SPECTRAL FACTORIZATION OF BI-INFINITE BLOCK TOEPLITZ MATRICES WITH APPLICATIONS

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Abstract. In this paper we review some numerical methods for the computation of the spectral factorization of bi-infinite Toeplitz matrices with both scalar and block symbols. We also discuss the spectral factorization of the inverse of bi-infinite block Toeplitz matrices with a finite norm with respect to suitable weight sequences. The results on the spectral factors and their inverses are then applied to the solution of semi-infinite block Toeplitz linear systems and to the identification of the limiting profile in the Gram-Schmidt orthonormalization process, applied to increasing intervals, of a sequence of function vectors generated by integer translates of a given function vector. As applications of the methods proposed, the numerical solution of the Poisson equation on an infinite strip and the identification of the limiting profile in the asymptotic orthonormalization of a vector of B-splines are then illustrated. Finally we discuss the extension of the theory to the spectral factorization of bi-infinite block Toeplitz matrices.

1. Introduction. In this article we give a review of numerical methods for the spectral factorization for bi-infinite block Toeplitz matrices and some of its applications. As a start we list some definitions.

Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_+$ the set of non-negative integers. By a semi-infinite (resp., bi-infinite) block Toeplitz matrix of order k we mean a matrix $A = (A_{i-j})$, indexed by $\mathbb{Z}_+$ (resp., by $\mathbb{Z}$), whose elements are real $k \times k$ matrices where $k = 1, 2, \ldots$ does not depend on (i, j). Block Toeplitz matrices of order 1 are called Toeplitz matrices. Next, let $A = (A_j)_{j\in\mathbb{Z}}$ be a sequence of real $k \times k$ matrices, where $k \in \mathbb{N}$ does not depend on j. Taking a sequence of positive numbers $\beta = (\beta_j)_{j\in\mathbb{Z}}$ satisfying $\beta_{i+j} \le \beta_i\beta_j$ for all $i, j \in \mathbb{Z}$, by $\ell^k_{1,\beta}(\mathbb{Z})$ we mean the space of all bi-infinite sequences $A = (A_j)_{j\in\mathbb{Z}}$ of real $k \times k$ matrices for which

$$\|A\|_{1,\beta} := \sum_{j\in\mathbb{Z}} \beta_j \|A_j\| < \infty, \tag{1.1}$$

where $\|\cdot\|$ denotes an arbitrary $k \times k$ matrix norm. The most common choices of β are $\beta_j = (1 + |j|)^\rho$ for $\rho > 0$ and $\beta_j = g^{|j|}$ for $g > 1$, which correspond to algebraic and exponential weights, respectively. We write $\ell^k_1(\mathbb{Z})$ if $\beta_i = 1$ for all $i \in \mathbb{Z}$.

Spectral factorizations play a crucial role in the solution of semi-infinite Toeplitz systems of the type

$$\sum_{j\in\mathbb{Z}_+} T_{i-j}x_j = b_i, \quad i \in \mathbb{Z}_+. \tag{1.2}$$

Given a weight sequence β and a sequence of $k \times k$ matrices $(T_h)_{h\in\mathbb{Z}}$, with finite norm $\|(T_h)_{h\in\mathbb{Z}_+}\|_{1,\beta}$, we seek solutions $(x_h)_{h\in\mathbb{Z}}$ of (1.2) satisfying $\|(x_h)_{h\in\mathbb{Z}_+}\|_{1,\beta} < \infty$ for right-hand sides $(b_h)_{h\in\mathbb{Z}}$ satisfying $\|(b_h)_{h\in\mathbb{Z}_+}\|_{1,\beta} < \infty$. Here b_h and x_h are real column vectors of length k for every $h \in \mathbb{Z}_+$.

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The paper is organized as follows. In Section 2 we define the spectral factorization of bi-infinite block Toeplitz matrices. In Section 3 we discuss various numerical factorization methods for Toeplitz and block Toeplitz matrices. In Section 4 we recall Krein's method for solving the linear system (1.2), emphasizing the crucial role played by the spectral factorization of the inverse of a bi-infinite block Toeplitz matrix. In Section 5 we show that the identification of the limiting profile in the Gram-Schmidt orthonormalization process, applied to the uniform translates of a vector of functions defined on increasing intervals, essentially depends on the spectral factorization of a bi-infinite block Toeplitz matrix. In Section 6 we illustrate the application of the spectral factorization methods to the numerical solution of some semi-infinite block Toeplitz systems. In Section 7 we discuss generalizations to multi-index Toeplitz and block Toeplitz matrices.

2. Preliminaries. In this section we first introduce the Banach algebra $\mathcal{W}^k$ and indicate its invertible elements. We then go on to define spectral factorizations of bi-infinite block Toeplitz matrices and relate them to factorization properties in $\mathcal{W}^k$. Finally, we discuss factorization results for certain classes of bi-infinite and semi-infinite block Toeplitz matrices.

The class of matrix functions

$$\widehat{T}(z) = \sum_{j\in\mathbb{Z}} z^j T_j \tag{2.1}$$

on the unit circle $\mathcal{T}$ for which

$$\|\widehat{T}\|_{1,\beta} := \|(T_h)_{h\in\mathbb{Z}}\|_{1,\beta} = \sum_{j\in\mathbb{Z}} \beta_j \|T_j\| < \infty,$$

is a Banach algebra [10] denoted as $\mathcal{W}^k_\beta$. This means that its norm satisfies the inequality

$$\|\widehat{T}\|_{1,\beta} \leq \|\widehat{T}^{(1)}\|_{1,\beta} \|\widehat{T}^{(2)}\|_{1,\beta}; \qquad T_i = \sum_{j\in Z} T^{(1)}_{i-j} T^{(2)}_j, \quad i \in \mathbb{Z}.$$

We write $\mathcal{W}^k$ if $\beta_i = 1$ for all $i \in \mathbb{Z}$; this particular algebra is called the Wiener algebra (of order k). We call $\widehat{T}(z)$ the *symbol* of the block Toeplitz matrix $T = (T_{i-j})_{i,j\in\mathbb{Z}}$.

For $k = 1$ the invertible elements of $\mathcal{W}^k_\beta$ are completely described by the Gelfand theory of commutative Banach algebras [20, 10]. These results extend to matrix functions [10]. We have the following result.

THEOREM 2.1. *Let $\hat{T}(z)$ belong to $\mathcal{W}^k_\beta$ and have the form* (2.1). *Put $\beta_\pm = \lim_{j\to\pm\infty} \beta_j^{1/|j|}$. Then if $\hat{T}(z)$ is a nonsingular $k \times k$ matrix for all z satisfying $(1/\beta_-) \leq |z| \leq \beta_+$, $\hat{T}(z)^{-1}$ belongs to $\mathcal{W}^k_\beta$.*

In particular, if $\beta_j = g^{|j|}$ for some $g > 1$, then $\hat{T}(z)$ is an invertible element of $\mathcal{W}^k_\beta$ if and only if $\hat{T}(z)$ is a nonsingular matrix for $1/g \leq |z| \leq g$. Analogously, if $\beta_j = (1 + |j|)^\alpha$ for some $\alpha \geq 0$, then $\hat{T}(z)$ is an invertible element of $\mathcal{W}^k_\beta$ if and only if $\hat{T}(z)$ is a nonsingular matrix for $|z| = 1$.

By a *spectral factorization* of T (with respect to the weight sequence β) we mean a representation of T in any of the two forms

$$T = LDM^T \quad \text{and} \quad T = UDV^T, \tag{2.2}$$

where the superscript T denotes matrix transposition and L, M, D, U and V are bi-infinite Toeplitz matrices of order k having the following properties:

1. $L_0 = M_0 = U_0 = V_0 = I_k$ (the $k \times k$ identity matrix);
2. $D_s = 0$ for $s \neq 0$ and $L_s = M_s = 0$ for $s < 0$ and $U_s = V_s = 0$ for $s > 0$;
3. the inverses L^{-1}, M^{-1}, U^{-1} and V^{-1} of L, M, U and V are bi-infinite block Toeplitz matrices satisfying $[L^{-1}]_s = [M^{-1}]_s = 0$ for $s < 0$ and $[U^{-1}]_s = [V^{-1}]_s = 0$ for $s > 0$;
4. the matrix sequences with entries L_s, M_s, U_s, V_s, $[L^{-1}]_s$, $[M^{-1}]_s$, $[U^{-1}]_s$ and $[V^{-1}]_s$ belong to $\ell^k_{1,\beta}(\mathbb{Z})$.

A given bi-infinite block Toeplitz matrix has at most one spectral factorization of the type $T = LDM^T$ and at most one spectral factorization of the type $T = UDV^T$ as in (2.2).

Given a positive definite bi-infinite block Toeplitz matrix, setting $\mathcal{L}_s = L_s D_0^{1/2}$ or $\mathcal{U}_s = U_s D_0^{1/2}$, we obtain the *block Cholesky factorization* $T = \mathcal{L}\mathcal{L}^T$ or the *block Wiener-Hopf factorization* $T = \mathcal{U}\mathcal{U}^T$, where $\mathcal{L} = (\mathcal{L}_{i-j})_{i,j\in\mathbb{Z}}$ and $\mathcal{U} = (\mathcal{U}_{i-j})_{i,j\in\mathbb{Z}}$ are invertible on $\ell^k_{1,\beta}(\mathbb{Z})$ with inverses $\mathcal{L}^{-1}$ and $\mathcal{U}^{-1}$, the sequences $(\mathcal{L}_s)_{s\in\mathbb{Z}}$, $([\mathcal{L}^{-1}]_s)_{s\in\mathbb{Z}}$, $(\mathcal{U}_s)_{s\in\mathbb{Z}}$ and $([\mathcal{U}^{-1}]_s)_{s\in\mathbb{Z}}$ belong to $\ell^k_{1,\beta}(\mathbb{Z})$, and $\mathcal{L}_0$ and $\mathcal{U}_0$ are positive definite hermitian.

Passing to the symbols $\widehat{L}(z)$, $\widehat{D}(z) \equiv D_0$ and $\widehat{M}(z)$ of L, D and M, [resp., $\widehat{U}(z)$, $\widehat{D}(z) \equiv D_0$ and $\widehat{V}(z)$ of U, D and V], we have

$$\widehat{T}(z) = \widehat{L}(z) D_0 \widehat{M}(z)^T, \quad \left[\widehat{T}(z) = \widehat{U}(z) D_0 \widehat{V}(z)^T\right], \quad |z| = 1, \tag{2.3}$$

where $\widehat{L}(z)$ and $\widehat{M}(z)$ [$\widehat{U}(z)$ and $\widehat{V}(z)$] extend to matrix functions that are continuous on the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ and analytic on the open unit disk. Further, $\widehat{L}(z)$ and $\widehat{M}(z)$ [resp., $\widehat{U}(z)$ and $\widehat{V}(z)$] are nonsingular matrices for $|z| \leq 1$ and $\widehat{L}(0) = \widehat{M}(0) = I_k$ [resp., $\widehat{U}(0) = \widehat{V}(0) = I_k$].

Positive definite matrix functions in $\mathcal{W}^k$, matrix functions $\hat{T}(z)$ in $\mathcal{W}^k$ such that $\hat{T}(z) + \hat{T}(z)^*$ is positive definite for all $z \in \mathcal{T}$ (where the asterisk denotes the conjugate transpose), and matrix functions $\widehat{T}(z)$ in $\mathcal{W}^k$ such that $\|I - \widehat{T}(z)\| < 1$ for all $z \in \mathcal{T}$ (the norm representing the largest singular value of $I - \widehat{T}(z)$) have factorizations of the type (2.3) (cf. [7]). When the Toeplitz matrix T of order k is banded (i.e., if $T_{i-j} = 0$ whenever $|i-j| \geq m$ for some integer m), necessary and sufficient conditions for the existence of the spectral factorization (2.3) as well as a numerical procedure for its calculation have been given [24], based on matrix polynomial theory (e.g., [29]).

We have the following result.

