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Boundedness in a nonlinear attraction-repulsion Keller–Segel system with production and consumption

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A B S T R A C T

This paper is focused on the zero-flux attraction-repulsion chemotaxis model

$$\begin{cases} u_t = \nabla\cdot((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v & \text{in } \Omega\times(0,T_{max}),\\ \qquad + \xi u(u+1)^{m_3-1}\nabla w) & \\ v_t = \Delta v - f(u)v & \text{in } \Omega\times(0,T_{max}),\\ 0 = \Delta w - \delta w + g(u) & \text{in } \Omega\times(0,T_{max}), \end{cases} \tag{$\diamond$}$$

defined in Ω, which is a bounded and smooth domain of $\mathbb{R}^n$, for $n\geq 2$, with $\chi,\xi,\delta>0$, $m_1,m_2,m_3\in\mathbb{R}$, and $f(u)$ and $g(u)$ reasonably regular functions generalizing the prototypes $f(u)=Ku^\alpha$ and $g(u)=\gamma u^l$, with $K,\gamma>0$ and appropriate $\alpha,l>0$. Moreover T_{max} is finite or infinite and $(0,T_{max})$ stands for the maximal temporal interval where solutions to the related initial problem exist. Our main interest is to identify constellations of the impacts m_1,m_2 and m_3 of the diffusion and drift terms, as well as of the growth l of the production g for the chemorepellent (i.e., w) and the rate α of the consumption f for the chemoattractant (i.e., v), which ensure boundedness of cell densities (i.e., u). Precisely, for any fixed $\alpha\in\left(0,\frac{1}{2}+\frac{1}{n}\right)$ and $l\geq 1$, we prove that whenever

$$m_1 > \min\left\{2m_2+1-(m_3+l), \max\left\{2m_2, \frac{n-2}{n}\right\}\right\},$$

any sufficiently smooth initial data $u(x,0)=u_0(x)\geq 0$ and $v(x,0)=v_0(x)\geq 0$ produce a unique classical solution (u,v,w) to problem ($\diamond$) such that its life span $T_{max}=\infty$ and, moreover, u,v and w are uniformly bounded in $\Omega\times(0,\infty)$.

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1. Introduction: presentation of the model in the framework of the literature

We consider a chemotaxis model connecting two classical Keller–Segel systems, widely employed in biological processes. We refer to the landmark models ([16–18]) idealizing motility phenomena in situations where certain cells (populations, organisms) spread while they are attracted by a signal they themselves, in one case, produce and, in the other case, absorb. More precisely, in order to define a proper coupling of the erstwhile models, another chemical signal, with opposite impact to the first, is considered; subsequently, the dynamics of the cell density is governed at the same time by attractive and repulsive effects, coming from the drift terms involving the two chemicals and the cells. (Below, we give more precise motivations and references on real models of this type.) Moreover, besides this combination, we also introduce nonlinearities in the aforementioned diffusive and cross terms. More precisely, if $u = u(x, t)$ is used to denote the population density of the cells at the position x and at the time t, and $v = v(x, t)$ and $w = w(x, t)$ stand, respectively, for the concentration of the attractive and repulsive chemical signals (chemoattractant and chemorepellent), we study the initial-boundary value problem given by

$$\begin{cases} u_t = \nabla \cdot ((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w) & \text{in } \Omega \times (0, T_{max}), \\ v_t = \Delta v - f(u)v & \text{in } \Omega \times (0, T_{max}), \\ 0 = \Delta w - \delta w + g(u) & \text{in } \Omega \times (0, T_{max}), \\ u_\nu = v_\nu = w_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\ u(x, 0) = u_0(x),\ v(x, 0) = v_0(x) & x \in \bar{\Omega}, \end{cases} \tag{1}$$

defined in a bounded and smooth domain Ω of $\mathbb{R}^n$, with $n \geq 2$, $\chi, \xi, \delta > 0$, $m_1, m_2, m_3 \in \mathbb{R}$ and some functions $f = f(s)$ and $g = g(s)$, sufficiently regular in their argument $s \geq 0$, and further regular initial data $u_0(x) \geq 0$ and $v_0(x) \geq 0$. Additionally, the subscript ν in $(\cdot)_\nu$ indicates the outward normal derivative on $\partial\Omega$, whereas T_{max} is the maximal time up to which solutions to the system are defined. We aim at extending the mathematical comprehension of this attraction-repulsion Keller–Segel system by analyzing aspects which, as far as we know, were not yet faced.

Having in mind the biological interpretation of the original chemotaxis systems, by the purely intuitive standpoint, problem (1) idealizes an attraction-repulsion model with production and consumption for which: (**a**) the cells, whose initial distribution obeys the law of u_0, move inside an insulated domain (zero-flux on the border) accordingly to the competition between the aggregation/repulsion impact from the drift terms $-\chi u(u+1)^{m_2-1}\nabla v/\xi u(u+1)^{m_3-1}\nabla w$ (increasing for larger sizes of χ and ξ as well as m_2 and m_3) and the diffusion of the cells (stronger and stronger as m_1 in $(u+1)^{m_1-1}\nabla u$ increases); (**b**) the initial signal v_0 dissipates, w diffuses as well but much faster than v, which is consumed by the cells with rate $f(u)$ whereas, conversely, w is produced with rate $g(u)$; (**c**) consumption of v and proliferation of w are higher the more the cell density increases. Naturally, all these cross-actions in model (1) might lead to different scenarios for the cellular movement, as global stabilization and convergence to equilibrium of the cell distribution u, or *chemotactic collapse*, for which aggregation processes of u, eventually blowing up/exploding at finite time, appear. This aspect, mathematically interpreted, means that solutions (u, v, w) can be defined and are bounded for all (x, t) in $\Omega \times (0, \infty)$, or (u, v, w) might cease to exist for larger values than some finite (blow-up) time T_{max}; in this case, the particle density becomes unbounded approaching T_{max}.

In this regard, for the classical signal-production (u produces v throughout time) Keller–Segel model,

$$u_t = \Delta u - \chi \nabla \cdot (u \nabla v) \quad \text{and} \quad v_t = \Delta v - v + u \quad \text{in } \Omega \times (0, T_{max}), \tag{2}$$

it is well-established that this secretion of v may break the natural homogenization process of the cells, especially in terms of aggregation impact associated to the size of χ, the initial mass of the particle distribution, i.e., $m = \int_\Omega u_0(x)dx$, and the space dimension. On this subject, in [10,14,32,35,49] (and references therein cited), the interested reader can find several discussions dealing with the existence and properties of global, uniformly bounded or blow-up (local) solutions to the Cauchy problem associated to (2).

Conversely, for the signal-absorption (u consumes v throughout time) Keller–Segel model,

$$u_t = \Delta u - \chi\nabla\cdot(u\nabla v) \quad \text{and} \quad v_t = \Delta v - uv \quad \text{in } \Omega\times(0,T_{max}), \tag{3}$$

it is difficult to know whether unbounded solutions to the corresponding initial boundary-value problem of (3) can be constructed. In fact, only in two-dimensional settings (as a combination of the results in [51] and [52], where a more general coupled chemotaxis-fluid model is studied), we know that classical solutions (u,v) emanating from any sufficiently regular initial data (u_0,v_0) are uniformly bounded; for $n\geq 3$, oppositely, the smallness assumption $\chi\|v_0\|_{L^\infty(\Omega)}\leq \frac{1}{6(n+1)}$ is required ([39]). Nevertheless, this condition does not exclude the possibility that solutions emanating from other initial data may collapse in finite time.

From the perspective of blow-up prevention (or boundedness enforcement, when explosion scenarios are not known), the introduction of different external agents is quite efficient in several (*but not* in all) cases. Since the related literature abounds enormously, a detailed description does not seem appropriate, at least in this context: despite that, we can say that in this respect the more common variants of models (2) and (3) may include logistic type sources with dampening effects on the cells' increasing ([20,23,42,43,55]), involve nonlinear diffusion with higher cells' spread ([5,6,9,12,13,21,41,45,48] and [50]), consider weaker (stronger) laws for the production (consumption) of the chemical signal ([22,27,54]) or, more generally, suitable combinations of (some of) these actions ([4,33,47,58]). Moreover, for the same boundedness purpose, other models take into account an additional chemical signal, produced or absorbed by the same cells but repelling them, thus influencing the overall dynamics: in this way, as explained at the beginning of the section, model (1) is encompassed in the above casuistry.

