash, A.A.: Nonlinear stage of modulational instability. *Phys. Rev. Lett.* **111**, 054101 (2013)
- Zakharov, V.E., Popkov, A.F.: Contribution to the nonlinear theory of magnetostatic spin waves. *Sov. Phys. JETP* **57**, 350–355 (1983)
- Zakharov, V.E., Shabat, A.B.: Exact theory of two-dimensional self-focusing and one dimensional self-modulation of waves in nonlinear media. *Sov. Phys. JETP* **34**, 62–69 (1972)
- Zakharov, V.E., Shabat, A.B.: Interaction between solitons in a stable medium. *Sov. Phys. JETP* **37**, 823–828 (1973)

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