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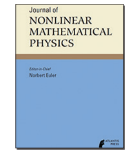

Research Article

Focusing NLS Equations with Nonzero Boundary Conditions: Triangular Representations and Direct Scattering

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ABSTRACT

In this article we derive the triangular representations of the fundamental eigensolutions of the focusing 1 + 1 AKNS system with symmetric nonvanishing boundary conditions. Its continuous spectrum equals $\mathbb{R} \cup [-i\mu, i\mu]$, where μ is the absolute value of the AKNS solution at spatial infinity. We also study the behavior of the scattering coefficients near the endpoints $\pm i\mu$ of the branch cut, where we distinguish between the generic case and the exceptional case.

1. INTRODUCTION

Nonlinear Schrödinger (NLS) equations have been fundamental in modeling nonlinear wave phenomena in plasmas [22,34], deep water surfaces [1,4,38], optical fibres [1,18,28], ferromagnetic materials [11,37], and Bose–Einstein condensates [26,27]. NLS equations with solutions decaying at infinity have been studied in detail [2–4,9,16]. After finding the Peregrine solutions [25], significant results on NLS equations with nonvanishing boundary conditions have been reported in Akhmediev et al. [5,6], Akhmediev and Korneev [7], Its et al. [19], Mihalache et al. [23], Tajiri and Watanabe [29], Zakharov and Gelash [35,36]. The direct and inverse scattering theory of the focusing NLS equation with nonvanishing boundary conditions has been studied systematically in Biondini and Kovačić [8], Demontis et al. [14]. We shall frequently refer to Demontis et al. [14] for some of the direct scattering results.

In this article we consider the focusing 1 + 1 AKNS system

$$v_x = (-ik\sigma_3 + \mathbf{Q})v, \tag{1.1}$$

where v is a function of $x \in \mathbb{R}$ with values in $\mathbb{R}^2$, $\sigma_3 = \mathrm{diag}(1, -1)$, and $\mathbf{Q} = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}$ is the potential. We assume that there exist two distinct 2×2 matrices $\mathbf{Q}_r = \begin{pmatrix} 0 & q_r \\ -q_r^* & 0 \end{pmatrix}$ and $\mathbf{Q}_l = \begin{pmatrix} 0 & q_l \\ -q_l & 0 \end{pmatrix}$ satisfying $\mu = |q_r| = |q_l| > 0$ such that for some $s \geq 0$ the integrability condition

$$(\boldsymbol{H}_s) \qquad \int_0^\infty dy(1+|y|)^s\left(\|\mathbf{Q}(-y)-\mathbf{Q}_l\|+\|\mathbf{Q}(y)-\mathbf{Q}_r\|\right)<+\infty \tag{1.2}$$

holds. Condition (1.2) is usually assumed for $s = 0, 1$. The Lax pair equations whose compatibility condition is equivalent to the focusing NLS equation, are discussed in detail in Appendix B (*cf.* [13]).

Under condition $(\boldsymbol{H}_0)$ and for $k \in \mathbb{R} \cup (-i\mu, i\mu)$, there exist two *fundamental eigensolutions* $\widetilde{\Phi}(x, k)$ and $\widetilde{\Psi}(x, k)$ of (1.1) satisfying the asymptotic conditions

$$\widetilde{\Phi}(x, k) = e^{x\Lambda_l(k)}[I_2 + o(1)], \qquad x \to -\infty, \tag{1.3a}$$

$$\widetilde{\Psi}(x, k) = e^{x\Lambda_r(k)}[I_2 + o(1)], \qquad x \to +\infty, \tag{1.3b}$$

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where

$$\Lambda_l(k) = -ik\sigma_3 + \mathbf{Q}_l, \qquad \Lambda_r(k) = -ik\sigma_3 + \mathbf{Q}_r. \tag{1.4}$$

Under condition $(\boldsymbol{H}_1)$ the fundamental eigensolutions can be defined for $k \in \mathbb{R} \cup [-i\mu, i\mu]$. Their existence can easily be proven by iterating the Volterra integral equations [14]

$$\widetilde{\Phi}(x,k) = e^{x\Lambda_l(k)} + \int_{-\infty}^{x} dy\, e^{(x-y)\Lambda_l(k)} \left[\mathbf{Q}(y) - \mathbf{Q}_l\right] \widetilde{\Phi}(y,k) e^{y\Lambda_l(k)}, \tag{1.5a}$$

$$\widetilde{\Psi}(x,k) = e^{x\Lambda_r(k)} - \int_{x}^{\infty} dy\, e^{-(y-x)\Lambda_r(k)} \left[\mathbf{Q}(y) - \mathbf{Q}_r\right] \widetilde{\Psi}(y,k) e^{y\Lambda_r(k)}. \tag{1.5b}$$

We observe that the corresponding matrix groups are given by

$$e^{x\Lambda_{r,l}(k)} = \left(\cos(\lambda x) I_2 + \frac{\sin(\lambda x)}{\lambda} \mathbf{Q}_{r,l}\right) - ik \frac{\sin(\lambda x)}{\lambda} \sigma_3, \tag{1.6}$$

where

$$\lambda(k) = \sqrt{k^2 + \mu^2} \tag{1.7}$$

appears in (1.6) only as the argument of even functions and hence the sign indeterminacy in defining $\lambda(k)$ by means of a square root does not affect (1.6).

In this article we derive in a rigorous way the triangular representations

$$\widetilde{\Phi}(x,k) = e^{x\Lambda_l(k)} + \int_{-\infty}^{x} dy\, \boldsymbol{J}(x,y) e^{y\Lambda_l(k)}, \tag{1.8a}$$

$$\widetilde{\Psi}(x,k) = e^{x\Lambda_r(k)} + \int_{x}^{\infty} dy\, \boldsymbol{K}(x,y) e^{y\Lambda_r(k)}, \tag{1.8b}$$

where for each $x \in \mathbb{R}$ the integrability condition

$$\int_{-\infty}^{x} dy\, \|\boldsymbol{J}(x,y)\| + \int_{x}^{\infty} \mathrm{d}y \|\boldsymbol{K}(x,y)\| < +\infty \tag{1.9}$$

holds. Triangular representations are well-known under vanishing boundary conditions [13,27–29] but have never been derived under integrability conditions of the form (1.9), with one notable exception. In Demontis et al. [14] the representations (1.8) have been derived under the conditions $(\boldsymbol{H}_1)$ and $q_x \in L^1(\mathbb{R})$ at the expense of replacing (1.9) by the quadratic integrability condition

$$\int_{-\infty}^{x} dy \|\boldsymbol{J}(x,y)\|^2 + \int_{x}^{\infty} dy \|\boldsymbol{K}(x,y)\|^2 < +\infty. \tag{1.10}$$

After establishing the triangular representations of the fundamental eigensolutions, we introduce the conformal mapping $\lambda(k)$ by (1.7) and distinguish between left and right versions of $k \in (-i\mu, i\mu)$ as boundary points of the analytic manifolds $k \in \mathbb{K}^{\pm}$ in 1,1-correspondence with the complex half-planes $\lambda \in \mathbb{C}^{\pm}$. We then go on to define the Jost functions and to derive their triangular representations. We also study in detail the conjugate transposition and conjugate non-transposition symmetries of the various quantities.

Once the triangular representations have been established, we go on studying the asymptotic behavior of the scattering coefficients as the spectral parameter k tends to $\pm i\mu$. In analogy with the case of the Schrödinger equation on the line [10,12,15], we shall make the distinction between the generic case where the corresponding Jost solutions are linearly independent, and the exceptional case where these solutions are proportional.

This paper is organized as follows. In Section 2 we establish the triangular representations of the fundamental eigensolutions. Their asymptotic behavior at either end of the real x-line will be the topic of Section 3. We then introduce the conformal mapping $\lambda(k)$, define the Jost functions, and derive their triangular representations in Section 4. Section 5 is devoted to the conjugation symmetry properties of the various quantities. Next, in Section 6 we investigate the asymptotic behavior of the scattering coefficients near the endpoints of the branch cut $k \in [-i\mu, i\mu]$ and prove the integrability of the Fourier transforms of the reflection coefficients.

Finally, we discuss the contents of the various appendices. In Appendix A we discuss the Wiener algebra of constants plus Fourier transforms of L^1-functions and invertibility within this algebra, well-known material treated at length in Gelfand et al. [17]. In this appendix we introduce the notation of writing $\mathcal{Z}^{n\times m}$ as the set of $n \times m$ matrices with entries in $\mathcal{Z}$. The time dependence of the scattering data is discussed in Appendix B. To avoid clogging notations in the main body of the paper, we do not indicate any time dependence unless it is absolutely necessary.

2. TRIANGULAR REPRESENTATIONS

In this section we derive the triangular representations of the fundamental eigensolutions in a rigorous way. We also relate the potential to the integral kernels appearing in the triangular representations.

Factoring out the asymptotic behavior of the fundamental eigensolutions, we write

$$\widetilde{\Phi}(x,k)=\tilde{M}(x,k)e^{x\Lambda_l(k)},\quad \widetilde{\Psi}(x,k)=\tilde{N}(x,k)e^{x\Lambda_r(k)}. \tag{2.1}$$

Then, under condition $(\boldsymbol{H}_1)$ and for $k\in\mathbb{R}\cup[-i\mu,i\mu]$, we easily derive from (1.5) the Volterra integral equations

$$\tilde{M}(x,k)=I_2+\int_0^\infty d\alpha\, e^{\alpha\Lambda_l(k)}\left[\mathbf{Q}(x-\alpha)-\mathbf{Q}_l\right]\tilde{M}(x-\alpha,k)e^{-\alpha\Lambda_l(k)}, \tag{2.2a}$$

$$\tilde{N}(x,k)=I_2-\int_0^\infty d\alpha\, e^{-\alpha\Lambda_r(k)}\left[\mathbf{Q}(x+\alpha)-\mathbf{Q}_r\right]\tilde{N}(x+\alpha,k)e^{\alpha\Lambda_r(k)}. \tag{2.2b}$$

These equations can be written in the form

$$\tilde{M}(x,k)-I_2=+\int_0^\infty d\alpha\, e^{\alpha\Lambda_l(k)}\left[\mathbf{Q}(x-\alpha)-\mathbf{Q}_l\right]e^{-\alpha\Lambda_l(k)}+\int_0^\infty d\alpha\, e^{\alpha\Lambda_l(k)}\left[\mathbf{Q}(x-\alpha)-\mathbf{Q}_l\right]\left(\tilde{M}(x-\alpha,k)-I_2\right)e^{-\alpha\Lambda_l(k)}, \tag{2.3a}$$

$$\tilde{N}(x,k)-I_2=-\int_0^\infty d\alpha\, e^{-\alpha\Lambda_r(k)}\left[\mathbf{Q}(x+\alpha)-\mathbf{Q}_r\right]e^{\alpha\Lambda_r(k)}-\int_0^\infty d\alpha\, e^{-\alpha\Lambda_r(k)}\left[\mathbf{Q}(x+\alpha)-\mathbf{Q}_r\right]\left(\tilde{N}(x+\alpha,k)-I_2\right)e^{\alpha\Lambda_r(k)}, \tag{2.3b}$$

where the triangular representations (1.8) are putatively written as

$$\tilde{M}(x,k)=I_2+\int_0^\infty d\alpha\, \boldsymbol{J}(x,x-\alpha)e^{-\alpha\Lambda_l(k)}, \tag{2.4a}$$

$$\tilde{N}(x,k)=I_2+\int_0^\infty d\alpha\, \boldsymbol{K}(x,x+\alpha)e^{\alpha\Lambda_r(k)}. \tag{2.4b}$$

We may thus convert the integral equations (2.3) into Volterra integral equations for the *integral kernels* $\boldsymbol{J}(x,x-\alpha)$ and $\boldsymbol{K}(x,x+\alpha)$. As in the vanishing case [3,13], we shall derive estimates for the integral kernels from the Volterra integral equations they satisfy.

Although at first sight the procedure explained in the past few lines may seem circular, below we shall in fact prove the existence of the integral kernels satisfying (1.9) by applying Gronwall's inequality to the putative integral equations obtained by Fourier transforming the Volterra integral equations (2.3). By Fourier transformation of their unique solutions, we then arrive at the triangular representations for the fundamental eigensolutions satisfying (2.3), thus completing their proof.

The triangular representations (2.4) can be viewed as integral transforms of the type described by the following result.

Proposition 2.1. *Suppose the entries of F belong to* $L^1(\mathbb{R}^+;(1+\alpha)d\alpha)$. *Then the integral transform*

$$\begin{aligned}\hat{F}(k)&=\int_0^\infty d\alpha\, F(\alpha)e^{\pm\alpha\Lambda_{r,l}(k)}\\&=\int_0^\infty d\alpha\cos(\lambda\,\alpha)F_e(\alpha)\mp ik\int_0^\infty d\alpha\,\cos(\lambda\alpha)F_o(\alpha)\end{aligned} \tag{2.5}$$

allows the inversion formula

$$F(\alpha)=\frac{4}{\pi^2}\int_0^\infty d\lambda\cos(\lambda\alpha)\left[\frac{\hat{F}(k)!+\hat{F}(-k)}{2}-\frac{\hat{F}(k)-\hat{F}(-k)}{2ik}\sigma_3\mathbf{Q}_{r,l}\right]. \tag{2.6}$$

Moreover,

$$F(\alpha)=F_e(\alpha)\mp F_o(\alpha)\,\sigma_3\mathbf{Q}_{r,l}. \tag{2.7}$$

Let us first convert the sinc and sinc^2 transforms into cosine transforms, where $L^1_s(\mathbb{R}^+)=L^1(\mathbb{R}^+;(1+|y|)^s dy)$. For $F\in L^1_1(\mathbb{R}^+)$ we have

$$\begin{aligned}\int_0^\infty dy\,\frac{\sin(\lambda y)}{\lambda}F(y)&=\left[-\frac{\sin(\lambda y)}{\lambda}\int_y^\infty dz\,F(z)\right]_0^\infty+\int_0^\infty dy\cos(\lambda y)\int_y^\infty dz\,F(z)\\&=\int_0^\infty dy\cos(\lambda y)\int_y^\infty dz\,F(z).\end{aligned} \tag{2.8}$$

Furthermore, for $F \in L_2^1(\mathbb{R}^+)$ we have

$$\int_0^\infty dy \left(\frac{\sin(\lambda y)}{\lambda}\right)^2 F(y) = \left[-\left(\frac{\sin(\lambda y)}{\lambda}\right)^2 \int_y^\infty dz\, F(z)\right]_0^\infty + \int_0^\infty dy \frac{\sin(2\lambda y)}{\lambda} \int_y^\infty dz\, F(z) = \left[-\frac{\sin(2\lambda y)}{\lambda}\int_y^\infty dw \int_w^\infty dz\, F(z)\right]_0^\infty$$
$$+2\int_0^\infty dy \cos(2\lambda y)\int_y^\infty dw \int_w^\infty dz\, F(z)$$
$$= \int_0^\infty dy \cos(\lambda y)\int_{y/2}^\infty dw \int_w^\infty dz\, F(z)$$
$$= \int_0^\infty dy \cos(\lambda y)\int_{y/2}^\infty dz \left(z - \frac{y}{2}\right) F(z).$$

