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# INTEGRAL EQUATIONS IN INVERSE SCATTERING

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**Abstract**

This paper deals with the explicit solution of the so-called Marchenko integral equations in the inverse scattering theory of the matrix Schrödinger equation if the reflection coefficient is rational, using methods from linear systems theory.

Key words. Integral equations, Inverse scattering.
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## 1. Formulation of the Marchenko integral equations

Inverse scattering of certain (systems of) ordinary differential equations on the line often leads to integral equations of Marchenko type. These integral equations have the following form:

$$B(x,\alpha)=\int_0^\infty d\beta\, B(x,\beta)K(2x+\alpha+\beta)+K(2x+\alpha), \tag{1.1}$$

where $\alpha \in I\!R^+$, $x \in I\!R$ is a parameter, $B(x,\cdot)$ is an $n\times n$ matrix function and $K(\cdot)$ is an $n\times n$ matrix kernel function with $L^1$ entries. In certain cases, the integral kernel has separated variables and hence the equation can be solved in closed form. Once the solution is found, the unknown potential in the differential equation is obtained from $B(x,0^+)$, either directly or by differentiating with respect to $x$. In this article we illustrate the solution of (1.1) by two examples, both of them worked out in more detail elsewhere. Because of space limitations, we will be brief when referencing; for a more complete account of the history of these problems we refer to [1, 2].

Let us present the first example (See [2] for details). For $0 \neq k \in I\!R$, the $n\times n$ matrix Schrödinger equation

$$\psi''(k,x)+k^2\psi(k,x)=Q(x)\psi(k,x)$$

has the so-called Jost solutions

$$f_l(k,x)=\begin{cases} e^{ikx}[I_n+o(1)], & x\to+\infty,\\ a_l(k)e^{ikx}+b_l(k)e^{-ikx}+o(1), & x\to-\infty,\end{cases}$$

$$f_r(k,x)=\begin{cases} a_r(k)e^{-ikx}+b_r(k)e^{ikx}+o(1), & x\to+\infty,\\ e^{-ikx}[I_n+o(1)], & x\to-\infty,\end{cases}$$

$I_n$ denoting the unit matrix of order $n$. When the potential $Q(x)$ is selfadjoint, i.e., if $Q(x)^\dagger=Q(x)$, the dagger indicating the conjugate transpose, the $2n\times 2n$ matrices

$$A_l(k) = \begin{bmatrix} a_l(k) & b_l(-k) \\ b_l(k) & a_l(-k) \end{bmatrix}, \qquad A_r(k) = \begin{bmatrix} a_r(k) & b_r(-k) \\ b_r(k) & a_r(-k) \end{bmatrix},$$

satisfy the relations

$$A_l(k)^{-1} = A_r(-k) = J_{2n}A_l(k)^\dagger J_{2n}, \qquad A_r(k)^{-1} = A_l(-k) = J_{2n}A_r(k)^\dagger J_{2n},$$

where $J_{2n} = I_n \oplus (-I_n)$. Hence, there exist, for $0 \neq k \in \mathbb{R}$, the transmission coefficients

$$T_l(k) = a_l(k)^{-1}, \qquad T_r(k) = a_r(k)^{-1},$$

and the reflection coefficients

$$L(k) = b_l(k)a_l(k)^{-1}, \qquad R(k) = b_r(k)a_r(k)^{-1}.$$

We organize the reflection and transmission coefficients in the scattering matrix

$$\mathbf{S}(k) = \begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{bmatrix}, \tag{1.2}$$

which is a unitary matrix if $0 \neq k \in \mathbb{R}$. Then the inverse scattering problem is to find $Q(x)$ from one of $R(k)$ or $L(k)$ plus the bound state data. We assume that the entries of $(1+|x|)Q(x)$ are in $L^1(\mathbb{R})$.

The inverse scattering problem for the matrix Schrödinger equation was first considered in [15] and studied in detail in [12]. In [2] the small-$k$ asymptotics of its scattering solutions and scattering coefficients were derived, along with the unique solvability of the corresponding Marchenko integral equations. A system theory approach to get explicit solutions in the case of rational reflection coefficients, different from the one to be sketched below, was presented in [4]. The more completely developed inverse scattering theory for the scalar ($n = 1$) Schrödinger equation appears in [6, 8].