To the best of our knowledge, a general n-dimensional understanding of the attraction-repulsion chemotaxis system in the form of (1), is not yet available in the literature; hereafter, we aim at providing some partial results in this direction. To be precise, problem (1) is a possible counterpart of the situation where the chemoattractant and chemorepellent are both produced; this model has a real application ([29]) in the description of aggregation phenomena of microglia observed in Alzheimer's disease. More exactly, for the attraction-repulsion system with only production (herein referred to production-production models), equipped with Neumann boundary conditions,

$$\begin{cases} u_t = \nabla\cdot((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v + \xi u(u+1)^{m_3-1}\nabla w) \\ \tau v_t = \Delta v - \beta v + f(u) \quad \text{and} \quad 0 = \Delta w - \delta w + g(u) \end{cases},$$
$$\text{in } \Omega\times(0,T_{max}), \quad \chi,\xi,\beta,\delta>0, \tau\in\{0,1\}, \tag{4}$$

the following is known.

▷ Linear diffusion and drift terms: $m_1 = m_2 = m_3 = 1$.

- When $\tau = 1$ and $f(u) = g(u) = u$, an attraction-repulsion Stokes system is studied in two-dimensional settings, and boundedness of classical solutions is achieved for any initial data ([28]);
- When $\tau = 0$, $f(u) = \alpha u$, $\alpha > 0$, and $g(u) = \gamma u$, $\gamma > 0$, if $\xi\gamma > \chi\alpha$ (repulsion prevails over attraction), in any dimension all solutions are globally bounded, whereas for $\xi\gamma < \chi\alpha$ (attraction prevails over repulsion) and $n = 2$ unbounded solutions can be constructed ([8,24,40,44,57]. See also [26] for the case where also the equation for w is parabolic);
- When $\tau = 0$ and f and g generalize the prototypes $f(u) = \alpha u^s$, $s > 0$, and $g(u) = \gamma u^r$, $r \geq 1$, if $r > s \geq 1$ (resp. $s > r \geq 1$), there exists $\xi^* > 0$ (resp. $\xi_* > 0$) such that if $\xi > \xi^*$ (resp. $\xi \geq \xi_*$), any (resp. small) initial data produce a unique classical and bounded solution ([46]).

▷ Nonlinear diffusion and linear drift terms: $m_1 \in \mathbb{R}$ and $m_2 = m_3 = 1$. When $\tau = 1$, $f(u) = \alpha u$, $\alpha > 0$ and $g(u) = \gamma u$, $\gamma > 0$:

- Boundedness is achieved, in any dimension and for any initial data provided $m_1 > 2 - \frac{2}{n}$, while for $m_1 \leq 2 - \frac{2}{n}$ blow-up solutions can be detected ([25]);
- Whenever repulsion dominates or cancels attraction, if $m_1 > 2 - \frac{4}{n+2}$, boundedness of solutions is established, in any dimension and for any initial data ([15]).

Finally, for the sake of completeness, we also have to refer to [36–38,56]. In these articles, attraction-repulsion chemotaxis systems with general production and/or consumption laws, and similarly formulated as in (4), are discussed when the evolution of the cell density is as well influenced by logistic terms. To be more precise, conditions on involved parameters are established so to address questions on global existence of (classical and/or weak) solutions and their long time behavior under specific interplays between the diffusion and the chemosensitivities (linear, nonlinear and even degenerate). In this sense, inspired by these researches, alike studies for our model (1) may surely represent interesting future projects.

2. Motivations and presentation of the theorem

Conforming to what we have discussed above, specifically with respect to the coaction of diffusion and drift terms, together with the laws for the chemorepellent and the chemoattractant, in this research we will derive criteria on the data of problem (1) ensuring that the life span T_{max} of its solutions is infinity and that, moreover, the solutions are bounded. Since, as specified above, a chemorepellent impact and a nonlinear diffusion do not suffice, at least in production-production models, to prevent blow-up, we aim at establishing how the scenario changes in the investigated consumption-production model. Another motivating reason supporting this analysis, is the consideration that if instead of (1) one studies the zero-flux consumption-logistic model

$$\begin{cases} u_t = \nabla \cdot \big((u+1)^{m_1-1}\nabla u - \chi u(u+1)^{m_2-1}\nabla v\big) + ku - \mu u^2 \\ v_t = \Delta v - uv \end{cases} \quad \chi, k, \mu > 0 \quad \text{and} \quad m_1, m_2 \in \mathbb{R}, \tag{5}$$

uniform boundedness of classical solutions (u, v) to the related initial value problem is ensured ([31, Theorem 2.2], proved in a slightly more general context) under two assumptions: not only $m_1 > 2m_2 - 1$, but also μ large with respect to some function of $\|\chi v_0\|_{L^\infty(\Omega)}$ (i.e., diffusion and dampening effects strong enough).

Even though none of systems (1), (4) and (5) can be strictly reformulated as a particular case or an extension of the other, the following questions appear natural:

$\mathcal{Q}.1$ *To what extent are nonlinear diffusive terms more effective toward boundedness in attraction-repulsion models with consumption/production of chemoattractant/chemorepellent than in those with only production?*

$\mathcal{Q}.2$ *And, again in terms of boundedness, may one expect that in attraction-repulsion models with consumed chemoattractant and produced chemorepellent, the same chemorepellent acts more efficaciously than a logistic source does in only attraction models?*

As a matter of fact, as a consequence of the forthcoming Theorem 2.1, we can give a quantification to the first question and open discussions on the second. (See Remark 1 and Remark 2.)

To this scope, if these assumptions are satisfied,

$$f, g \in C^1(\mathbb{R}) \quad \text{with} \quad 0 \leq f(s) \leq Ks^\alpha \text{ and } \gamma s^l \leq g(s) \leq \gamma s(s+1)^{l-1},$$

$$\text{for some} \quad K, \gamma, \alpha > 0, l \geq 1 \quad \text{and all } s \geq 0, \tag{6}$$

this result is proved:

Theorem 2.1. *Let Ω be a smooth and bounded domain of $\mathbb{R}^n$, with $n \geq 2$, and χ, ξ, δ positive. Moreover, let f and g fulfill (6), respectively with $\alpha \in \left(0, \frac{1}{2} + \frac{1}{n}\right)$ and $l \geq 1$. Then for any $m_1, m_2, m_3 \in \mathbb{R}$ satisfying*

$$m_1 > \min\left\{2m_2 + 1 - (m_3 + l), \max\left\{2m_2, \frac{n-2}{n}\right\}\right\}, \tag{7}$$

this conclusion holds true: for any initial data $(u_0, v_0) \in (W^{1,\infty}(\Omega))^2$, with $u_0, v_0 \geq 0$ on $\bar{\Omega}$, there exists a unique triplet of nonnegative functions

$$u, v \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty)) \quad \textit{and} \quad w \in C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,0}(\bar{\Omega} \times (0, \infty)),$$

solving problem (1) and such that for some $C > 0$

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{L^\infty(\Omega)} + \|w(\cdot, t)\|_{L^\infty(\Omega)} \leq C \quad \textit{for all} \quad t \in (0, \infty).$$

Remark 1 *(On question Q.1).* For the nonlinear diffusion, and linear drift terms and production rate case, i.e., $m_1 \in \mathbb{R}$ and $m_2 = m_3 = l = 1$, from (7) we have boundedness of solutions to problem (1) provided $m_1 > 1$ (uniformly with respect to the space dimension). Seeing that for any $n \geq 2$ we have $2 - \frac{4}{n+2} \geq 1$, the result is consistent with [15, Theorem 1.1] (and also with [25, Theorem 1.1], both already mentioned when we dealt with model (4)), since for models with saturated attractive chemical signal an evolution toward boundedness is more conceivable than for models with produced chemoattractant.

Remark 2 *(On question Q.2).* As to the role of the logistic source in (5) and the chemorepellent in (1), we observe that the condition $m_1 > 2m_2 - 1$ is sharper than $m_1 > 2m_2$. Nevertheless, $m_1 > 2m_2 - 1$ on its own does not imply boundedness; as said, a largeness requirement on μ has to be imposed also. Conversely, $m_1 > 2m_2$ suffices in this respect, whenever $\xi > 0$ is arbitrarily small. Moreover, and in this case we can also omit to emphasize the size of μ, for $m_3 > 2 - l$, the relation $m_1 > 1 + 2m_2 - m_3 - l$ improves $m_1 > 2m_2 - 1$. Furthermore, if we consider that for high values of the cell concentration in problem (1) the chemoattractant is consumed with a weaker law than that in (5), an affirmative response to our question might be expected; despite that, due to the different ranges for α in these problems (i.e., $0 < \alpha < \frac{1}{2} + \frac{1}{n}$ vs. $\alpha = 1$, respectively), at least by the theoretical point of view, a direct comparison does not seem well-founded and a deeper insight is required.

Remark 3. Let us also discuss some other implications resulting from assumption (7):

(1) For the linear diffusion and drift terms case, i.e., $m_1 = m_2 = m_3 = 1$, at least a superlinear rate for the production g of the chemorepellent w is required to ensure boundedness of u. (See [7] for the linear rate case.)
(2) To higher impacts of the drift terms associated to the chemorepellent w and/or to a stronger segregation rate of the same chemorepellent, corresponds an increase of the term $(m_3 + l)$; in turn this allows one to consider smaller values of the parameter m_1 responsible for settling tendencies of the cells throughout time.