Next, write the first line of (2.5) as

$$\hat{F}(k) = \int_0^\infty d\alpha\, F(\alpha) e^{\pm\alpha\Lambda_{r,l}(k)}$$
$$= \int_0^\infty d\alpha \cos(\lambda\alpha) F_e(\alpha) \mp ik \int_0^\infty d\alpha \cos(\lambda\alpha) F_o(\alpha),$$

where

$$\int_0^\infty d\alpha \cos(\lambda\alpha) F_e(\alpha) = \int_0^\infty d\alpha\, F(\alpha)\left(\cos(\lambda\alpha) I_2 \pm \frac{\sin(\lambda\alpha)}{\lambda} \mathbf{Q}_{r,l}\right),$$
$$\int_0^\infty d\alpha \cos(\lambda\alpha) F_o(\alpha) = \int_0^\infty d\alpha\, \frac{\sin(\lambda\alpha)}{\lambda} F(\alpha)\sigma_3.$$

Then (2.7) is true, while (2.8) implies $F_e(\alpha) = F(\alpha) \pm \int_\alpha^\infty d\beta\, F(\beta)\mathbf{Q}_{r,l}$ and $F_o(\beta) = \int_\alpha^\infty d\beta\, F(\beta)\sigma_3$. Consequently,

$$F(\alpha) = F_e(\alpha) \pm F_o(\alpha)\sigma_3 \mathbf{Q}_{r,l}. \tag{2.9}$$

We now observe that $\hat{F}(k)$ does not change when changing the sign of λ while keeping k invariant. Decomposing $\hat{F}(k)$ into k-even and k-odd functions of $k \in \mathbb{R} \cup [-i\mu, i\mu]$, we obtain

$$\frac{\hat{F}(k) + \hat{F}(-k)}{2} = \int_0^\infty d\alpha \cos(\lambda\alpha) F_e(\alpha),$$
$$\frac{\hat{F}(k) - \hat{F}(-k)}{\mp 2ik} = \int_0^\infty d\alpha \cos(\lambda\alpha) F_o(\alpha).$$

Applying Fourier cosine transform inversion and using (2.9) we obtain the inversion formula (2.6), as claimed.

Let us now derive the triangular representations of the fundamental eigensolutions following the procedure explained above.

Theorem 2.1. *Let condition ($\boldsymbol{H}_1$) be satisfied. Then there exist integral kernels $\boldsymbol{J}(x, y)$ and $\boldsymbol{K}(x, y)$ satisfying*

$$\int_{-\infty}^x dy\, \|\boldsymbol{J}(x, y)\| + \int_x^\infty dy\, \|\boldsymbol{K}(x, y)\| < +\infty,\ x \in \mathbb{R}, \tag{2.10}$$

such that the triangular representations

$$\tilde{\Phi}(x, k) = e^{x\Lambda_l(k)} + \int_{-\infty}^x dy\, \boldsymbol{J}(x, y) e^{y\Lambda_l(k)}, \tag{2.11a}$$
$$\tilde{\Psi}(x, k) = e^{x\Lambda_r(k)} + \int_x^\infty dy\, \boldsymbol{K}(x, y) e^{y\Lambda_r(k)}, \tag{2.11b}$$

hold true. Moreover, if condition ($\mathbf{H}_{s+1}$) is true for some $s \geq 0$, then for $x \in \mathbb{R}$ the integral kernels $\boldsymbol{J}(x, y)$ and $\boldsymbol{K}(x, y)$ satisfy the estimates

$$\int_{-\infty}^x dy (1 + x - y)^s \|\boldsymbol{J}(x, y)\| + \int_x^\infty dy (1 + y - x)^s \|\boldsymbol{K}(x, y)\| < +\infty. \tag{2.12}$$

We prove the triangular representation (2.11a) first, then use a symmetry argument to derive (2.11b) within a few lines, and then establish the second part of the theorem involving condition ($\boldsymbol{H}_{s+1}$) for $s = 0, 1, 2, \ldots$ by going through the modifications required in the estimates. The second part for noninteger $s \geq 0$ then follows by an interpolation argument.

Let us now prove (2.11a). Substituting (2.4a) into (2.4b) we get

$$\begin{aligned}\int_0^\infty d\hat{\alpha}\,\boldsymbol{J}(x,x-\hat{\alpha})e^{-\hat{\alpha}\Lambda_l(k)} &= \int_0^\infty d\hat{\alpha}\,e^{\hat{\alpha}\Lambda_l(k)}\left[\mathbf{Q}(x-\hat{\alpha})-\mathbf{Q}_l\right]e^{-\hat{\alpha}\Lambda_l(k)} + \int_0^\infty d\hat{\alpha}\,e^{\hat{\alpha}\Lambda_l(k)}\left[\mathbf{Q}(x-\hat{\alpha})-\mathbf{Q}_l\right]\int_0^\infty d\hat{\beta}\,\boldsymbol{J}(x-\hat{\alpha},x-\hat{\alpha}-\hat{\beta})e^{-(\hat{\alpha}+\hat{\beta})\Lambda_l(k)}\\ &= \int_0^\infty d\hat{\alpha}\,F_{\text{inh}}(\hat{\alpha})e^{-\hat{\alpha}\Lambda_l(k)} + \int_0^\infty d\hat{\alpha}\,F_h(\hat{\alpha})e^{-\hat{\alpha}\Lambda_l(k)}.\end{aligned}$$

We shall derive L^1-estimates for the inhomogeneous term $F_{\text{inh}}(\alpha)$ and the $\boldsymbol{J}$-dependent term $F_h(\alpha)$ and then apply Gronwall's inequality to obtain an L^1-estimate for $\boldsymbol{J}$.

Let us compute the inhomogeneous term on the right-hand side, splitting it into k-even and k-odd parts. Using $\frac{k^2}{\lambda^2}=1-\frac{\mu^2}{\lambda^2}$, we get under condition $(\boldsymbol{H}_2)$ for the k-even component of the inhomogeneous term

$$\begin{aligned}&\int_0^\infty d\hat{\alpha}\left[\cos(\lambda\hat{\alpha})I_2+\frac{\sin(\lambda\hat{\alpha})}{\lambda}\mathbf{Q}_l\right]\left[\mathbf{Q}(x-\hat{\alpha})-\mathbf{Q}_l\right]\left[\cos(\lambda\hat{\alpha})I_2-\frac{\sin(\lambda\hat{\alpha})}{\lambda}\mathbf{Q}_l\right]+\int_0^\infty d\hat{\alpha}\left(1-\frac{\mu^2}{\mu^2}\right)\sin^2(\lambda\hat{\alpha})\sigma_3\left[\mathbf{Q}(x-\hat{\alpha})-\mathbf{Q}_l\right]\sigma_3\\ &=\frac{1}{2}\int_0^\infty d\alpha\cos(\lambda\alpha)\left[\mathbf{Q}\left(x-\frac{\alpha}{2}\right)-\mathbf{Q}_l\right]+\frac{1}{4}\int_0^\infty d\alpha\frac{\sin(\lambda\alpha)}{\lambda}\left(\mathbf{Q}_l\left[\mathbf{Q}\left(x-\frac{\alpha}{2}\right)-\mathbf{Q}_l\right]-\left[\mathbf{Q}\left(x-\frac{\alpha}{2}\right)-\mathbf{Q}_l\right]\mathbf{Q}_l\right)\\ &+\int_0^\infty d\alpha\left(\frac{\sin(\lambda\alpha)}{\lambda}\right)^2\left(\mu^2\left[\mathbf{Q}(x-\alpha)-\mathbf{Q}_l\right]-\mathbf{Q}_l\left[\mathbf{Q}(x-\alpha)-\mathbf{Q}_l\right]\mathbf{Q}_l\right).\end{aligned}$$

Similarly, under condition $(\boldsymbol{H}_2)$ we obtain for the k-odd component of the inhomogeneous term divided by ik

$$\begin{aligned}&\int_0^\infty d\hat{\alpha}\left[-\frac{\sin(\lambda\hat{\alpha})}{\lambda}\sigma_3\right]\left[\mathbf{Q}(x-\hat{\alpha})-\mathbf{Q}_l\right]\left[\cos(\lambda\hat{\alpha})I_2-\frac{\sin(\lambda\hat{\alpha})}{\lambda}\mathbf{Q}_l\right]+\int_0^\infty d\hat{\alpha}\left[\cos(\lambda\hat{\alpha})I_2+\frac{\sin(\lambda\hat{\alpha})}{\lambda}\mathbf{Q}_l\right]\left[\mathbf{Q}(x-\hat{\alpha})-\mathbf{Q}_l\right]\left[\frac{\sin(\lambda\hat{\alpha})}{\lambda}\sigma_3\right]\\ &=\frac{1}{2}\int_0^\infty d\alpha\,\frac{\sin(\lambda\alpha)}{\lambda}\left[\mathbf{Q}\left(x-\frac{\alpha}{2}\right)-\mathbf{Q}_l\right]\sigma_3+\int_0^\infty d\alpha\left(\frac{\sin(\lambda\alpha)}{\lambda}\right)^2\left(\sigma_3\left[\mathbf{Q}(x-\alpha)-\mathbf{Q}_l\right]\mathbf{Q}_l+\mathbf{Q}_l\left[\mathbf{Q}(x-\alpha)-\mathbf{Q}_l\right]\sigma_3\right).\end{aligned}$$

Applying (2.9) to compute the Fourier cosine transform of $F_{\text{inh}}(\alpha)$, we see that the terms containing factors of the form $\left(\frac{\sin(\lambda\alpha)}{\lambda}\right)^2$ cancel out. Using (2.8) we get

$$\int_0^\infty d\alpha\cos(\lambda\alpha)F_{\text{inh}}(\alpha)=\int_0^\infty d\alpha\cos(\lambda\alpha)\left\{\frac{1}{2}\left[Q\left(x-\frac{\alpha}{2}\right)-\mathbf{Q}_l\right]+\frac{1}{4}\int_\alpha^\infty d\beta\left(\mathbf{Q}_l\left[\mathbf{Q}\left(x-\frac{\beta}{2}\right)-\mathbf{Q}_l\right]+\left[\mathbf{Q}\left(x-\frac{\beta}{2}\right)-\mathbf{Q}_l\right]\mathbf{Q}_l\right)\right\}. \tag{2.13}$$

Consequently, under condition $(\boldsymbol{H}_2)$ we get

$$\int_0^\infty d\alpha\,\|F_{\text{inh}}(\alpha)\|\le\int_{-\infty}^x dz\left(1+\mu(x-z)\right)\|\mathbf{Q}(z)-\mathbf{Q}_l\|. \tag{2.14}$$

Using an approximation argument, the estimate (2.14) is easily shown to hold under the more general condition $(\boldsymbol{H}_1)$.

Let us now write the $\boldsymbol{J}$-dependent term on the right-hand side of (2.3a) as a Fourier cosine transform. To do so, we use the trigonometric formulae

$$\begin{aligned}\cos(\lambda(\hat{\alpha}+\hat{\beta}))\cos(\lambda\hat{\alpha})&=\frac{1}{2}\left(\cos(\lambda\alpha)+\cos(\lambda\beta)\right),\\ \sin(\lambda(\hat{\alpha}+\hat{\beta}))\sin(\lambda\hat{\alpha})&=-\frac{1}{2}\left(\cos(\lambda\alpha)-\cos(\lambda\beta)\right),\\ \frac{\sin(\lambda(\hat{\alpha}+\beta))}{\lambda}\cos(\lambda\hat{\alpha})&=\frac{1}{2}\left(\frac{\sin(\lambda\alpha)}{\lambda}+\frac{\sin(\lambda\beta)}{\lambda}\right),\\ \cos(\lambda(\hat{\alpha}+\beta))\frac{\sin(\lambda\hat{\alpha})}{\lambda}&=\frac{1}{2}\left(\frac{\sin(\lambda\alpha)}{\lambda}-\frac{\sin(\lambda\beta)}{\lambda}\right),\end{aligned}$$

where $\alpha=2\hat{\alpha}+\hat{\beta}$, $\beta=\hat{\beta}$, $\int_0^\infty d\hat{\alpha}\int_0^\infty d\hat{\beta}=\frac{1}{2}\int_0^\infty d\alpha\int_0^\alpha d\beta=\frac{1}{2}\int_0^\infty d\beta\int_\beta^\infty d\alpha$, as well as the trigonometric formula

$$\begin{aligned}\frac{\sin(\lambda(\hat{\alpha}+\hat{\beta}))}{\lambda}\,\frac{\sin(\lambda\hat{\alpha})}{\lambda}&=\left(\frac{\sin(\lambda\alpha)}{\lambda}\right)^2\cos^2(\lambda\beta)-\cos^2(\lambda\alpha)\left(\frac{\sin(\lambda\beta)}{\lambda}\right)^2\\ &=\left(\frac{\sin(\lambda\alpha)}{\lambda}\right)^2-\left(\frac{\sin(\lambda\beta)}{\lambda}\right)^2,\end{aligned}$$

where $\alpha=\widehat{\alpha}+\frac{1}{2}\widehat{\beta},\ \beta=\frac{1}{2}\widehat{\beta},\ \int_0^\infty d\widehat{\alpha}\int_0^\infty d\widehat{\beta}=2\int_0^\infty d\alpha\int_0^\alpha d\beta=2\int_0^\infty d\beta\int_\beta^\infty d\alpha$. We thus get for the k-even component of the $\mathbf{J}$-dependent term

$$\begin{aligned}
&\int_0^\infty d\widehat{\alpha}\int_0^\infty d\widehat{\beta}\left[\cos(\lambda\widehat{\alpha})I_2+\frac{\sin(\lambda\widehat{\alpha})}{\lambda}\mathbf{Q}_l\right]\left[\mathbf{Q}(x-\widehat{\alpha})-\mathbf{Q}_l\right]\times\boldsymbol{J}(x-\widehat{\alpha},x-\widehat{\alpha}-\widehat{\beta})\left[\cos(\lambda(\widehat{\alpha}+\widehat{\beta}))I_2-\frac{\sin(\lambda(\widehat{\alpha}+\widehat{\beta}))}{\lambda}\mathbf{Q}_l\right]\\
&+\int_0^\infty d\widehat{\alpha}\int_0^\infty d\widehat{\beta}\left(1-\frac{\mu^2}{\lambda^2}\right)\sin(\lambda\widehat{\alpha})\sin(\lambda(\widehat{\alpha}+\widehat{\beta}))\sigma_3\left[\mathbf{Q}(x-\widehat{\alpha})-\mathbf{Q}_l\right]\boldsymbol{J}(x-\widehat{\alpha},x-\widehat{\alpha}-\widehat{\beta})\sigma_3\\
&=\frac{1}{4}\int_0^\infty d\alpha\int_0^\alpha d\beta\big(\cos(\lambda\alpha)+\cos(\lambda\beta)\big)\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)\\
&-\frac{1}{4}\int_0^\infty d\alpha\int_0^\alpha d\beta\big(\cos(\lambda\alpha)-\cos(\lambda\beta)\big)\sigma_3\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)\sigma_3\\
&+\frac{1}{4}\int_0^\infty d\alpha\int_0^\alpha d\beta\left[\frac{\sin(\lambda\alpha)}{\lambda}-\frac{\sin(\lambda\beta)}{\lambda}\right]\mathbf{Q}_l\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)\\
&-\frac{1}{4}\int_0^\infty d\alpha\int_0^\alpha d\beta\left[\frac{\sin(\lambda\alpha)}{\lambda}+\frac{\sin(\lambda\beta)}{\lambda}\right]\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)\mathbf{Q}_l\\
&-2\int_0^\infty d\alpha\int_0^\alpha d\beta\left[\left(\frac{\sin(\lambda\alpha)}{\lambda}\right)^2-\left(\frac{\sin(\lambda\beta)}{\lambda}\right)^2\right]\Big(+\mathbf{Q}_l\left[\mathbf{Q}(x-\alpha+\beta)-\mathbf{Q}_l\right]\\
&\times\boldsymbol{J}(x-\alpha+\beta,x-\alpha-\beta)\mathbf{Q}_l+\mu^2\sigma_3\left[\mathbf{Q}(x-\alpha+\beta)-\mathbf{Q}_l\right]\boldsymbol{J}(x-\alpha+\beta,x-\alpha-\beta)\sigma_3\Big).
\end{aligned}$$

