PERGAMON

Nonlinear Analysis 47 (2001) 2113–2123

Nonlinear Analysis

www.elsevier.nl/locate/na

Dead Cores for Time Dependent Reaction-diffusion Equations

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Abstract

We investigate a two-species reaction-diffusion problem described by a system of two semilinear parabolic equations with suitable initial-boundary conditions. We find restrictions on data to realize regions of zero concentration (so-called dead cores) for either species.

Key words: reaction-diffusion equation, system of parabolic equations, dead core
1991 MSC: 35K57, 35K60

1 Introduction

The existence of subregions in which the solution of an (initial-)boundary value problem vanishes identically was investigated for a parabolic equation by Bandle, Nanbu and Stakgold [2] and for an elliptic equation by Bandle, Sperb and Stakgold [3] and by Bobisud and Stakgold [6]. In this article we want to generalize the treatment of these so-called dead cores from a single semilinear PDE to a system of semilinear parabolic equations and, as in [2], establish comparison theorems that imply the existence or non-existence of dead cores under suitable conditions on the initial and boundary data. We remark that the class of such problems originated from the seminal paper by Stakgold [7] where their physical and chemical background is explained in detail.

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[1] Research partially supported by INDAM and MURST.

0362-546X/01/$ - see front matter © 2001 Published by Elsevier Science Ltd.
PII: S0362-546X(01)00337-6

Let us explain the contents of the various sections. In Section 2 we formulate the model problem and present suitable transformations to rewrite it in a more convenient form. In Section 3 some comparison theorems are proved to get monotonicity of solutions from either the monotonicity of the reaction terms or from the monotonicity of the intial-boundary data functions. We then go on to analyze the corresponding diffusion-free initial value problem, where the terms involving spatial derivatives have been omitted. The solutions coming out of this simplified problem are then utilized as super- and subsolutions of the original problem, thus leading to existence and nonexistence results for dead cores. We conclude with an illustrative example.

2 Formulation and Model Problem

We consider the following initial-boundary value problem for $u(x,t)$ and $v(x,t)$:

$$
\begin{aligned}
u_t - \Delta u &= -\lambda f(u) g(v) && \text{in } Q = \Omega \times \mathbb{R}^+, && (2.1)\\
v_t - \Delta v &= -k\lambda f(u) g(v) && \text{in } Q = \Omega \times \mathbb{R}^+, && (2.2)\\
u(x,t) &= \chi(x) && \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, && (2.3)\\
v(x,t) &= \eta(x) && \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, && (2.4)\\
u(x,0) &= u_0(x) && \text{in } \Omega \text{ with } 0 \le u_0(x) \le \alpha_0, && (2.5)\\
v(x,0) &= v_0(x) && \text{in } \Omega \text{ with } 0 \le v_0(x) \le \beta_0. && (2.6)
\end{aligned}
$$

Here $\chi(x)$ and $\eta(x)$ are continuous and nonnegative on Γ and $u_0(x)$ and $v_0(x)$ can be extended to nonnegative continuous functions on $\overline{\Omega}$, satisfying the compatibility conditions

$$
u_0(x) = \chi(x) \text{ and } v_0(x) = \eta(x), \qquad x \in \partial\Omega. \tag{2.7}
$$

The physical domain Ω is either an open interval of $\mathbb{R}^1$ or a bounded open connected set in $\mathbb{R}^N$ $(N > 1)$ whose boundary is a surface of class C^3, $k, \lambda > 0$, and Δ denotes the N-dimensional Laplace operator. For the absorption functions f, g, we impose the following conditions:

$$
\begin{cases}
f, g \in C[0,\infty) \cap C^1(0,\infty) & \\
f(0) = g(0) = 0, & f'(s) > 0 \text{ and } g'(s) > 0 \ (s > 0).
\end{cases}
$$

Following [6], we introduce

$$
w = -v + ku. \tag{2.8}
$$

From (2.1)-(2.6) we then obtain the initial-boundary value problem

$$\begin{aligned} w_t - \Delta w &= 0 && \text{in } Q = \Omega \times \mathbb{R}^+, && (2.9)\\ w(x,t) &= -\eta(x) + k\chi(x) && \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, && (2.10)\\ w(x,0) &= -v_0(x) + ku_0(x) && \text{in } \Omega \text{ with } 0 \le u_0(x) \le \alpha_0, && (2.11) \end{aligned}$$

which is the heat equation on Ω with suitable initial and boundary values.