THEOREM 2.2. *Suppose the bi-infinite block Toeplitz matrix $T = (T_i)_{i\in\mathbb{Z}}$ with symbol in $\mathcal{W}^k$ has a spectral factorization of the type* (2.1) *and $\hat{T}(z)$ is a nonsingular matrix for $(1/\beta_-) \leq |z| \leq \beta_+$. Then the factors and their inverses have symbols belonging to $\mathcal{W}^k_\beta$.*

The above theorem is most relevant to banded matrices T where $T_i = 0$ for $|i| > m$. Suppose one can find an annulus $1/g \leq |z| \leq g$ with $g > 1$ on which $\det \hat{T}(z) \neq 0$. Then if T has a spectral factorization, the factors and their inverses have symbols belonging to $\mathcal{W}^k_\beta$ for $\beta_j = g^{|j|}$.

3. Numerical factorization methods. In this section we discuss five well-known numerical factorization methods for bi-infinite banded Toeplitz matrices with positive definite symbol. We then go on to describe the corresponding methods for block Toeplitz matrices. Finally, we illustrate two factorization methods for computing the inverses of block Toeplitz matrices that we will use later to solve a differential boundary value problem on a semi-infinite strip and to find the limiting profile in the Gram-Schmidt orthonormalization process applied to a semi-infinite sequence of function vectors.

3.1. The scalar Toeplitz case. Let us review five methods for the spectral factorization of banded and real symmetric Toeplitz matrices [18, 23].

Let A be a bi-infinite, real symmetric and banded Toeplitz matrix

$$A = \text{Toeplitz}(A_{-m}, \ldots, A_0, \ldots, A_m),$$

where $A_j = \bar{A}_j = A_{-j}$, $j = 0, 1, 2, \ldots, m$. When its symbol $\hat{A}(z) > 0$ for $z \in \mathcal{T}$, the problem is to find real numbers $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$ such that

$$\hat{\Gamma}(z) = \sum_{j=0}^{m} \Gamma_j z^j$$

has all of its roots outside the unit disk and such that

$$\hat{\Gamma}(z)\hat{\Gamma}(z^{-1}) = \hat{A}(z), \quad z \in \mathcal{T}. \tag{3.1}$$

To guarantee the uniqueness of the factorization we impose the normalization $\hat{\Gamma}(0) > 0$ on the polynomial $\hat{\Gamma}(z)$. The existence of such a factorization is well-known and generally attributed to Fejér [cf. [27], Sec. 53].

The first method we wish to describe is due to F. Bauer [2, 3]. As $\hat{A}(z) > 0$ for $z \in \mathcal{T}$, the matrix A is positive definite on $\ell_2(\mathbb{Z})$. Moreover, the factorization (3.1) corresponds to the factorization

$$A = \Gamma\Gamma^T, \qquad \Gamma = (\Gamma_{i-j})_{i,j\in\mathbb{Z}}.$$

Let A_+ be the semi-infinite compression of A given by

$$A_+ = (A_{ij})_{i,j\in\mathbb{Z}_+}$$

and A_n, $n \in \mathbb{Z}_+$, be the sequence of finite compressions of A given by

$$A_n = (A_{ij})_{i,j=0,1,\ldots,n}, \quad n = 0, 1, 2, \ldots.$$

Then each A_n has a unique Cholesky factorization $L_n L_n^T = A_n$, where L_n is a lower triangular matrix with positive diagonal elements. As we replace n by $n+1$, the matrix A_{n+1} agrees with A_n on its first $n+1$ rows and columns. Likewise L_{n+1} has its first $n+1$ rows and columns equal to those of L_n. Thus we may consider L_n as the n-th finite section of a semi-infinite matrix L which is the unique Cholesky factorization of A_+, i.e. $A_+ = LL^T$. Bauer proved that $(L_n)_{ij} \to \Gamma_{i-j}$ as $n \to \infty$, and in [17] it has been proved that these elements approach those of Γ exponentially, i.e., there exist $c > 0$ and $\rho \in (0,1)$ such that for all $i, j \in \mathbb{Z}_+$

$$|L_{ij} - \Gamma_{i-j}| \leq c\rho^i.$$

Based on this observation, it is straightforward to justify the corresponding algorithm.

The second method we consider is due to Wilson [31] (cf. [18]). His idea is to write equation (3.1) as the equivalent system of quadratic equations

$$\sum_{j=0}^{m-i} \Gamma_j \Gamma_{j+i} = A_i, \quad i = 0, 1, \ldots, m, \tag{3.2}$$

and to use a Newton-Raphson method for solving the system (3.2). Wilson proved that, if $\hat{A}(z)$ has no zeros on the unit circle, for a suitable and easily made choice of the starting values $\Gamma_0^{(0)}, \Gamma_1^{(0)}, \ldots, \Gamma_m^{(0)}$, the iteration is self-correcting and converges quadratically to the required solution. For details on the algorithm we refer to [18].

The third method is based on the computation of the zeros of the Laurent polynomial $\hat{A}(z)$. To improve the effectiveness of the method, in view of the symmetry of its coefficients, we write $\hat{A}(z)$ in the form

$$\hat{A}(z) = A_0 + \sum_{j=1}^{m} A_j \left(z^j + z^{-j}\right)$$

and make the change of variable $w = z + z^{-1}$ to obtain the polynomial

$$\hat{C}(w) := \hat{A}(z(w)) = \sum_{j=0}^{m} C_j w^j$$

whose coefficients $C_0, C_1, \ldots, C_m$ can be expressed recursively in the original coefficients $A_0, A_1, \ldots, A_m$. The zeros w_i, $i = 1, 2, \ldots, p$, of the polynomial $\hat{C}(w)$ can then be evaluated by computing the eigenvalues of its companion matrix by a QR method [16]. Next, we solve the m quadratic equations

$$z + z^{-1} = w_i, \quad i = 1, 2, \ldots, m,$$

and then, taking into account their multiplicity, we obtain the $2m$ roots z_i, $i = 1, 2, \ldots, 2m$, of $\hat{A}(z)$ ordered by decreasing modulus. The polynomial $\hat{\Gamma}(z)$ we are looking for can then be constructed by multiplying the linear factors corresponding to all the roots z_j (including $\overline{z}_j$) outside the unit circle.

The fourth method tested in [18] is the minimum phase factorization. The basic idea of the algorithm is based on the following observation. Let $\hat{A}(z)$ have no zeros on the unit circle. We consider the minimization problem

$$\min \int_{-\pi}^{\pi} \frac{|G(e^{i\theta})|^2}{\hat{A}(e^{i\theta})}\, d\theta,$$

where G runs through all polynomials of degree $\leq m$ and it is normalized by $G(0) = 1$. Then the solution of this minimization problem is given by

$$G_{\text{opt}}(z) = \frac{\hat{\Gamma}(z)}{\hat{\Gamma}(0)},$$

where $\hat{\Gamma}$ is the polynomial in (3.1) we seek.

The final method we review is very popular in signal processing and goes by the name of the *cepstral* algorithm [4, 26]. It is based on some results on the factorization of an absolute convergent Fourier series on the unit circle, discovered independently by M.G. Krein [22] and by A. Calderón et al. [6]. Let us discuss the basic idea underlying this method.

As the Laurent polynomial $\hat{A}(z)$ is strictly positive on the unit circle, by the Wiener-Lévy theorem on trigonometric series, the function $\log \hat{A}(z)$ can be written as an absolutely convergent Laurent series

$$B(z) := \log \hat{A}(z) = \sum_{j \in \mathbb{Z}} B_j z^j.$$

Moreover, as those of $\hat{A}(z)$, the coefficients of $B(z)$ are even in j and decay exponentially, since $\log \hat{A}(z)$ has an extension as an analytic function in some annulus $r^{-1} < |z| < r$, $r > 1$. Splitting the Laurent series in the form

$$B(z) = B_-(z) + B_+(z),$$

where

$$B_\pm(z) := \sum_{j=0}^{\infty} \left(1 - \frac{1}{2}\delta_{j,0}\right) B_{\pm j} z^{\pm j}, \quad |z^\pm| \leq 1,$$

we obtain the factorization (3.1) of the required type, where

$$\hat{\Gamma}(z) := \exp(B_+(z))$$

is analytic and zero free in the disk $\{z : |z| \leq r\}$ and normalized so that $\hat{\Gamma}(0) > 0$. Note that $\hat{\Gamma}$ above is indeed a polynomial of degree at most m. A Matlab implementation of this method has been proposed in [1].

In [18] it was found that there is a huge disparity between the methods, and that all of them except for the Wilson method are significantly affected by the variation of the coefficients of the Laurent polynomial in magnitude, by the vicinity of the zeros of this polynomial to the unit circle and by their spacing.

3.2. The block Toeplitz case. Among the spectral factorization methods for scalar positive definite symbols, only Bauer's method allows for a straightforward generalization to bi-infinite block Toeplitz matrices with positive definite symbols. The generalization of Wilson's method to the block Toeplitz case, as formulated in [32], cannot be implemented in a straightforward way, since the algebraic equations for symbols appearing in [32] cannot be easily converted to linear matrix equations for their coefficients. Further, no extension of the fourth method above to the block Toeplitz case has ever been numerically implemented. Finally, the cepstral method cannot be generalized to the block Toeplitz case either, since it is ultimately based on the additivity property of the logarithm.

Let us now discuss the UDL factorization of the banded bi-infinite block Toeplitz matrix A of the form

$$A = L^T DR, \tag{3.3}$$

where $L = (L_{i-j})_{i,j\in\mathbb{Z}}$, $R = (R_{i-j})_{i,j\in\mathbb{Z}}$ and $D = (D_{i-j})_{i,j\in\mathbb{Z}}$ are banded block Toeplitz matrices with $L_0 = R_0 = I_k$ (the $k \times k$ identity matrix), $L_s = -\mathcal{L}_s$ ($s = 1, \ldots, m$) and $R_s = -\mathcal{R}_s$ ($s = 1, 2, \ldots, n$), $D_0 = \mathcal{D}$, $D_s = 0$ for $s \neq 0$, $L_s = 0$ for $s \neq 0, 1, \ldots, m$, and $R_s = 0$ for $s \neq 0, 1, \ldots, n$. In other words, L and R are lower block-triangular matrices and D is a block-diagonal matrix. Further, let the matrix function

$$\Sigma(z) = \sum_{j=-m}^{n} z^j A_j \tag{3.4}$$

be the *symbol* associated to the matrix A, and let us assume A_n to be nonsingular. Then $\mathcal{P}(z) = z^m A_n^{-1} \Sigma(z)$ is a matrix polynomial of degree $m + n$ whose leading coefficient is the identity matrix I_k.

Let us consider a simple closed positively oriented rectifiable Jordan curve Γ, with $0 \notin \Gamma$, dividing the complex plane into an *interior* bounded domain Ω_+ with $0 \in \Omega_+$ and an *exterior* domain Ω_-, and assume that $\det(z^m\Sigma(z))$ does not vanish for $z \in \Gamma$. Then [29] the matrix factorization (3.3) is equivalent to the factorization

$$\Sigma(z) = \left(I_k - \sum_{i=1}^{m} z^{-i}\mathcal{L}_i\right)^T \mathcal{D} \left(I_k - \sum_{j=1}^{n} z^j \mathcal{R}_j\right), \quad z \in \Gamma, \tag{3.5}$$

of the symbol of A. The nonsingularity of A_n implies that both $\mathcal{R}_n$ and $\mathcal{D}$ are nonsingular.

Before describing the algorithm to factorize A, we recall some basic properties of matrix polynomials [13, 14, 29]. Consider the matrix polynomial

$$\mathcal{P}(z) = z^\ell I_k + z^{\ell-1}A_{\ell-1} + \ldots + zA_1 + A_0,$$

of degree ℓ whose coefficients are $k \times k$ matrices. We call $z_j \in \mathbb{C}$ an *eigenvalue* of $\mathcal{P}$ if $\det \mathcal{P}(z_j) = 0$. The corresponding *eigenvector* $\mathbf{x}_{j1}$ is a nontrivial vector in $\mathbb{C}^k$ satisfying $\mathcal{P}(z_j)\mathbf{x}_{j1} = 0$. Then, obviously, $\mathcal{P}$ has a *spectrum* of exactly $k\ell$ eigenvalues, taking into account their multiplicities as zeros of $\det \mathcal{P}(z)$. The complex vectors $\{\mathbf{x}_{j1}, \mathbf{x}_{j2}, \ldots, \mathbf{x}_{jr}\}$ constitute a *Jordan chain* at z_j of length r, if $\mathbf{x}_{j1} \neq 0$ and the lower triangular linear system of equations

$$\sum_{v=1}^{u} \frac{\mathcal{P}^{(u-v)}(z_j)}{(u-v)!}\mathbf{x}_{jv} = 0, \quad u = 1, 2, \ldots, r,$$

is valid. The lengths of the Jordan chains at the eigenvalue z_j in a system of maximal Jordan chains are called the *partial multiplicities* of $\mathcal{P}$ at z_j. The sum of the partial multiplicities at z_j coincides with the order of z_j as a zero of $\det \mathcal{P}(z)$ and is called its *algebraic multiplicity*; the dimension of the kernel of $\mathcal{P}(z_j)$ is called its *geometric multiplicity* and is equal to the number of linearly independent maximal Jordan chains at z_j.