For the k-odd component of the $\boldsymbol{J}$-dependent term divided by ik we obtain

$$\begin{aligned}
&-\int_0^\infty d\widehat{\alpha}\int_0^\infty d\widehat{\beta}\,\frac{\sin(\lambda\widehat{\alpha})}{\lambda}\sigma_3\left[\mathbf{Q}(x-\widehat{\alpha})-\mathbf{Q}_l\right]\boldsymbol{J}(x-\widehat{\alpha},x-\widehat{\alpha}-\widehat{\beta})\\
&\times\left[\cos(\lambda(\widehat{\alpha}+\widehat{\beta}))I_2-\frac{\sin(\lambda(\widehat{\alpha}+\widehat{\beta}))}{\lambda}\mathbf{Q}_l\right]+\int_0^\infty d\widehat{\alpha}\int_0^\infty d\widehat{\beta}\left[\cos(\lambda\widehat{\alpha})I_2+\frac{\sin(\lambda\widehat{\alpha})}{\lambda}\mathbf{Q}_l\right]\times\left[\mathbf{Q}(x-\widehat{\alpha})-\mathbf{Q}_l\right]\boldsymbol{J}(x-\widehat{\alpha},x-\widehat{\alpha}-\widehat{\beta})\sigma_3\frac{\sin(\lambda(\widehat{\alpha}+\widehat{\beta}))}{\lambda}\\
&=-\frac{1}{4}\int_0^\infty d\alpha\int_0^\alpha d\beta\left(\frac{\sin(\lambda\alpha)}{\lambda}-\frac{\sin(\lambda\beta)}{\lambda}\right)\sigma_3\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)+\frac{1}{4}\int_0^\infty d\alpha\int_0^\alpha d\beta\left(\frac{\sin(\lambda\alpha)}{\lambda}+\frac{\sin(\lambda\beta)}{\lambda}\right)\\
&\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)\sigma_3+2\int_0^\infty d\alpha\int_0^\alpha d\beta\left[\left(\frac{\sin(\lambda\alpha)}{\lambda}\right)^2-\left(\frac{\sin(\lambda\beta)}{\lambda}\right)^2\right]\Big(+\sigma_3\left[\mathbf{Q}(x-\alpha+\beta)-\mathbf{Q}_l\right]\\
&\times\boldsymbol{J}(x-\alpha+\beta,x-\alpha-\beta)\mathbf{Q}_l+\mathbf{Q}_l\left[\mathbf{Q}(x-\alpha+\beta)-\mathbf{Q}_l\right]\mathbf{J}(x-\alpha+\beta,x-\alpha-\beta)\sigma_3\Big).
\end{aligned}$$

Applying (2.9) to compute the Fourier cosine transform of $F_h(\alpha)$, we see that the terms containing factors of the form $\left(\frac{\sin(\lambda\alpha)}{\lambda}\right)^2$ cancel out. Unfortunately, only half the terms containing factors $\frac{\sin(\lambda\alpha)}{\lambda}$ do. Using (2.8) we get

$$\begin{aligned}
\int_0^\infty d\alpha\cos(\lambda\alpha)F_h(\alpha)&=\frac{1}{4}\int_0^\infty d\alpha\int_0^\alpha d\beta\big(\cos(\lambda\alpha)+\cos(\lambda\beta)\big)\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)\\
&-\tfrac{1}{4}\int_0^\infty d\alpha\int_0^\alpha d\beta\big(\cos(\lambda\alpha)-\cos(\lambda\beta)\big)\sigma_3\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)\sigma_3\\
&+\tfrac{1}{4}\int_0^\infty d\alpha\int_0^\alpha d\beta\left[\frac{\sin(\lambda\alpha)}{\lambda}-\frac{\sin(\lambda\beta)}{\lambda}\right]\Big(\mathbf{Q}_l\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)\\
&-\left[\mathbf{Q}\left(x-\frac{\alpha-\beta}{2}\right)-\mathbf{Q}_l\right]\sigma_3\boldsymbol{J}\left(x-\frac{\alpha-\beta}{2},x-\frac{\alpha+\beta}{2}\right)\sigma_3\mathbf{Q}_l\Big).
\end{aligned}\tag{2.15}$$

We now strip off the Fourier cosine transform using (2.8), directly in the terms containing a factor $\cos(\lambda\alpha)$ or $\frac{\sin(\lambda\alpha)}{\lambda}$ and indirectly after interchanging α and β in the terms containing a factor $\cos(\lambda\beta)$ or $\frac{\sin(\lambda\beta)}{\lambda}$. To avoid repeating nearly similar estimates, we write the cosine transform terms as the Fourier cosine transforms of various matrix functions of the form

$$\left(\int_0^{\alpha} d\beta \pm \int_{\alpha}^{\infty} d\beta\right)\mathcal{Q}\left(x-\frac{|\alpha-\beta|}{2}\right)\mathcal{J}\left(x-\frac{|\alpha-\beta|}{2}, x-\frac{\alpha+\beta}{2}\right) = 2\int_{x-\frac{\alpha}{2}}^{x} dz\mathcal{Q}(z)\mathcal{J}(z, 2x-z-\alpha) \pm 2\int_{-\infty}^{x} dz\mathcal{Q}(z)\mathcal{J}(z, z-\alpha) \tag{2.16a}$$

and

$$\left(\int_{\alpha}^{\infty} d\gamma \int_0^{\gamma} d\beta - \int_{\alpha}^{\infty} d\gamma \int_{\gamma}^{\infty} d\beta\right)\mathcal{Q}\left(x-\frac{|\gamma-\beta|}{2}\right)\mathcal{J}\left(x-\frac{|\gamma-\beta|}{2}, x-\frac{\gamma+\beta}{2}\right) = 2\int_{-\infty}^{x} dz\mathcal{Q}(z)\int_{z-\alpha}^{\min(z,2x-z-\alpha)} dw\,\mathcal{J}(z, w), \tag{2.16b}$$

where $z-\alpha \le z$ and $z-\alpha \le 2x-z-\alpha$ for each $\alpha \ge 0$. In fact, $\mathcal{Q} = \mathbf{Q}-\mathbf{Q}_l$ and $\mathcal{J} = \boldsymbol{J}$ in the first three lines of (2.15), $\mathcal{Q} = \mathbf{Q}_l[\mathbf{Q}-\mathbf{Q}_l]$ and $\mathcal{J} = \boldsymbol{J}$ in the fourth line of (2.15), and $\mathcal{Q} = \mathbf{Q}-\mathbf{Q}_l$ and $\mathcal{J} = \sigma_3\boldsymbol{J}\sigma_3\mathbf{Q}_l$ in the fifth line of (2.15). Integrating (2.16a) and (2.16b) with respect to $\alpha \in \mathbb{R}^+$ we obtain the upper bounds

$$\int_0^{\infty} d\alpha \left\| 2\int_{x-\frac{\alpha}{2}}^{x} dz\mathcal{Q}(z)\mathcal{J}(z, 2x-z-\alpha) \pm 2\int_{-\infty}^{x} dz\mathcal{Q}(z)\mathcal{J}(z, z-\alpha)\right\| \le 4\int_{-\infty}^{x} dz\|\mathcal{Q}(z)\|\int_{-\infty}^{z} dw\|\mathcal{J}(z, w)\|, \tag{2.17a}$$

as well as

$$2\int_0^{\infty} d\alpha \int_{-\infty}^{x} dz\|\mathcal{Q}(z)\|\int_{z-\alpha}^{\min(z,2x-z-\alpha)} dw\,\|\mathcal{J}(z, w)\| \le 4\int_{-\infty}^{x} dz(x-z)\|\mathcal{Q}(z)\|\int_{-\infty}^{z} dw\|\mathcal{J}(z, w)\|. \tag{2.17b}$$

Using (2.14) and the various meanings of $\mathcal{Q}$ and $\mathcal{J}$, under condition $(\boldsymbol{H}_2)$ we obtain for the $\boldsymbol{J}$-dependent term

$$\int_0^{\infty} d\alpha\|F_h(\alpha)\| \le 2\int_{-\infty}^{x} dz[1+\mu(x-z)]\|\mathbf{Q}(z)-\mathbf{Q}_l\|\int_{-\infty}^{z} dw\|\boldsymbol{J}(z, w)\|, \tag{2.18}$$

which is easily shown to hold under the more general condition $(\boldsymbol{H}_1)$.

Applying Gronwall's inequality [14] to the inequality

$$\int_{-\infty}^{x} dw\|\boldsymbol{J}(x, w)\| \le \int_{-\infty}^{x} dz[1+\mu(x-z)]\|\mathbf{Q}(z)-\mathbf{Q}_l\| + 2\int_{-\infty}^{x} dz[1+\mu(x-z)]\|\mathbf{Q}(z)-\mathbf{Q}_l\|\int_{-\infty}^{z} dw\|\boldsymbol{J}(z, w)\|$$

following from (2.14) and (2.18), we obtain

$$\int_{-\infty}^{x} dw\|\boldsymbol{J}(x, w)\| \le \left[\int_{-\infty}^{x} dz[1+\mu(x-z)]\|\mathbf{Q}(z)-\mathbf{Q}_l\|\right] \times \exp\left(2\int_{x}^{\infty} dz[1+\mu(x-z)]\|\mathbf{Q}(z)-\mathbf{Q}_l\|\right), \tag{2.19}$$

thus proving the triangular representation (2.11a).

The proof of the triangular representation (2.11b) is based on a simple parity symmetry argument. In fact, letting $\mathbf{Q}^{(\#)}(x) = \mathbf{Q}(-x)$ we switch the roles of $\mathbf{Q}_r$ and $\mathbf{Q}_l$ by using $\mathbf{Q}_{r,l}^{(\#)} = \mathbf{Q}_{l,r}$ and obtain the following symmetry relations for the fundamental eigensolutions:

$$\widetilde{\Psi}^{(\#)}(x, k) = \sigma_3\widetilde{\Phi}(-x, -k)\sigma_3,\ \widetilde{\Phi}^{(\#)}(x, k) = \sigma_3\widetilde{\Psi}(-x, -k)\sigma_3.$$

We thus get the triangular representation

$$\begin{aligned}\widetilde{\Psi}(k, x) &= \sigma_3\widetilde{\Phi}^{(\#)}(-x, -k)\sigma_3 \\ &= \sigma_3 e^{-x\Lambda_l^{(\#)}(-k)} + \int_{-\infty}^{-x} dw\,\sigma_3\boldsymbol{J}^{(\#)}(-x, w)\sigma_3\sigma_3 e^{w\Lambda_l^{(\#)}(-k)}\sigma_3 \\ &= e^{x\Lambda_r(k)} + \int_{x}^{\infty} dz\,\sigma_3\boldsymbol{J}^{(\#)}(-x, -z)\sigma_3 e^{z\Lambda_r(k)},\end{aligned}$$

so that

$$\boldsymbol{K}(x, y) = \sigma_3\boldsymbol{J}^{(\#)}(-x, -y)\sigma_3 \tag{2.20}$$

has the integrability properties (2.10).

Let us now prove the second part of the theorem for $s = 0, 1, 2, \ldots$. Under the hypothesis $(\boldsymbol{H}_{s+1})$, we modify the estimates (2.14), (2.17a), and (2.17b), where $s = 0, 1, 2, \ldots$. Instead of (2.14) we get

$$\begin{aligned}\int_0^{\infty}(1+\alpha)^s\|F_{\text{inh}}(\alpha)\| &\le \frac{1}{2}\int_0^{\infty} d\alpha(1+\alpha)^s\left\|\mathbf{Q}\left(x-\frac{\alpha}{2}\right)-\mathbf{Q}_l\right\| + \frac{\mu}{2}\int_0^{\infty} d\alpha(1+\alpha)^s\int_{\alpha}^{\infty} d\beta\left\|\mathbf{Q}\left(x-\frac{\beta}{2}\right)-\mathbf{Q}_l\right\| \\ &= \int_{-\infty}^{x} dz(1+2(x-z))^s\|\mathbf{Q}(z)-\mathbf{Q}_l\| + \frac{\mu}{s+1}\sum_{j=0}^{s} 2^{j+1}\int_{-\infty}^{x} dz(x-z)^{j+1}\|\mathbf{Q}(z)-\mathbf{Q}_l\|.\end{aligned}$$

Instead of (2.17a) we get

$$\int_0^\infty d\alpha(1+\alpha)^s \left\| 2\int_{x-\frac{\alpha}{2}}^x dz\mathcal{Q}(z)\mathcal{J}(z,2x-z-\alpha) \pm 2\int_{-\infty}^x dz\mathcal{Q}(z)\mathcal{J}(z,z-\alpha)\right\|$$
$$\leq 2\int_{-\infty}^x dz\|\mathcal{Q}(z)\|\left[\int_{2(x-z)}^\infty d\alpha(1+\alpha)^s\|\mathcal{J}(z,2x-z-\alpha)\| + \int_0^\infty d\alpha(1+\alpha)^s\|\mathcal{J}(z,z-\alpha)\|\right]$$
$$= 2\int_{-\infty}^x dz\|\mathcal{Q}(z)\|\int_{-\infty}^z dw\left[(1+2(x-z)+z-w)^s + (1+z-w)^s\right]\|\mathcal{J}(z,w)\|,$$

where $(1+2(x-z)+z-w)^s = \sum_{j=0}^s \binom{s}{j} 2^j (x-z)^j (1+z-w)^{s-j}$. Instead of (2.17b) we get

$$2\int_0^\infty d\alpha(1+\alpha)^s \int_{-\infty}^x dz\|\mathcal{Q}(z)\|\int_{z-\alpha}^{\min(z,2x-z-\alpha)} dw\|\mathcal{J}(z,w)\| = 2\int_{-\infty}^x dz\|\mathcal{Q}(z)\|\int_{-\infty}^z dw\int_{z-w}^{2x-z-w} d\alpha(1+\alpha)^s\|\mathcal{J}(z,w)\|$$
$$= \frac{2}{s+1}\sum_{j=0}^s 2^{j+1}\int_{-\infty}^x dz(x-z)^{j+1}\|\mathcal{Q}(z)\|\int_{-\infty}^z dw(1+z-w)^{s-j}\|\mathcal{J}(z,w)\|.$$

With the help of Gronwall's inequality we then derive the final result for $s = 0, 1, 2, \ldots$.