The initial-boundary value problem (2.1)-(2.6) has a unique weak solution which turns out to be a classical solution [1]. Using w as in (2.9)-(2.11), we obtain the two initial-boundary value problems

$$\begin{aligned} u_t - \Delta u &= -\lambda f(u) g(ku - w) && \text{in } Q = \Omega \times \mathbb{R}^+, && (2.12)\\ u(x,t) &= \chi(x) && \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, && (2.3)\\ u(x,0) &= u_0(x) && \text{in } \Omega \text{ with } 0 \le u_0(x) \le \alpha_0, && (2.5) \end{aligned}$$

and

$$\begin{aligned} v_t - \Delta v &= -k\lambda f\Big(\frac{v+w}{k}\Big) g(v) && \text{in } Q = \Omega \times \mathbb{R}^+, && (2.13)\\ v(x,t) &= \eta(x) && \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, && (2.4)\\ v(x,0) &= v_0(x) && \text{in } \Omega \text{ with } 0 \le v_0(x) \le \beta_0. && (2.6) \end{aligned}$$

Clearly, a unique weak (and hence classical) solution w of the heat equation exists. Now the existence of a unique weak solution (u, v) follows from Theorem 0.1 of [5] applied separately to (2.12), (2.3) and (2.5), and (2.13), (2.4) and (2.6).

For comparison reasons, we let h_+ and h_- be the harmonic functions in Ω such that $h_+|_{\partial\Omega} = \chi$ and $h_-|_{\partial\Omega} = \eta$. Then $h_+ \ge 0$ and $h_- \ge 0$. Putting $\tilde u = u - h_+$ and $\tilde v = v - h_-$ we obtain the initial-boundary value problems

$$\begin{aligned} \tilde u_t - \Delta \tilde u &= -\lambda f(\tilde u + h_+) g(k\tilde u + h_-) && \text{in } Q = \Omega \times \mathbb{R}^+, && (2.14)\\ \tilde u(x,t) &= 0 && \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, && (2.15)\\ \tilde u(x,0) &= u_0(x) - h_+(x) && \text{in } \Omega, && (2.16) \end{aligned}$$

and

$$\begin{aligned} \tilde v_t - \Delta \tilde v &= -k\lambda f(\frac{1}{k}\tilde v + h_+) g(\tilde v + h_-) && \text{in } Q = \Omega \times \mathbb{R}^+, && (2.17)\\ \tilde v(x,t) &= 0 && \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, && (2.18)\\ \tilde v(x,0) &= v_0(x) - h_-(x) && \text{in } \Omega. && (2.19) \end{aligned}$$

When $0 \le h_+ \le u_0$ (resp. $0 \le h_- \le v_0$), the dead core problem for (2.14)-(2.16) (resp. (2.17)-(2.19)) can be solved as in [2].

Let us define a *supersolution* $(\overline{u}, \overline{v})$ (resp. *subsolution* $(\underline{u}, \underline{v})$) of (2.1)-(2.6) to be a pair of funcions (u, v) such that (2.1)-(2.6) hold with the inequality sign $\geq$ (resp. $\leq$) instead of the equality sign.

3 Monotonicity and other comparison theorems

In analogy with [2], consider the problem (2.1)-(2.6) when only one part of the data is changed. We then have the following monotonicity properties, where w_1 and w_2 are the solutions of the corresponding heat equations (2.9)-(2.11):

(a) Let (u_1, v_1) and (u_2, v_2) be the solutions corresponding to λ_1 and λ_2, respectively, with $\lambda_1 \leq \lambda_2$; then $u_2 \leq u_1$ and $v_2 \leq v_1$ in Q.
(b) If $f_1 \leq f_2$, $g_1 \leq g_2$ and $w_2 \leq w_1 \leq 0$, then $u_2 \leq u_1$ in Q. Analogously, if $f_1 \leq f_2$, $g_1 \leq g_2$ and $0 \leq w_1 \leq w_2$, then $v_2 \leq v_1$ in Q.
(c) If any of the initial and/or boundary data is decreased while w remains the same, either of the solutions u and v is decreased.