Let us consider the second example (See [1] for details). The so-called canonical system is as follows:

$$-iJ\frac{dX}{dx}(x,\lambda) + V(x)X(x,\lambda) = \lambda X(x,\lambda), \tag{1.3}$$

where $\lambda \in \mathbb{R}$, $J = I_n \oplus (-I_n)$ and $V(x) = V(x)^\dagger = -JV(x)J$ has only $L^1$ entries. In other words,

$$V(x) = \begin{bmatrix} 0 & k(x) \\ k(x)^\dagger & 0 \end{bmatrix}.$$

Consider the Jost solutions

$$e^{-i\lambda Jx}F_l(x,\lambda) = \begin{cases} I_{2n} + o(1), & x \to +\infty, \\ A_l(\lambda) + o(1), & x \to -\infty, \end{cases}$$



$$e^{-i\lambda Jx}F_r(x,\lambda) = \begin{cases} A_r(\lambda) + o(1), & x \to +\infty, \\ I_{2n} + o(1), & x \to -\infty. \end{cases}$$

Then

$$A_r(\lambda) = A_l(\lambda)^{-1} = JA_l(\lambda)^\dagger J, \qquad A_l(\lambda) = A_r(\lambda)^{-1} = JA_r(\lambda)^\dagger J.$$

Let us write the $2 \times 2$ matrices with $n \times n$ matrix entries

$$A_l(\lambda) = \begin{bmatrix} a_{l1}(\lambda) & a_{l2}(\lambda) \\ a_{l3}(\lambda) & a_{l4}(\lambda) \end{bmatrix}, \qquad A_r(\lambda) = \begin{bmatrix} a_{r1}(\lambda) & a_{r2}(\lambda) \\ a_{r3}(\lambda) & a_{r4}(\lambda) \end{bmatrix},$$

introduce the transmission coefficients

$$T_l(\lambda) = a_{l1}(\lambda)^{-1}, \qquad T_r(\lambda) = a_{r4}(\lambda)^{-1},$$

and the reflection coefficients

$$R(\lambda) = a_{r2}(\lambda)a_{r4}(\lambda)^{-1} = -a_{l1}(\lambda)^{-1}a_{l2}(\lambda),$$

$$L(\lambda) = a_{l3}(\lambda)a_{l1}(\lambda)^{-1} = -a_{r4}(\lambda)^{-1}a_{r3}(\lambda),$$

and define the scattering matrix as in (1.2), with $k$ replaced by $\lambda$. Then the inverse scattering problem is to find $k(x)$ from one of the reflection coefficients $R(\lambda)$ or $L(\lambda)$. Since the differential operator $-iJ(d/dx) + V(x)$ appearing in (1.3) is selfadjoint and has the whole real axis as its spectrum, it does not have isolated eigenvalues (i.e., there are no bound states).

The inverse scattering theory for the canonical system, including explicit solutions for rational reflection coefficients, appears in [2]. The corresponding inverse spectral problem on the half-line has been solved in [11] (without explicit solutions) and that on the full line in [9].

The inverse scattering theories for (1.1), (1.3) and similar differential equations are usually developed according to the following outline:

1. We define the Jost solutions and the scattering coefficients. Splitting off the asymptotic part as $\lambda \to \pm\infty$ as a factor, we then introduce the Faddeev functions, for which we establish the continuity and analyticity properties, using Volterra integral equations.
2. We write the Faddeev functions and the scattering coefficients in the form $F(\lambda) = F_\infty + \int_{-\infty}^{\infty} d\alpha\, e^{i\lambda\alpha}\hat{F}(\alpha)$, where $\hat{F}$ has $L^1$ entries.
3. We formulate the matrix Riemann-Hilbert problem and convert it to a Fredholm integral equations of Marchenko type of the form (1.1).
4. We pass from the solution of the integral equation at $\alpha = 0^+$ to the potential.

We refer to the various publications (e.g., [1, 2, 6, 8, 12]) for details.