A pair of matrices (X, T), where X is of size $k \times k\ell$ and T is of size $k\ell \times k\ell$, is called a *right spectral pair* for the polynomial $\mathcal{P}(z)$ if the matrix

$$\mathrm{col}[XT^j]_{j=0}^{\ell-1} := \begin{bmatrix} X \\ XT \\ \vdots \\ XT^{\ell-1} \end{bmatrix}$$

is invertible and the following equality holds

$$XT^\ell + \sum_{j=0}^{\ell-1} A_j XT^j = 0.$$

By a *left spectral pair* for the polynomial $\mathcal{P}(z)$ we mean a right spectral pair for $\mathcal{P}(z)^T$. The *right canonical form* for $\mathcal{P}(z)$ is given by

$$\mathcal{P}(z) = z^\ell I_k - XT^\ell(V_1 + zV_2 + \ldots + z^{\ell-1}V_\ell),$$

where $V_1, \ldots, V_\ell$ are the $k\ell \times k$ matrices defined by

$$\begin{bmatrix} V_1 & V_2 & \ldots & V_\ell \end{bmatrix} = \left(\mathrm{col}[XT^j]_{j=0}^{\ell-1}\right)^{-1}.$$

For a right spectral pair we have the inversion formula

$$\mathcal{P}(z)^{-1} = X(z-T)^{-1}V_\ell. \tag{3.6}$$

A similar definition holds for the *left canonical form.*

Denoting by $z_1, \dots, z_p$ the *distinct* eigenvalues of $\mathcal{P}(z)$, each of algebraic multiplicity m_j, it can be shown [29] that a right spectral pair (X,T) for $\mathcal{P}(z)$ is given by

$$X = \begin{bmatrix} X_1 & X_2 & \dots & X_p \end{bmatrix}, \quad T = T_1 \oplus T_2 \oplus \dots \oplus T_p, \tag{3.7}$$

where the matrices X_j, of size $k \times m_j$, and T_j, of size $m_j \times m_j$, are given by

$$\begin{aligned} X_j &= \begin{bmatrix} \mathbf{x}_{11}^{(j)} & \dots & \mathbf{x}_{1r_{j1}}^{(j)} & \mathbf{x}_{21}^{(j)} & \dots & \mathbf{x}_{2r_{j2}}^{(j)} & \dots & \mathbf{x}_{q_j1}^{(j)} & \dots & \mathbf{x}_{q_jr_{jq_j}}^{(j)} \end{bmatrix}, \\ T_j &= J_{r_{j1}}(z_j) \oplus J_{r_{j2}}(z_j) \oplus \dots \oplus J_{r_{jq_j}}(z_j). \end{aligned}$$

Here $\mathbf{x}_{s1}^{(j)}, \mathbf{x}_{s2}^{(j)}, \dots, \mathbf{x}_{sr_{js}}^{(j)}$, $s = 1, \dots, q_j$, are the maximal Jordan chains for $\mathcal{P}(z)$ corresponding to z_j, $J_{r_s}(z_j)$ is the $r_s \times r_s$ upper triangular Jordan block with eigenvalue z_j, and $r_{j1} + r_{j2} + \dots + r_{jq_j} = m_j$ $(j = 1, \dots, p)$.

To obtain the factorization the following theorem is crucial.

THEOREM 3.1. *Let $z_1, \dots, z_s$ be the distinct zeros of $\det(z^m\Sigma(z))$ in Ω_- and let $z_{s+1}, \dots, z_{s+t}$ be its distinct zeros in Ω_+. Moreover, let (X,T) be a right spectral pair of the matrix polynomial $\mathcal{P}(z) = z^m A_n^{-1}\Sigma(z)$. Then there exists a factorization of $\Sigma(z)$ of the type* (3.5) *where $\det(I_k - \sum_{i=1}^m z^{-i}\mathcal{L}_i) \neq 0$ for $z \in \Omega_-$ and $\det(I_k - \sum_{j=1}^n z^j\mathcal{R}_j) \neq 0$ for $z \in \Omega_+$, if and only if*

$$m_1 + \dots + m_s = nk, \quad m_{s+1} + \dots + m_{s+t} = mk,$$

and the restriction of $\mathrm{col}[XT^j]_{j=0}^{n-1}$ to the linear span of the eigenvectors and generalized eigenvectors of T in Ω_+ is invertible. This factorization is unique and is called the spectral factorization of A.

Following the proof of Theorem 2.1 of [24], the explicit construction of the factorization (3.5) has been given in [25] and will be presented here in abbreviated form. Indeed, as specified in (3.7), we construct the right spectral pair (X_R, T_R) of the matrix polynomial $\mathcal{P}(z) = z^m A_n^{-1}\Sigma(z)$ by using its Jordan chains, which coincide with the Jordan chains of $z^m\Sigma(z)$, and its left spectral pair (X_L, T_L) in a similar fashion. Partitioning the $k\times(m+n)k$ matrices X_L and X_R into a $k \times nk$ block and a $k \times mk$ block, we have

$$\begin{aligned} X_L &= \begin{bmatrix} ? & V \end{bmatrix}, \quad X_R = \begin{bmatrix} W & ? \end{bmatrix}, \\ T_L = T_R &= \bigoplus_{j=1}^{s+t} \left(J_{r_{j1}}(z_j) \oplus \dots \oplus J_{r_{jq_j}}(z_j) \right), \end{aligned}$$

where the matrices at the question marks are irrelevant and $r_{j1} + \dots + r_{jq_j} = m_j$ $(j = 1, \dots, s+t)$. We now set

$$\begin{aligned} \Lambda_L &= \bigoplus_{j=s+1}^{s+t} \left(J_{r_{j1}}(z_j) \oplus \dots \oplus J_{r_{jq_j}}(z_j) \right), \\ \Lambda_R &= \bigoplus_{j=1}^{s} \left(J_{r_{j1}}(z_j) \oplus \dots \oplus J_{r_{jq_j}}(z_j) \right), \end{aligned}$$

and construct the $mk \times k$ matrices $W_1, \dots, W_m$ and the $nk \times k$ matrices $V_1, \dots, V_n$ by putting

$$\begin{bmatrix} W_1 & W_2 & \dots & W_m \end{bmatrix} = \left(\mathrm{col}[V\Lambda_L^j]_{j=0}^{m-1}\right)^{-1},$$
$$\begin{bmatrix} V_1 & V_2 & \dots & V_n \end{bmatrix} = \left(\mathrm{col}[W\Lambda_R^j]_{j=0}^{n-1}\right)^{-1}.$$

The coefficients of the factorization in (3.3) are then given by

$$\begin{aligned}
\mathcal{L}_i &= (A_n^T)^{-1}\, V\Lambda_L^m W_{m-i+1}\, A_n^T, && i = 1, \dots, m,\\
\mathcal{R}_j &= -\,(W\Lambda_R^n V_1)^{-1}\, W\Lambda_R^n V_{j+1}, && j = 1, \dots, n-1,\\
\mathcal{R}_n &= (W\Lambda_R^n V_1)^{-1}, &&\\
\mathcal{D} &= -(\mathcal{L}_m^T)^{-1} A_{-m} = -A_n \mathcal{R}_n^{-1}, &&
\end{aligned}$$

which corresponds to taking right spectral pairs (V, Λ_L) and (W, Λ_R) for the matrix polynomials $\mathcal{L}(z)(A_n^T)^{-1}$ and $\mathcal{R}(z)$, respectively, and recovering the spectral factors from their canonical forms.

Assuming $\Gamma = \{z \in \mathbb{C} : |z| = \rho\}$, $\Omega_+ = \{z \in \mathbb{C} : |z| < \rho\}$ and $\Omega_- = \{z \in \mathbb{C} : |z| > \rho\}$, $\Sigma(z)$ has the factorization (3.5) for $z \in \Gamma$, where

$$\begin{aligned}
\left(I_k - \sum_{i=1}^{m} z^{-i}\mathcal{L}_i\right)^{-1} &= z^m (A_n^T)^{-1} V (z - \Lambda_L)^{-1} W_m A_n^T\\
&= \sum_{\mu=0}^{\infty} z^{m-\mu-1} (A_n^T)^{-1} V \Lambda_L^{\mu} W_m A_n^T
\end{aligned}$$

for $z \in \Omega_-$ and

$$\left(I_k - \sum_{j=1}^{n} z^j \mathcal{R}_j\right)^{-1} = W(I_k - z\Lambda_R^{-1})^{-1}\Lambda_R^{-1} V_n \mathcal{R}_n^{-1} = \sum_{\mu=0}^{\infty} z^{\mu} W \Lambda_R^{-(\mu+1)} V_n \mathcal{R}_n^{-1}$$

for $z \in \Omega_+$.

When $\Sigma(z)$ is the symbol corresponding to a real matrix A which is symmetric ($m = n$, $A_{-j} = A_j^T$), positive definite, block Toeplitz and banded, we can assume the curve Γ to be the unit circle, so that Theorem 3.1 assures that there exists only one factorization of A of the form

$$A = UU^T,$$

where $U = L^T D^{1/2}$. This is often referred to as the *Wiener-Hopf factorization* of A. In this case the matrix D in (3.3) is real positive definite, as well as A_0 and $\mathcal{D}$. Moreover, $\mathcal{L}_j = \mathcal{R}_j$ and the symbol

$$\mathcal{U}(z) := \sum_{j=0}^{m} z^{-j} \mathcal{U}_j$$

of the matrix U has coefficients defined by

$$\mathcal{U}_0 = \mathcal{D}^{1/2},$$
$$\mathcal{U}_j = -\mathcal{R}_j^T \mathcal{D}^{1/2} = -\mathcal{L}_j^T \mathcal{D}^{1/2}, \qquad j = 1, \ldots, m.$$

The factorization method just discussed has the disadvantage that it can only be used for banded block Toeplitz matrices. We now present a method based on band extension (cf. Section XXXV.3 of [10]) which applies only to positive definite semi-infinite block Toeplitz matrices.

Given the real $k \times k$ matrices $A_{-p}, A_{-p+1}, \ldots, A_p$, we seek a bi-infinite block Toeplitz matrix $\Phi = (\Phi_{i-j})_{i,j\in\mathbb{Z}}$ which is positive definite, satisfies $\Phi_j = A_j$ for $|j| \leq p$, and has the property $\sum_{j\in\mathbb{Z}} \|\Phi_j\| < +\infty$. In other words, we seek a positive definite band extension Φ of the banded bi-infinite block Toeplitz matrix with symbol

$$\sum_{j=-p}^{p} z^j A_j. \tag{3.8}$$

The following result indicates when such a so-called *Carathéodory-Toeplitz extension* exists. Moreover, in the theorem we present an algorithm [10] for constructing one particular such extension as well as all such extensions.