As to (a), note that the initial-boundary value problem (2.9)-(2.11) does not depend on λ and hence $w_1 = w_2$, which allows a straightforward application of the super- and subsolution technique to either of (2.12) and (2.13). As to (b), we need the comparison conditions on w_1 and w_2 to be able to apply this technique to either (2.12), (2.3) and (2.5), or (2.13), (2.4) and (2.6). Statement (c) follows by this technique, because the data are such that w remains the same.

Prior to analyzing the existence of dead cores of either species, we make the following observations. Consider the unique solution (z, y) of the initial value problem

$$z_t = -\lambda f(z) g(y) \qquad \text{in } \mathbb{R}^+, \tag{3.1}$$
$$y_t = -k\lambda f(z) g(y) \qquad \text{in } \mathbb{R}^+, \tag{3.2}$$
$$z(0) = z_0, \tag{3.3}$$
$$y(0) = y_0. \tag{3.4}$$

Then it is easily seen that $(\tilde{u}, \tilde{v})$ with

$$\tilde{u}(x,t) = z(t), \qquad \tilde{v}(x,t) = y(t),$$

is the unique solution of (2.1)-(2.6), where $\tilde{u}|_{\partial\Omega} \equiv z_0$, $\tilde{u}|_{t=0} \equiv z_0$, $\tilde{v}|_{\partial\Omega} \equiv y_0$, and $\tilde{v}|_{t=0} \equiv y_0$. In other words, if the initial and boundary values of u (resp. v) are equal to the same positive constant z_0 (resp. y_0), then u and v do not depend on x.

Let us now look at (3.1)-(3.4). As long as $z(t) > 0$ we integrate (3.1) to obtain

$$\lambda \int_0^t g(y(t))\,dt = \int_{z(t)}^{z_0} \frac{ds}{f(s)}. \tag{3.5}$$

Analogously, as long as $y(t) > 0$ we integrate (3.2) to obtain

$$k\lambda \int_0^t f(z(t))\,dt = \int_{y(t)}^{y_0} \frac{ds}{g(s)}. \tag{3.6}$$

In analogy with (2.8)-(2.11), introducing $\xi = -y + kz$ we obtain the initial value problem

$$\xi_t = 0, \qquad \xi(0) = \xi_0 = -y_0 + kz_0,$$

which has the unique constant solution $\xi(t) \equiv \xi_0$. Thus we have the decoupled initial value problems

$$z_t = -\lambda f(z) g\left(k\left(z - \frac{\xi_0}{k}\right)\right), \qquad z(0) = z_0 > 0, \tag{3.7}$$

$$y_t = -k\lambda f\left(\frac{y+\xi_0}{k}\right) g(y), \qquad y(0) = y_0 > 0. \tag{3.8}$$

We will adopt (3.7) if $\xi_0 < 0$, and (3.8) if $\xi_0 > 0$.

We now distinguish three cases.

The case $\xi_0 > 0$. We then integrate (3.8) to obtain

$$k\lambda t = \int_{y(t)}^{y_0} \frac{ds}{f\left(\frac{s+\xi_0}{k}\right) g(s)}, \tag{3.9}$$

in which $f((s+\xi_0)/k) \geq f(\xi_0/k) > 0$ and hence the convergence of the integral as $y(t) \downarrow 0$ is determined by g. In fact, if $I_g = \int_0^{y_0} (ds/g(s))$ is infinite (weak absorption of the second species), (3.9) provides a solution $y(t) > 0$ for all t, with y vanishing as $t \to \infty$. If, however, $I_g < \infty$ (strong absorption of the second species), then $y(t) > 0$ for $0 \leq t < t_\#$ and $y(t) \equiv 0$ for $t \geq t_\#$, where

$$t_\# = \frac{1}{k\lambda} \int_0^{y_0} \frac{ds}{f\left(\frac{s+\xi_0}{k}\right) g(s)} < \frac{I_g}{k\lambda f(\xi_0/k)} < \infty.$$