For noninteger $s \geq 0$ we apply an interpolation argument [34] based on the Hölder estimate

$$\int_0^\infty d\alpha(1+\alpha)^s\|F(\alpha)\| \leq \left[\int_0^\infty d\alpha(1+\alpha)^{N+1}\|F(\alpha)\|\right]^{s-N} \times \left[\int_0^\infty d\alpha(1+\alpha)^N\|F(\alpha)\|\right]^{N+1-s},$$

where $N \leq s \leq N+1$.

Let us now derive expressions to pass from the (1, 2)-elements of the integral kernel $\boldsymbol{J}(x, y)$ and $\boldsymbol{K}(x, y)$ to the potential $\boldsymbol{Q}(x)$. These expressions have been derived by different means in Demontis et al. [14, Eq. (3.5)] under the assumption that $(\boldsymbol{H}_2)$ is valid and $q_x \in L^1(\mathbb{R})$.

Theorem 2.2. *Under condition* $(\boldsymbol{H}_1)$ *we have*

$$\boldsymbol{J}_{12}(x,x) = \frac{1}{2}[q(x)-q_l], \qquad \boldsymbol{K}_{12}(x,x) = -\frac{1}{2}[q(x)-q_r]. \tag{2.21}$$

It suffices to extend the expression obtained in Demontis et al. [14] to general potentials satisfying $(\boldsymbol{H}_1)$. Taking $\alpha \to 0^+$ in the expression

$$\boldsymbol{J}(x, x-\alpha) = F_{\text{inh}}(\alpha) + F_{\text{h}}(\alpha),$$

we obtain using (2.13) and (2.15)

$$\boldsymbol{J}(x,x) = \frac{1}{2}[\mathbf{Q}(x)-\mathbf{Q}_l] + \frac{1}{4}\int_0^\infty d\beta\left(\mathbf{Q}_l\left[\mathbf{Q}\left(x-\frac{\beta}{2}\right)-\mathbf{Q}_l\right] + \left[\mathbf{Q}\left(x-\frac{\beta}{2}\right)-\mathbf{Q}_l\right]\mathbf{Q}_l\right)$$
$$+\frac{1}{4}\int_0^\infty d\beta[\mathbf{Q}\left(x-\frac{\beta}{2}\right)-\mathbf{Q}_l]\boldsymbol{J}\left(x-\frac{\beta}{2}, x-\frac{\beta}{2}\right) + \frac{1}{4}\int_0^\infty d\beta\sigma_3[\mathbf{Q}\left(x-\frac{\beta}{2}\right)-\mathbf{Q}_l]\boldsymbol{J}\left(x-\frac{\beta}{2}, x-\frac{\beta}{2}\right)\sigma_3$$
$$-\frac{1}{4}\int_0^\infty d\beta\int_0^\beta d\gamma\left(\mathbf{Q}_l\left[\mathbf{Q}\left(x-\frac{\beta-\gamma}{2}\right)-\mathbf{Q}_l\right]\boldsymbol{J}\left(x-\frac{\beta-\gamma}{2}, x-\frac{\beta+\gamma}{2}\right)\right.$$
$$\left.-[\mathbf{Q}\left(x-\frac{\beta-\gamma}{2}\right)-\mathbf{Q}_l]\sigma_3\boldsymbol{J}\left(x-\frac{\beta-\gamma}{2}, x-\frac{\beta+\gamma}{2}\right)\sigma_3\mathbf{Q}_l\right).$$

Using (2.19) we see that, under condition $(\boldsymbol{H}_1)$, $\boldsymbol{J}(x,x)-\frac{1}{2}[\mathbf{Q}(x)-\mathbf{Q}_l]$ is a continuous function of $x \in \mathbb{R}$. Utilizing a continuity argument we easily extend (2.21) to arbitrary potentials satisfying $(\boldsymbol{H}_1)$.

The proof for $\boldsymbol{K}(x, x)$ can in fact be obtained from the result for $\boldsymbol{J}(x, x)$ by using (2.20).

3. RELATING FUNDAMENTAL EIGENSOLUTIONS

To study the asymptotic behavior of the fundamental eigensolutions as $x \to \pm\infty$, we write the Volterra integral equations (1.5) as follows [14, Eqs. (2.12)]:

$$\widetilde{\Phi}(x,k) = \mathcal{G}(x,0;k) + \int_{-\infty}^{x} dy\, \mathcal{G}(x,y;k)[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\widetilde{\Phi}(y,k), \tag{3.1a}$$

$$\widetilde{\Psi}(x,k) = \mathcal{G}(x,0;k) - \int_{x}^{\infty} dy\, \mathcal{G}(x,y;k)[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\widetilde{\Psi}(y,k), \tag{3.1b}$$

where

$$\mathbf{Q}_f(x) = \begin{cases} \mathbf{Q}_l, & x \in \mathbb{R}^-, \\ \mathbf{Q}_r, & x \in \mathbb{R}^+, \end{cases}$$

and

$$\mathcal{G}(x,y;k) = \begin{cases} e^{(x-y)\Lambda_l(k)}, \; x, y \in \mathbb{R}^-, \\ e^{(x-y)\Lambda_r(k)}, \; x, y \in \mathbb{R}^+, \\ e^{x\Lambda_l(k)} e^{-y\Lambda_r(k)}, \; x \le 0 \le y, \\ e^{x\Lambda_r(k)} e^{-y\Lambda_l(k)}, \; x \ge 0 \ge y, \end{cases}$$

is the evolution system associated with the first order system (1.1) associated with the piecewise constant potential $\mathbf{Q}_f$. Then [14, Eqs. (2.22) and (2.23)]

$$\widetilde{\Phi}(x,k) = \widetilde{\Psi}(x,k) B_l(k), \quad \widetilde{\Psi}(x,k) = \widetilde{\Phi}(x,k) B_r(k), \tag{3.2}$$

where

$$\boldsymbol{B}_l(k) = I_2 + \int_{-\infty}^{\infty} dy\, \mathcal{G}(0,y;k)[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\widetilde{\Phi}(y,k), \tag{3.3a}$$

$$\boldsymbol{B}_r(k) = I_2 - \int_{-\infty}^{\infty} dy\, \mathcal{G}(0,y;k)[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\widetilde{\Psi}(y,k), \tag{3.3b}$$

are each other's inverses.

Letting $V(x,t)$ be a square matrix solution of the AKNS system (1.1), we easily derive for V^{-1} the "inverse" AKNS system

$$[V^{-1}]_x = -V^{-1}V_xV^{-1} = -V^{-1}(-ik\sigma_3 + \mathbf{Q})VV^{-1} = V^{-1}(ik\sigma_3 - \mathbf{Q}).$$

Consequently, in analogy with (1.5) we obtain the Volterra integral equations

$$\widetilde{\Phi}(x,k)^{-1} = e^{-x\Lambda_l(k)} - \int_{-\infty}^{x} dy\, \widetilde{\Phi}(y,k)^{-1}[\mathbf{Q}(y) - \mathbf{Q}_l]e^{-(x-y)\Lambda_l(k)}, \tag{3.4a}$$

$$\widetilde{\Psi}(x,k)^{-1} = e^{-x\Lambda_r(k)} + \int_{x}^{\infty} dy\, \widetilde{\Psi}(y,k)^{-1}[\mathbf{Q}(y) - \mathbf{Q}_r]e^{(y-x)\Lambda_r(k)}. \tag{3.4b}$$

We can also write the Volterra integral equations in the form

$$\widetilde{\Phi}(x,k)^{-1} = \mathcal{G}(0,x;k) - \int_{-\infty}^{x} dy\, \widetilde{\Phi}(y,k)^{-1}[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\mathcal{G}(y,x;k), \tag{3.5a}$$

$$\widetilde{\Psi}(x,k)^{-1} = \mathcal{G}(0,x;k) + \int_{x}^{\infty} dy\, \widetilde{\Psi}(y,k)^{-1}[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\mathcal{G}(y,x;k), \tag{3.5b}$$

in analogy with (3.1). Taking the limits as $x \to \pm\infty$ and using (3.2) we get

$$B_r(k) = I_2 - \int_{-\infty}^{\infty} dy\, \widetilde{\Phi}(y,k)^{-1}[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\mathcal{G}(y,0;k), \tag{3.6a}$$

$$B_l(k) = I_2 + \int_{-\infty}^{\infty} dy\, \widetilde{\Psi}(y,k)^{-1}[\mathbf{Q}(y) - \mathbf{Q}_f(y)]\mathcal{G}(y,0;k). \tag{3.6b}$$

Mimicking the proof of Theorem 2.1, we can derive the triangular representations

$$\widetilde{\Phi}(x,k)^{-1} = e^{-x\Lambda_l(k)} + \int_{-\infty}^{x} dy\, e^{-y\Lambda_l(k)} \tilde{\boldsymbol{J}}(y,x), \tag{3.7a}$$

$$\widetilde{\Psi}(x,k)^{-1} = e^{-x\Lambda_r(k)} + \int_{x}^{\infty} dy\, e^{-y\Lambda_r(k)} \tilde{\boldsymbol{K}}(y,x), \tag{3.7b}$$

where

$$\int_{-\infty}^{x} dy \| \tilde{\boldsymbol{J}}(y,x) \| + \int_{x}^{\infty} dy \| \tilde{\boldsymbol{K}}(y,x) \| < +\infty, \; x \in \mathbb{R}.$$

Equations (3.7) can also be written in the form

$$\tilde{\boldsymbol{M}}(x,k)^{-1} = I_2 + \int_0^\infty d\alpha\, e^{\alpha\Lambda_l(k)}\tilde{\boldsymbol{J}}(x-\alpha,x), \tag{3.8a}$$

$$\tilde{\boldsymbol{N}}(x,k)^{-1} = I_2 + \int_0^\infty d\alpha\, e^{-\alpha\Lambda_r(k)}\tilde{\boldsymbol{K}}(x+\alpha,x), \tag{3.8b}$$

in analogy with (2.4).

4. JOST FUNCTIONS AND SCATTERING COEFFICIENTS

In this section we view (1.7) as a conformal mapping from a suitable k-manifold to a suitable λ-manifold and define the Jost functions. We also derive the triangular representations of the Jost functions. Finally, we introduce the scattering coefficients and the reflection coefficients and derive their representations as Fourier transforms.

4.1. Conformal Mapping

Let us now view

$$\lambda(k) = \sqrt{k^2+\mu^2}$$

as the conformal mapping from the complex k-plane $\mathbb{K}$ cut along $[-i\mu, i\mu]$ onto the complex λ plane $\mathbb{C}$ that satisfies $\lambda \sim k$ at infinity. Allowing each $k \in (-i\mu, i\mu)$ to have a left and a right copy to be put into 1, 1-correspondence with $\lambda \in (-\mu, 0)$ and $\lambda \in (0, \mu)$, respectively, we create diffeomorphisms between the analytic manifolds $\mathbb{K}^\pm$ and the open complex half-planes $\mathbb{C}^\pm$ and between the analytic manifolds with boundary $\mathbb{K}^\pm \cup \partial\mathbb{K}^\pm$ and the closed complex half-planes $\mathbb{C}^\pm \cup \mathbb{R}$. Doing it this way, many functions can be interchangeably viewed as functions of k and as functions of λ (Figure 1).

The following Fourier representation is true [24, 10.22.61]:

$$\int_0^\infty dt\, e^{i\lambda t}\frac{J_1(\mu t)}{\mu t} = \int_{-\infty}^0 dt\, e^{-i\lambda t}\frac{J_1(\mu t)}{\mu t} = \frac{i}{\lambda+k}, \tag{4.1}$$

where $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$ and $k \in \mathbb{K}^+ \cup \partial\mathbb{K}^+$. Here

$$J_1(z) = \sum_{k=0}^\infty \frac{(-1)^k (z/2)^{2k+1}}{k!(k+1)!} = \begin{cases} \frac{z}{2}[1+O(z^2)], & z \to 0^+, \\ \sqrt{\frac{2}{\pi z}}\cos\left(z - \frac{3}{4}\pi\right)\left[1 + O\left(\frac{1}{z}\right)\right], & z \to +\infty, \end{cases}$$

is the Bessel function of order one. Obviously, the left-hand side of (4.1) is the Fourier transform of a function in $L^1(\mathbb{R}^+)$ and hence is continuous in $\lambda \in \mathbb{C}^+ \cup \mathbb{R}$, is analytic in $\lambda \in \mathbb{C}^+$, and vanishes as $\lambda \to \infty$ from within the closed upper half complex λ-plane. For $\lambda \in \mathbb{C}^- \cup \mathbb{R}$ and $k \in \mathbb{K}^- \cup \partial\mathbb{K}^-$ we obtain by complex conjugation

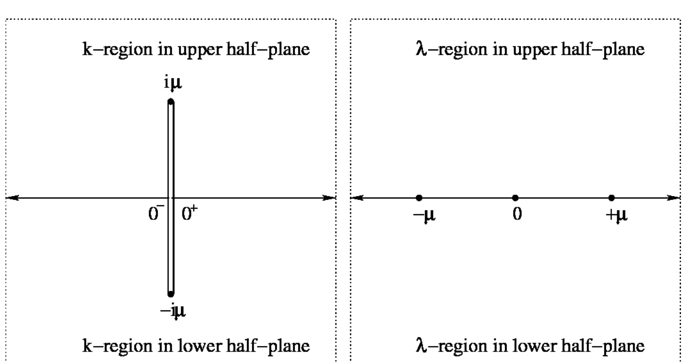

Figure 1 | The regions $k \in \mathbb{K}^\pm$ and $\lambda \in \mathbb{C}^\pm$ with manifold boundary. Note that $\mathbb{K}^\pm$ have the common boundary $(-\infty, -\mu] \cup [\mu, +\infty)$ and $\mathbb{C}^\pm$ have the real line as their common boundary.