THEOREM 3.2. *Given the real $k \times k$ matrices $A_{-p}, A_{-p+1}, \ldots, A_p$, there exists a Carathéodory-Toeplitz extension of the banded bi-infinite block Toeplitz matrix with symbol* (3.8) *if and only if the $k(p+1) \times k(p+1)$ matrix*

$$\Gamma = \begin{bmatrix} A_0 & A_{-1} & \ldots & A_{-p} \\ A_1 & A_0 & \ldots & A_{-p+1} \\ \vdots & \vdots & & \vdots \\ A_p & A_{p-1} & \ldots & A_0 \end{bmatrix} \tag{3.9}$$

is positive definite. In that case there exists a unique solution Φ_b with the property that the inverse of the corresponding bi-infinite block Toeplitz matrix is banded. Defining the $k(p+1) \times k$ matrices

$$\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_p \end{bmatrix} = \Gamma^{-1} \begin{bmatrix} I_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} Y_{-p} \\ \vdots \\ Y_{-1} \\ Y_0 \end{bmatrix} = \Gamma^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_k \end{bmatrix}, \tag{3.10}$$

the bi-infinite block Toeplitz matrices $U = (X_{i-j})_{i,j\in\mathbb{Z}_+}$ and $V = (Y_{i-j})_{i,j\in\mathbb{Z}_+}$ using $X_j = 0$ for $j < 0$ and $j > p$ and $Y_j = 0$ for $j < -p$ and $j > 0$, and the block-diagonal block Toeplitz matrices $\mathcal{D}_X$ and $\mathcal{D}_Y$ with respective diagonal entries X_0 and Y_0, we have

$$\Phi_b^{-1} = U\mathcal{D}_X U^T = V\mathcal{D}_Y V^T. \tag{3.11}$$

Moreover, each real solution Φ of the Carathéodory-Toeplitz extension problem has the property

$$\Phi^{-1} = (VG + U)(I - G^T G)^{-1}(G^T V^T + U^T), \tag{3.12}$$

where $G = (G_{i-j})_{i,j\in\mathbb{Z}_+}$ is a bi-infinite real block Toeplitz matrix satisfying $G_j = 0$ for $j \leq p$ which is a strict contraction on $\ell_2(\mathbb{Z}_+)$. There is a 1,1-correspondence between real solutions Φ and such matrices G.

The above theorem leads to the following factorization of positive definite bi-infinite block Toeplitz matrices. Let A be a positive definite bi-infinite block Toeplitz matrix. For $p \in \mathbb{Z}_+$ large enough, the bi-infinite block Toeplitz matrix with symbol (3.8) has a positive definite banded extension Φ such that the bi-infinite inverse Φ^{-1} is banded. The factors of the inverse Φ^{-1}, as given by (3.11), are approximations of the factors of the inverse of the original positive definite block Toeplitz matrix.

4. Spectral factorization and the solution of semi-infinite block Toeplitz systems. Let us now recall Krein's method [22] for solving the semi-infinite block Toeplitz system (1.2). Denoting by $\widehat{T}(z)$ the symbol associated to the Toeplitz matrix T, the system (1.2) has a unique solution with discrete Fourier transform in $\mathcal{W}_\beta^k$ if and only if $\widehat{T}(z) \neq 0$ for all z with $(1/\beta_-) \leq |z| \leq \beta_+$ and $\beta_\pm$ as above. Then $\widehat{T}(z)^{-1}$ has the factorization

$$\widehat{T}(z)^{-1} = \Gamma_+(z)\Gamma_-(z) \tag{4.1}$$

in $\mathcal{W}_\beta^k$, where

$$\Gamma_+(z) = \sum_{j\in\mathbb{Z}_+} z^j\Gamma_j^{(1)}, \qquad \Gamma_-(z) = \sum_{j\in\mathbb{Z}_+} z^{-j}\Gamma_j^{(2)}, \tag{4.2}$$

with $\|\Gamma^{(1)}\|_{1,\beta} < \infty$ and $\|\Gamma^{(2)}\|_{1,\beta} < \infty$. One then has the solution formula

$$x_\ell = \sum_{s\in\mathbb{Z}_+} \Gamma_{\ell s} b_s, \quad \ell \in \mathbb{Z}_+, \tag{4.3}$$

for the system of equations (1.2), where

$$\Gamma_{\ell s} = \sum_{h=0}^{\min(\ell,s)} \Gamma_{\ell-h}^{(1)}\Gamma_{s-h}^{(2)}. \tag{4.4}$$

Note that the mentioned properties of the sequences $\{\Gamma_j^{(1)}\}$ and $\{\Gamma_j^{(2)}\}$ allows us to approximate properly x_ℓ by equation (4.3). If T is block banded, for example, then $\Gamma^{(1)}$, $\Gamma^{(2)}$ and Γ are exponentially decaying and, as a result, we can obtain a good approximation by considering a small number of terms in the series representation.

We now have the following result.

THEOREM 4.1. *Consider the semi-infinite block Toeplitz system* (1.2), *where, for some fixed weight sequence β, $\|(T_h)_{h\in\mathbb{Z}_+}\|_{1,\beta} < \infty$ and $\left\|(b_i)_{i\in\mathbb{Z}_+}\right\|_{1,\beta} < \infty$. Suppose $\widehat{T}(z)$ is invertible for every z with $(1/\beta_-) \leq |z| \leq \beta_+$. Then each solution $(x_i)_{i\in\mathbb{Z}_+}$ of* (1.2) *with $\left\|(x_i)_{i\in\mathbb{Z}_+}\right\|_1 < \infty$ satisfies $\left\|(x_i)_{i\in\mathbb{Z}_+}\right\|_{1,\beta} < \infty$.*

Let us now illustrate some asymptotic properties of the factors of a sequence of increasing compressions of an infinite block Toeplitz matrix. Consider the bi-infinite banded block Toeplitz matrix A with symbol given by (3.8), where each A_i is a $k \times k$ matrix. We define the semi-infinite block Toeplitz matrices A_+ and A_- as follows:

$$A_+ = (A_{i-j})_{i,j\in\mathbb{Z}_+}, \qquad A_- = (A_{j-i})_{i,j\in\mathbb{Z}_+}, \tag{4.5}$$

where $A_{i-j} = 0$ if $|i-j| > p$. Then A_+ (resp. A_-) are boundedly invertible on the Hilbert space ℓ_2 of square summable complex sequences $(x_i)_{i\in\mathbb{Z}_+}$ if and only if the symbol $\Sigma(z) = \sum_{j=-p}^{p} z^j A_j$ (resp., $\Sigma(z^{-1}) = \sum_{j=-p}^{p} z^j A_{-j}$) has a spectral factorization. Now we define on ℓ_2 the projections P_n of rank nk as follows: $P_n(x_i)_{i=0}^{\infty} = (x_0, x_1, \dots, x_{nk}, 0, 0, 0, \dots)$. Define the $(nk)\times(nk)$ matrices A_{n+} and A_{n-} as the corresponding left upper blocks of A_+ and A_-, respectively. Then, according to Theorem 2.1 of [11] (also [9, 28]), the matrices A_{n+} and A_{n-} are nonsingular for sufficiently large n and for every $y \in \ell_2$ the vector $(A_{n+}^{-1}y, 0, 0, 0, \dots)$ converges to $A_+^{-1}y$ in the norm of ℓ_2. Similarly, for every $y \in \ell_2$ the vector $(A_{n-}^{-1}y, 0, 0, 0, \dots)$ converges to $A_-^{-1}y$ in the norm of ℓ_2. In fact, these two convergence properties together are equivalent to the bounded invertibility of both A_+ and A_- on ℓ_2.

The semi-infinite matrices $A_\pm$ given by (4.5) have the LDU-factorizations $A_\pm = L_\pm D_\pm M_\pm^T$ if and only if they are boundedly invertible on ℓ_2. Then, for n large enough, the matrices $A_{n\pm}$ corresponding to their $(nk)\times(nk)$ left upper corners have LDU-factorizations of the form

$$A_{n+} = L_{n+}D_{n+}M_{n+}^T, \quad A_{n-} = L_{n-}D_{n-}M_{n-}^T,$$

where D_{n+} and D_{n-} are block diagonal matrices and L_{n+}, L_{n-}, M_{n+} and M_{n-} are lower block triangular matrices having I_k as their diagonal blocks. Moreover, writing $\mathcal{J}_n[T]$ as the semi-infinite matrix obtained from the $(nk)\times(nk)$ matrix T by adding zero entries, the extended $(nk)\times(nk)$ matrices $\mathcal{J}_n[L_{n\pm}]$, $\mathcal{J}_n[L_{n\pm}^{-1}]$, $\mathcal{J}_n[M_{n\pm}]$, $\mathcal{J}_n[M_{n\pm}^{-1}]$, $\mathcal{J}_n[D_{n\pm}]$ and $\mathcal{J}_n[D_{n\pm}^{-1}]$ converge to the respective semi-infinite matrices $L_\pm$, $L_\pm^{-1}$, $M_\pm$, $M_\pm^{-1}$, $D_\pm$ and $D_\pm^{-1}$ in the strong operator topology as $n\to\infty$. To prove these convergence properties, one applies the theory of multiplicative LU-decompositions [10] with respect to the chain of orthogonal projections $\{0, P_0, P_1, \dots, I_{\ell_2}\}$ to the semi-infinite matrices $A_\pm$, where I_{ℓ_2} is the identity operator on ℓ_2. Using the same multiplicative LU-decompositions with respect to the same chain of projections in the Hilbert space of complex sequences $(x_i)_{i\in\mathbb{Z}_+}$ endowed with the weighted norm $\left[\sum_{i\in\mathbb{Z}_+} r^{2i}|x_i|^2\right]^{1/2}$ for a suitable $r > 1$, one proves that the above convergence properties also hold with respect to the weighted norm.

Similar results hold for the LDM^T-factorization of a bi-infinite banded block Toeplitz matrix A. Indeed, let $\ell_2(\mathbb{Z})$ stand for the Hilbert space of square summable sequences indexed by the integers and let us define on $\ell_2(\mathbb{Z})$ the projection $P_n^{\leftrightarrow}$ of rank $(2n+1)k$ as follows: $P_n^{\leftrightarrow}(x_i)_{i=-\infty}^{\infty} = (\dots, 0, 0, x_{-n}, \dots, x_n, 0, 0, \dots)$. Next, define the $(2n+1)k\times(2n+1)k$ matrices A_n as the corresponding central blocks of A. Then, according to [11], Theorem 4.1, applied for $R = S$, A_n is nonsingular for sufficiently large n and for every $y \in \ell_2(\mathbb{Z})$ the vector $(\dots, 0, 0, A_n^{-1}y, 0, 0, \dots)$ converges to $A^{-1}y$ in the norm of $\ell_2(\mathbb{Z})$. Actually, this convergence property is equivalent to the invertibility of both A_+ and A_- on ℓ_2. Moreover, for sufficiently large n the matrices A_n have an LDU-factorization of the form

$$A = L_n D_n M_n^T,$$

where D_n is a block diagonal matrix and L_n and M_n are lower triangular banded block Toeplitz matrices having I_k as their diagonal blocks. Moreover, writing $\mathcal{J}_n^{\leftrightarrow}[T]$ as the bi-infinite matrix obtained from the $(2n+1)k\times(2n+1)k$ matrix T by adding zero entries, the extended matrices $\mathcal{J}_n^{\leftrightarrow}[L_n]$, $\mathcal{J}_n^{\leftrightarrow}[L_n^{-1}]$, $\mathcal{J}_n^{\leftrightarrow}[M_n]$, $\mathcal{J}_n^{\leftrightarrow}[M_n^{-1}]$, $\mathcal{J}_n^{\leftrightarrow}[D_n]$ and $\mathcal{J}_n^{\leftrightarrow}[D_n^{-1}]$ converge to the respective bi-infinite matrices L, L^{-1}, M, M^{-1}, D and D^{-1} in the strong operator topology as $n\to\infty$. The proof of these convergence properties follows from the theory of multiplicative LU-decompositions [10] with respect to a suitable chain of orthogonal projections.

When A is real positive definite and hence A_0 is real positive definite and $A_1^T = A_{-1}$, A_+ and A_- are both boundedly invertible on ℓ_2, since their LDU-factorizations both exist. In this case the unique solution of the equation $Ax = y$ can be approximated as indicated in the previous two paragraphs.

5. Spectral factorization and limiting profile. The properties of the Cholesky factorization of positive definite bi-infinite block matrices, that we use here, are a generalization of analogous results for scalar matrices obtained in [19], where they were applied to studying the asymptotic behavior of shifts of an exponentially decaying function. In this section we summarize the results from [24] on the asymptotic behavior of shifts of a vector of exponentially decaying functions.