Strong absorption of the second species therefore leads to its extinction in finite time, the extinction time $t_\#$ being inverse proportional to λ. Now (3.9) implies that for $t \geq t_\#$

$$\lambda \int_0^{t_\#} g(y(s))\,ds = \int_{z(t)}^{z_0} \frac{ds}{f(s)}. \tag{3.10}$$

Consequently, $z(t)$ is constant for $t \geq t_\#$. Again we must distinguish various cases. If $I_f = \int_0^{z_0} (ds/f(s))$ is infinite (weak absorption of the first species), there is a unique $z_\infty > 0$ such that

$$\lambda \int_0^{t_\#} g(y(s))\, ds = \int_{z_\infty}^{z_0} \frac{ds}{f(s)}. \tag{3.11}$$

In this case z decreases from z_0 to z_∞ if $0 \leq t \leq t_\#$ and $z(t) \equiv z_\infty$ for $t \geq t_\#$. However, if $I_f < \infty$ (strong absorption of the first species), then there is a unique $z_\infty > 0$ such that (3.11) holds if and only if $I_f > \lambda \int_0^{t_\#} g(y(s))\, ds$, in which case $z(t) \equiv z_\infty$ for $t \geq t_\#$. But if $I_f \leq \lambda \int_0^{t_\#} g(y(s))\, ds$, the first species gets extinct first (at the finite time t_* ($\leq t_\#$)). But then

$$\lambda \int_0^t g(y(s))\, ds = I_f, \qquad t \geq t_*,$$

implying $y(t) \equiv 0$ for $t \geq t_*$; hence $t_* = t_\#$.

The case $\xi_0 < 0$. We then integrate (3.7) to obtain

$$\lambda t = \int_{z(t)}^{z_0} \frac{ds}{f(s) g\left(k\left(s - \frac{\xi_0}{k}\right)\right)},$$

and repeat the preceding reasoning with the following modifications. Putting $\mu = k\lambda$, $\ell = (1/k)$, $\eta_0 = -z_0 + \ell y_0$ and $\eta = -z + \ell y$, we convert the initial value problem (3.1)-(3.4) into the modified initial value problem

$$\begin{aligned} y_t &= -\mu g(y) f(z) && \text{in } \mathbb{R}^+, \\ z_t &= -\ell\mu g(y) f(z) && \text{in } \mathbb{R}^+, \\ y(0) &= y_0, \\ z(0) &= z_0, \end{aligned}$$

while $\eta_0 = -\ell(-y_0 + kz_0) > 0$ and $\eta(t) \equiv \eta_0$. We then have the following results:

- If $I_f = \infty$ (weak absorption of the first species), then $z(t) > 0$ for all t, with z vanishing as $t \to \infty$.
- If $I_f < \infty$ (strong absorption of the first species), then $z(t) > 0$ for $t < t_*$ and $z(t) \equiv 0$ for $t \geq t_*$, where

$$t_* = \frac{1}{\lambda} \int_0^{z_0} \frac{ds}{f(s) g\left(k\left(z - \frac{\xi_0}{k}\right)\right)} < \frac{I_f}{\lambda g(-\xi_0)} < \infty.$$

- If $I_f < \infty$ and $\infty \geq I_g > k\lambda \int_0^{t_*} f(z(s))\, ds$, then y decreases if $0 \leq t \leq t_*$

and stabilizes at the positive value $y(t) \equiv y_\infty$ determined by

$$\lambda \int_0^{t_*} f(z(s))\,ds = \int_{y_\infty}^{y_0} \frac{ds}{g(s)}.$$

- If I_f and I_g are both finite and $I_g \leq k\lambda \int_0^{t_*} f(z(s))\,ds$, then y and z both decrease if $0 \leq t \leq t_*$ and $y(t) \equiv z(t) \equiv 0$ if $t \geq t_*$.

The case $\xi_0 = 0$. In this case $y_0 = kz_0$ and hence $y = kz$, where

$$z_t = -\lambda f(z)g(kz) \qquad \text{in}\,\mathbb{R}^+, \tag{3.12}$$

$$z(0) = z_0, \tag{3.13}$$

which can be treated as in [2]. More precisely, if $I = \int_0^{z_0} (ds/(f(s)g(ks))$ is infinite, the solutions $z(t)$ and $y(t)$ are positive for all t, tending to zero as $t \to \infty$. If, however, I is finite, then $z(t)$ and $y(t)$ are positive if $0 \leq t < t_*$ and vanish identically for $t \geq t_*$, where

$$t_* = \frac{1}{\lambda} \int_0^{z_0} \frac{ds}{f(s)g(ks)}.$$

In this case both species get extinct at the same finite time t_* that is inverse proportional to λ.