3. Organization of the paper and background material

3.1. Structure of the paper

In the remaining part of the paper, we first focus on the existence of local classical solutions (u, v, w) to problem (1), their main properties and the crucial "passage" ensuring uniform-in-time boundedness of these solutions toward their L^p-boundedness, for some suitable $p > 1$ (§4). Then, by studying the evolutive behavior of the functional $y(t) := \int_\Omega (u+1)^p + \int_\Omega |\nabla v|^{2p}$, we derive this L^p bound, and prove our main result (§5).

3.2. Supporting tools

In our reasoning, we will make use of the following listed relations, some of which are only "formally" presented; their rigorous hypotheses and related proofs can be found in the corresponding references.

Let $\Omega \subset \mathbb{R}^n$, with $n \geq 2$, be a bounded and smooth domain. Then

- ([31, Lemma 3.1] and [23, Lemma 2.2], respectively.) For all $p > 1$, and ψ such $\psi_\nu = 0$ on $\partial\Omega$, if $D^2\psi$ represents the Hessian matrix of ψ and $|D^2\psi|^2 = \sum_{i,j=1}^{n} \psi^2_{x_i x_j}$, we have

$$|D^2\psi \nabla\psi|^2 \leq |D^2\psi|^2 |\nabla\psi|^2, \tag{8}$$

$$\int_\Omega |\nabla\psi|^{2p+2} \leq 2(4p^2+n)\|\psi\|^2_{L^\infty(\bar\Omega)} \int_\Omega |\nabla\psi|^{2p-2}|D^2\psi|^2. \tag{9}$$

- ([46, Lemma 3.1] and also [53, Lemma 2.2].) For all $\delta > 0$ and $g \geq 0$, the solution ψ of the problem

$$0 = \Delta\psi - \delta\psi + g \ \text{ in } \Omega \quad \text{and} \quad \psi_\nu = 0 \ \text{ on } \partial\Omega,$$

has the following property: For any $\hat{c}, \sigma > 0$ and $\bar{p} \in (1, \infty)$, there exists $\tilde{c} > 0$ such that

$$\hat{c}\int_\Omega \psi^{\bar{p}+1} \leq \sigma \int_\Omega g^{\bar{p}+1} + \tilde{c}\Big(\int_\Omega g\Big)^{\bar{p}+1}. \tag{10}$$

4. Local-in-time classical solutions and their consequential uniform-in-time boundedness

Let us show that system (1) is classically solvable, at least locally. The main idea of the proof follows that given, for instance, in [2,11]; nevertheless, since further details are required, we reproduce the proof here, also in order to make the research more self-contained.

Lemma 4.1 *(*Local existence*). Let Ω be a bounded and smooth domain of $\mathbb{R}^n$, with $n \geq 2$, $\chi, \xi, \delta > 0$, $m_1, m_2, m_3 \in \mathbb{R}$ and $0 \leq f, g \in C^1(\mathbb{R})$. Then, for any nontrivial $(u_0, v_0) \in (W^{1,\infty}(\Omega))^2$, with $u_0 \geq 0$ and $v_0 \geq 0$ on $\bar\Omega$, there exist $T_{max} \in (0, \infty]$ and a unique triplet of nonnegative functions*

$$u, v \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,1}(\bar\Omega \times (0, T_{max})) \quad \textit{and} \quad w \in C^0(\bar\Omega \times [0, T_{max})) \cap C^{2,0}(\bar\Omega \times (0, T_{max})),$$

solving problem (1), and such that for the life span T_{max} this dichotomy criterion holds true:

$$\textit{either } T_{max} = \infty \textit{ or } \limsup_{t \to T_{max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{11}$$

Moreover, the u-component obeys the mass conservation property, i.e.

$$\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx = m > 0 \quad \textit{for all } t \in (0, T_{max}), \tag{12}$$

whilst the v-component is such that

$$0 \le v \le \|v_0\|_{L^\infty(\Omega)} \quad \textit{in} \quad \Omega \times (0, T_{max}). \tag{13}$$

Proof. For any $R > 0$ and $0 \not\equiv u_0 \in W^{1,\infty}(\Omega)$ and $0 \not\equiv v_0 \in W^{1,\infty}(\Omega)$, such that $\|u_0\|_{L^\infty(\Omega)} \le R$, let us consider for fixed $T \in (0, 1)$ the Banach space $X = (C^0([0, T]; C^0(\bar\Omega)))^2$ and its closed convex subset

$$S_T = \{0 \le u \text{ and } 0 \le v \in X : \|u(\cdot, t)\|_{L^\infty(\Omega)} \le R + 1, \text{ for all } t \in [0, T]\}.$$

Once an element $(\tilde u, \tilde v)$ of S_T is picked, from properties of g, the solution w to

$$\begin{cases} -\Delta w + \delta w = g(\tilde u) & \text{in } \Omega \times (0, T), \\ w_\nu = 0 & \text{on } \partial\Omega \times (0, T), \end{cases} \tag{14}$$

thanks to classical elliptic regularity results ([3, Theorem 9.33]), belongs to $C^{1+\delta_1}(\bar\Omega)$, for all $\delta_1 \in (0, 1)$ and $t \in (0, T)$; in particular, this implies that $\nabla w \in L^\infty(\Omega)$ for all $t \in (0, T)$. On the other hand, from $v_0 \in W^{1,\infty}(\Omega)$ (embedded in all the Hölder continuous functions spaces) and the regularity of f, [19, Theorem V 1.1] (see also [30, Theorem 5.1.17]) applied to problem

$$\begin{cases} v_t - \Delta v + f(\tilde u)v = 0 & \text{in } \Omega \times (0, T), \\ v_\nu = 0 & \text{on } \partial\Omega \times (0, T), \\ v(x, 0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{15}$$

gives for some $\tilde\delta \in (0, 1)$ that $v \in C^{\tilde\delta, \frac{\tilde\delta}{2}}(\bar\Omega \times [0, T])$, so that $\nabla v \in L^\infty(\Omega)$ for all $t \in (0, T)$ as well. Now, from these gained properties of ∇w and ∇v, using $u_0 \in W^{1,\infty}(\Omega)$ and the smoothness of $\zeta \longmapsto (1 + \zeta)^\vartheta$ for all $\vartheta \in \mathbb{R}$ and $\zeta \ge 0$, we have again from [19, Theorem V. 1.1] that $u \in C^{\hat\delta, \frac{\hat\delta}{2}}(\bar\Omega \times [0, T])$, for proper $\hat\delta \in (0, 1)$, solves

$$\begin{cases} u_t = \nabla \cdot \left((\tilde u + 1)^{m_1 - 1}\nabla u - \chi u(\tilde u + 1)^{m_2 - 1}\nabla v + \xi u(\tilde u + 1)^{m_3 - 1}\nabla w\right) & \text{in } \Omega \times (0, T), \\ u_\nu = 0 & \text{on } \partial\Omega \times (0, T), \\ u(x, 0) = u_0(x) & x \in \overline{\Omega}. \end{cases} \tag{16}$$

In particular, $(u, v) \in (C^{\delta, \frac{\delta}{2}}(\bar\Omega \times [0, T]))^2$, with $\delta = \min\{\hat\delta, \tilde\delta\}$, so that this produces some positive constant c such that

$$u(x, t) \le u_0(x) + ct^{\frac{\delta}{2}}, \qquad (x, t) \in \Omega \times (0, T), \quad \text{or} \quad \max_{t \in [0,T]} \|u(\cdot, t)\|_{L^\infty(\Omega)} \le \|u_0\|_{L^\infty(\Omega)} + cT^{\frac{\delta}{2}}.$$

In this way, for $T < c^{\frac{-2}{\delta}}$ we deduce that

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \le R + 1 \qquad t \in (0, T).$$

Moreover, since $\underline{u} \equiv 0$ is a subsolution of the first equation in (16), and $f \ge 0$ in (15), the parabolic comparison principle warrants the nonnegativity of u and $0 \le v \le \|v_0\|_{L^\infty(\Omega)}$ on $\Omega \times (0, T)$. So, the map

$\Phi(\tilde{u}, \tilde{v}) = (u, v)$, u solving (16) and v (15), is such that $\Phi(S_T) \subset S_T$ and Φ is compact, because the [3, Ascoli–Arzelà Theorem 4.25] implies that the natural embedding of $(C^{\delta,\frac{\delta}{2}}(\bar{\Omega} \times [0,T]))^2$ into X is a compact linear operator. Let (u, v) be a fixed point of Φ; first, the elliptic maximum principle and $g \geq 0$ in (14) also imply $w \geq 0$ in $\Omega \times (0, T)$. Secondly, by employing the elliptic and parabolic regularity theory to problems (14), and (15) and (16) ([3, Theorem 9.33] and [1, Theorem 14.6]), we have $w \in C^{2+\delta,\frac{\delta}{2}}(\bar{\Omega} \times [0,T])$, and $u, v \in C^0(\bar{\Omega} \times [0,T]) \cap C^{2,1}(\bar{\Omega} \times (0,T])$. On the other hand, by integrating over Ω the first equation of (16), we easily have $\int_\Omega u = \int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx = m$ on $(0, T)$. Moreover, since the choice of T depends only on $\|u_0\|_{L^\infty(\Omega)}$ and $\|v_0\|_{L^\infty(\Omega)}$, by standard bootstrap arguments, the solution (u, v, w) may be prolonged in the interval $[0, T_{max})$, with $T_{max} \leq \infty$, T_{max} being finite if and only if (11) holds. In this way, the obtained nonnegativity of $u(\cdot, t)$ in $[0, T]$, as well as that of v and w, in conjunction with the mass conservation property (12) and bound (13) remain preserved up to T_{max}.