Let $\varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_\ell(x))^T$, $x \in \mathbb{R}$, be a vector of real functions defined on $\mathbb{R}$. We say that φ is an exponentially decaying vector function if there exist constants $\kappa > 0$ and $\lambda \in (0,1)$ such that

$$\|\varphi(x)\|_\infty := \max_{i=1,\ldots,\ell} |\varphi_i(x)| \le \kappa \lambda^{|x|}, \quad x \in \mathbb{R}. \tag{5.1}$$

Now consider the integer translates of φ, i.e., the function vectors

$$\varphi_i := \varphi(\cdot - i), \quad i \in \mathbb{Z},$$

where $\varphi_0 := \varphi$. Furthermore, for $r, s \in \mathbb{Z}$, let the symbol $\langle \varphi_r, \varphi_s \rangle_{[a,b]}$ denote the $\ell \times \ell$ matrix

$$\left(\langle \varphi_{i+r}, \varphi_{j+s} \rangle_{[a,b]} \right)_{i,j=1,\ldots,\ell},$$

where $\langle \cdot, \cdot \rangle_{[a,b]}$ is the usual scalar product in $L^2[a,b]$. According to this definition, φ_r is orthonormal to φ_s on the interval $[a,b]$ if and only if $\langle \varphi_r, \varphi_s \rangle_{[a,b]} = I_\ell \, \delta_{rs}$.

For any such function vector φ, the bi-infinite block matrix sequence

$$T_h := \langle \varphi_0, \varphi_h \rangle_{\mathbb{R}}, \quad h \in \mathbb{Z},$$

is even in the sense that T_h and T_{-h} are each other's transposes, $h \in \mathbb{Z}$, and it is exponentially decaying since

$$\|T_h\|_\infty := \max_{i=1,\ldots,\ell} \sum_{j=1}^{\ell} |[T_h]_{ij}| \le \kappa^2 \ell \, \lambda^{|h|} \left(|h| + \frac{1}{\log(\lambda^{-1})} \right).$$

Hence ([24], Lemma 3.1) the bi-infinite block Toeplitz matrix of order ℓ

$$T = (T_{rs})_{r,s\in\mathbb{Z}} = (T_{r-s})_{r,s\in\mathbb{Z}},$$

is a bounded operator on $\ell_2(\mathbb{Z})$.

Now consider the semi-infinite block Gram matrix

$$G = (G_{rs})_{r,s\in\mathbb{Z}}, \qquad G_{rs} := \langle \varphi_r, \varphi_s \rangle_{\mathbb{R}_+}, \quad r, s \in \mathbb{Z}_+,$$

generated by φ.

LEMMA 5.1. *If φ is exponentially decaying, then there exist $c > 0$ and $\lambda \in (0,1)$ such that*

$$\|G_{ij} - T_{ij}\| \le c\lambda^{i+j}, \quad i, j \in \mathbb{Z}_+.$$

Now consider the finite block Gram matrices

$$[\widehat{G}_n]_{rs} := \langle \varphi_r, \varphi_s \rangle_{[0,n]}, \quad r,s = 0,1,\ldots,n, \quad n \in \mathbb{Z}_+,$$

whose block entries for large enough n, r and s are very close to the corresponding entries of G, as specified by the following lemma.

LEMMA 5.2. *Take $\rho \in (0,1)$ and consider the truncation $G_n = (G_{ij})_{i,j=0}^n$ of G. Then there exist $C > 0$ and $\mu \in (0,1)$ such that*

$$\|[G_n]_{rs} - [\widehat{G}_n]_{rs}\| \le C\mu^{r+s}, \qquad 0 \le r,s \le [\rho n].$$

Moreover, the sequence of inverses to the matrices $\left([\widehat{G}_n]_{rs}\right)_{r,s=0}^{[\rho n]}$ is bounded.

Now suppose that the matrices T and G are positive definite on $\ell_2(\mathbb{Z})$ and $\ell_2(\mathbb{Z}_+)$, respectively. As specified in Theorem 3.11 of [24], under this hypothesis there exist the block Cholesky factorizations

$$T = LL^T \quad \text{and} \quad G = MM^T$$

of T and G where L and L^{-1} are bi-infinite lower triangular block Toeplitz matrices with blocks of order ℓ, M and M^{-1} are semi-infinite lower triangular block matrices, and L, L^{-1}, M and M^{-1} decay exponentially. Consider the block Cholesky factorization

$$\widehat{G}_n = \widehat{M}_n \widehat{M}_n^T, \quad n \in \mathbb{Z}_+.$$

It is straightforward to prove that for all $i = 0,1,\ldots,n$ and $x \in \mathbb{R}$ the function vectors ψ_i^n defined by

$$\psi_i^n(x) = \sum_{j=0}^{i} \left[\widehat{M}_n^{-1}\right]_{ij} \varphi_j(x) \tag{5.2}$$

are the ones generated by the block Gram-Schmidt orthonormalization process applied to $\{\varphi_j\}_{j=0,1,\ldots,n}$ on $[0,n]$.

Under the above hypothesis on the Toeplitz matrix T, the function vector

$$\psi(x) := \sum_{j\in\mathbb{Z}_+} [L^{-1}]_j \varphi_{-j}(x), \quad x \in \mathbb{R}, \tag{5.3}$$

decays exponentially. Furthermore, its integer translates are orthonormal on $\mathbb{R}$, i.e.,

$$\langle \psi_r, \psi_s \rangle_{\mathbb{R}} = I_\ell \, \delta_{rs}, \quad r,s \in \mathbb{Z},$$

with $\psi_h := \psi(\cdot - h)$, $h \in \mathbb{Z}$.

The main result of Subsection 5, which is a generalization of Theorem 2.1 of [19], is that, under the conditions claimed in Theorem 5.3, the function vector ψ defined in (5.3) supplies the limiting profile of the block Gram-Schmidt orthonormalization process applied to the interval $[0,n]$, with large enough n.

Using formulae (5.1) and (5.2)-(5.3) we easily obtain that, for $x \in \mathbb{R}$ and all $i = 0,1,\ldots,n$,

$$\|\psi_i^n(x) - \psi_i(x)\| \le \kappa \sum_{j=0}^{i} \|[\widehat{M}_n^{-1}]_{ij} - [L^{-1}]_{ij}\| + \kappa \sum_{j=-\infty}^{-1} \|[L^{-1}]_{ij}\|.$$

Since $[L^{-1}]_{ij} = [L^{-1}]_{i-j}$ and the sequence $[L^{-1}]_h$, $h \in \mathbb{Z}_+$, decays exponentially as $h \to \infty$, there are constants $c > 0$ and $\alpha \in (0,1)$ such that

$$\|[L^{-1}]_{ij}\| \leq c\alpha^{|i-j|}, \quad i,j = 0,1,\dots,n.$$

This inequality implies that, for $x \in \mathbb{R}$,

$$\|\psi_i^n(x) - \psi_i(x)\| \leq \kappa \sum_{j=0}^{i} \|[\widehat{M}_n^{-1}]_{ij} - [L^{-1}]_{ij}\| + \kappa \frac{\alpha^i}{1-\alpha}, \quad i = 0,1,\dots,n.$$

THEOREM 5.3. *Suppose φ is a real function vector on $\mathbb{R}$ which satisfies the inequality* (5.1) *and has the property that the corresponding block Toeplitz matrix T and the Gram matrix G are positive definite. Then for every constant $\rho \in (0,1)$ there are constants $C > 0$, $\mu \in (0,1)$ and $N \in \mathbb{Z}_+$ such that for all $i, n \in \mathbb{Z}_+$, with $0 \leq i \leq \rho n$ and $n \geq N$ we have*

$$\|\psi_i^n(x) - \psi_i(x)\| \leq C\mu^i, \quad x \in \mathbb{R},$$

where ψ is defined by (5.3).

Let us now give a practical illustration of the procedure for the identification of the limiting profile proposed above. To this end we consider the following two B-splines, already introduced in [24]:

$$\begin{aligned} B_1(x) &= -\frac{11}{36}(-x)_+^3 - \frac{1}{2}(-x)_+^2 + \frac{1}{2}(1-x)_+^3 \\ &\quad -\frac{1}{4}(2-x)_+^3 + \frac{1}{18}(3-x)_+^3, \\ B_2(x) &= \frac{1}{12}(3-x)_+^4 - 2(2-x)_+^3 + \frac{3}{4}(2-x)_+^4 \\ &\quad -\frac{3}{4}(1-x)_+^4 - 2(1-x)_+^3 - \frac{1}{12}(-x)_+^4. \end{aligned}$$

Now, take $\varphi(x) = (B_1(x), B_2(x))^T$, $x \in \mathbb{R}$, and consider its integer translates $\varphi_i(x) := \varphi(x-i)$, $i \in \mathbb{Z}$.

Further, let T, with entries $T_{ij} = \langle \varphi_i, \varphi_j \rangle$, $i,j \in \mathbb{Z}$, be the Gram matrix associated to them. It is a 5-diagonal block Toeplitz matrix of order 2, whose nonzero blocks $T_{ij} = T_{i-j}$, $|i-j| \leq 2$, are

$$T_0 = \frac{1}{\alpha}\begin{bmatrix} 13176 & 10179 \\ 10179 & 11304 \end{bmatrix}, \quad T_1 = \frac{1}{\alpha}\begin{bmatrix} 4634 & 6573 \\ 1275 & 1688 \end{bmatrix}, \quad T_2 = \frac{1}{\alpha}\begin{bmatrix} 124 & 111 \\ 6 & 4 \end{bmatrix}$$

with $\alpha = 362880$, $T_{-2} = T_2^T$ and $T_{-1} = T_1^T$.

Hence, the symbol corresponding to T is

$$\Sigma(z) = \frac{1}{\alpha}\left(T_2 z^2 + T_1 z + T_0 + T_1^T z^{-1} + T_2^T z^{-2}\right),$$

where the superscript T denotes the matrix transpose. It is positive for $|z| = 1$ and further $\det(z^2\Sigma(z))$ has four zeros inside the unit circle and four outside of it, so that T is positive definite. Since in this case $G = T_+$, the hypotheses of Theorem 5.3 are satisfied.

Now let $\psi_i^n(x)$, $i = 0,1,\dots,n$, be the function vectors (5.2) generated by the Gram-Schmidt process applied to the translates of the function $\varphi(x)$ on the interval $[0,n]$. Then Figure 5.1 depicts the behavior of the two components of $\psi_i^{20}(x)$ for some values of i and Figure 5.2 shows the graph of the limiting profile $\psi(x)$ arising from the asymptotic orthonormalization process applied to the aforementioned vector functions ϕ_i, $i \in \mathbb{Z}_+$.

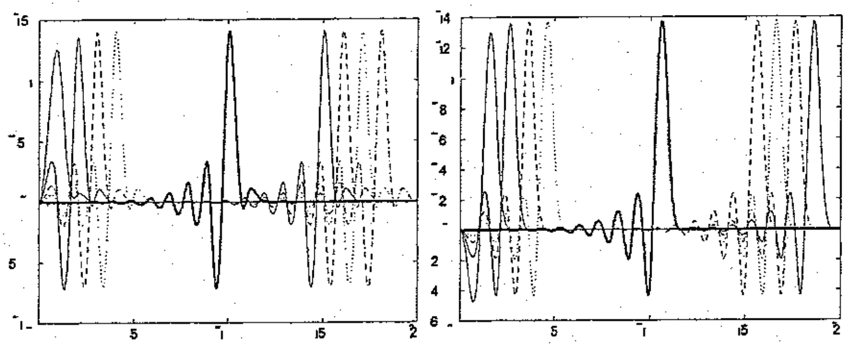

FIG. 5.1. *Components of* $\psi_i^{20}(x)$, $i = 0, 1, 2, 3, 9, 14, 15, 16, 17$

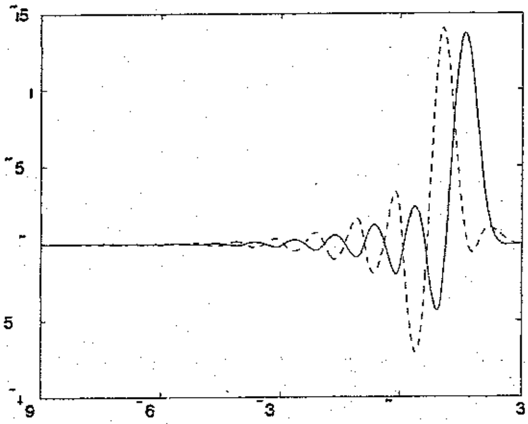

FIG. 5.2. *Limiting profile* $\psi(x)$

6. Applications. As a first application of the previous method, let us consider the numerical solution of the Poisson equation on a semi-infinite strip. More precisely, setting $\Omega = \{(x, y) : 0 < x < \infty,\ 0 < y < 1\}$ and denoting by $\partial\Omega$ the boundary of Ω, we consider the numerical solution, by a finite differences method, of the differential boundary value problem

$$\begin{cases} \Delta u(x,y) = -f(x,y), & (x,y) \in \Omega, \\ u(x,y) = 0, & (x,y) \in \partial\Omega, \end{cases} \tag{6.1}$$

where Δ is the Laplace operator.