Our first result is almost immediate from Theorems 3.1 and 4.1 of [2].

Theorem 3.1 *Let* $v_0 = ku_0$ *in* $\overline{\Omega}$. *Put* $H(s) = \int_0^s f(t)g(kt)\,dt$, *and let* ϕ *stand for the solution of the steady-state problem* $\Delta\phi = \lambda f(\phi)g(k\phi)$, $\phi|_{\partial\Omega} = \chi$. *Then the following statements are true:*

1. *If* $\chi = \eta = 0$ *in* $\partial\Omega$ *and* $\int_0^1 \frac{ds}{f(s)g(ks)} < \infty$ *(strong absorption), then there is simultaneous extinction of both species in finite time, i.e.* $u(x,t) = v(x,t) = 0$ *for* $t \geq t_*$ *and* $x \in \overline{\Omega}$.
2. *If* $\min_{\overline{\Omega}} v_0 = k \min_{\overline{\Omega}} u_0 > 0$ *and* $\int_0^1 \frac{ds}{f(s)g(ks)} = \infty$ *(weak absorption), then* $v(x,t) = ku(x,t) > 0$ *for all* $(x,t) \in Q$.
3. *If* $\int_0^1 \frac{ds}{\sqrt{H(s)}}$ *and* $\int_0^1 \frac{ds}{f(s)g(ks)}$ *are finite and* $\lambda_0 = \inf_\lambda \{\phi(x_0,\lambda) = 0\}$, *then* $v(x_0,t) = ku(x_0,t) = 0$ *whenever* $(\lambda - \lambda_0)t \geq I$.
4. *If* $\int_0^1 \frac{ds}{\sqrt{H(s)}} < \infty$ *and* $\int_0^1 \frac{ds}{f(s)g(ks)} = \infty$, *then* $v(x,t) = ku(x,t) > 0$ *for all* $(x,t) \in Q$.

Proof. Under the conditions of Theorem 3.1, $w \equiv 0$ in Q and Eq. (2.12) with conditions (2.3) and (2.5) reduces to the initial-boundary value problem

$$u_t - \Delta u = -\lambda f(u)g(ku) \qquad \text{in } Q = \Omega \times \mathbb{R}^+, \tag{3.14}$$

$$u(x,t) = \chi(x) \geq 0 \qquad \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, \tag{3.15}$$

$$u(x,0) = u_0(x) \qquad \text{in } \Omega, \tag{3.16}$$

while $v = ku$. To prove part 1, one observes that the solution z of (3.12)-(3.13) for $z_0 = \max_{\overline{\Omega}} u_0$ is a supersolution of (3.14)-(3.16). Then part 1 follows from the case $\xi_0 = 0$ discussed above with I finite. Part 2 follows in the same way, noting that the solution z of (3.12)-(3.13) for $z_0 = \min u_0$ is a subsolution of (3.14)-(3.16).

Note that the steady-state problem $\Delta\phi = \lambda f(\phi)g(k\phi)$, $\phi|_{\partial\Omega} = \chi$, has a dead core if the first integral in the statement of part 3 is finite [3]. Hence λ_0 exists if $x_0 \in \Omega$, and part 3 follows from Theorem 4.1(a) of [2]. Part 4 is immediate from Theorem 4.1(b) of [2].