Uniqueness. By absurdity, let (u_1, v_1, w_1) and (u_2, v_2, w_2) be two different nonnegative classical solutions of (1) in $\Omega \times (0, T_{max})$ with the same initial data $u_1(\cdot, 0) = u_2(\cdot, 0) = u_0(x)$ and $v_1(\cdot, 0) = v_2(\cdot, 0) = v_0(x)$. We, thus, have from (14)-(16) and $i = 1, 2$ these six problems:

$$\begin{cases} -\Delta w_i + \delta w_i = g(u_i) & \text{in } \Omega \times (0, T_{max}), \\ (w_i)_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}), \end{cases} \tag{17}$$

$$\begin{cases} (v_i)_t = \Delta v_i - f(u_i)v_i & \text{in } \Omega \times (0, T_{max}), \\ (v_i)_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\ v_i(x,0) = v_0(x) & \text{in } \Omega, \end{cases} \tag{18}$$

and

$$\begin{cases} (u_i)_t = \nabla \cdot ((u_i+1)^{m_1-1}\nabla u_i - \chi u_i(u_i+1)^{m_2-1}\nabla v_i + \xi u_i(u_i+1)^{m_3-1}\nabla w_i) & \text{in } \Omega \times (0, T_{max}), \\ (u_i)_\nu = 0 & \text{on } \partial\Omega \times (0, T_{max}), \\ u_i(x,0) = u_0(x) & x \in \overline{\Omega}. \end{cases} \tag{19}$$

Now, for all arbitrary $T_0 < T_{max}$, we set

$$s_1 = s_1(T_0) = \min\left\{\|u_1\|_{L^\infty(\Omega\times(0,T_0))}, \|u_2\|_{L^\infty(\Omega\times(0,T_0))}\right\},$$
$$s_2 = s_2(T_0) = \max\left\{\|u_1\|_{L^\infty(\Omega\times(0,T_0))}, \|u_2\|_{L^\infty(\Omega\times(0,T_0))}\right\},$$

and introduce for $s \geq 0$ the function

$$\theta(s) = \begin{cases} \log(s+1) & \text{if } m_1 = 0, \\ (s+1)^{m_1}/m_1 & \text{if } m_1 \neq 0, \end{cases} \quad \text{with } \; m_1 \in \mathbb{R}.$$

Noting that $\theta'(s) > 0$ for all $s \geq 0$, we define, in view of forthcoming applications of the Mean Value Theorem, the positive constants

$$\begin{cases} C_1 = C_1(T_0) = \max_{[s_1,s_2]} |g'|, \\ C_2 = C_2(T_0) = \max_{[s_1,s_2]} |f'|, \end{cases} \quad \text{and} \;\; C_3 = C_3(T_0) = \begin{cases} \max_{[s_1,s_2]} \theta' & \text{if } (u_1-u_2)\Delta(u_1-u_2) \geq 0, \\ \min_{[s_1,s_2]} \theta' & \text{if } (u_1-u_2)\Delta(u_1-u_2) \leq 0. \end{cases} \tag{20}$$

Under such circumstances, by considering problems (17), we have that $(w_1 - w_2)$ solves $-\Delta(w_1 - w_2) + \delta(w_1 - w_2) = g(u_1) - g(u_2)$, with Neumann boundary conditions. In light of this, through Young's inequality, an integration by parts provides, taking into account (20), the following estimate

$$\int_\Omega |\nabla(w_1-w_2)|^2+\delta\int_\Omega (w_1-w_2)^2=\int_\Omega (g(u_1)-g(u_2))(w_1-w_2)\le C_1\int_\Omega |(u_1-u_2)(w_1-w_2)|$$
$$\le \frac{\delta}{4}\int_\Omega (w_1-w_2)^2+\frac{C_1^2}{\delta}\int_\Omega (u_1-u_2)^2 \quad \text{on } t\in(0,T_0),$$

inferring

$$\int_\Omega |\nabla(w_1-w_2)|^2\le -\frac{3}{4}\delta\int_\Omega (w_1-w_2)^2+\frac{C_1^2}{\delta}\int_\Omega (u_1-u_2)^2\le \frac{C_1^2}{\delta}\int_\Omega (u_1-u_2)^2 \quad \text{for all } t\in(0,T_0). \tag{21}$$

By reasoning as before and by recalling that $0\le v_i\le \|v_0\|_{L^\infty(\Omega)}$ on $\Omega\times(0,T_0)$, for $i=1,2$, we can write considering the problems in (18) and definitions (20)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (v_1-v_2)^2&=\int_\Omega (v_1-v_2)(v_1-v_2)_t=\int_\Omega (v_1-v_2)\left(\Delta(v_1-v_2)-f(u_1)v_1+f(u_2)v_2\right)\\
&=-\int_\Omega |\nabla(v_1-v_2)|^2+\int_\Omega (v_1-v_2)(f(u_2)-f(u_1))v_1-\int_\Omega f(u_2)(v_1-v_2)^2\\
&\le -\int_\Omega |\nabla(v_1-v_2)|^2+\|v_0\|_{L^\infty(\Omega)}\int_\Omega (v_1-v_2)f'(\overline{u})(u_2-u_1)\\
&\le -\int_\Omega |\nabla(v_1-v_2)|^2+\int_\Omega (v_1-v_2)^2+\frac{C_2^2\|v_0\|^2_{L^\infty(\Omega)}}{4}\int_\Omega (u_1-u_2)^2 \quad \text{on } (0,T_0).
\end{aligned} \tag{22}$$

We also get from problems (19)

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega (u_1-u_2)^2&=\int_\Omega (u_1-u_2)(u_1-u_2)_t\\
&=-\int_\Omega \nabla(u_1-u_2)\cdot\big((u_1+1)^{m_1-1}\nabla u_1-(u_2+1)^{m_1-1}\nabla u_2\big)\\
&\quad+\chi\int_\Omega \nabla(u_1-u_2)\cdot\big(u_1(u_1+1)^{m_2-1}\nabla v_1-u_2(u_2+1)^{m_2-1}\nabla v_2\big)\\
&\quad-\xi\int_\Omega \nabla(u_1-u_2)\cdot\big(u_1(u_1+1)^{m_3-1}\nabla w_1-u_2(u_2+1)^{m_3-1}\nabla w_2\big) \quad \text{for all } t\in(0,T_0),
\end{aligned}$$

where the second, third, and fourth lines are denoted, for simplicity, by I_1, I_2 and I_3, respectively. We now estimate these separate terms as follows (recall (20), again, and the definition of θ):

$$\begin{aligned}
I_1&=-\int_\Omega \nabla(u_1-u_2)\cdot\nabla\left(\theta(u_1)-\theta(u_2)\right)\\
&=\int_\Omega \Delta(u_1-u_2)(\theta(u_1)-\theta(u_2))=\int_\Omega \Delta(u_1-u_2)\theta'(\overline{u})(u_1-u_2)\\
&\le C_3\int_\Omega (u_1-u_2)\Delta(u_1-u_2)=-C_3\int_\Omega |\nabla(u_1-u_2)|^2 \quad \text{for all } t\in(0,T_0).
\end{aligned} \tag{23}$$