Discretizing the differential problem (6.1) by a 5-points scheme on the mesh points

$$(x_i, y_j) = (ih, jh), \quad j = 0, 1, \ldots, n+1, \quad i = 0, 1, \ldots$$

with the stepsize $h = 1/(n+1)$ and choosing the usual order for the unknowns u_{ij}, we obtain a semi-infinite linear system of block Toeplitz type whose matrix is of the form

$$T = \begin{bmatrix} T_0 & -I_n & & \\ -I_n & T_0 & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \tag{6.2}$$

where I_n is the identity matrix of order n and T_0 is the $n \times n$ tridiagonal Toeplitz matrix

$$\begin{bmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}.$$

The symbol associated to the matrix T is the Laurent matrix polynomial

$$\widehat{T}(z) = -z^{-1}I_n + T_0 - zI_n.$$

In this case the eigenvalues $\{\lambda_j\}$ of the corresponding monic matrix polynomial $\mathcal{P}(z) = -z\widehat{T}(z)$ can be obtained analytically. Indeed, it is straightforward to prove that

$$\lambda_j = \begin{cases} \frac{1}{2}\left(\mu_j + \sqrt{\mu_j^2 - 4}\right), & j = 1, \ldots, n \\ 1/\lambda_{2n-j+1}, & j = n+1, \ldots, 2n, \end{cases}$$

where

$$\mu_j = 4 + 2\cos\frac{j\pi}{n+1}, \quad j = 1, \ldots, n.$$

The first n eigenvalues lie outside the unit circle and the last n inside, as shown in Figure 6.1 for $n = 20$. As a consequence, a very accurate spectral factorization of $\mathcal{P}(z)^{-1}$ can be obtained using the first method, even for moderately high values of n.

Denoting by u_{ij}, for $i = 0, 1, \ldots$ and $j = 0, 1, \ldots, n+1$, the solution of the discretized problem, we assess the accuracy of the results, with respect to n, by the following two error estimates:

$$E_n^{(k)} = \max\left\{|u(x_i, y_j) - u_{ij}^{(k)}|,\ i \in \mathcal{I}_{10n},\ j \in \mathcal{I}_n\right\} \tag{6.3}$$

$$E_{r,n}^{(k)} = \max\left\{\frac{|u(x_i, y_j) - u_{ij}^{(k)}|}{|u(x_i, y_j)|},\ i \in \mathcal{I}_{10n},\ j \in \mathcal{I}_n,\ |u(x_i, y_j)| > 10^{-16}\right\}, \tag{6.4}$$

where $\mathcal{I}_n = \{1, \ldots, n\}$ and $k = 1, 2$ specifies if the spectral factorization has been carried out by the first or the second method, respectively.

As a first example, take

$$f(x, y) = \left[2\pi\cos\pi x + (2\pi^2 - 1)\sin\pi x\right] e^{-x} \sin\pi y,$$

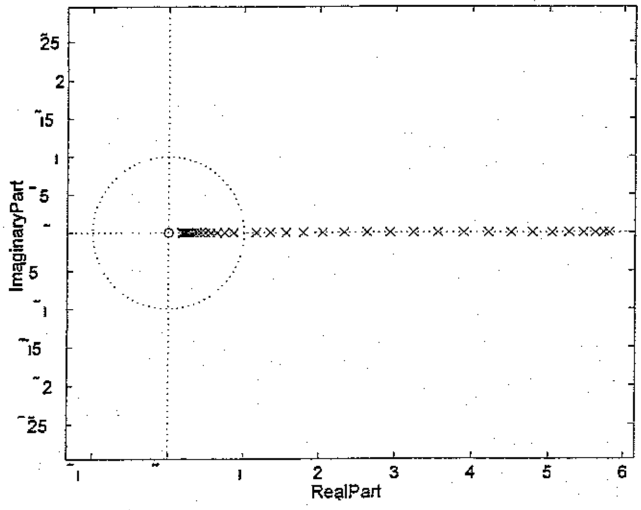

Fig. 6.1

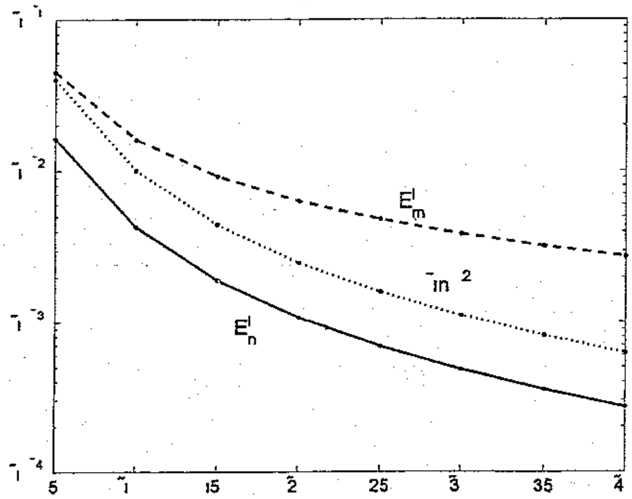

Fig. 6.2

so that the exact solution of (6.1) is

$$u(x,y) = \mathrm{e}^{-x} \sin \pi x \ \sin \pi y.$$

Figure 6.2 shows $E_n^{(1)}$, $E_{r,n}^{(1)}$ and $1/n^2$ in the range $5 \leq n \leq 40$. Note that, as we expect solving the Poisson equation by the 5-points discretization method [30], $E_n^{(1)} = O(n^{-2})$.

The high level of accuracy of the results essentially depends on the exact knowledge of the eigenvalues of the matrix polynomial $\mathcal{P}(z) = -z\widehat{T}(z)$. Indeed, our experience suggests that the precision attainable by the first method primarily depends on the accurate evaluation of the eigenvalues, on their separation and on their distance from the curve Γ mentioned in Section 3.

Figure 6.3 reports the error $E_n^{(k)}$, $k = 1, 2$, for several values of n and for two values

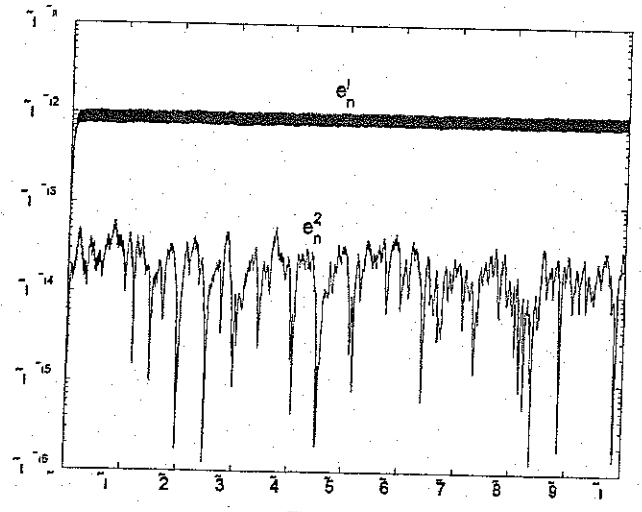

Fig. 6.3

of the extension parameter p. It shows, in particular, that for small values of n the two methods are equally effective. Furthermore, it highlights that the value of the extension parameter depends on the size of the blocks n. In fact, as n gets larger it is necessary to increase the value of p to obtain, by the band extension method, the same accuracy as in the first factorization method. For example, when $n = 25$ and $p = 60$ we need to solve a linear block Toeplitz system of dimension $n(p+1) = 1525$. This fact poses no particular problem from the point of view of computational complexity, the matrix being banded, but implies a larger propagation of roundoff errors with respect to the first method. However, as we will see soon, the first algorithm is not always more accurate than the second one.

As a second example on the solution of the same boundary value problem, let us consider

$$u(x,y) = \mathrm{e}^{-x/2} xy(1-y)$$

so that

$$f(x,y) = \mathrm{e}^{-x/2}\left[(1-y)(1-x/4)y + 2x\right].$$

To show the accuracy of the results, we depict in Figures 6.4 and 6.5 the errors $u(x_i, y_j) - u_{ij}$ corresponding to the first and the second example, respectively. In both cases, the results are obtained via the first spectral factorization method.

Now, as a second application, let T be the 5-diagonal block Toeplitz matrix of order 2 introduced in the previous section, and let T_+ be the semi-infinite block Toeplitz matrix given by $(T_+)_{ij} = T_{ij}$, $i, j \in \mathbb{Z}_+$. Furthermore, set $\mathrm{b} = T_+\mathrm{x}$, with $x_n = \left(\frac{8}{9}\right)^n$, $n \in \mathbb{Z}_+$, and solve the semi-infinite block Toeplitz linear system

$$T_+\mathrm{x} = \mathrm{b}$$

by the two spectral factorization methods illustrated above.

Denoting by $x^{(1)}$ and $x^{(2)}$ the solution vectors obtained by using the two spectral factorization methods, let

$$e_n^{(k)} = \frac{|x_n - x_n^{(k)}|}{|x_n|}, \tag{6.5}$$

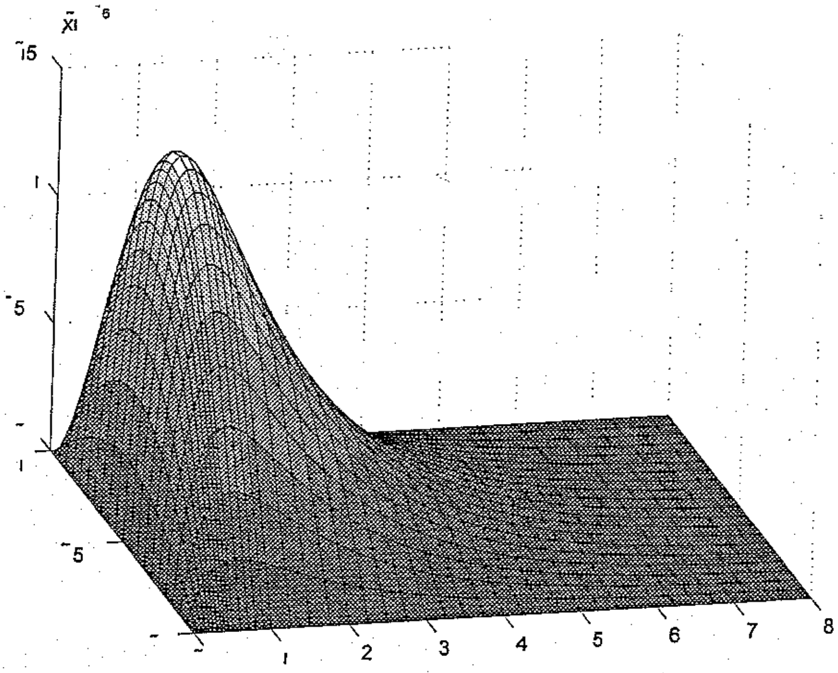

Fig. 6.5

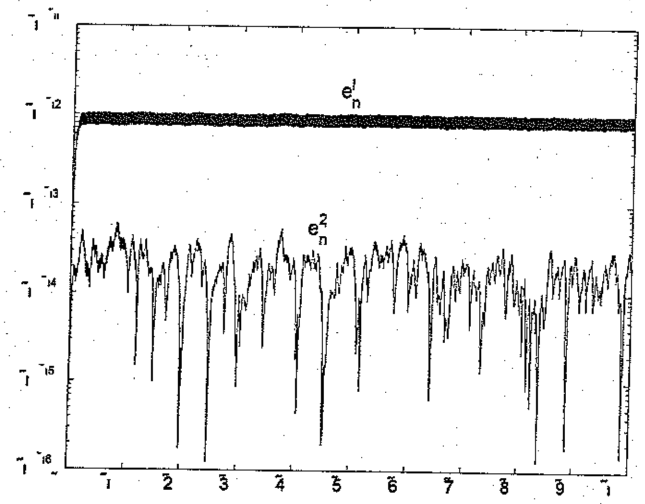

Fig. 6.6

satisfying $\beta_{i+j} \leq \beta_i \beta_j$ for all $i, j \in \mathbb{Z}^m$, by $\ell^k_{1,\beta}(\mathbb{Z}^m)$ we mean the space of all bi-infinite sequences $A = (A_j)_{j \in \mathbb{Z}^m}$ of real $k \times k$ matrices for which

$$\|A\|_{1,\beta} := \sum_{j \in \mathbb{Z}^m} \beta_j \|A_j\| < \infty, \tag{7.1}$$