Theorem 3.2 *Let $-v_0 + ku_0 \not\equiv 0$ in $\overline{\Omega}$ and be nonnegative there. Put $G(s) = \int_0^s g(t)\,dt$, let $I_G = \int_0^1 \frac{ds}{\sqrt{G(s)}} < \infty$, and let w_∞ be the harmonic function in Ω that extends $-\eta + k\chi$. Then the following statements are true:*

1. *If $I_g = \int_0^1 \frac{ds}{g(s)} < \infty$, then for all $x_0 \in \Omega$ there exists λ_0 such that $v(x_0,t) = 0$ whenever $(\lambda - \lambda_0)t \geq I_g$.*
2. *If $I_g = \infty$, then $u(x,t) > 0$ and $v(x,t) > 0$ for all $(x,t) \in Q$, where $u \to (w_\infty/k)$ and $v \to 0$ as $t \to \infty$.*

Analogously, let $-v_0 + ku_0 \not\equiv 0$ in $\overline{\Omega}$ and be nonpositive there. Put $F(s) = \int_0^s f(t)\,dt$, let $I_F = \int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty$, and let w_∞ be as above. Then the following statements are true:

1′. *If $I_f = \int_0^1 \frac{ds}{f(s)} < \infty$, then for all $x_0 \in \Omega$ there exists λ_0 such that $u(x_0,t) = 0$ whenever $(\lambda - \lambda_0)t \geq I_f$.*

2′. *If $I_f = \infty$, then $u(x,t) > 0$ and $v(x,t) > 0$ for all $(x,t) \in Q$, where $u \to 0$ and $v \to -w_\infty$ as $t \to \infty$.*

Proof. Now suppose that $-v_0 + ku_0 \geq 0$ (but $\not\equiv 0$) in $\overline{\Omega}$. Then $-\eta + k\chi \geq 0$ in $\partial\Omega$ [cf. (2.7)] and the solution w of (2.9)-(2.11) is positive in Ω. Further, $w \downarrow w_\infty \geq 0$ as $t \to \infty$, where w_∞ is the harmonic function in Ω extending $-\eta + k\chi$. So it suffices to study the solution v of (2.13), (2.4) and (2.6).

To prove part 1, for every compact subset K of Ω, we define

$$m_K = \min f\Big(\frac{v+w}{k}\Big) > 0$$

and study the initial-boundary value problem

$$\begin{aligned} V_t - \Delta V &= -k\lambda m_K g(V) && \text{in } Q = \Omega \times \mathbb{R}^+, \\ V(x,t) &= \eta(x) && \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, \\ V(x,0) &= v_0(x) && \text{in } \Omega \text{ with } 0 \leq v_0(x) \leq \beta_0. \end{aligned}$$

Then $0 \leq v \leq V$ in K. If $\int_0^1 \frac{ds}{\sqrt{G(s)}} < \infty$, the corresponding stationary problem has a dead core for sufficiently large λ and hence λ_0 exists. For $x_0 \in \Omega$ we then choose K so that $x_0 \in K$; letting Φ stand for the solution of the steady-state problem $\Delta\Phi = k\lambda m_K g(\Phi)$, $\Phi|_{\partial\Omega} = \eta$, we put $\lambda_0 = \min\{\lambda : \Phi(x_0, \lambda) = 0\}$ and derive part 1 as in the proof of Theorem 4.1(a) of [2]. In fact, $V(x_0, t) = 0$ if $(\lambda - \lambda_0)t \geq I_g$, which implies $v(x_0, t) = 0$ if $(\lambda - \lambda_0)t \geq I_g$. Part 2 is proved using the same comparison argument, but this time m_K stands for the **maximum** of $f((v + w)/k)$ in K. This then leads to a subsolution V of (2.13), (2.4) and (2.6). Assuming $\int_0^1 \frac{ds}{g(s)} = \infty$, we now get $V(x, t) > 0$ for all $(x, t) \in Q$ and hence $v(x, t) > 0$ for all $(x, t) \in Q$. Finally, since $u = (v + w)/k$ with $w > 0$ in Ω and $w \to w_\infty$ as $t \to \infty$, we get the statement in part 2 involving the asymptotic behavior of u.

Analogously, if $-v_0 + ku_0 \leq 0$ (but $\not\equiv 0$) in $\overline{\Omega}$, the solution w of (2.9)-(2.11) is negative in Ω. Further, $w \uparrow w_\infty \leq 0$ as $t \to \infty$, where w_∞ is the harmonic function in Ω extending $-\eta + k\chi$. So it suffices to study the solution u of (2.12), (2.3) and (2.5).