On the other hand, once more the smoothness of $\zeta \longmapsto (1+\zeta)^{\vartheta}$ for all $\vartheta \in \mathbb{R}$ and $\zeta \geq 0$, makes that for some $C_4 = C_4(T_0) > 0$ and $C_5 = C_5(T_0) > 0$ (which we will now use)

$$\begin{cases} |u_1(u_1+1)^{m_2-1} - u_2(u_2+1)^{m_2-1}| \leq C_4|u_1-u_2| & \text{in } \Omega \times (0,T_0), \\ u_2(u_2+1)^{m_2-1} \leq C_4 & \text{in } \Omega \times (0,T_0), \\ |\nabla v_1| \leq C_4 \text{ and } |\nabla v_2| \leq C_4 & \text{in } \Omega \times (0,T_0), \end{cases}$$

$$\begin{cases} |u_1(u_1+1)^{m_3-1} - u_2(u_2+1)^{m_3-1}| \leq C_5|u_1-u_2| & \text{in } \Omega \times (0,T_0), \\ u_2(u_2+1)^{m_3-1} \leq C_5 & \text{in } \Omega \times (0,T_0), \\ |\nabla w_1| \leq C_5 \text{ and } |\nabla w_2| \leq C_5 & \text{in } \Omega \times (0,T_0). \end{cases}$$

Henceforth, starting from the identity

$$I_2 = \chi \int_\Omega \nabla(u_1-u_2) \cdot \big((u_1(u_1+1)^{m_2-1} - u_2(u_2+1)^{m_2-1})\nabla v_1 + u_2(u_2+1)^{m_2-1}\nabla(v_1-v_2)\big) \quad \text{on } (0,T_0),$$

and using Schwartz' inequality we obtain

$$\begin{aligned} I_2^2 &\leq \chi^2 \int_\Omega |(u_1(u_1+1)^{m_2-1} - u_2(u_2+1)^{m_2-1})\nabla v_1 + u_2(u_2+1)^{m_2-1}\nabla(v_1-v_2)|^2 \int_\Omega |\nabla(u_1-u_2)|^2 \\ &\leq 2\chi^2\Big(\int_\Omega |(u_1(u_1+1)^{m_2-1} - u_2(u_2+1)^{m_2-1})\nabla v_1|^2 + \int_\Omega |u_2(u_2+1)^{m_2-1}\nabla(v_1-v_2)|^2\Big) \int_\Omega |\nabla(u_1-u_2)|^2 \\ &\leq 2\chi^2 C_4^2\Big(C_4^2 \int_\Omega (u_1-u_2)^2 + \int_\Omega |\nabla(v_1-v_2)|^2\Big) \int_\Omega |\nabla(u_1-u_2)|^2 \quad \text{for all } t \in (0,T_0). \end{aligned}$$

Taking the square root, we get by Young's inequality

$$\begin{aligned} I_2 &\leq \chi C_4 \sqrt{2} \left(C_4 \left(\int_\Omega (u_1-u_2)^2 \right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla(v_1-v_2)|^2 \right)^{\frac{1}{2}} \right) \left(\int_\Omega |\nabla(u_1-u_2)|^2 \right)^{\frac{1}{2}} \\ &\leq \frac{C_3}{2} \int_\Omega |\nabla(u_1-u_2)|^2 + \frac{2\chi^2 C_4^4}{C_3} \int_\Omega (u_1-u_2)^2 + \frac{2\chi^2 C_4^2}{C_3} \int_\Omega |\nabla(v_1-v_2)|^2 \quad \text{on } (0,T_0). \end{aligned} \tag{24}$$

Similarly, from

$$\begin{aligned} I_3 = \xi \int_\Omega \nabla(u_1-u_2) \cdot \big(&\left(u_2(u_2+1)^{m_3-1} - u_1(u_1+1)^{m_3-1}\right) \nabla w_2 \\ &+ u_1(u_1+1)^{m_3-1}\nabla(w_2-w_1)\big) \quad \text{on } (0,T_0), \end{aligned}$$

we obtain, also by employing estimate (21),

$$
\begin{aligned}
I_3 &\le \xi C_5 \sqrt{2}\left(C_5\left(\int_\Omega (u_1-u_2)^2\right)^{\frac{1}{2}} + \left(\int_\Omega |\nabla(w_1-w_2)|^2\right)^{\frac{1}{2}}\right)\left(\int_\Omega |\nabla(u_1-u_2)|^2\right)^{\frac{1}{2}} \\
&\le \frac{C_3}{2}\int_\Omega |\nabla(u_1-u_2)|^2 + \frac{2\xi^2 C_5^4}{C_3}\int_\Omega (u_1-u_2)^2 + \frac{2\xi^2 C_5^2}{C_3}\int_\Omega |\nabla(w_1-w_2)|^2 \\
&\le \frac{C_3}{2}\int_\Omega |\nabla(u_1-u_2)|^2 + \frac{2\xi^2 C_5^2}{C_3}\left(C_5^2 + \frac{C_1^2}{\delta}\right)\int_\Omega (u_1-u_2)^2 \quad \text{on } (0,T_0).
\end{aligned} \tag{25}
$$

Now for $K_0 = 2\chi^2 C_4^2/C_3$ let us consider the sum

$$
\mathcal{F}(t) := \frac{1}{2}\frac{d}{dt}\int_\Omega (u_1-u_2)^2 + \frac{K_0}{2}\frac{d}{dt}\int_\Omega (v_1-v_2)^2 \quad \text{on } (0,T_0). \tag{26}
$$

Henceforth, given the gained bounds (22)-(25), expression (26) is estimated above by

$$
\left(\frac{2\chi^2 C_4^4}{C_3} + \frac{2\xi^2 C_5^4}{C_3} + \frac{2\xi^2 C_5^2 C_1^2}{C_3\delta} + \frac{C_2^2 K_0 \|v_0\|^2_{L^\infty(\Omega)}}{4}\right)\int_\Omega (u_1-u_2)^2 + K_0\int_\Omega (v_1-v_2)^2 \quad \text{for all } t\in(0,T_0),
$$

so that (26) becomes for a suitable positive constant $\tilde{C}$

$$
\frac{d}{dt}\mathcal{F}(t) \le \tilde{C}\mathcal{F}(t), \qquad t\in(0,T_0).
$$

We then exploit the nonnegativity of $\mathcal{F}(t)$, the initial condition $\mathcal{F}(0)=0$ (based on $u_1(\cdot,0)=u_2(\cdot,0)$ and $v_1(\cdot,0)=v_2(\cdot,0)$), and Gronwall's inequality to prove that $\mathcal{F}(t)\equiv 0$. As a result, $u_1-u_2=0$ and $v_1-v_2=0$ on $\Omega\times(0,T_0)$ and consequently, recalling again problems (17), we manifestly get $w_1-w_2=0$, as well on $\Omega\times(0,T_0)$. Since $T_0<T_{max}$ is arbitrary, the proof is concluded. □

With a local solution (u,v,w) at our disposal, its uniform-in-time boundedness is ensured whenever a uniform-in-time estimate for the L^p-norm of u, for some suitable $p>1$, is obtainable.

Lemma 4.2. *Under the hypotheses of Lemma 4.1, if the u-component belongs to $L^\infty((0,T_{max});L^p(\Omega))$, with $p>1$ arbitrarily large, then the life span $T_{max}=\infty$, i.e. (u,v,w) is global in time. In addition, u,v and w are uniformly bounded in $\Omega\times(0,\infty)$, in the sense that for some $C>0$*

$$
\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{L^\infty(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \le C \quad \textit{for all} \quad t\in(0,\infty).
$$

Proof. From our hypotheses on u and assumptions on f and g, we also have $f,g\in L^\infty((0,T_{max});L^p(\Omega))$ for arbitrarily large $p>1$; henceforth, by invoking classical regularity results on elliptic and parabolic equations, $\nabla v,\nabla w\in L^\infty((0,T_{max});L^p(\Omega))$, as well. Subsequently, for $F(x,t)=-\chi u(u+1)^{m_2-1}\nabla v+\xi u(u+1)^{m_3-1}\nabla w$ and $D(x,t,u)=(u+1)^{m_1-1}$, the first equation of problem (1) reads $u_t=\nabla\cdot(D(x,t,u)\nabla u)+\nabla\cdot F(x,t)$. Since $F\in L^\infty((0,T_{max});L^{q_1}(\Omega))$ for every $q_1>1$, the conclusion $u\in L^\infty((0,T_{max});L^\infty(\Omega))$ follows directly by relying on [41, Lemma A.1]. Finally, $u\in L^\infty((0,T_{max});L^\infty(\Omega))$ and $v\in L^\infty((0,T_{max});L^\infty(\Omega))$ imply from the dichotomy criterion (11) that necessarily we must have $T_{max}=\infty$, so that actually $u,v,w\in L^\infty((0,\infty);L^\infty(\Omega))$. □

5. A priori estimates and proof of the theorem

In the light of what has been established, the key step is controlling the L^p-norm of u, for $p > 1$: this is attained by constructing an absorptive differential inequality for the functional $y(t) := \int_\Omega (u+1)^p + \int_\Omega |\nabla v|^{2p}$, defined on $(0, T_{max})$. In this direction, some a priori estimates are crucial.

We start with this result, which is the only place where the restriction $0 < \alpha < \frac{1}{2} + \frac{1}{n}$ on the function f, imposing in Theorem 2.1, is needed; indeed, this allows us to control the L^2-norm of ∇v, successively necessary in the applications of the Gagliardo–Nirenberg inequality.