Now return to the above examples. Consider the matrix Schrödinger equation. Let

$$R(k)=\int_{-\infty}^{\infty} d\alpha\, e^{-ik\alpha}\hat{R}(\alpha), \qquad L(k)=\int_{-\infty}^{\infty} d\alpha\, e^{-ik\alpha}\hat{L}(\alpha). \tag{1.4}$$

For $x\in I\!R$ we then have the Marchenko integral equations

$$B_l(x,y)+\int_0^{\infty} dz\, B_l(x,z)S_l(2x+y+z)=-S_l(2x+y), \tag{1.5}$$

$$B_r(x,y)+\int_0^{\infty} dz\, B_r(x,z)S_r(-2x+y+z)=-S_r(-2x+y), \tag{1.6}$$

where $y>0$ and

$$S_l(x,y)=\hat{R}(y)+\sum_j e^{-\kappa_j y}C_{lj}, \qquad S_r(x,y)=\hat{L}(y)+\sum_j e^{-\kappa_j y}C_{rj};$$

here $C_{lj}$ and $C_{rj}$ are suitable positive selfadjoint matrices constructed from the bound state date [2, 12]. Then

$$Q(x)=-2\frac{d}{dx}B_l(x,0^+)=2\frac{d}{dx}B_r(x,0^+). \tag{1.7}$$

Since

$$R(-k)=R(k)^\dagger, \qquad L(-k)=L(k)^\dagger, \qquad k\in I\!R,$$

the matrices $\hat{R}(y)$ and $\hat{L}(y)$ are selfadjoint and hence so are the integral operators in (1.5) and (1.6). It can be proved in a standard way [2] that these equations are uniquely solvable in the direct sum of $n$ copies of $L^p(I\!R^+)$, $1\le p<+\infty$.

Next consider the canonical system. Defining $R(\lambda)$ and $L(\lambda)$ as in (1.4), with $k$ replaced by $\lambda$, we have

$$B_{l2}(x,\alpha)=-\hat{R}(\alpha+2x)+\int_0^{\infty} d\beta\int_0^{\infty} d\gamma\, B_{l2}(x,\gamma)\,\hat{R}(\beta+\gamma+2x)^\dagger\,\hat{R}(\alpha+\beta+2x), \tag{1.8}$$

$$B_{l3}(x,\alpha)=-\hat{R}(\alpha+2x)^\dagger+\int_0^{\infty} d\beta\int_0^{\infty} d\gamma\, B_{l3}(x,\gamma)\,\hat{R}(\beta+\gamma+2x)\,\hat{R}(\alpha+\beta+2x)^\dagger, \tag{1.9}$$

$$B_{r2}(x,\alpha)=-\hat{L}(\alpha-2x)^\dagger+\int_0^{\infty} d\beta\int_0^{\infty} d\gamma\, B_{r2}(x,\gamma)\,\hat{L}(\beta+\gamma-2x)\,\hat{L}(\alpha+\beta-2x)^\dagger, \tag{1.10}$$

$$B_{r3}(x,\alpha)=-\hat{L}(\alpha-2x)+\int_0^{\infty} d\beta\int_0^{\infty} d\gamma\, B_{r3}(x,\gamma)\,\hat{L}(\beta+\gamma-2x)^\dagger\,\hat{L}(\alpha+\beta-2x). \tag{1.11}$$

Then

$$k(x)=2i\,B_{l2}(x,0^+)=-2i\,B_{r2}(x,0^+)=2i\,B_{l3}(x,0^+)^\dagger=-2i\,B_{r3}(x,0^+)^\dagger.$$

Clearly, the integral operators in (1.8)–(1.11) are selfadjoint. Using contraction mapping and compactness arguments [1], one derives the unique solvability of the above Marchenko integral equations in the direct sum of $n$ copies of $L^p(I\!R^+)$, $1\le p<+\infty$.



## 2. Marchenko equations in the rational case

In this section we solve the inverse scattering problem for the matrix Schrödinger equation without bound states, if the reflection coefficients are rational matrix functions. The inverse problem with bound states is a straightforward generalization, where the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ below are replaced by larger matrices that also incorporate bound state information (See e.g. [3, 14] for such an extension).

If $R(k)$ is a rational matrix function, it allows the realization [5, 10]

$$\begin{aligned} R(k) &= i\,\mathcal{C}(k-i\mathcal{A})^{-1}\mathcal{B}=i\,\mathcal{C}_+(k-i\mathcal{A}_+)^{-1}\mathcal{B}_+ + i\,\mathcal{C}_-(k-i\mathcal{A}_-)^{-1}\mathcal{B}_- \\ &= -\int_{-\infty}^{\infty} dt\, e^{-ikt}\,\mathcal{C}\,E(t;-\mathcal{A})\,\mathcal{B}, \end{aligned} \tag{2.1}$$

where

$$\mathcal{A}=\begin{bmatrix}\mathcal{A}_+ & 0\\ 0 & \mathcal{A}_-\end{bmatrix}, \quad \mathcal{B}=\begin{bmatrix}\mathcal{B}_+\\ \mathcal{B}_-\end{bmatrix}, \quad \mathcal{C}=[\,\mathcal{C}_+ \quad \mathcal{C}_-\,]. \tag{2.2}$$