$$\int_0^\infty dt\, e^{-i\lambda t}\frac{J_1(\mu t)}{\mu t} = \int_{-\infty}^0 dt\, e^{i\lambda t}\frac{J_1(\mu t)}{\mu t} = \frac{-i}{\lambda + k}. \tag{4.2}$$

4.2. Definition of Jost Functions

Letting

$$W_{r,l}(k) = \begin{pmatrix} w_{r,l}^{(1)}(k) & w_{r,l}^{(2)}(k) \end{pmatrix} = I_2 - \frac{i}{\lambda + k}\sigma_3 Q_{r,l}$$

stand for the 2×2 matrix whose columns are the eigenvectors of $\Lambda_{r,l}(k)$, i.e., letting

$$e^{x\Lambda_{r,l}(k)} W_{r,l}(k) = W_{r,l}(k) e^{-i\lambda x\sigma_3}, \tag{4.3}$$

we define the *Jost functions* as $\phi(x, k)$ and $\psi(x, k)$ for $k \in \mathbb{K}^+ \cup \partial\mathbb{K}^+$ and $\overline{\psi}(x, k)$ and $\overline{\phi}(x, k)$ for $k \in \mathbb{K}^- \cup \partial\mathbb{K}^-$ as follows:

$$\phi(x, k) = \widetilde{\Phi}(x, k) w_l^{(1)}(k), \psi(x, k) = \widetilde{\Psi}(k, x) w_r^{(2)}(k), \quad k \in \mathbb{K}^+ \cup \partial\mathbb{K}^+, \tag{4.4a}$$
$$\overline{\psi}(x, k) = \overline{\Psi}(x, k) w_r^{(1)}(k), \overline{\phi}(x, k) = \tilde{\Phi}(x, k) w_l^{(2)}(k), \quad k \in \mathbb{K}^- \cup \partial\mathbb{K}^-. \tag{4.4b}$$

The Jost functions in (4.4a) can also be defined for $k \in \mathbb{K}^- \cup \partial\mathbb{K}^-$ and those in (4.4b) for $k \in \mathbb{K}^+ \cup \partial\mathbb{K}^+$ by computing the corresponding columns of $W_{r,l}(k)$ for k in the complementary manifold. For $0 \neq k \in \mathbb{R}$ (and hence for $\lambda \in (-\infty, -\mu] \cup [\mu, +\infty)$ the Jost functions coincide when defined either way. For $k \in \partial\mathbb{K}^+ \cup \partial\mathbb{K}^-$ we call

$$\Phi(x, k) = \begin{pmatrix} \phi(x, k) & \overline{\phi}(x, k) \end{pmatrix}, \quad \Psi(x, k) = \begin{pmatrix} \overline{\psi}(x, k) & \psi(x, k) \end{pmatrix},$$

the *Jost matrices*.

4.3. Definition of Scattering Coefficients

Putting

$$S(k) = W_r(k)^{-1} \boldsymbol{B}_l(k) W_l(k), \quad \overline{S}(k) = W_l(k)^{-1} \boldsymbol{B}_r(k) W_r(k),$$

where $S(k)$ and $\overline{S}(k)$ are each other's inverses, we obtain

$$\Phi(x, k) = \Psi(x, k) S(k), \qquad \Psi(x, k) = \Phi(x, k) \overline{S}(k), \tag{4.5}$$

where

$$S(k) = \begin{pmatrix} a(k) & \overline{b}(k) \\ b(k) & \overline{a}(k) \end{pmatrix}, \quad \overline{S}(k) = \begin{pmatrix} \overline{c}(k) & d(k) \\ \overline{d}(k) & c(k) \end{pmatrix},$$

are written in terms of the traditional a, b, c, and d functions [3, Ch. 2]. Consequently,

$$\frac{2\lambda}{\lambda + k} S(k) = \left[\frac{2\lambda}{\lambda + k} \Psi(x, k)^{-1} \right] \Phi(x, k). \tag{4.6a}$$
$$\frac{2\lambda}{\lambda + k} \overline{S}(k) = \left[\frac{2\lambda}{\lambda + k} \Phi(x, k)^{-1} \right] \Psi(x, k). \tag{4.6b}$$

Using the identity

$$\frac{2\lambda}{\lambda + k} W_{r,l}(k)^{-1} = \sigma_3 W_{r,l}(k) \sigma_3 = \sigma_3 \begin{pmatrix} w_{r,l}^{[1]}(k) \\ w_{r,l}^{[2]}(k) \end{pmatrix} \sigma_3, \tag{4.7}$$

where $w_{r,l}^{[1]}(k)$ and $w_{r,l}^{[2]}(k)$ are the rows of $W_{r,l}(k)$, we obtain

$$\frac{2\lambda}{\lambda+k}a(k)=w_r^{[1]}(k)\sigma_3\boldsymbol{B}_l(k)w_l^{(1)}(k),$$
$$\frac{2\lambda}{\lambda+k}b(k)=-w_r^{[2]}(k)\sigma_3\boldsymbol{B}_l(k)w_l^{(1)}(k),$$
$$\frac{2\lambda}{\lambda+k}\bar{b}(k)=w_r^{[1]}(k)\sigma_3\boldsymbol{B}_l(k)w_l^{(2)}(k),$$
$$\frac{2\lambda}{\lambda+k}\bar{a}(k)=-w_r^{[2]}(k)\sigma_3\boldsymbol{B}_l(k)w_l^{(2)}(k),$$

and similarly for the entries of $\bar{S}(k)$. Thus the scattering coefficients $a(k)$ and $c(k)$ are well-defined for $k\in\mathbb{K}^+\cup\partial\mathbb{K}^+$ and for $k\in\partial\mathbb{K}^-$, $\bar{a}(k)$ and $\bar{c}(k)$ are well-defined for $k\in\mathbb{K}^-\cup\partial\mathbb{K}^-$ and for $k\in\partial\mathbb{K}^+$, and the off-diagonal scattering coefficients for $k\in\mathbb{R}\cup\partial\mathbb{K}^+\cup\partial\mathbb{K}^-$, with the possible exception of $k=\pm i\mu$. The entries of $s(k)$ and $\bar{S}(k)$ may not be defined for $k=\pm i\mu$ but they are when multiplied by $\frac{2\lambda}{\lambda+k}$.

4.4. Triangular Representations of Jost Solutions

Using the triangular representations (2.11) and the Fourier representations (4.1) and (4.2), we get

$$e^{i\lambda x}\phi(x,k)=w_l^{(1)}(k)+\int_0^\infty d\alpha\, e^{i\lambda\alpha}J(x,x-\alpha)w_l^{(1)}(k)=\begin{pmatrix}1\\0\end{pmatrix}+\int_0^\infty d\alpha\, e^{i\lambda\alpha}\boldsymbol{J}(x,x-\alpha), \tag{4.8a}$$

$$e^{-i\lambda x}\psi(x,k)=w_r^{(2)}(k)+\int_0^\infty d\alpha\, e^{i\lambda\alpha}K(x,x+\alpha)w_r^{(2)}(k)=\begin{pmatrix}0\\1\end{pmatrix}+\int_0^\infty d\alpha\, e^{i\lambda\alpha}\boldsymbol{K}(x,x+\alpha), \tag{4.8b}$$

for $k\in\mathbb{K}^+\cup\partial\mathbb{K}^+$, and

$$e^{i\lambda x}\bar{\psi}(x,k)=w_r^{(1)}(k)+\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\boldsymbol{K}(x,x+\alpha)w_r^{(1)}(k)=\begin{pmatrix}1\\0\end{pmatrix}+\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\bar{K}(x,x+\alpha), \tag{4.8c}$$

$$e^{-i\lambda x}\bar{\phi}(x,k)=w_l^{(2)}(k)+\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\boldsymbol{J}(x,x-\alpha)w_l^{(2)}(k)=\begin{pmatrix}0\\1\end{pmatrix}+\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\bar{J}(x,x-\alpha), \tag{4.8d}$$

for $k\in\mathbb{K}^-\cup\partial\mathbb{K}^-$, where

$$\bar{K}(x,y)=\boldsymbol{K}(x,y)\begin{pmatrix}1\\0\end{pmatrix}+\frac{J_1(\mu[y-x])}{\mu[y-x]}q_r^*\begin{pmatrix}0\\1\end{pmatrix}+\int_x^y dz\frac{J_1(\mu[y-z])}{\mu[y-z]}q_r^*\boldsymbol{K}(x,z)\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.9a}$$

$$K(x,y)=\boldsymbol{K}(x,y)\begin{pmatrix}0\\1\end{pmatrix}-\frac{J_1(\mu[y-x])}{\mu[y-x]}q_r\begin{pmatrix}1\\0\end{pmatrix}-\int_x^y dz\frac{J_1(\mu[y-z])}{\mu[y-z]}q_r\boldsymbol{K}(x,z)\begin{pmatrix}1\\0\end{pmatrix}, \tag{4.9b}$$

$$J(x,y)=\boldsymbol{J}(x,y)\begin{pmatrix}1\\0\end{pmatrix}-\frac{J_1(\mu[x-y])}{\mu[x-y]}q_l^*\begin{pmatrix}0\\1\end{pmatrix}-\int_y^x dz\frac{J_1(\mu[z-y])}{\mu[z-y]}q_l^*\boldsymbol{J}(x,z)\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.9c}$$

$$\bar{J}(x,y)=\boldsymbol{J}(x,y)\begin{pmatrix}0\\1\end{pmatrix}+\frac{J_1(\mu[x-y])}{\mu[x-y]}q_l\begin{pmatrix}1\\0\end{pmatrix}+\int_y^x dz\frac{J_1(\mu[z-y])}{\mu[z-y]}q_l\boldsymbol{J}(x,z)\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.9d}$$

Consequently, using that $J_1(w)=w+O(w^3)$ as $w\to 0^+$, we get

$$\left(\bar{K}(x,x)\,K(x,x)\right)=\boldsymbol{K}(x,x)-\boldsymbol{Q}_r, \tag{4.10a}$$

$$\left(J(x,x)\,\bar{J}(x,x)\right)=\boldsymbol{J}(x,x)+\boldsymbol{Q}_l. \tag{4.10b}$$

Writing (4.8a) and (4.8b) as Fourier transforms for $k \in \partial\mathbb{K}^-$ and using (4.2) we get

$$e^{i\lambda x}\phi(x,k)=\begin{pmatrix}1\\0\end{pmatrix}-\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\frac{J_1(\mu\alpha)}{\mu\alpha}q_l^*\begin{pmatrix}0\\1\end{pmatrix}+\int_0^\infty d\alpha\, e^{i\lambda\alpha}\boldsymbol{J}(x,x-\alpha)\begin{pmatrix}1\\0\end{pmatrix}-\int_{-\infty}^\infty d\alpha\, e^{-i\lambda\alpha}\int_0^\infty d\beta\frac{J_1(\mu[\alpha+\beta])}{\mu[\alpha+\beta]}q_l^*\boldsymbol{J}(x,x-\beta)\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.11a}$$

$$e^{-i\lambda x}\psi(x,k)=\begin{pmatrix}0\\1\end{pmatrix}-\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\frac{J_1(\mu\alpha)}{\mu\alpha}q_l\begin{pmatrix}1\\0\end{pmatrix}+\int_0^\infty d\alpha\, e^{i\lambda\alpha}\boldsymbol{K}(x,x+\alpha)\begin{pmatrix}0\\1\end{pmatrix}-\int_{-\infty}^\infty d\alpha\, e^{-i\lambda\alpha}\int_0^\infty d\beta\frac{J_1(\mu[\alpha+\beta])}{\mu[\alpha+\beta]}q_r\boldsymbol{K}(x,x+\beta)q_r\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.11b}$$

On the other hand, writing (4.8c) and (4.8d) as Fourier transforms for $k \in \partial\mathbb{K}^+$ and using (4.1) we get

$$e^{i\lambda x}\overline{\psi}(x,k)=\begin{pmatrix}1\\0\end{pmatrix}+\int_0^\infty d\alpha\, e^{i\lambda\alpha}\frac{J_1(\mu\alpha)}{\mu\alpha}q_r^*\begin{pmatrix}0\\1\end{pmatrix}+\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\boldsymbol{K}(x,x+\alpha)\begin{pmatrix}1\\0\end{pmatrix}+\int_{-\infty}^\infty d\alpha\, e^{i\lambda\alpha}\int_0^\infty d\beta\frac{J_1(\mu[\alpha+\beta])}{\mu[\alpha+\beta]}q_r^*\boldsymbol{K}(x,x+\beta)\begin{pmatrix}0\\1\end{pmatrix}, \tag{4.11c}$$

$$e^{-i\lambda x}\overline{\phi}(x,k)=\begin{pmatrix}0\\1\end{pmatrix}+\int_0^\infty d\alpha\, e^{i\lambda\alpha}\frac{J_1(\mu\alpha)}{\mu\alpha}q_l\begin{pmatrix}1\\0\end{pmatrix}+\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\boldsymbol{J}(x,x-\alpha)\begin{pmatrix}0\\1\end{pmatrix}+\int_{-\infty}^\infty d\alpha\, e^{-i\lambda\alpha}\int_0^\infty d\beta\frac{J_1(\mu[\alpha+\beta])}{\mu[\alpha+\beta]}q_l\boldsymbol{J}(x,x-\beta)\begin{pmatrix}1\\0\end{pmatrix}. \tag{4.11d}$$

Let us now apply the projections Π_+ and Π_- defined by (A.1) to (4.11). Applying the projections Π_+ to (4.11a) and (4.11b) and Π_- to (4.11c) and (4.11d), we obtain with the help of (A.1)

$$\Pi_+\left[e^{i\lambda x}\phi(x,k)\right]=\int_0^\infty d\alpha\, e^{i\lambda\alpha}\left\{\boldsymbol{J}(x,x-\alpha)\begin{pmatrix}1\\0\end{pmatrix}-\int_0^\infty d\beta\,\frac{J_1(\mu[\beta-\alpha])}{\mu[\beta-\alpha]}\boldsymbol{J}(x,x-\beta)\begin{pmatrix}0\\1\end{pmatrix}q_l^*\right\}, \tag{4.12a}$$

$$\Pi_+\left[e^{-i\lambda x}\psi(x,k)\right]=\int_0^\infty d\alpha\, e^{i\lambda\alpha}\left\{\boldsymbol{K}(x,x+\alpha)\begin{pmatrix}0\\1\end{pmatrix}-\int_0^\infty d\beta\frac{J_1(\mu[\beta-\alpha])}{\mu[\beta-\alpha]}\boldsymbol{K}(x,x+\beta)\begin{pmatrix}1\\0\end{pmatrix}q_r\right\}, \tag{4.12b}$$

$$\Pi_-\left[e^{i\lambda x}\overline{\psi}(x,k)\right]=\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\left\{\boldsymbol{K}(x,x+\alpha)\begin{pmatrix}1\\0\end{pmatrix}+\int_0^\infty d\beta\,\frac{J_1(\mu[\beta-\alpha])}{\mu[\beta-\alpha]}\boldsymbol{K}(x,x+\beta)\begin{pmatrix}0\\1\end{pmatrix}q_r^*\right\}, \tag{4.12c}$$

$$\Pi_-\left[e^{-i\lambda x}\overline{\phi}(x,k)\right]=\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\left\{\boldsymbol{J}(x,x-\alpha)\begin{pmatrix}0\\1\end{pmatrix}+\int_0^\infty d\beta\frac{J_1(\mu[\beta-\alpha])}{\mu[\beta-\alpha]}\boldsymbol{J}(x,x-\beta)\begin{pmatrix}1\\0\end{pmatrix}q_l\right\}. \tag{4.12d}$$