Lemma 5.1. *For any $n \geq 2$ and $\alpha \in (0, \frac{1}{2} + \frac{1}{n})$, let f comply with assumptions (6). Then, under the remaining hypotheses of Lemma 4.1, for some $c_0 > 0$ the v-component is such that*

$$\int_\Omega |\nabla v(\cdot, t)|^2 \leq c_0 \quad \text{on } (0, T_{max}). \tag{27}$$

Proof. We distinguish the cases $0 < \alpha \leq \frac{1}{2}$ and $\frac{1}{2} < \alpha < \frac{1}{2} + \frac{1}{n}$. For $0 < \alpha \leq \frac{1}{2}$, from the second equation of (1), we have that an integration over Ω, the Young inequality, the bound for v given in (13) and the hypotheses on f in (6) lead to

$$\begin{aligned} \frac{d}{dt}\int_\Omega |\nabla v|^2 &= 2\int_\Omega \nabla v \cdot \nabla(\Delta v - f(u)v) = -2\int_\Omega (\Delta v)^2 + 2\int_\Omega f(u)v\Delta v \\ &= -2\int_\Omega (\Delta v)^2 + 2\int_\Omega v(f(u)-1)\Delta v - 2\int_\Omega |\nabla v|^2 \\ &\leq -\int_\Omega (\Delta v)^2 - 2\int_\Omega |\nabla v|^2 + \int_\Omega v^2(f(u)-1)^2 \\ &\leq -2\int_\Omega |\nabla v|^2 + \|v_0\|^2_{L^\infty(\Omega)} K^2 \int_\Omega u^{2\alpha} \\ &\quad + 2\|v_0\|^2_{L^\infty(\Omega)} K \int_\Omega u^\alpha + \|v_0\|^2_{L^\infty(\Omega)} |\Omega| \quad \text{on } (0, T_{max}). \end{aligned} \tag{28}$$

Now, from the mass conservation property (12), we have thanks to Hölder's inequality the uniform-in-time finiteness of both $\int_\Omega u^\alpha$ and $\int_\Omega u^{2\alpha}$; in this way, there is $c_1 > 0$ such that

$$\|v_0\|^2_{L^\infty(\Omega)} K^2 \int_\Omega u^{2\alpha} + 2\|v_0\|^2_{L^\infty(\Omega)} K \int_\Omega u^\alpha + \|v_0\|^2_{L^\infty(\Omega)} |\Omega| \leq c_1 \quad \text{with } t \in (0, T_{max}),$$

so that (28) reads

$$\frac{d}{dt}\int_\Omega |\nabla v|^2 \leq -2\int_\Omega |\nabla v|^2 + c_1 \quad \text{on } (0, T_{max}),$$

and a comparison argument entails $\int_\Omega |\nabla v|^2 \leq \max\{\frac{c_1}{2}, \int_\Omega |\nabla v_0|^2\}$ for all $t \in (0, T_{max})$.

When, indeed, $\frac{1}{2} < \alpha < \frac{1}{2} + \frac{1}{n}$, we can pick $\frac{1}{2} < \rho < 1 - \frac{n}{2}(\alpha - \frac{1}{2})$ and set $\zeta = 1 - \rho - \frac{n}{2}(\alpha - \frac{1}{2}) > 0$; subsequently, the Gamma function Γ ensures the convergence of this integral:

$$\int_0^t (t-s)^{-\rho-\frac{n}{2}(\alpha-\frac{1}{2})} e^{-\lambda_1(t-s)} ds. \tag{29}$$

On the other hand, through the Hölder inequality, by invoking (6) and once (12), we get

$$\|f(u(\cdot,t))\|^{\frac{1}{\alpha}}_{L^{\frac{1}{\alpha}}(\Omega)} = \int_\Omega f(u)^{\frac{1}{\alpha}} \le K^{\frac{1}{\alpha}} \int_\Omega u = K^{\frac{1}{\alpha}} m \quad \text{for all } t < T_{max}. \tag{30}$$

As a consequence, from the representation formula for v, we have

$$v(\cdot,t) = e^{t\Delta} v_0 - \int_0^t e^{(t-s)\Delta} f(u(\cdot,s)) v(\cdot,s) ds \quad \text{for all } \ t \in (0, T_{max}),$$

and smoothing properties related to the Neumann heat semigroup $(e^{t\Delta})_{t\ge 0}$ (see Section 2 of [11] and Lemma 1.3 of [49]), provide some $\lambda_1 > 0$, $C_S > 0$ and $c_2 > 0$, such that, once the bounds $v \le \|v_0\|_{L^\infty(\Omega)}$ on $\bar{\Omega} \times (0, T_{max})$ and (30) are considered,

$$\begin{aligned}
\|v(\cdot,t)\|_{W^{1,2}(\Omega)} &\le \|e^{t\Delta} v_0\|_{W^{1,2}(\Omega)} + \int_0^t \|e^{(t-s)\Delta} f(u(\cdot,s)) v(\cdot,s)\|_{W^{1,2}(\Omega)} ds \\
&\le C_S \|v_0\|_{W^{1,2}(\Omega)} + C_S \int_0^t \|(-\Delta+1)^\rho e^{(t-s)\Delta} f(u(\cdot,s)) v(\cdot,s)\|_{L^2(\Omega)} ds \\
&\le C_S \|v_0\|_{W^{1,2}(\Omega)} + C_S \|v_0\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{2}} \int_0^t (t-s)^{-\rho-\frac{n}{2}(\alpha-\frac{1}{2})} e^{-\lambda_1(t-s)} \|f(u(\cdot,s))\|_{L^{\frac{1}{\alpha}}(\Omega)} ds \\
&\le c_2 \Big(1 + \int_0^t (t-s)^{-\rho-\frac{n}{2}(\alpha-\frac{1}{2})} e^{-\lambda_1(t-s)} ds \Big).
\end{aligned}$$

By recalling the finiteness of (29), relation in (27) is achieved with some computable $c_0 > 0$. □

Remark 4. We observe that the estimate in (27) is crucial when controlling the functional $y(t) = \int_\Omega (u+1)^p + \int_\Omega |\nabla v|^{2p}$; indeed, a proper version of the Gagliardo–Nirenberg inequality, as well as the proof of Lemma 5.3, strongly rely on this bound. The case $\alpha = 1$, for which relation (27) does not hold true, cannot be discussed in the frame of our computations, and subsequently it is herein excluded.

Let us now analyze, separately in the two following lemmas, the evolution of $\int_\Omega (u+1)^p$ and $\int_\Omega |\nabla v|^{2p}$, for $p > 1$ and over $(0, T_{max})$.

Lemma 5.2. *For any integer $n \ge 2$, $m_1, m_2, m_3 \in \mathbb{R}$ and $l \ge 1$ complying with either $m_1 > 1 + 2m_2 - m_3 - l$ or $m_1 > \max\{2m_2, \frac{n-2}{n}\}$, let g fulfill assumption (6). Then, under the remaining hypotheses of Lemma 4.1, there exists $p_0 > 1$ such that for any $p > p_0$ and some $C(p, \xi, \gamma, m_3)$ and $c_{14} > 0$, (u, v, w) satisfies*

$$\begin{aligned}
\frac{d}{dt} \int_\Omega (u+1)^p \le& -\frac{2p(p-1)}{(m_1+p-1)^2} \int_\Omega |\nabla (u+1)^{\frac{m_1+p-1}{2}}|^2 + \frac{p}{8(4p^2+n)\|v_0\|^2_{L^\infty(\Omega)}} \int_\Omega |\nabla v|^{2(p+1)} \\
&- C(p,\xi,\gamma,m_3) \int_\Omega (u+1)^{m_3+p+l-1} + c_{14} \qquad \textit{for all } t \in (0, T_{max}).
\end{aligned}$$

Proof. Let $p > p_0 > 1 - m_3$; henceforth, upon enlarging p_0 when necessary, all the following implications are justified for $p > p_0$. Testing the first equation of problem (1) by $p(u+1)^{p-1}$ and using its boundary conditions provide on $(0, T_{max})$

$$\frac{d}{dt}\int_\Omega (u+1)^p = \int_\Omega p(u+1)^{p-1}u_t = -p(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 + p(p-1)\chi\int_\Omega u(u+1)^{m_2+p-3}\nabla u\cdot\nabla v$$
$$- p(p-1)\xi\int_\Omega u(u+1)^{m_3+p-3}\nabla u\cdot\nabla w.$$

By putting $h_{p,m_3}(u) = p(p-1)\int_0^u \hat{u}(\hat{u}+1)^{m_3+p-3}\,d\hat{u}$ and recalling the trivial inequality $u < (u+1)$ we get

$$\frac{p(p-1)}{p+m_3-1}u^{p+m_3-1} \le h_{p,m_3}(u) \le \frac{p(p-1)}{p+m_3-1}[(u+1)^{p+m_3-1} - 1] \quad \text{in} \quad \Omega\times(0,T_{max}). \tag{31}$$

