12

TRANSPORT THEORY AND STATISTICAL PHYSICS, 11(3), 245-248 (1982-83)

UNITARY EQUIVALENCE OF LINEAR TRANSPORT MODELS

C.V.M. van der Mee
Dept. of Physics and Astronomy
Vrije Universiteit
De Boelelaan 1081
1081 HV Amsterdam, The Netherlands

Abstract

One simple unitary transformation is provided between the linear transport model of R. Beals and a model presented elsewhere. Special attention is paid to similar relationships in electron transport theory.

During the past few years the investigation of the abstract differential equation

$$(T\psi)'(x) = -A\psi(x) \quad , \; 0<x<\tau(\leq\infty), \tag{1}$$

where T and A are self-adjoint operators on a Hilbert space H and "partial range" boundary conditions are imposed, has grown to be a popular theme. Being the offspring of Hangelbroeks thesis[1] the subject has been subjected to study by e.g. Beals[2,3,4], Hangelbroek[5], Kaper and Lekkerkerker[6,7,8], van der Mee[9,10], and recently Greenberg and Zweifel[10], and has been applied in several branches of physics[1,2,8,9,10]. The main purpose of this short note is to present one simple transformation, which yields a short derivation of the main results of Ref.7 from the results of

Beals[3] and makes explicit the connection between ideas of Refs. 3,4 and 9 and ideas presented in Ref.8. This transformation is an extension of T to a unitary operator.

Let us suppose that T is bounded with zero null space and A is positive with closed range. Following Beals[3] we define H_T to be the completion of H with respect to the inner product

$$(x,y)_{|T|} = (|T|x,y) \qquad (x,y\varepsilon H). \tag{2}$$

According to Kaper and Lekkerkerker[7,8] $H_{T^{-1}}$ is the completion of ImT = {Tx/xεH} with respect to the inner product

$$(x,y)_{|T|^{-1}} = (|T|^{-1}x,y) \qquad (x,y\varepsilon \mathrm{Im}T). \tag{3}$$

Clearly T extends to a unitary operator from H_T onto $H_{T^{-1}}$[11], which establishes a natural relationship between half-space and finite slab results of Ref.3 (formulated in H_T) and their analogues of Ref.7 (formulated in $H_{T^{-1}}$): any operator K of Ref.3 (such as the Larsen-Habetler[12] albedo operator) is connected to its analogue K^+ of Ref.7 by the formula

$$TK = K^+T : H_T \to H_{T^{-1}} . \tag{4}$$

The transformation T is readily suggested on comparing Eqs.(3.8), (3.9) and (3.11) of Ref. 3 to Eqs.(4.4) and (4.5) and the bottom formula at page 358 of Ref.7. In this way the Beals[3] solvability results for the half-space and finite slab problems (formulated through H_T) can be transformed into those of Kaper and Lekkerkerker[7], as exemplified by Lemma 3.2 in combination with Lemma 3.1 of Ref.3 versus the invertibility of V and V_T in Section 4 of Ref.7.

In Ref.8 the electron transport half-space problem is stated and some ideas for its solution are considered worth presenting.

In this problem A has a one-dimensional null space and has a compact resolvent.[4] Denoting the (two-dimensional) zero generalized eigenvector spaces of $T^{-1}A$ and AT^{-1} by H_0 and H_0^+, respectively, one finds

$$T[H_0] = H^+ \subset \text{Im } T. \qquad (5)$$

In this way T acts as a unitary operator from H_0 onto H_0^+, provided one endows H_0 with the indefinite inner product[13]

$$(x,y)_T = (T\,x,y) \qquad (6)$$

and H_0^+ with the indefinite inner product[13]

$$(x,y)_{T^{-1}} = (T^{-1}\,x,y). \qquad (7)$$

Maximal positive/negative subspaces of H_0 (with respect to (6)) are mapped by T onto maximal positive/negative subspaces of H_0^+ (with respect to (7)). Half-space problems with non-injective A were studied rigorously in Refs. 6, 3 and 9. The idea to exploit the indefinite inner product (6) to solve half-space problems with non-injective A was first published by van der Mee.[9] For electron transport a parallel idea, through the "T-transform" (7) of (6), turned up in Ref.8 together with the suggestion to study "connecting" transformations on $H_{T^{-1}}$. Though in a not completely correct way[10], "T^{-1}-transforms" of such connecting transformations were investigated before by Beals[4]. Again a unitary extension of T could be applied to make the connection between different papers.

The forementioned relationships are based on a general principle. For neutron transport the solution $\psi(x)$ of Eq.(1) represents a neutron angular density, whereas $T\,\psi(x)$ represents a current density[14]. In radiative transfer a similar pair of physical concepts is involved, namely the intensity and radiative flux. One could say that T transforms angular densities (resp. intensities) into current densities (resp. radiative fluxes). In both applications T is the multiplication operator by the cosine of the direction of propagation.

References

(1) R.J. Hangelbroek, T.T.S.P. 5, 1 (1976)

(2) R. Beals, J. Math. Anal. Appl. 58, 32 (1977)

(3) R. Beals, J. Funct. Anal. 34, 1 (1979)

(4) R. Beals, J. Math. Phys. 22, 954 (1981)

(5) R.J. Hangelbroek, Report 7720, Nijmegen University (1978)

(6) C.G. Lekkerkerker, Proc. Edinburgh Math. Soc. 75A, 259, 283 (1975)

(7) H.G. Kaper (Joint work with C.G. Lekkerkerker), "Factorization Methods in Transport Theory". In: I. Gohberg (Ed.), "Toeplitz Centennial", OT4, Birkhäuser, Basel (1982), p.343

(8) H.G. Kaper, C.G. Lekkerkerker and A. Zettl, "Linear Transport Theory and an Indefinite Sturm-Liouville Problem". To appear in: Proc. Conference on Ordinary and Partial Differential Equations held at Dundee, Scotland, March 29-April 2, 1982, Lecture Notes in Mathematics, Springer Verlag, Heidelberg

(9) C.V.M. van der Mee, "Semigroup and Factorization Methods in Transport Theory", Mathematical Centre Tract 146, Amsterdam (1981)

(10) W. Greenberg, C.V.M. van der Mee and P.F. Zweifel, "Generalized Kinetic Equations." Submitted to: Integral Equations and Operator Theory

(11) C.V.M. van der Mee, Private Communication, being a letter to H.G. Kaper and C.G. Lekkerkerker

(12) E.W. Larsen and G.J. Habetler, Comm. Pure Appl. Math. 26, 525 (1973)

(13) J. Bognár, "Indefinite Inner Product Spaces", Springer, Berlin (1974)

(14) H.G. Kaper, Lecture at the Seventh International Conference on Transport Theory. Journalist report in: T.T.S.P. 10, 115 (1981)