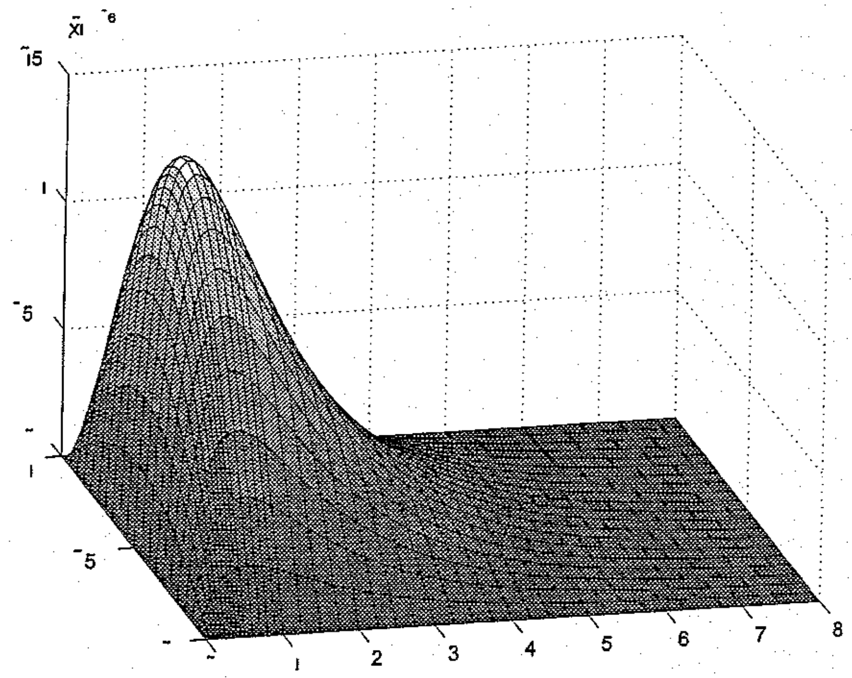

Fig. 6.5

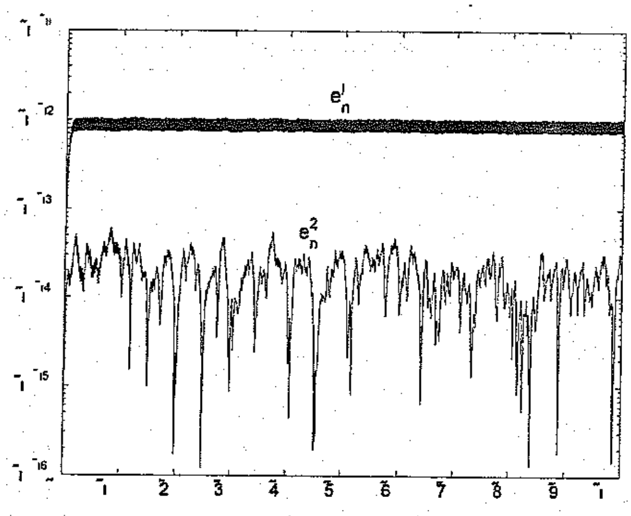

Fig. 6.6

satisfying $\beta_{i+j} \le \beta_i \beta_j$ for all $i, j \in \mathbb{Z}^m$, by $\ell^k_{1,\beta}(\mathbb{Z}^m)$ we mean the space of all bi-infinite sequences $A = (A_j)_{j\in\mathbb{Z}^m}$ of real $k \times k$ matrices for which

$$\|A\|_{1,\beta} := \sum_{j\in\mathbb{Z}^m} \beta_j \|A_j\| < \infty, \tag{7.1}$$

where $\|\cdot\|$ denotes an arbitrary $k \times k$ matrix norm. The most common choices* of β are $\beta_j = (1+|j|)^\rho$ for $\rho > 0$ and $\beta_j = g^{|j|}$ for $g > 1$, which correspond to algebraic and exponential weights, respectively. We write $\ell_1^k(\mathbb{Z}^m)$ if $\beta_i = 1$ for all $i \in \mathbb{Z}^m$. Further, for $k, m \in \mathbb{N}$, let $\ell_2^k(\mathbb{Z}^m)$ be the Hilbert space of all square summable functions $x : \mathbb{Z}^m \to \mathbb{R}^k$ with norm

$$\|x\|^2 = \sum_{j\in\mathbb{Z}^m} \|x_j\|^2, \qquad x = (x_j)_{j\in\mathbb{Z}^m},$$

and scalar product

$$\langle x, y\rangle = \sum_{j\in\mathbb{Z}^m} \langle x_j y_j\rangle, \qquad x = (x_j)_{j\in\mathbb{Z}^m}, \qquad y = (y_j)_{j\in\mathbb{Z}^m},$$

where $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ denote the Euclidean norm and the usual scalar product in $\mathbb{R}^k$, respectively.

For any m-index block Toeplitz matrix $(A_{i-j})_{i,j\in\mathbb{Z}^m}$ with $k \times k$ matrix entries for which the expression in (7.1) is finite for a suitable weight sequence $(\beta_j)_{j\in\mathbb{Z}^m}$, we define the *symbol* by

$$\hat{A}(z) = \sum_{j\in\mathbb{Z}^m} z^j A_j, \qquad z = (z_1, \cdots, z_m) \in \mathbb{T}^m, \tag{7.2}$$

where $z^j = z_1^{j_1} \dots z_m^{j_m}$ for $j = (j_1, \dots, j_m)$ and $\mathbb{T}^m = \{z = (z_1, \dots, z_m) \in \mathbb{C}^m : |z_1| = \dots = |z_m| = 1\}$ is the m-dimensional torus. Clearly, since necessarily $\beta_j \geq 1$ for any $j \in \mathbb{Z}^m$, the symbol of such a bi-infinite block Toeplitz matrix is a continuous $k \times k$ matrix function defined on $\mathbb{T}^m$. The symbols of the bi-infinite block Toeplitz matrices $(A_{i-j})_{i,j\in\mathbb{Z}^m}$ such that

$$\sum_{j\in\mathbb{Z}^m} \beta_j \|A_j\| < +\infty,$$

form a Banach algebra with respect to the pointwise matrix product, denoted as $\mathcal{W}_\beta^k$; here we note that $\hat{A}(z)\hat{B}(z)$ is the symbol of the bi-infinite Toeplitz matrix given by the convolution product

$$(A * B)_j = \sum_{l\in\mathbb{Z}^m} A_{j-l} B_l.$$

This Banach algebra is commutative if and only if $k = 1$.

In the Banach algebra $\mathcal{W}_\beta^k$ the multiplicative linear projections (in the sense of [15]) [or, for $k = 1$, the multiplicative linear functionals] are exactly the maps $A \mapsto \hat{A}(z)$ for which z belongs to the set

$$\begin{aligned} \mathcal{C}_\beta &= \left\{ z \in \mathbb{Z}^m : \sup_{j\in\mathbb{Z}^m} \frac{|z|^j}{\beta_j} < +\infty \right\} \\ &= \left\{ (z_1, \dots, z_m) \in \mathbb{Z}^m : \sup_{j_1,\dots,j_m\in\mathbb{Z}} \frac{|z_1|^{j_1} \dots |z_m|^{j_m}}{\beta_{(j_1,\dots,j_m)}} < +\infty \right\}. \end{aligned}$$

*The addition in $\mathbb{Z}^m$ is defined componentwise. For $j = (j_1, \dots, j_m) \in \mathbb{Z}^m$ we put $|j| = |j_1| + \dots + |j_m|$.

Note that $\mathcal{C}_\beta$ contains $\mathbb{T}^m$ (as a result of $\beta_j \geq 1$ for each $j \in \mathbb{Z}^m$), does not contain $(0, \ldots, 0) \in \mathbb{C}^m$, is compact, and is closed with respect to the operation $z \mapsto (c_1 z_1, \ldots, c_m z_m)$ for any $(c_1, \ldots, c_m) \in \mathbb{T}^m$. When

$$\beta_{(j_1,\ldots,j_m)} = \beta_{j_1}^{(1)} \cdots \beta_{j_m}^{(m)}$$

for m weight sequences $\beta_j^{(l)}$ of positive numbers satisfying $\beta_{i+j}^{(l)} = \beta_i^{(l)}\beta_j^{(l)}$ for $i, j \in \mathbb{Z}$ and $l = 1, \ldots, m$, we have

$$\mathcal{C}_\beta = \prod_{l=1}^{m} \left\{ z \in \mathbb{C} : (1/g_-^{(l)}) \leq |z| \leq g_+^{(l)} \right\},$$

where $g_\pm^{(l)} = \lim_{j\to+\infty} (\beta_{\pm j}^{(l)})^{1/j}$. In particular, if $\beta_j = (1+|j|)^\rho$ for some $\rho \geq 0$, we have $\mathcal{C}_\beta = \mathbb{T}^m$, and if $\beta_j = g^{|j|}$ for some $g \geq 1$, we have $\mathcal{C}_\beta = \{z = (z_1, \ldots, z_m) \in \mathbb{C}^m : (1/g) \leq |z_l| \leq g, l = 1, \ldots, m\}$.

For $k = 1$ the following result follows using standard Banach algebra theory [10], Chapter XXX. When $k \geq 2$, it follows instead from the main result of [15].

THEOREM 7.1. *Let $\hat{A}(z)$ belong to $\mathcal{W}_\beta^k$ and have the form (7.1). Then if $\hat{T}(z)$ is a nonsingular $k \times k$ matrix for all $z \in \mathcal{C}_\beta$, $\hat{T}(z)^{-1}$ belongs to $\mathcal{W}_\beta^k$.*

To define the spectral factorization of an m-index bi-infinite block Toeplitz matrix with $m \geq 2$ one needs a linear order $\preceq$ on $\mathbb{Z}^m$ which allows one to call a Toeplitz matrix A lower (resp. upper) triangular if $A_j = 0$ for all $j \in \mathbb{Z}^m$ with $j \prec (0, \cdots, 0)$ (resp. $j \succ (0, \cdots, 0)$). This linear order must have the following properties:

(i) $i \preceq j \Longrightarrow i + l \preceq j + l$ for all $l \in \mathbb{Z}^m$, and

(ii) $(i \preceq j$ and $c \geq 0) \Longrightarrow ci \preceq cj$.

The main problem is that for $m \geq 2$ such a so-called term ordering is by no means unique. In fact, the lexicographical order on $\mathbb{Z}^m$ with respect to any order of the "letters" $1, \cdots, m$ within the "alphabet" $\{1, \cdots, m\}$ will do. With respect to $\preceq$, we call $A = (a_{i-j})_{i,j\in\mathbb{Z}^m}$ *lower triangular* if $a_j = 0$ for all $j \in \mathbb{Z}^m$ with $j \prec (0, \cdots, 0)$, *upper triangular* if $a_j = 0$ for all $j \in \mathbb{Z}^m$ with $j \succ (0, \cdots, 0)$, and *diagonal* if $a_j = 0$ for all $j \in \mathbb{Z}^m$ different from $(0, \cdots, 0)$.