Therefore, exploiting the third equation of (1) and the growth of g mentioned in (6) imply

$$\begin{aligned}
\frac{d}{dt}\int_\Omega (u+1)^p &= -p(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2\\
&\quad + p(p-1)\chi\int_\Omega u(u+1)^{m_2+p-3}\nabla u\cdot\nabla v + \xi\int_\Omega h_{p,m_3}\Delta w\\
&\le -p(p-1)\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 + p(p-1)\chi\int_\Omega u(u+1)^{m_2+p-3}\nabla u\cdot\nabla v\\
&\quad + \frac{p(p-1)\xi\delta}{m_3+p-1}\int_\Omega (u+1)^{p+m_3-1}w - \frac{p(p-1)\xi\gamma}{m_3+p-1}\int_\Omega u^l\\
&\quad - \frac{p(p-1)\xi\gamma}{2^{p-1}(m_3+p-1)}\int_\Omega (u+1)^{p+m_3+l-1}\\
&\quad + \frac{p(p-1)\xi\gamma|\Omega|}{m_3+p-1} \quad \text{for all } t\in(0,T_{max}),
\end{aligned} \tag{32}$$

where in the last implication we have used that $\delta\int_\Omega w = \int_\Omega g(u)$ (as a result of an integration over Ω of the third equation in (1)), estimate (31) and

$$(A+B)^p \le 2^{p-1}(A^p + B^p) \quad \text{with} \ \ A, B \ge 0 \quad \text{ and } p > 1.$$

Now we focus on the third integral on the right-hand side of (32). Thanks again to the assumptions on g, Young's inequality and (10) with $\psi = w$ and $\bar{p} = \frac{m_3+p-1}{l}$, give for $\epsilon_1, \sigma, \tilde{\sigma} > 0$ and some positive c_3

$$\begin{aligned}
\frac{p(p-1)\xi\delta}{m_3+p-1}\int_\Omega (u+1)^{m_3+p-1}w &\le \epsilon_1\int_\Omega (u+1)^{m_3+p+l-1} + \hat{c}\int_\Omega w^{\frac{m_3+p-1}{l}+1}\\
&\le \epsilon_1\int_\Omega (u+1)^{m_3+p+l-1} + \sigma\int_\Omega (g(u))^{\frac{m_3+p-1}{l}+1} + \tilde{c}\left(\int_\Omega g(u)\right)^{\frac{m_3+p-1}{l}+1}\\
&\le (\epsilon_1+\tilde{\sigma})\int_\Omega (u+1)^{m_3+p+l-1} + c_3\left(\int_\Omega (u+1)^l\right)^{\frac{m_3+p-1}{l}+1} \quad \text{for } t\in(0,T_{max}).
\end{aligned} \tag{33}$$

By exploiting the Gagliardo–Nirenberg interpolation inequality ([34]), we arrive for $p > p_0$ at

$$\theta_1 = \frac{\frac{n(m_1+p-1)}{2}\left(1-\frac{1}{l}\right)}{1-\frac{n}{2}+\frac{n(m_1+p-1)}{2}} \in (0,1),$$

and some $c_4 > 0$ provides

$$\begin{aligned} c_3\left(\int_\Omega (u+1)^l\right)^{\frac{m_3+p-1+l}{l}} &= c_3\|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2l}{m_1+p-1}}(\Omega)}^{\frac{2(m_3+p-1+l)}{m_1+p-1}} \\ &\le c_4\|\nabla(u+1)^{\frac{m_1+p-1}{2}}\|_{L^2(\Omega)}^{\frac{2(m_3+p-1+l)}{m_1+p-1}\theta_1}\|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2}{m_1+p-1}}(\Omega)}^{\frac{2(m_3+p-1+l)}{m_1+p-1}(1-\theta_1)} \\ &+ c_4\|(u+1)^{\frac{m_1+p-1}{2}}\|_{L^{\frac{2}{m_1+p-1}}(\Omega)}^{\frac{2(m_3+p-1+l)}{m_1+p-1}} \quad \text{on } (0,T_{max}). \end{aligned}$$

Hence, by recalling the mass conservation property (12), and in view of $\frac{(m_3+p-1+l)}{m_1+p-1}\theta_1 < 1$, the Young and above inequalities entail for $c_5, c_6 > 0$ and any $\epsilon_2 > 0$

$$\begin{aligned} c_3\left(\int_\Omega (u+1)^l\right)^{\frac{m_3+p-1+l}{l}} &\le c_5\Big(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\Big)^{\frac{(m_3+p-1+l)}{m_1+p-1}\theta_1} + c_5 \\ &\le \epsilon_2\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 + c_6 \quad \text{on } (0,T_{max}). \end{aligned} \tag{34}$$

In turn, the cases $m_3+l-2m_2+m_1 > 1$ and $m_1 > \max\{2m_2, \frac{n-2}{n}\}$ are, respectively, addressed. By applying twice the Young inequality to the second integral on the right-hand side of (32), we get on $(0,T_{max})$

$$\begin{aligned} p(p-1)\chi\int_\Omega u(u+1)^{m_2+p-3}\nabla u\cdot\nabla v &\le \frac{p(p-1)}{4}\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 + c_7\int_\Omega (u+1)^{p+2m_2-m_1-1}|\nabla v|^2 \\ \le \frac{p(p-1)}{4}\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 &+ \frac{p}{8(4p^2+n)\|v_0\|^2_{L^\infty(\Omega)}}\int_\Omega |\nabla v|^{2(p+1)} + c_8\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)(p+1)}{p}}, \end{aligned} \tag{35}$$

with positive c_7, c_8. Since $m_3+l-2m_2+m_1 > 1$, we deduce that $\frac{(p+2m_2-m_1-1)(p+1)}{p} < (m_3+p+l-1)$, and Young's inequality infers for every $\epsilon_3 > 0$, some positive $c_9 > 0$ giving for all $p > p_0$

$$c_8\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)(p+1)}{p}} \le \epsilon_3\int_\Omega (u+1)^{m_3+p+l-1} + c_9 \quad \text{on } (0,T_{max}). \tag{36}$$

On the other hand, when $m_1 > \max\{2m_2, \frac{n-2}{n}\}$, we note (recall that we can enlarge p_0 when necessary) that $m_1 > 2m_2$ and any $p > p_0$ imply $\frac{(p+2m_2-m_1-1)(p+1)}{p} < p$, so by means of the Young inequality estimate (36) can alternatively read

$$c_8\int_\Omega (u+1)^{\frac{(p+2m_2-m_1-1)(p+1)}{p}} \le \epsilon_4\int_\Omega (u+1)^p + c_{10} \quad \text{on } (0,T_{max}), \tag{37}$$

with $\epsilon_4 > 0$ and positive c_{10}. In turn, a further application of the Gagliardo–Nirenberg inequality yields, for all p larger than some appropriate $p_0 > 1$,

$$\theta_2 = \frac{\frac{n(m_1+p-1)}{2}\left(1-\frac{1}{p}\right)}{1-\frac{n}{2}+\frac{n(m_1+p-1)}{2}} \in (0,1), \tag{38}$$

so entailing for certain $c_{11} > 0$

$$\int_\Omega (u+1)^p = \|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2p}{m_1+p-1}}_{L^{\frac{2p}{m_1+p-1}}(\Omega)} \le c_{11}\|\nabla(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2p}{m_1+p-1}\theta_2}_{L^2(\Omega)}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2p}{m_1+p-1}(1-\theta_2)}_{L^{\frac{2}{m_1+p-1}}(\Omega)}$$
$$+ c_{11}\|(u+1)^{\frac{m_1+p-1}{2}}\|^{\frac{2p}{m_1+p-1}}_{L^{\frac{2}{m_1+p-1}}(\Omega)} \quad \text{for all } t \in (0, T_{max}).$$

Similarly to what was done in some previous lines, from (12), and $\frac{p}{m_1+p-1}\theta_2 < 1$ for $m_1 > \frac{n-2}{n}$, the Young inequality implies for any positive ϵ_5 and some positive $c_{12}, c_{13} > 0$

$$\epsilon_4 \int_\Omega (u+1)^p \le c_{12}\Big(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\Big)^{\frac{p}{m_1+p-1}\theta_2} + c_{12} \le \epsilon_5 \int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2 + c_{13} \quad \text{on } (0, T_{max}). \tag{39}$$

By plugging estimates (33) and (35) into relation (32), as well as taking into account (34) and (36) (or, alternatively to (36), bounds (37) and (39)), imply for proper ϵ_i, $i = 1, \dots, 5$, $C(p, \xi, \gamma, m_3) = \frac{p(p-1)\xi\gamma}{2^{p+1}(m_3+p-1)}$ and some $c_{14} > 0$ the claim, having also exploited that $\int_\Omega (u+1)^{p+m_1-3}|\nabla u|^2 = \frac{4}{(m_1+p-1)^2}\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2$ on $(0, T_{max})$. □

Lemma 5.3. *For any integer $n \ge 2$ and $\alpha \in (0, \frac{1}{2} + \frac{1}{n})$, let f comply with assumptions (6). Then, under the remaining hypotheses of Lemma 4.1, there exists $p_0 > 1$ such that any $p > p_0$ implies that for (u, v, w) the estimate*

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 \le \epsilon_6 \int_\Omega (u+1)^{m_3+p+l-1}$$
$$+ \frac{p}{8(4p^2+n)\|v_0\|^2_{L^\infty(\Omega)}}\int_\Omega |\nabla v|^{2(p+1)} + c_{19} \quad \textit{on } (0, T_{max}),$$

holds true for every $\epsilon_6 > 0$ and some positive constant c_{19}.