Here the spectra $\sigma(\mathcal{A}_+)\subseteq\{\mathrm{Re}\, k>0\}$ and $\sigma(\mathcal{A}_-)\subseteq\{\mathrm{Re}\, k<0\}$. Then

$$\hat{R}(t)=-\mathcal{C}\,E(t;-\mathcal{A})\,\mathcal{B}, \qquad \hat{R}(t)^\dagger=-\mathcal{B}^\dagger\,E(t;-\mathcal{A}^\dagger)\,\mathcal{C}^\dagger,$$

where

$$E(t;-\mathcal{A})=\begin{bmatrix} e^{-t\mathcal{A}_+} & 0\\ 0 & 0\end{bmatrix} \text{ for } t>0, \quad E(t;-\mathcal{A})=\begin{bmatrix}0 & 0\\ 0 & -e^{-t\mathcal{A}_-}\end{bmatrix} \text{ for } t<0.$$

If the order of the matrix $\mathcal{A}$ in (2.1) is minimal, then $R(k)=R(-k)^\dagger$ implies [13] the existence of a nonsingular selfadjoint matrix $H$ such that

$$H\mathcal{A}=\mathcal{A}^\dagger H, \qquad H\mathcal{B}=\mathcal{C}^\dagger, \qquad \mathcal{C}H^{-1}=\mathcal{B}^\dagger,$$

where $H=H_-\oplus H_+$ with $H_+$ and $H_-$ selfadjoint if $\mathcal{A}=\mathcal{A}_-\oplus\mathcal{A}_+$. We write $p_\pm$ for the minimal matrix order of $\mathcal{A}_\pm$ and put $p=p_++p_-$.

Introducing the $n\times n$ matrix

$$\mathcal{D}=\int_0^{\infty} dt\, e^{-t\mathcal{A}_+}\mathcal{B}_+\mathcal{C}_+\,e^{-t\mathcal{A}_+},$$

we obtain for the unique solution of the integral equation (1.5)

$$B_l(x,\alpha)=\mathcal{C}_+\left[I_{p_+}-e^{-2x\mathcal{A}_+}\mathcal{D}\right]^{-1}e^{-(\alpha+2x)\mathcal{A}_+}\mathcal{B}_+,$$

whenever $x>0$. Consequently [cf. (1.7)], for $x>0$ we obtain for the potential

$$Q(x)=4\,\mathcal{C}_+\left[I_{p_+}-e^{-2x\mathcal{A}_+}\mathcal{D}\right]^{-1}\mathcal{A}_+e^{-2x\mathcal{A}_+}\left[I_{p_+}-e^{-2x\mathcal{A}_+}\mathcal{D}\right]^{-1}\mathcal{B}_+.$$

The potential $Q(x)$ also contains the term $[\lim_{k\to\infty} 2ikR(k)]\,\delta(x)$, where $\delta(x)$ is Dirac's delta function. Hence it remains to compute $Q(x)$ for $x < 0$.

The unitarity of the scattering matrix $\mathbf{S}(k)$ implies

$$T_l(k)T_l(k^*)^\dagger = I_n - R(k)R(k^*)^\dagger, \qquad L(k)T_l(k^*)^\dagger = -T_r(k)R(k^*)^\dagger,$$
$$T_r(k^*)^\dagger T_r(k) = I_n - R(k^*)^\dagger R(k), \qquad T_r(k^*)^\dagger L(k) = -R(k^*)^\dagger T_l(k).$$

Here $T_l(k)$ and $T_r(k)$ are rational matrix functions without zeros and poles in $\overline{\mathbf{C}^+}\setminus\{0\}$ that tend to $I_n$ as $k \to \infty$ and generically have a (simple) zero at $k = 0$. Hence, starting from a rational matrix function $R(k)$ satisfying

$$\|R(k)\| < 1 \text{ for } 0 \neq k \in \mathbb{R}, \qquad R(\infty) = 0,$$

there exist unique such $T_l(k)$ and $T_r(k)$, and hence a unique such $L(k)$. Finding $\mathbf{S}(k)$ from $R(k)$ is a unitary extension problem known as the Darlington synthesis problem (See, e.g., [7]). The problem is to start from a representation (2.1) of $R(k)$ and to employ the solution of the Darlington synthesis problem to write $L(k)$ in a similar form. The potential for $x < 0$ can then be computed by solving the Marchenko integral equation (1.11).