To prove part 1′, for every compact subset K of Ω, we define

$$n_K = \min g(ku - w) > 0$$

and study the initial-boundary value problem

$$\begin{aligned} U_t - \Delta U &= -\lambda n_K f(U) && \text{in } Q = \Omega \times \mathbb{R}^+, \\ U(x,t) &= \chi(x) && \text{in } \Gamma = \partial\Omega \times \mathbb{R}^+, \\ U(x,0) &= u_0(x) && \text{in } \Omega \text{ with } 0 \leq u_0(x) \leq \alpha_0. \end{aligned}$$

Then $0 \leq u \leq U$ in K. If $\int_0^1 \frac{ds}{\sqrt{F(s)}} < \infty$, the corresponding stationary problem has a dead core for sufficiently large λ and hence λ_0 exists. For $x_0 \in \Omega$ we then choose K so that $x_0 \in K$; letting Φ stand for the solution of the steady-state problem $\Delta\Phi = \lambda n_K f(\Phi)$, $\Phi|_{\partial\Omega} = \chi$, we put $\lambda_0 = \min\{\lambda : \Phi(x_0, \lambda) = 0\}$ and derive part 1′ as in the proof of Theorem 4.1(a) of [2]. In fact, $U(x_0, t) = 0$ if $(\lambda - \lambda_0)t \geq I_f$, which implies $u(x_0, t) = 0$ if $(\lambda - \lambda_0)t \geq I_f$. Part 2′ is proved using the same comparison argument, but this time n_K stands for the **maximum** of $g(ku - w)$ in K. This then leads to a subsolution U of (2.12), (2.3) and (2.5). Assuming $\int_0^1 \frac{ds}{f(s)} = \infty$, we now get $U(x, t) > 0$ for all $(x, t) \in Q$ and hence $u(x, t) > 0$ for all $(x, t) \in Q$. Finally, since $v = ku - w$ with $w < 0$ in Ω and $w \to w_\infty$ as $t \to \infty$, we get the statement in part 2′ involving the asymptotic behavior of v.

The treatment of the case where $-v_0 + ku_0$ changes sign, is more complicated, even in the steady-state situation [6]. As in [6], we divide Ω into the region Ω_+ where $w_\infty > 0$, and the region Ω_- where $w_\infty < 0$. Here, as we recall, w_∞ is the harmonic function in Ω extending $-\eta + k\chi$. We now consider the solution w of (2.9)-(2.11), which changes sign for every $t \in \mathbb{R}^+$. However, as $t \to \infty$ the distance $|w(x,t) - w_\infty(x)| \to 0$ monotonically, for every $x \in \Omega$. As a result, $w > 0$ in Ω_+ and $w < 0$ in Ω_-, no matter the choice of $t \in \mathbb{R}^+$. If we assume Ω to be connected, the unique continuation property of harmonic functions implies that w_∞ vanishes on a closed subset of $\overline{\Omega}$ of measure zero.

We can now invoke Theorem 3.2 separately for Ω_+ and Ω_-, as done before in [6]. The first half of this theorem then pertains to Ω_+ and the second half to Ω_-. Thus any dead core for v must necessarily be a subset of Ω_+ and any dead core of u must be a subset of Ω_-.

Example 3.3 In analogy with [6], let us discuss the following example in $\Omega = (-1, 1)$:

$$\begin{aligned} u_t - u'' &= -\lambda f(u)g(v) && \text{in } Q = (-1,1) \times \mathbb{R}^+, \\ v_t - v'' &= -k\lambda f(u)g(v) && \text{in } Q = (-1,1) \times \mathbb{R}^+, \\ u(-1,t) &= 1,\ u(1,t) = 0, && v(-1,t) = 0,\ v(1,t) = 1, \\ u(x,0) &= \frac{1-x}{2}, && -1 < x < 1, \\ v(x,0) &= \frac{1+x}{2}, && -1 < x < 1. \end{aligned}$$

Then

$$w(x,t) = w_\infty(x) = \frac{k-1}{2} - \frac{k+1}{2}x$$

is time independent and hence $\Omega_+ = (-1, x_1)$ and $\Omega_- = (x_1, 1)$, where $x_1 = (k-1)/(k+1)$. Hence any dead core for u is contained in $(x_1, 1)$ and any dead core for v is contained in $(-1, x_1)$.

Acknowledgments

The authors are greatly indebted to Ivar Stakgold for useful discussions on elliptic and parabolic problems leading to dead cores.

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