Proof. The conclusion can be obtained, taking into account the equation for v in (1), by controlling the term $(|\nabla v|^2)_t$ and by analyzing the temporal evolution of $\int_\Omega |\nabla v|^{2p}$, for some $p > p_0 > 1$. In particular, an adaptation to our case of the general framework in [23, Lemma 4.2] (see also [31, Lemma 5.2]), allows one to deduce that for certain $c_{15}, c_{16} > 0$ (and relying also on estimate (27))

$$\frac{d}{dt}\int_\Omega |\nabla v|^{2p} + p\int_\Omega |\nabla v|^{2p-2}|D^2 v|^2 \le c_{15}\int_\Omega u^{2\alpha}|\nabla v|^{2p-2} + c_{16} \quad \text{on } (0, T_{max}). \tag{40}$$

Further, a double application of the Young inequality gives for all $p > p_0 > 1$ (properly large), arbitrary $\epsilon_6 > 0$ and some $c_{17}, c_{18} > 0$

$$
\begin{aligned}
c_{15}\int_\Omega u^{2\alpha}|\nabla v|^{2p-2} &\le \epsilon_6\int_\Omega (u+1)^{m_3+p+l-1} + c_{17}\int_\Omega |\nabla v|^{\frac{2(p-1)(m_3+p+l-1)}{m_3+p+l-1-2\alpha}}\\
&\le \epsilon_6\int_\Omega (u+1)^{m_3+p+l-1}\\
&\quad+\frac{p}{8(4p^2+n)\|v_0\|^2_{L^\infty(\Omega)}}\int_\Omega |\nabla v|^{2(p+1)}+c_{18}\quad\text{for all}\quad t\in(0,T_{max}),
\end{aligned}\tag{41}
$$

where we used that, by virtue of $\alpha<1$, one has $\frac{2(p-1)(m_3+p+l-1)}{m_3+p+l-1-2\alpha}<2(p+1)$. We have the claim by inserting relation (41) into (40). □

We are now in a position to show that $y(t)$ is uniformly bounded in $(0,T_{max})$, as a consequence of the fact that $\int_\Omega u^p$ is actually bounded by a time independent constant on $(0,T_{max})$.

Lemma 5.4. *For any integer $n\ge 2$, $\alpha\in(0,\frac{1}{2}+\frac{1}{n})$, $m_1,m_2,m_3\in\mathbb{R}$ and $l\ge 1$ complying with either $m_1>1+2m_2-m_3-l$ or $m_1>\max\{2m_2,\frac{n-2}{n}\}$, let f and g fulfill assumptions (6). Then, under the remaining hypotheses of Lemma 4.1, there exists $p_0>1$ such that for all $p>p_0$ the u-component belongs to $L^\infty((0,T_{max});L^p(\Omega))$.*

Proof. Once the functional $y(t):=\int_\Omega(u+1)^p+\int_\Omega|\nabla v|^{2p}$ is considered again, by summing up the expressions of Lemma 5.2 and Lemma 5.3, we deduce for $c_{20}>0$ the following estimate

$$
\begin{aligned}
&y'(t)+\frac{2p(p-1)}{(m_1+p-1)^2}\int_\Omega|\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2+p\int_\Omega|\nabla v|^{2p-2}|D^2v|^2\\
&\le(\epsilon_6-C(p,\xi,\gamma,m_3))\int_\Omega(u+1)^{m_3+p+l-1}\\
&\quad+\frac{p}{4(4p^2+n)\|v_0\|^2_{L^\infty(\Omega)}}\int_\Omega|\nabla v|^{2(p+1)}+c_{20}\quad\text{on }(0,T_{max}).
\end{aligned}\tag{42}
$$

Moreover, from bounds (9), used with $\psi=v$, and (13) we have

$$
\int_\Omega|\nabla v|^{2(p+1)}\le 2(4p^2+n)\|v_0\|^2_{L^\infty}\int_\Omega|\nabla v|^{2p-2}|D^2v|^2\qquad\text{for all }(0,T_{max}),
$$

so that by virtue of (recall (8))

$$
|\nabla|\nabla v|^p|^2=\frac{p^2}{4}|\nabla v|^{2p-4}|\nabla|\nabla v|^2|^2=p^2|\nabla v|^{2p-4}|D^2v\nabla v|^2\le p^2|\nabla v|^{2p-2}|D^2v|^2,
$$

for $\epsilon_6=C(p,\xi,\gamma,m_3)$, we can rephrase (42) as

$$
y'(t)+\frac{2p(p-1)}{(m_1+p-1)^2}\int_\Omega|\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2+\frac{1}{2p}\int_\Omega|\nabla|\nabla v|^p|^2\le c_{20}\qquad\text{on }(0,T_{max}).\tag{43}
$$

Successively, for a suitable $p_0>1$, let us set for $p>p_0$ and θ_2 already defined in (38)

$$
\kappa_1=\frac{p}{m_1+p-1}\theta_2\in(0,1)\qquad\text{and}\qquad\kappa_2=\frac{\frac{np}{2}\left(1-\frac{1}{p}\right)}{1-\frac{n}{2}+\frac{np}{2}}\in(0,1).
$$

Then, by relying again on the Gagliardo–Nirenberg inequality, there exist positive constants c_{21}, c_{22} such that

$$\int_\Omega (u+1)^p \le c_{21}\Big(\int_\Omega |\nabla(u+1)^{\frac{m_1+p-1}{2}}|^2\Big)^{\kappa_1} + c_{21} \quad \text{on } (0,T_{max}), \tag{44}$$

(as already done in (39)), and

$$\int_\Omega |\nabla v|^{2p} = |||\nabla v|^p||^2_{L^2(\Omega)} \le c_{22}||\nabla|\nabla v|^p||^{2\kappa_2}_{L^2(\Omega)}|||\nabla v|^p||^{2(1-\kappa_2)}_{L^{\frac{2}{p}}(\Omega)} + c_{22}|||\nabla v|^p||^2_{L^{\frac{2}{p}}(\Omega)} \quad \text{on } (0,T_{max}).$$

From this, the L^2-bound for ∇v in (27) infers some $c_{23} > 0$ leading to

$$\int_\Omega |\nabla v|^{2p} \le c_{23}\Big(\int_\Omega |\nabla|\nabla v|^p|^2\Big)^{\kappa_2} + c_{23} \quad t \in (0,T_{max}). \tag{45}$$

As a consequence of all of the above, by arranging inequalities (44) and (45) and successively using the results into (43), we can observe also by virtue of (see [31, Lemma 3.3])

$$A^{d_1} + B^{d_2} \ge 2^{-d}(A+B)^d - d_3 \quad \text{for any} \quad A,B \ge 0, d_1,d_2 > 0 \quad \text{and some } d,d_3 > 0,$$

that for appropriate positive constants c_{24} and c_{25}, and $\kappa = \min\{\frac{1}{\kappa_1}, \frac{1}{\kappa_2}\}$, the functional y solves this initial problem

$$\begin{cases} y'(t) \le c_{24} - c_{25}y^\kappa(t) & \text{for all } t \in (0,T_{max}), \\ y(0) = \int_\Omega (u_0+1)^p + \int_\Omega |\nabla v_0|^{2p}. \end{cases}$$

Finally, by relying on ODE comparison principles, we have that $\int_\Omega u^p \le y(t) \le \max\{y(0), \big(\frac{c_{24}}{c_{25}}\big)^{\frac{1}{\kappa}}\}$ for all $t \in (0,T_{max})$. □

Collecting the derived information, we conclude.

Proof of Theorem 2.1. Since Lemma 5.4 ensures that the u-component of the local solution (u,v,w) to problem (1) belongs to $L^\infty((0,T_{max}); L^p(\Omega))$ for arbitrarily large $p > 1$, we have the claim by invoking Lemma 4.2. □

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