Consider the linear order $\preceq$ of $\mathbb{Z}^m$ as specified above. Let us study representations of A in the form

$$A = LDU, \tag{7.3}$$

where $L = (L_{i-j})_{i,j\in\mathbb{Z}^m}$ is lower triangular with $\ell_{(0,\cdots,0)} = 1$, $U = (U_{i-j})_{i,j\in\mathbb{Z}^m}$ is block upper triangular with $U_{(0,\cdots,0)} = I_k$, and $D = (D_{i-j})_{i,j\in\mathbb{Z}^m}$ is a block diagonal matrix. This factorization, which can be proven to exist under very general conditions on A, is not unique. To make it unique, we also require D to be invertible (i.e., D_0 to be nonsingular) and L and U to be boundedly invertible on $\ell_2^k(\mathbb{Z}^m)$ with inverses L^{-1} and U^{-1} that are block lower and block upper triangular matrices, respectively. A representation of A in the form (7.3) where L, D and U have the above properties, is called an *LDU*-factorization of A. In that case, A has to be boundedly invertible on $\ell_2^k(\mathbb{Z}^m)$, but in general this is not sufficient for the existence of an *LDU*-factorization. It is now easily seen that *LDU*-factorization of A with the symbols of its factors and inverses in the Banach algebra $\mathcal{W}_\beta^k$ amounts to the factorization

$$\hat{A}(z) = \hat{L}(z) D_0 \hat{U}(z), \qquad z \in \mathcal{C}_\beta, \tag{7.4}$$

in terms of the corresponding symbols, where D_0 is a nonsingular $k \times k$ matrix, $\hat{L}(z) - 1$ and $\hat{L}(z)^{-1} - 1$ belong to $\mathcal{W}_\beta^k$ and have Fourier series where all terms proportional to z^j with $j \prec (0, \cdots, 0)$ vanish, and $\hat{U}(z) - 1$ and $\hat{U}(z)^{-1} - 1$ belong to $\mathcal{W}_\beta^k$ and have Fourier series where all terms proportional to z^j with $j \succ (0, \cdots, 0)$ vanish.

If A is a bi-infinite Toeplitz matrix with symbol in the so-called Wiener class (i.e., in $\mathcal{W}_\beta^k$ for $\beta_j \equiv 1$) and its symbol $\hat{A}(z)$ is positive definite selfadjoint for all $z \in \mathbb{T}^m$, i.e., if A is positive definite selfadjoint on $\ell_2^k(\mathbb{Z}^m)$, the factorization (7.4) exists. In that case, D_0 is positive definite selfadjoint, $\hat{U}(z) = \hat{L}(\bar{z})^*$, and therefore

$$\hat{A}(z) = \mathrm{L}(z)\mathrm{L}(\bar{z})^*, \qquad \mathrm{L}(z) = \hat{L}(z)D_0^{1/2},$$

leads to a Cholesky factorization of A. Here $D_0^{1/2}$ is the (unique) positive definite selfadjoint square root of D_0. It is easily seen that if $\hat{A}(z)$ is positive definite selfadjoint for all $z \in \mathbb{T}^m$ and belongs to $\mathcal{W}_\beta^k$ for some weight sequence $(\beta_j)_{j\in\mathbb{Z}^m}$, then its LDU factors and their inverses belong to $\mathcal{W}_\beta^k$.

In fact, we have the following result.

THEOREM 7.2. *Suppose the bi-infinite block Toeplitz matrix $A = (A_i)_{i\in\mathbb{Z}^m}$ with symbol in $\mathcal{W}^k$ has a spectral factorization of the type* (7.4) *and $\hat{A}(z)$ is a nonsingular matrix for all $z \in C_\beta$ for some weight sequence $(\beta_j)_{j\in\mathbb{Z}^m}$. Then the factors and their inverses have symbols belonging to $\mathcal{W}_\beta^k$.*

Generalizing the existence theory for LDU-factorizations by using the Wiener-Hopf factorization theory for their symbols is not obvious if $m \geq 2$. First of all, there is no obvious geometrical criterion for the existence of the factorization (7.4) and the construction of the factors as in the case $m = 1$ where the winding number turned out to be the key to the solution of the problem. Secondly, the factorization (7.4) depends in an essential way on the choice of the linear order $\preceq$ on $\mathbb{Z}^m$.

The generalization of the previous results for $m = 1$ can be based on the closely related theories of chains of projections in a Hilbert space [12, 10] and of nest algebras [8], which have the additional advantage of yielding an LDU factorization theory for arbitrary bounded linear operators on $\ell_2^k(\mathbb{Z}^m)$ or on general separable Hilbert spaces. For $k = 1$ these concepts have been introduced in [23], where the main results have been discussed. The definitions and results generalize in a straightforward way to the case $k \geq 2$.

Finally, when the term ordering $\preceq$ is lexicographical, m-index block Toeplitz matrices can be identified in a natural way with one index block Toeplitz matrices, where the entries themselves are $(m-1)$-index block Toeplitz matrices, as observed in [33] for *finite* m-index Toeplitz matrices. Indeed, let us consider $m = 2$ for simplicity and let $\preceq$ be the lexicographical order defined by

$$(i_1, i_2) \preceq (j_1, j_2) \iff \begin{cases} i_1 \leq j_1, & \text{or} \\ \text{if } i_1 = j_1, \text{ then } i_2 \leq j_2. \end{cases}$$

Now consider the two-index bi-infinite block Toeplitz matrix $A = (A_{i-j})_{i,j\in\mathbb{Z}^2}$ where the entries are real $k \times k$ matrices. Then A can be identified in a natural way with the one-index block Toeplitz matrix $B = (B_{i-j})_{i,j\in\mathbb{Z}}$, where B_l is the one-index block Toeplitz matrix defined by $(B_l)_{i-j} = A_{(l,i-j)}$, i.e., the blocks of B have infinite order. An LDU-factorization of A corresponds exactly to an LDU-factorization of B, where the "central" block B_0 itself (which is defined by $(B_0)_{i-j} = A_{(0,i-j)}$) undergoes an LDU-factorization. Hopefully, this observation can be used to obtain numerical spectral factorizations of multi-index block Toeplitz matrices with respect to a lexicographical order in the near future.

REFERENCES

[1] K. Amaratunga. A fast Matlab routine for calculating Daubechies filters. *Wavelet Digest* 4(4), 1995.

[2] F.L. Bauer. Ein direktes Iterationsverfahren zur Hurwitz-zerlegung eines Polynoms. *Arch. Elektr. Uebertragung* 9:285–290, 1955.

[3] F.L. Bauer. Beiträge zur Entwicklung numerischer Verfahren für programmgesteuerte Rechenanlagen. II. Direkte Faktorizierung eines Polynoms. Sitz. Ber. Bayer. Akad. Wiss. 163–203, 1956.

[4] B.P. Bogert, M.J.R. Healy, and J.W. Tukey. The quefrency alanysis of time serier for echoes: cepstrum pseudo-autocovariance, cross-sepstrum and saphe cracking. In: M. Rosenblatt (ed.), *Proc. Symposium Time Series Analysis.* John Wiley and Sons, New York, 1963, pp. 209–243.

[5] A. Böttcher and B. Silbermann. Operator-valued Szegő-Widom theorems. In: E.L. Basor and I. Gohberg (Eds.), *Toeplitz Operators and Related Topics*, volume 71 of *Operator Theory: Advances and Applications.* Birkhäuser, Basel-Boston, 1994, pp. 33–53.

[6] A. Calderón, F. Spitzer, and H. Widom. Inversion of Toeplitz matrices. *Illinois J. Math.*, 3:490–498, 1959.

[7] K. Clancey and I. Gohberg. *Factorization of Matrix Functions and Singular Integral Operators*, volume 3 of *Operator Theory: Advances and Applications.* Birkhäuser, Basel-Boston, 1981.

[8] K.R. Davidson. *Nest Algebras.* Pitman, Research Notes in Mathematics 191, London, 1988.

[9] I.C. Gohberg and I.A. Feldman. *Convolution Equations and Projection Methods for their Solution.* Transl. Math. Monographs 41, Amer. Math. Soc., Providence, R.I., 1974.

[10] I. Gohberg, S. Goldberg, and M.A. Kaashoek. *Classes of Linear Operators, Vol. II*, volume 63 of *Operator Theory: Advances and Applications.* Birkhäuser, Basel-Boston, 1993.

[11] I. Gohberg and M.A. Kaashoek, Projection method for block Toeplitz operators with operator-valued symbols. In: E.L. Basor and I. Gohberg (Eds.), *Toeplitz Operators and Related Topics*, volume 71 of *Operator Theory: Advances and Applications.* Birkhäuser, Basel-Boston, 1994, pp. 79–104.

[12] I.C. Gohberg and M.G. Krein, *Theory and Applications of Volterra Operators in Hilbert Space.* AMS, Providence, R.I., Transl. Math. Monographs 24, 1970.

[13] I. Gohberg, P. Lancaster, and L. Rodman. Spectral analysis of matrix polynomials. I. Canonical forms and divisors. *Linear Algebra and its Applications*, 20:1–44, 1978.

[14] I. Gohberg, P. Lancaster, and L. Rodman. *Matrix Polynomials*, Academic Press, New York, 1982.

[15] I.C. Gohberg and J. Leiterer. Factorization of operator functions with respect to a contour. II. Canonical factorization of operator functions close to the identity. *Math. Nachrichten* 54:41–74, 1972 [Russian] (appendix).

[16] G.H. Golub and C.F. Van Loan. *Matrix Computations.* Third ed., The John Hopkins University Press, Baltimore, 1996.

[17] T.N.T. Goodman, C.A. Micchelli, G. Rodriguez, and S. Seatzu. On the Cholesky factorization of the Gram matrix of locally supported functions. *BIT* 35:233–257, 1995.

[18] T.N.T. Goodman, C.A. Micchelli, G. Rodriguez, and S. Seatzu. Spectral factorization of Laurent polynomials. *Adv. in Comp. Math.*, 7:429–454, 1997.

[19] T.N.T. Goodman, C.A. Micchelli, G. Rodriguez, and S. Seatzu. On the limiting profile arising from orthonormalizing shifts of exponentially decaying functions. *IMA J. Num. Anal.* 18:331–354, 1998.

[20] I. Gelfand, D. Raikov, and G. Shilov. *Commutative Normed Rings*, AMS Chelsea Publ., Providence, R.I., 1964.

[21] S. Jaffard. Propriétés des matrices "bien localisées" près de leur diagonale et quelques applications. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 7(5):461–476, 1990.

[22] M.G. Krein. Integral equations on the half-line with kernel depending upon the difference of the arguments. *Uspehi Mat. Nauk.*, 13(5):3–120, 1958. (Russian, translated in *AMS Translations*, 22:163–288, 1962).

[23] C.V.M. van der Mee, G. Rodriguez, and S. Seatzu. *LDU* factorization results for bi-infinite and semi-infinite scalar and block Toeplitz matrices. *Calcolo*, 33:307–335, 1996.

[24] C.V.M. van der Mee, G. Rodriguez, and S. Seatzu. Block Cholesky factorization of infinite matrices, and orthonormalization of vectors of functions. In Z. Chen, Y. Li, C.A. Micchelli, and Y. Xu, editors, *Advances in Computational Mathematics*, volume 202 of *Lecture Notes in Pure and Applied Mathematics*, pages 423–455, New York and Basel, 1998. M. Dekker Inc.

[25] C.V.M. van der Mee, G. Rodriguez, and S. Seatzu. Solution methods for perturbed semi-infinite linear systems of block-Toeplitz type. In: D. Bini, E.E. Tyrtyshnikov and P. Yalamov (eds.), *Structured*

Matrices: Recent Developments in Theory and Computations. Nova Science Publisher Inc., to appear in January 2000.

[26] A.V. Oppenheim and R.W. Schafer. *Discrete-Time Signal Processing.* Prentice Hall, Englewood Cliffs, NJ, 1989.

[27] F. Riesz and B. Sz.-Nagy. *Functional Analysis.* Frederick Ungar, New York, 1955 [also: Dover Publ., New York, 1990].

[28] S. Roch and B. Silbermann. C^*-algebra techniques in numerical analysis. *J. Operator Theory* 35:241–280, 1996.

[29] L. Rodman. *An Introduction to Operator Polynomials*, volume 38 of *Operator Theory: Advances and Applications.* Birkhäuser, Basel-Boston, 1989.

[30] J. Stoer and R. Bulirsch *Introduction to Numerical Analysis, second edition*, volume 12 of *Texts in Applied Mathematics.* Springer-Verlag, New York, 1991.

[31] G. Tunnicliffe Wilson. Factorization of the covariance generating function of a pure moving average process. *SIAM J. Numer. Anal.* 6:1–7, 1969.

[32] G. Tunnicliffe Wilson. A convergence theorem for spectral factorization. *J. Multivariate Anal.* 8:222–232, 1978.

[33] E.E. Tyrtyshnikov and N.L. Zamarashkin. Spectra of multilevel Toeplitz matrices: Advanced theory via simple matrix relationships. *Linear Algebra and its Applications* 270:15–27, 1998.