From (2.1) we obtain by straightforward calculation

$$T_l(k)\,T_l(k^*)^\dagger = I_n - i\,[\mathcal{C} \quad 0\,]\,(k - i\mathcal{K}_l)^{-1}\begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix},$$

$$T_r(k^*)^\dagger\, T_r(k) = I_n + i\,[\,0 \quad \mathcal{B}^\dagger\,]\,(k - i\mathcal{K}_r)^{-1}\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix},$$

where

$$\mathcal{K}_l = \begin{bmatrix} \mathcal{A} & -\mathcal{B}\mathcal{B}^\dagger \\ 0 & -\mathcal{A}^\dagger \end{bmatrix}, \qquad \mathcal{K}_r = \begin{bmatrix} \mathcal{A} & 0 \\ \mathcal{C}^\dagger\mathcal{C} & -\mathcal{A}^\dagger \end{bmatrix}.$$

The inverses have the following form [5]:

$$\left[T_l(k^*)^\dagger\right]^{-1} T_l(k)^{-1} = I_n + i\,[\mathcal{C} \quad 0\,]\,(k - i\mathcal{E})^{-1}\begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix},$$

$$T_r(k)^{-1}\left[T_r(k^*)^\dagger\right]^{-1} = I_n - i\,[\,0 \quad \mathcal{B}^\dagger\,]\,(k - i\mathcal{E})^{-1}\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix},$$

where $\mathcal{E}$ is the so-called *state characteristic matrix* given by

$$\mathcal{E} = \begin{bmatrix} \mathcal{A} & -\mathcal{B}\mathcal{B}^\dagger \\ \mathcal{C}^\dagger\mathcal{C} & -\mathcal{A}^\dagger \end{bmatrix}.$$

Now let $R(k)$ have the representation (2.1), where $\mathcal{A}$ is a matrix of minimal order $p$. Then there exist unique hermitian solutions $\mathcal{X}$ and $\mathcal{Y}$ of the Riccati equations [10]

$$A\mathcal{X} + \mathcal{X}A^\dagger = \mathcal{B}\mathcal{B}^\dagger + \mathcal{X}\mathcal{C}^\dagger\mathcal{C}\mathcal{X};$$



$$A^\dagger\mathcal{Y} + \mathcal{Y}A = \mathcal{C}^\dagger\mathcal{C} + \mathcal{Y}\mathcal{B}\mathcal{B}^\dagger\mathcal{Y},$$

such that

$$\mathcal{E}_r = \mathcal{A} - \mathcal{B}\mathcal{B}^\dagger\mathcal{Y}, \qquad \mathcal{E}_l = \mathcal{A} - \mathcal{X}\mathcal{C}^\dagger\mathcal{C},$$

have their eigenvalues in the left half-plane and possibly at $k = 0$ with only Jordan blocks of order 1 at $k = 0$ (See [13]; here note that $-i\mathcal{E}$ is $\begin{bmatrix} 0 & iI_p \\ -iI_p & 0 \end{bmatrix}$ selfadjoint).

Let $\mathcal{M}_{l,r}$ be the direct sum of the null space of $\mathcal{E}$ and the invariant subspace of $\mathcal{E}$ spanned by the eigenvectors and generalized eigenvectors corresponding to its eigenvalues in the left (resp. right) half plane, and let $\mathcal{N}_{r,l}$ be the invariant subspace of $\mathcal{K}_r$ (resp. $\mathcal{K}_l$) spanned by the eigenvectors and generalized eigenvectors corresponding to its eigenvalues in the right (resp. left) half plane. Then [10]

$$\mathcal{M}_l = \operatorname{Im}\begin{bmatrix} I_p \\ \mathcal{Y} \end{bmatrix}, \qquad \mathcal{M}_r = \operatorname{Im}\begin{bmatrix} \mathcal{X} \\ I_p \end{bmatrix},$$

and hence

$$(k - i\mathcal{E})^{-1} = \Sigma(k - i[\mathcal{E}_r \oplus (-\mathcal{E}_l^\dagger)])^{-1}\Sigma^{-1}, \quad \Sigma = \begin{bmatrix} I_p & \mathcal{Y} \\ \mathcal{X} & I_p \end{bmatrix}.$$