In a similar way we get by applying the projections Π_- to (4.11a) and (4.11b) and Π_+ to (4.11c) and (4.11d)

$$\Pi_-\left[e^{i\lambda x}\phi(x,k)\right]=-\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\left\{\frac{J_1(\mu\alpha)}{\mu\alpha}q_l^*\begin{pmatrix}0\\1\end{pmatrix}+\int_0^\infty d\beta\frac{J_1(\mu[\alpha+\beta])}{\mu[\alpha+\beta]}\boldsymbol{J}(x,x-\beta)\begin{pmatrix}0\\1\end{pmatrix}q_l^*\right\}, \tag{4.13a}$$

$$\Pi_-\left[e^{-i\lambda x}\psi(x,k)\right]=-\int_0^\infty d\alpha\, e^{-i\lambda\alpha}\left\{\frac{J_1(\mu\alpha)}{\mu\alpha}q_r\begin{pmatrix}1\\0\end{pmatrix}+\int_0^\infty d\beta\frac{J_1(\mu[\alpha+\beta])}{\mu[\alpha+\beta]}\boldsymbol{K}(x,x+\beta)\begin{pmatrix}1\\0\end{pmatrix}q_r\right\}, \tag{4.13b}$$

$$\Pi_+\left[e^{i\lambda x}\overline{\psi}(x,k)\right]=\int_0^\infty d\alpha\, e^{i\lambda\alpha}\left\{\frac{J_1(\mu\alpha)}{\mu\alpha}q_r^*\begin{pmatrix}0\\1\end{pmatrix}+\int_0^\infty d\beta\,\frac{J_1(\mu[\alpha+\beta])}{\mu[\alpha+\beta]}\boldsymbol{K}(x,x+\beta)\begin{pmatrix}0\\1\end{pmatrix}q_r^*\right\}, \tag{4.13c}$$

$$\Pi_+\left[e^{-i\lambda x}\overline{\phi}(x,k)\right]=\int_0^\infty d\alpha\, e^{i\lambda\alpha}\left\{\frac{J_1(\mu\alpha)}{\mu\alpha}q_l\begin{pmatrix}1\\0\end{pmatrix}+\int_0^\infty d\beta\frac{J_1(\mu[\alpha+\beta])}{\mu[\alpha+\beta]}\boldsymbol{J}(x,x-\beta)\begin{pmatrix}1\\0\end{pmatrix}q_l\right\}. \tag{4.13d}$$

4.5. Scattering Coefficients as Fourier Transforms

Writing

$$\Phi(x,k)^{-1}=\begin{pmatrix}\overset{\smile}{\overline{\phi}}(x,k)\\ \breve{\phi}(x,k)\end{pmatrix},\ \Psi(x,k)^{-1}=\begin{pmatrix}\breve{\psi}(x,k)\\ \overset{\smile}{\overline{\psi}}(x,k)\end{pmatrix}$$

for $0 \neq k \in \mathbb{R}$, using the identity (4.7), and the triangular representations (3.8) we obtain

$$\frac{2\lambda}{\lambda+k} e^{-i\lambda x} \bar{\phi}(x,k) = w_l^{[1]}(k)\sigma_3 + \int_0^\infty d\alpha\, e^{-i\lambda\alpha} w_l^{[1]}(k)\sigma_3 \tilde{\boldsymbol{J}}(x-\alpha, x), \tag{4.14a}$$

$$\frac{2\lambda}{\lambda+k} e^{i\lambda x} \bar{\psi}(x,k) = -w_r^{[2]}(k)\sigma_3 - \int_0^\infty d\alpha\, e^{-i\lambda\alpha} w_r^{[2]}(k)\sigma_3 \tilde{\boldsymbol{K}}(x+\alpha, x), \tag{4.14b}$$

$$\frac{2\lambda}{\lambda+k} e^{-i\lambda x} \breve{\psi}(x,k) = w_r^{[1]}(k)\sigma_3 + \int_0^\infty d\alpha\, e^{i\lambda\alpha} w_r^{[1]}(k)\sigma_3 \tilde{\boldsymbol{K}}(x+\alpha, x), \tag{4.14c}$$

$$\frac{2\lambda}{\lambda+k} e^{i\lambda x} \breve{\phi}(x,k) = -w_l^{[2]}(k)\sigma_3 - \int_0^\infty d\alpha\, e^{i\lambda\alpha} w_l^{[2]}(k)\sigma_3 \tilde{\boldsymbol{J}}(x-\alpha, x), \tag{4.14d}$$

where $k \in \mathbb{K}^- \cup \partial\mathbb{K}^-$ in (4.14a) and (4.14b) and $k \in \mathbb{K}^+ \cup \partial\mathbb{K}^+$ in (4.14c) and (4.14d).

Using the Wiener algebras defined in Appendix A, it is easily verified that

$$\frac{2\lambda}{\lambda+k} e^{-i\lambda x\sigma_3} S(k) e^{i\lambda x\sigma_3} = \left[\sigma_3 W_r(k)\sigma_3 + \int_0^\infty d\alpha\, e^{i\lambda\alpha\sigma_3}\sigma_3 W_r(k)\sigma_3 \tilde{\boldsymbol{K}}(x+\alpha, x)\right] \times \left[W_l(k) + \int_0^\infty d\beta\, \boldsymbol{J}(x, x-\beta) W_l(k) e^{i\lambda\beta\sigma_3}\right], \tag{4.15a}$$

$$\frac{2\lambda}{\lambda+k} e^{-i\lambda x\sigma_3} \bar{S}(k) e^{i\lambda x\sigma_3} = \left[\sigma_3 W_l(k)\sigma_3 + \int_0^\infty d\alpha\, e^{-i\lambda\alpha\sigma_3}\sigma_3 W_l(k)\sigma_3 \tilde{\boldsymbol{J}}(x-\alpha, x)\right] \times \left[W_r(k) + \int_0^\infty d\beta\, \boldsymbol{K}(x, x+\beta) W_r(k) e^{-i\lambda\beta\sigma_3}\right], \tag{4.15b}$$

belong to $\mathcal{W}^{2\times 2}$. The (1, 1)-element of (4.15a) and the (2, 2)-element of (4.15b) belong to $\mathcal{W}^+$. The (2, 2)-element of (4.15a) and the (1, 1)-element of (4.15b) belong to $\mathcal{W}^-$. Further, $\frac{2\lambda}{\lambda+k}a(k)-1$ and $\frac{2\lambda}{\lambda+k}c(k)-1$ belong to $\mathcal{W}^+$, $\frac{2\lambda}{\lambda+k}\bar{a}(k)-1$ and $\frac{2\lambda}{\lambda+k}\bar{c}(k)-1$ to $\mathcal{W}^-$, and $\frac{2\lambda}{\lambda+k}b(k)$, $\frac{2\lambda}{\lambda+k}\bar{b}(k)$, $\frac{2\lambda}{\lambda+k}d(k)$, and $\frac{2\lambda}{\lambda+k}\bar{d}(k)$ to $\mathcal{W}$.

Let us now define the *reflection coefficients*

$$\rho(k) = b(k)a(k)^{-1}, \quad r(k) = d(k)c(k)^{-1}, \quad k \in \partial\mathbb{K}^+, \tag{4.16a}$$

$$\bar{\rho}(k) = \bar{b}(k)\bar{a}(k)^{-1}, \quad \bar{r}(k) = \bar{d}(k)\bar{c}(k)^{-1}, \quad k \in \partial\mathbb{K}^-. \tag{4.16b}$$

Using that $S(k)\bar{S}(k) = I_2 = \bar{S}(k)S(k)$, we obtain the identities

$$\rho(k) = -c(k)^{-1}\bar{d}(k), \quad r(k) = -a(k)^{-1}\bar{b}(k), \tag{4.17a}$$

$$\bar{\rho}(k) = -\bar{c}(k)^{-1}d(k), \quad \bar{r}(k) = -\bar{a}(k)^{-1}b(k), \tag{4.17b}$$

where for the moment we leave open the existence of the reciprocals in (4.16) and (4.17). Hence, proving the reflection coefficients to belong to $\mathcal{W}_0$ is postponed to Section 6.

5. SYMMETRIES

In this section we derive the matrix conjugate transpose symmetry properties and nontranspose conjugate symmetry properties for Jost functions and scattering and reflection coefficients. The dagger denotes the matrix conjugate transpose.

a. Conjugate transposition symmetry. For $k \in \mathbb{R} \cup [-i\mu, i\mu]$ the matrix functions $\tilde{\Psi}(x,k)^{-1}$ and $\tilde{\Psi}(x,k^*)^\dagger$ both satisfy the differential equation

$$V_x = V(x,t)(ik\sigma_3 - \mathbf{Q}),$$

as do the matrix functions $\tilde{\Phi}(x,k)^{-1}$ and $\tilde{\Phi}(x,k^*)^\dagger$. Thus,

$$\tilde{\Phi}(x,k^*)^\dagger = \tilde{\Phi}(x,k)^{-1}, \quad \tilde{\Psi}(x,k^*)^\dagger = \tilde{\Psi}(x,k)^{-1}, \tag{5.1}$$

where $k \in \mathbb{R} \cup [-i\mu, i\mu]$. We observe that $k^* = k$ for $k \in \mathbb{R}$ and $k^* = -k$ for $k \in [-i\mu, i\mu]$. Thus for $k \in \mathbb{R}$ the fundamental eigensolutions $\widetilde{\Phi}(x,k)$ and $\widetilde{\Psi}(x,k)$ are unitary matrices of determinant 1. We also get

$$\widetilde{M}(x,k^*)^\dagger = \widetilde{M}(x,k)^{-1}, \quad \widetilde{N}(x,k^*)^\dagger = \widetilde{N}(x,k)^{-1}, \tag{5.2}$$

where $k \in \mathbb{R} \cup [-i\mu, i\mu]$. Using (3.2) we also obtain

$$\boldsymbol{B}_l(k^*)^\dagger = \boldsymbol{B}_r(k) = \boldsymbol{B}_l(k)^{-1}, \quad \boldsymbol{B}_r(k^*)^\dagger = \boldsymbol{B}_l(k) = \boldsymbol{B}_r(k)^{-1}, \tag{5.3}$$

where $k \in \mathbb{R} \cup [-i\mu, i\mu]$.

Next, we easily derive the identities

$$\Lambda_{r,l}(k^*)^\dagger = -\Lambda_{r,l}(k), \tag{5.4a}$$

$$W_{r,l}(k^*)^\dagger = \sigma_3 W_{r,l}(k)\sigma_3 = \frac{2\lambda}{\lambda+k} W_{r,l}(k)^{-1}, \tag{5.4b}$$

where in (5.4a) the choice of the sign in defining λ from k does not matter. In (5.4b) this choice is to be made consistently. Using (2.4), (3.8), and (5.4a) we obtain for the integral kernels

$$\boldsymbol{J}(x, x-\alpha)^\dagger = \tilde{\boldsymbol{J}}(x-\alpha, x), \quad \boldsymbol{K}(x, x+\alpha)^\dagger = \tilde{\boldsymbol{K}}(x+\alpha, x), \tag{5.5}$$

where $\alpha \in \mathbb{R}^+$.

Using (4.7) and (4.4) we obtain for the Jost matrices

$$\Phi(x,k^*)^\dagger = \frac{2\lambda}{\lambda+k} \Phi(x,k)^{-1}, \tag{5.6a}$$

$$\Psi(x,k^*)^\dagger = \frac{2\lambda}{\lambda+k} \Psi(x,k)^{-1}, \tag{5.6b}$$

where $k \in \mathbb{K}^+ \cup \partial\mathbb{K}^+$ as far as the second row of (5.6a) and the first row of (5.6b) are concerned and $k \in \mathbb{K}^- \cup \partial\mathbb{K}^-$ as far as the first row of (5.6b) and the second row of (5.6b) are concerned. Using (4.5) we obtain for the matrix of scattering coefficients

$$S(k^*)^\dagger = S(k)^{-1} = \bar{S}(k), \quad \bar{S}(k^*)^\dagger = \bar{S}(k) = S(k). \tag{5.7}$$

Thus $S(k)$ and $\bar{S}(k)$ are unitary matrices if $k \in \mathbb{R}$. Since $S(k)$ and $\bar{S}(k)$ both have unit determinant, we get

$$a(k^*)^* = \bar{a}(k), \quad c(k^*)^* = \bar{c}(k), \tag{5.8a}$$

$$b(k^*)^* = -\bar{b}(k), \; d(k^*)^* = -\bar{d}(k), \tag{5.8b}$$

where $-i\mu \neq k \in \mathbb{K}^- \cup \partial\mathbb{K}^-$ in (5.8a) and $-i\mu \neq k \in \partial\mathbb{K}^-$ in (5.8b). Equations (4.16) imply that the reflection coefficients satisfy the symmetry relations

$$\bar{\rho}(k) = -\rho(k^*)^*, \quad \bar{r}(k) = -r(k^*)^*, \tag{5.9}$$

provided the reciprocals in their definitions (4.16) exist.

b. Conjugation symmetry. Let $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ stand for the second Pauli matrix. Then it is easily verified that $\sigma_2\widetilde{\Psi}(x,k^*)^*\sigma_2$ and $\widetilde{\Psi}(x,k)$ both satisfy the differential equation (1.1). The same thing is true for the other fundamental eigensolution $\tilde{\Phi}$. We thus obtain

$$\sigma_2\tilde{\Phi}(x,k^*)^*\sigma_2 = \tilde{\Phi}(x,k), \quad \sigma_2\tilde{\Psi}(x,k^*)^*\sigma_2 = \tilde{\Psi}(x,k), \tag{5.10}$$

where $k \in \mathbb{R} \cup [-i\mu, i\mu]$. Furthermore,

$$\sigma_2\tilde{M}(x,k^*)^*\sigma_2 = \tilde{M}(x,k), \quad \sigma_2\tilde{N}(x,k^*)^*\sigma_2 = \tilde{N}(x,k). \tag{5.11}$$

where $k \in \mathbb{R} \cup [-i\mu, i\mu]$. Moreover,

$$\sigma_2 \boldsymbol{B}_{r,l}(k^*)^* \sigma_2 = \boldsymbol{B}_{r,l}(k), \tag{5.12}$$

where $k \in \mathbb{R} \cup [-i\mu, i\mu]$. Using (5.11) and (2.4) we get

$$\sigma_2 \boldsymbol{J}(x, x-\alpha)^* \sigma_2 = \boldsymbol{J}(x, x-\alpha), \quad \sigma_2 \boldsymbol{K}(x, x+\alpha)^* \sigma_2 = \boldsymbol{K}(x, x+\alpha), \tag{5.13}$$

where $\alpha \in \mathbb{R}^+$.

Next, we easily derive the identities

$$\Lambda_{r,l}(k^*)^* = \sigma_2 \Lambda_{r,l}(k) \sigma_2, \tag{5.14a}$$

$$W_{r,l}(k^*)^* = \sigma_2 W_{r,l}(k) \sigma_2, \tag{5.14b}$$

where in (5.14a) the choice of the sign in defining λ from k does not matter. In (5.14b) this choice is to be made consistently. Using (5.10) and (5.14b) we obtain for the Jost matrices

$$\Phi(x, k^*)^* = \sigma_2 \Phi(x, k) \sigma_2, \quad \Psi(x, k^*)^* = \sigma_2 \Psi(x, k) \sigma_2. \tag{5.15}$$

Consequently,

$$S(k^*)^* = \sigma_2 S(k) \sigma_2, \quad \bar{S}(k^*)^* = \sigma_2 \bar{S}(k) \sigma_2. \tag{5.16}$$

We immediately recover (5.8).

6. GENERIC AND EXCEPTIONAL CASES

It is well-known that in the scattering theory of the Schrödinger equation on the line with Faddeev class potential two cases can be distinguished [10,12,15]: the generic case where for $k = 0$ the two Jost functions are linearly independent, and the exceptional case where for $k = 0$ the two Jost functions are linearly dependent. The scattering theory in the exceptional case is more easily developed by strengthening the integrability condition on the potential (as done in Chadan and Sabatier [10], Deift and Trubowitz [12] and Faddeev [15]), though such strengthening can be avoided at the expense of more complicated mathematical arguments [20]. For reflectionless potentials we are always in the exceptional case.

In the theory of the Schrödinger equation on the line with Faddeev class potential we can actually prove that there are no spectral singularities [12]. In fact, for positive energy k^2 the two Jost functions can be proven to be linearly independent. In the present situation we actually need to assume absence of spectral singularities. Indeed, it is well-known that $a(k) = c(k)$ for $0 \neq \lambda \in \mathbb{C}^+ \cup \mathbb{R}$ and $\bar{a}(k) = \bar{c}(k)$ for $0 \neq \lambda \in \mathbb{C}^- \cup \mathbb{R}$. Therefore, throughout this article we assume *absence of spectral singularities*:

There do not exist any $0 \neq \lambda(k) \in \mathbb{R}$ where $a(k)$, $c(k)$, $\bar{a}(k)$, and $\bar{c}(k)$ vanish.

As a result, under this assumption the reflection coefficients $\rho(k)$ and $r(k)$ are well-defined for $i\mu \neq k \in \mathbb{K}^+ \cup \partial\mathbb{K}^+$ and the reflection coefficients $\bar{\rho}(k)$ and $\bar{r}(k)$ are well-defined for $-i\mu \neq k \in \mathbb{K}^- \cup \partial\mathbb{K}^-$. Moreover, $b(k)$, $\bar{b}(k)$, $d(k)$, and $\bar{d}(k)$ are well-defined for $k \in \partial\mathbb{K}^+ \cup \partial\mathbb{K}^-$. Their definitions for $k = \pm i\mu$ are a different matter to be pursued presently.

Under condition $(\boldsymbol{H}_1)$ we can define

$$\phi(x, i\mu) = \tilde{\Phi}(x, i\mu)\begin{pmatrix} 1 \\ -q_l^* / \mu \end{pmatrix}, \qquad \psi(x, i\mu) = \tilde{\Psi}(x, i\mu)\begin{pmatrix} -q_r / \mu \\ 1 \end{pmatrix},$$
$$\bar{\psi}(x, -i\mu) = \tilde{\Psi}(x, -i\mu)\begin{pmatrix} 1 \\ -q_r^* / \mu \end{pmatrix}, \qquad \bar{\phi}(x, -i\mu) = \tilde{\Phi}(x, -i\mu)\begin{pmatrix} -q_l / \mu \\ 1 \end{pmatrix}.$$

Since [see (5.15)]

$$\begin{pmatrix} \phi^{\text{up}}(x, i\mu) & \psi^{\text{up}}(x, i\mu) \\ \phi^{\text{dn}}(x, i\mu) & \psi^{\text{dn}}(x, i\mu) \end{pmatrix}^* = \sigma_2 \begin{pmatrix} \bar{\psi}^{\text{up}}(x, -i\mu) & \bar{\phi}^{\text{up}}(x, -i\mu) \\ \bar{\psi}^{\text{dn}}(x, -i\mu) & \bar{\phi}^{\text{dn}}(x, -i\mu) \end{pmatrix} \sigma_2,$$

it is clear that the Jost functions $\phi(x, i\mu)$ and $\psi(x, i\mu)$ are linearly independent iff the Jost functions $\overline{\psi}(x,-i\mu)$ and $\overline{\phi}(x,-i\mu)$ are linearly independent. As in the Schrödinger case, we can therefore make a distinction between the following two cases:

(a) the *generic case*: the Jost functions $\phi(x, i\mu)$ and $\psi(x, i\mu)$ are linearly independent. OR: the Jost functions $\overline{\phi}(x,-i\mu)$ and $\overline{\psi}(x,-i\mu)$ are linearly independent.

(b) the *exceptional case*: the Jost functions $\phi(x, i\mu)$ and $\psi(x, i\mu)$ are linearly dependent. OR: the Jost functions $\overline{\phi}(x,-i\mu)$ and $\overline{\psi}(x,-i\mu)$ are linearly dependent.

Theorem 6.1. *Suppose condition ($\boldsymbol{H}_1$) is satisfied. Then we are in the generic case if and only if*

$$\lim_{\substack{k\to i\mu\\ k\in\mathbb{K}^+\cup\partial\mathbb{K}^+}} \frac{2\lambda}{\lambda+k}a(k)$$

exists and is nonzero. If this limit vanishes, we are in the exceptional case.

Using (4.3) and (4.4) we get for $i\mu \neq k \in \mathbb{K}^+\cup\partial\mathbb{K}^+$

$$\phi(x,k)\simeq\begin{cases} e^{-i\lambda x}\begin{pmatrix}1\\ \dfrac{-i}{\lambda+k}q_l^*\end{pmatrix}, & x\to-\infty,\\ e^{-i\lambda x}\begin{pmatrix}a(k)-\dfrac{i}{\lambda+k}q_r e^{2i\lambda x}b(k),\\ e^{2i\lambda x}b(k)-\dfrac{i}{\lambda+k}q_r^* a(k)\end{pmatrix}, & x\to+\infty,\end{cases}$$

$$\psi(x,k)=\begin{cases} e^{i\lambda x}\begin{pmatrix}e^{-2i\lambda x}d(k)-\dfrac{i}{\lambda+k}q_l c(k)\\ c(k)-\dfrac{i}{\lambda+k}q_l^* e^{-2i\lambda x}d(k)\end{pmatrix}, & x\to-\infty,\\ e^{i\lambda x}\begin{pmatrix}\dfrac{-i}{\lambda+k}q_r\\ 1\end{pmatrix}, & x\to+\infty.\end{cases}$$

Since $\det(\phi(k, x)\ \psi(k, x))$ does not depend on $x \in \mathbb{R}$, we compute their determinants as $x \to \infty$ and $x \to +\infty$ and obtain

$$\frac{2\lambda}{\lambda+k}c(k)=\frac{2\lambda}{\lambda+k}a(k),$$

thus proving again that $a(k) = c(k)$. Since these determinants have the finite limit

$$\det\left(\tilde{\Phi}(x,i\mu)\begin{pmatrix}1\\ -q_l^*/\mu\end{pmatrix}\quad \tilde{\Psi}(x,i\mu)\begin{pmatrix}-q_r/\mu\\ 1\end{pmatrix}\right)$$

As $k \to i\mu$ from within $\mathbb{K}^+\cup\partial\mathbb{K}^+$, we arrive at the desired conclusion.

We now observe that

$$\phi(x,i\mu)\simeq\left[I_2+x\mathbf{Q}_l+\mu x\sigma_3\right]\begin{pmatrix}1\\ -q_l^*/\mu\end{pmatrix}=\begin{pmatrix}1\\ -q_l^*/\mu\end{pmatrix},\quad x\to-\infty, \tag{6.1a}$$

$$\psi(x,i\mu)\simeq\left[I_2+x\mathbf{Q}_r+\mu x\sigma_3\right]\begin{pmatrix}-q_r/\mu\\ 1\end{pmatrix}=\begin{pmatrix}-q_r/\mu\\ 1\end{pmatrix},\quad x\to+\infty. \tag{6.1b}$$

Hence, in the exceptional case the (proportional) Jost functions $\phi(x, i\mu)$ and $\psi(x, i\mu)$ are bounded in $x \in \mathbb{R}$ and have finite nonzero limits as $x \to \pm\infty$.

Theorem 6.2. *Let us assume condition ($\boldsymbol{H}_1$) in the generic case and condition ($\boldsymbol{H}_2$) in the exceptional case, as well as absence of spectral singularities. Let us also assume that $a(k)$ does not vanish as $k \to i\mu$. Then the reflection coefficients $\rho(k)$, $\overline{\rho}(k)$, $r(k)$, and $\overline{r}(k)$ are Fourier transforms of functions in $L^1(\mathbb{R})$. Moreover, there are only finitely many discrete eigenvalues.*

In the absence of spectral singularities and in the generic case, the four reflection coefficients are all Fourier transforms of functions in $L^1(\mathbb{R})$ [*cf.* Appendix A]. Moreover, in this case there are at most finitely many discrete eigenvalues.

It remains to consider the exceptional case in detail. To do so, we strengthen the integrability assumption on the potential by assuming condition (H_2). Since $\frac{2\lambda}{\lambda+k}a(k)$ and $\frac{2\lambda}{\lambda+k}b(k)$ can then easily be shown to be the Fourier transforms (in λ) of functions in $L^1(\mathbb{R}^+;(1+\alpha)d\alpha)$ and $L^1(\mathbb{R};(1+|\alpha|)d\alpha)$, respectively, we can then apply Taylor's theorem and write

$$a(k)=c(k)=\frac{a_{-1}}{\lambda}+a_0+o(1),\ k\to i\mu \text{ in } \mathbb{K}^+\cup\partial\ \mathbb{K}^+, \tag{6.2a}$$

$$b(k)=\frac{b_{-1}}{\lambda}+b_0+o(1), \qquad k\to i\mu \text{ in } \partial\mathbb{K}^+, \tag{6.2b}$$

where in the exceptional case we must have $a_{-1}=0$. We need to prove that, in the exceptional case, $a_0\neq 0$ and $b_{-1}=0$. Equations (5.7) and (5.8) imply

$$\begin{pmatrix} a(k) & -b(k^*)^* \\ b(k) & a(k^*)^* \end{pmatrix}\begin{pmatrix} c(k^*)^* & d(k) \\ -d(k^*)^* & c(k) \end{pmatrix}=I_2=\begin{pmatrix} c(k^*)^* & d(k) \\ -d(k^*)^* & c(k) \end{pmatrix}\begin{pmatrix} a(k) & -b(k^*)^* \\ b(k) & a(k^*)^* \end{pmatrix},$$

where $a(k)=c(k)$. Substituting (6.2) in the (2, 2)-element of either equation we get

$$b_{-1}d_{-1}=0,\quad b_{-1}d_0+b_0d_{-1}=0,\quad |a_0|^2+b_0d_0=|a_0|^2+b_0^*d_0^*=1.$$

Substituting (6.2) [with $\lambda^*=-\lambda$] in the (1, 2)-element of the left equation we get

$$a_0(d_{-1}+b_{-1}^*)=0,\quad a_0d_0=a_0b_0^*.$$

The Ansatz $b_{-1}=0$ and $d_{-1}\neq 0$ leads to $b_0=0$ and $|a_0|=1$, so that $d_{-1}+b_{-1}^*=0$, a contradiction. In the same way we arrive at a contradiction from the Ansatz $b_{-1}\neq 0$ and $d_{-1}=0$. We must therefore conclude that $b_{-1}=d_{-1}=0$. Instead of (6.2b), we thus arrive at the identities

$$b(k)=b_0+o(1),\quad d(k)=d_0+o(1), \tag{6.3}$$

where $b_0d_0=b_0^*d_0^*=1-|a_0|^2$ is a real number. Furthermore,

$$\begin{pmatrix} a_0 & -b_0^* \\ b_0 & a_0^* \end{pmatrix}\begin{pmatrix} a_0^* & d_0 \\ -d_0^* & a_0 \end{pmatrix}=I_2=\begin{pmatrix} a_0^* & d_0 \\ -d_0^* & a_0 \end{pmatrix}\begin{pmatrix} a_0 & -b_0^* \\ b_0 & a_0^* \end{pmatrix},$$

where the factors in either matrix product have unit determinant. Computing the two matrix products we get

$$\begin{pmatrix} |a_0|^2+b_0^*d_0^* & a_0(d_0-b_0^*) \\ a_0^*(b_0-d_0^*) & |a_0|^2+b_0d_0 \end{pmatrix}=I_2=\begin{pmatrix} |a_0|^2+b_0d_0 & a_0^*(d_0-b_0^*) \\ a_0(b_0-d_0^*) & |a_0|^2+b_0^*d_0^* \end{pmatrix}.$$

This leads to two mutually exclusive possibilities:

(a) $a_0\neq 0$, $d_0=b_0^*$, and $|a_0|^2+|b_0|^2=|a_0|^2+|d_0|^2=1$.

(b) $a_0=0$ and $b_0d_0=1$. Since $a(k)[d(k)+\bar{b}(k)]=0$ with $a(k)\neq 0$ for values of $k\in\partial\mathbb{K}^+$ approaching $i\mu$, the absence of spectral singularities assumption implies that $|b_0|=|d_0|$. Consequently, there exists a phase $\theta\in\mathbb{R}$ such that $b_0=d_0^*=e^{i\theta}$.

In the former case the reflection coefficients are Fourier transforms of functions in $L^1(\mathbb{R})$, whereas in the latter case the reflection coefficients blow up as $k\to i\mu$.

Now observe that

$$\bar{\psi}(x,i\mu)=\tilde{\Psi}(x,i\mu)\begin{pmatrix} 1 \\ -q_r^*/\mu \end{pmatrix}=\frac{-q_r^*}{\mu}\tilde{\Psi}(x,i\mu)\begin{pmatrix} -q_r/\mu \\ 1 \end{pmatrix}=\frac{-q_r^*}{\mu}\psi(x,i\mu)$$

is well-defined. Thus the identity $\phi = \overline{\psi}a + \psi b$ implies that in the exceptional case

$$\phi(x, i\mu) = \left(b_0 - \frac{q_r^*}{\mu}a_0\right)\psi(x, i\mu),$$

where the proportionality constant is nonzero. In the same way we prove that

$$\psi(x, i\mu) = \left(d_0 - \frac{q_l}{\mu}a_0\right)\phi(x, i\mu),$$

where the proportionality constant is nonzero as $k = \pm i\mu$. Since these two proportionality constants have product 1, $d_0 = b_0^*$, and $q_l q_r^* = \mu^2 e^{i(\theta_l - \theta_r)}$, we get

$$\left(b_0 - e^{-i\theta_r}a_0\right)\left(b_0^* - e^{i\theta_l}a_0\right) = 1.$$

The proof of Theorem 6.2 forced us to consider the mutually exclusive versions of the exceptional case, denoted by (a) and (b). In the generic case and in the exceptional case (a) the reflection coefficients are Fourier transforms of L^1-functions. Unfortunately this is no longer the case in the exceptional case (b), the so-called *hyperexceptional case* for want of a better term. At present we cannot exclude the occurrence of the hyperexceptional case, but we are not aware of any focusing potential leading to the hyperexceptional case either.

CONFLICTS OF INTEREST

The author declares no conflicts of interest.

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APPENDIX A

1. WIENER ALGEBRAS

By the (continuous) Wiener algebra $\mathcal{W}$ we mean the complex vector space of constants plus Fourier transforms of L^1-functions

$$\mathcal{W} = \{c + \hat{h} : c \in \mathbb{C}, h \in L^1(\mathbb{R})\}$$

endowed with the norm $|c| + \|h\|_1$. Here we define the Fourier transform as follows: $(\mathcal{F}h)(\lambda) = \hat{h}(\lambda) = \int_{-\infty}^{\infty} dy e^{i\lambda y} h(y)$. The invertible elements of the commutative Banach algebra $\mathcal{W}$ with unit element are exactly those $c + \hat{h} \in \mathcal{W}$ for which $c \neq 0$ and $c + \hat{h}(\lambda) \neq 0$ for each $\lambda \in \mathbb{R}$ [17].

The algebra $\mathcal{W}$ has the two closed subalgebras $\mathcal{W}^+$ and $\mathcal{W}^-$ consisting of those $c + \hat{h}$ such that h is supported on $\mathbb{R}^+$ and $\mathbb{R}^-$, respectively. The invertible elements of $\mathcal{W}^\pm$ are exactly those $c + \hat{h} \in \mathcal{W}^\pm$ for which $c \neq 0$ and $c + \hat{h}(\lambda) \neq 0$ for each $\lambda \in \mathbb{C}^\pm \cup \mathbb{R}$ [17]. Letting $\mathcal{W}_0^\pm$ and $\mathcal{W}_0$ stand for the (nonunital) closed subalgebras of $\mathcal{W}^\pm$ and $\mathcal{W}$ consisting of those $c + \hat{h}$ for which $c = 0$, we obtain the direct sum decomposition

$$\mathcal{W} = \mathbb{C} \oplus \mathcal{W}_0^+ \oplus \mathcal{W}_0^-, \qquad \mathcal{W}_0 = \mathcal{W}_0^+ \oplus \mathcal{W}_0^-.$$

By $\Pi_\pm$ we now denote the (bounded) projection of $\mathcal{W}$ onto $\mathcal{W}_0^\pm$ along $\mathbb{C} \oplus \mathcal{W}_0^\mp$. Then Π_+ and Π_- are complementary projections. In fact,

$$(\Pi_\pm f)(\lambda) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \mathrm{d}\zeta \frac{f(\zeta)}{\zeta - (\lambda \pm i0^+)}, \tag{A.1}$$

where $f \in \mathcal{W}_0 \cap L^p(\mathbb{R})$ for some $p \in (1, +\infty)$. These direct sum decompositions coupled by the Fourier transform can be schematically represented as follows:

$$\begin{array}{ccccc} L^1(\mathbb{R}) & = & L^1(\mathbb{R}^-) & \oplus & L^1(\mathbb{R}^+) \\ \downarrow \mathcal{F} & & \downarrow \mathcal{F} & & \downarrow \mathcal{F} \\ \mathcal{W}_0 & = & \mathcal{W}_0^- & \oplus & \mathcal{W}_0^+ \end{array}$$

Now observe that $\mathcal{F}$ acts as an isometric linear 1, 1-correspondence from $\boldsymbol{L}^1(\mathbb{R})$ onto $\mathcal{W}_0$. If we define the norm of $\mathbb{C} \oplus \boldsymbol{L}^1(\mathbb{R})$ as $\| c+h \| = |c| + \| h \|_1$, we obtain the direct sum decomposition

$$\boldsymbol{L}^1(\mathbb{R}) = \boldsymbol{L}^1(\mathbb{R}^+) \oplus \boldsymbol{L}^1(\mathbb{R}^-),$$

where the projection $\mathcal{F}^{-1}\Pi_\pm \mathcal{F}$ is the restriction of an arbitrary $h \in \boldsymbol{L}^1(\mathbb{R})$ to the half-line $\mathbb{R}^\pm$.

Throughout this article we denote the vector spaces of $n \times m$ matrices with entries in $\mathcal{W}$, $\mathcal{W}^\pm$, and $\mathcal{W}_0^\pm$ by $\mathcal{W}^{n\times m}$, $\mathcal{W}^{\pm n\times m}$, and $\mathcal{W}_0^{\pm n\times m}$, respectively. We write $\boldsymbol{L}^1(\mathbb{R})^{n\times m}$ and $\boldsymbol{L}^1(\mathbb{R}^\pm)^{n\times m}$ for the vector spaces of $n \times m$ matrices with entries in $\boldsymbol{L}^1(\mathbb{R})$ and $\boldsymbol{L}^1(\mathbb{R}^\pm)$, respectively. Using a suitable (i.e., submultiplicative) matrix norm, we can turn all of these vector spaces into Banach spaces. It is then clear that $\mathcal{W}^{n\times n}$ and $\mathcal{W}^{\pm n\times n}$ are noncommutative Banach algebras with unit element and $\mathcal{W}_0^{\pm n\times n}$ are (nonunital) noncommutative Banach algebras. The projections $\Pi^\pm$ can be extended in a natural way to matrices of Wiener algebra elements.

The following result is most easily proved using the Gelfand theory of commutative Banach algebras [17] but was known before to Wiener [32,33].

Theorem Appendix A.1. *If for some complex number* h_∞ *and some* $h \in \boldsymbol{L}^1(\mathbb{R})$ *the Fourier transform* $h_\infty + \int_{-\infty}^{\infty} dz e^{i\lambda z} h(z) \neq 0$ *for every* $\lambda \in \mathbb{R}$ *and if* $h_\infty \neq 0$, *then there exists* $k \in \boldsymbol{L}^1(\mathbb{R})$ *such that*

$$\frac{1}{h_\infty + \int_{-\infty}^{\infty} dz e^{i\lambda z} h(z)} = \frac{1}{h_\infty} + \int_{-\infty}^{\infty} dz e^{i\lambda z} k(z)$$

for every $\lambda \in \mathbb{R}$.

APPENDIX B

1. TIME DEPENDENCE OF THE SCATTERING DATA

The focusing NLS equation

$$i\sigma_3 \boldsymbol{Q}_t = \boldsymbol{Q}_{xx} - 2\boldsymbol{Q}^3, \tag{B.1}$$

where $\boldsymbol{Q} = \begin{pmatrix} 0 & \boldsymbol{Q} \\ -\boldsymbol{Q}^* & 0 \end{pmatrix}$, arises as the compatibility condition of the Lax pair equations [2,3,16]

$$v_x = (-ik\sigma_3 + \boldsymbol{Q})v, \quad v_t = (2ik^2\sigma_3 + i\sigma_3\boldsymbol{Q}^2 - 2k\boldsymbol{Q} - i\sigma_3\boldsymbol{Q}_x)v, \tag{B.2}$$

where v is a nonsingular 2×2 matrix function.

Using that Ψ, Φ, and v all satisfy the first order system (1.1), we can write $\Psi = vC_+$ and $\Phi = vC_-$, where $C_\pm$ do not depend on $x \in \mathbb{R}$ (but do depend on k and t). Since

$$\begin{aligned} \Psi_t &= v_t C_+^{-1} - vC_+^{-1}[C_+]_t C_+^{-1} \\ &= (2ik^2\sigma_3 + i\sigma_3\boldsymbol{Q}^2 - 2k\boldsymbol{Q} - i\sigma_3 Q_x)vC_+^{-1} - vC_+^{-1}[C_+]_t C_+^{-1} \\ &= (2ik^2\sigma_3 + i\sigma_3\boldsymbol{Q}^2 - 2k\boldsymbol{Q} - i\sigma_3\boldsymbol{Q}_x)\Psi - \Psi[C_+]_t C_+^{-1}, \end{aligned}$$

we obtain

$$[C_+]_t C_+^{-1} = \Psi^{-1}\left(2ik^2\sigma_3 + i\sigma_3\mathbf{Q}^2 - 2k\mathbf{Q} - i\sigma_3\mathbf{Q}_x\right)\Psi - \Psi^{-1}\Psi_t, \tag{B.3a}$$

and similarly

$$[C_-]_t C_-^{-1} = \Phi^{-1}\left(2ik^2\sigma_3 + i\sigma_3\mathbf{Q}^2 - 2k\mathbf{Q} - i\sigma_3\mathbf{Q}_x\right)\Phi - \Phi^{-1}\Phi_t. \tag{B.3b}$$

Using (B.1) we observe that

$$\begin{aligned}[\mathbf{Q}_{r,l}]_t = 2i\sigma_3\mathbf{Q}_{r,l}(t)^3 &= -2i\mu^2\sigma_3\mathbf{Q}_{r,l}(t) = 2i\mu^2\mathbf{Q}_{r,l}(t)\sigma_3 \\ &= i\mu^2\left\{\mathbf{Q}_{r,l}(t)\sigma_3 - \sigma_3\mathbf{Q}_{r,l}(t)\right\},\end{aligned} \tag{B.4}$$

so that

$$\mathbf{Q}_{r,l}(t) = e^{-i\mu^2 t\sigma_3}\mathbf{Q}_{r,l}(0)e^{i\mu^2 t\sigma_3}$$

and

$$[W_{r,l}(k)]_t = \frac{-i}{\lambda + k}\sigma_3[\mathbf{Q}_{r,l}]_t = \frac{-2\mu^2}{\lambda + k}\mathbf{Q}_{r,l}(t).$$

Since the left-hand sides of (B.3) do not depend on $x \in \mathbb{R}$, we can take the limits of the right-hand sides as $x \to \pm\infty$ and obtain

$$\begin{aligned}[\mathbf{C}_+]_t\mathbf{C}_+^{-1} &= e^{i\lambda x\sigma_3}\mathbf{W}_r(k)^{-1}\left[\left(2ik^2\sigma_3 - i\mu^2\sigma_3 - 2k\mathbf{Q}_r\right)\mathbf{W}_r(k) - [\mathbf{W}_r(k)]_t\right]e^{-i\lambda x\sigma_3} \\ &= e^{i\lambda x\sigma_3}\mathbf{W}_r(k)^{-1}\left[(2i\lambda k - i\mu^2)\mathbf{W}_r(k)\sigma_3\right]e^{-i\lambda x\sigma_3} = (2i\lambda k - i\mu^2)\sigma_3, \\ [\mathbf{C}_-]_t\mathbf{C}_-^{-1} &= e^{i\lambda x\sigma_3}\mathbf{W}_l(k)^{-1}\left[\left(2ik^2\sigma_3 - i\mu^2\sigma_3 - 2k\mathbf{Q}_l\right)\mathbf{W}_l(k) - [\mathbf{W}_l(k)]_t\right]e^{-i\lambda x\sigma_3} \\ &= e^{i\lambda x\sigma_3}\mathbf{W}_l(k)^{-1}\left[(2i\lambda k - i\mu^2)\mathbf{W}_l(k)\sigma_3\right]e^{-i\lambda x\sigma_3} = (2i\lambda k - i\mu^2)\sigma_3.\end{aligned}$$

Next, differentiating $S = \Psi^{-1}\Phi$ with respect to t and writing the second identity in (B.2) as $v_t = (\ldots)v$, we obtain

$$\begin{aligned}\mathbf{S}_t &= \Phi^{-1}\Phi_t - \Psi^{-1}\Psi_t\Psi^{-1}\Phi = \mathbf{S}\Phi^{-1}\Phi_t - \Psi^{-1}\Psi_t\mathbf{S} \\ &= \mathbf{S}\left(\Phi^{-1}(\ldots)\Phi - [\mathbf{C}_-]_t\mathbf{C}_-^{-1}\right) - \left(\Psi^{-1}(\ldots)\Psi - [\mathbf{C}_+]_t\mathbf{C}_+^{-1}\right)\mathbf{S} \\ &= -\mathbf{S}[\mathbf{C}_-]_t\mathbf{C}_-^{-1} + [\mathbf{C}_+]_t\mathbf{C}_+^{-1}\mathbf{S} \\ &= (2i\lambda k - i\mu^2)(\sigma_3\mathbf{S} - \mathbf{S}\sigma_3).\end{aligned}$$

Consequently,

$$\mathbf{S}(k,t) = e^{(2i\lambda k - i\mu^2)t\sigma_3}\mathbf{S}(k,0)e^{-(2i\lambda k - i\mu^2)t\sigma_3}.$$

As a result, the diagonal scattering coefficients $a(k)$ and $\bar{a}(k)$ are time independent, whereas

$$b(k,t) = e^{-(4i\lambda k - 2i\mu^2)t}b(k,0), \qquad \bar{b}(k,t) = e^{(4i\lambda k - 2i\mu^2)t}\bar{b}(k,0).$$

Using (4.17) we see that the reflection coefficients have the time evolution

$$\begin{aligned}\rho(k,t) &= e^{-(4i\lambda k - 2i\mu^2)t}\rho(k,0), &\qquad \bar{\rho}(k,t) &= e^{(4i\lambda k - 2i\mu^2)t}\bar{\rho}(k,0), \\ r(k,t) &= e^{(4i\lambda k - 2i\mu^2)t}r(k,0), &\qquad \bar{r}(k,t) &= e^{-(4i\lambda k - 2i\lambda^2)t}\bar{r}(k,0).\end{aligned}$$