Furthermore [10],

$$\mathcal{M}_l \oplus \mathcal{N}_r = \mathbf{C}^{2p}, \qquad \mathcal{M}_r \oplus \mathcal{N}_l = \mathbf{C}^{2p}.$$

Now let $\Pi$ be the projection of $\mathbb{C}^{2p}$ onto $\mathcal{M}_l$ along $\mathcal{N}_r$ and $\mathcal{Q}$ the projection of $\mathbb{C}^{2p}$ onto $\mathcal{M}_r$ along $\mathcal{N}_l$. Then [5]

$$\begin{aligned}
T_r(k^*)^\dagger &= I_n + i\,[\,0 \quad \mathcal{B}^\dagger\,]\,(k - i\mathcal{K}_r)^{-1}(I_{2p} - \Pi)\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \\
T_r(k) &= I_n + i\,[\,0 \quad \mathcal{B}^\dagger\,]\,\Pi\,(k - i\mathcal{K}_r)^{-1}\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \\
T_r(k)^{-1} &= I_n - i\,[\,0 \quad \mathcal{B}^\dagger\,]\,(k - i\mathcal{E})^{-1}\,\Pi\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \\
[T_r(k^*)^\dagger]^{-1} &= I_n - i\,[\,0 \quad \mathcal{B}^\dagger\,]\,(I_{2p} - \Pi)\,(k - i\mathcal{E})^{-1}\begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}, \\
T_l(k) &= I_n - i\,[\mathcal{C} \quad 0\,]\,(k - i\mathcal{K}_l)^{-1}\,\mathcal{Q}\begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \\
T_l(k^*)^\dagger &= I_n - i\,[\mathcal{C} \quad 0\,]\,(I_{2p} - \mathcal{Q})\,(k - i\mathcal{K}_l)^{-1}\begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \\
[T_l(k^*)^\dagger]^{-1} &= I_n + i\,[\mathcal{C} \quad 0\,]\,(k - i\mathcal{E})^{-1}(I_{2p} - \mathcal{Q})\begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}, \\
T_l(k)^{-1} &= I_n + i\,[\mathcal{C} \quad 0\,]\,\mathcal{Q}\,(k - i\mathcal{E})^{-1}\begin{bmatrix} 0 \\ \mathcal{C}^\dagger \end{bmatrix}.
\end{aligned}$$

For convenient $2p \times 2p$ matrices $\Phi_l$ and $\Phi_r$ we write

$$\operatorname{Im}\mathcal{Q} = \Phi_l^{-1}[\mathbf{C}^p \oplus \{0\}], \qquad \operatorname{Ker}\mathcal{Q} = \operatorname{Im}\begin{bmatrix} \mathcal{X} \\ I_p \end{bmatrix},$$

$$\operatorname{Ker}\Pi = \Phi_r^{-1}\left[\{0\} \oplus \mathbf{C}^p\right], \qquad \operatorname{Im}\Pi = \operatorname{Im}\begin{bmatrix} I_p \\ \mathcal{Y} \end{bmatrix},$$

where

$$\Phi_l^{-1} = \begin{bmatrix} I_{p-} & 0 & 0 & 0 \\ 0 & P_1 & 0 & -I_{p+} \\ 0 & 0 & I_{p-} & 0 \\ 0 & I_{p+} & 0 & 0 \end{bmatrix}, \qquad \Phi_r^{-1} = \begin{bmatrix} I_{p-} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{p+} \\ 0 & 0 & I_{p-} & 0 \\ 0 & I_{p+} & 0 & -P_2 \end{bmatrix},$$

and

$$P_1 = \int_0^\infty dt\, e^{-t\mathcal{A}_+}\mathcal{B}_+\mathcal{B}_+^\dagger e^{-t\mathcal{A}_+^\dagger}, \qquad P_2 = \int_0^\infty dt\, e^{-t\mathcal{A}_+^\dagger}\mathcal{C}_+^\dagger\mathcal{C}_+ e^{-t\mathcal{A}_+}.$$

Partitioning $\Phi_l^{-1}$ and $\Phi_r^{-1}$ into $p \times p$ blocks as

$$\Phi_l^{-1} = \begin{bmatrix} \Lambda_{l1} & \Lambda_{l2} \\ \Lambda_{l3} & \Lambda_{l4} \end{bmatrix}, \qquad \Phi_r^{-1} = \begin{bmatrix} \Lambda_{r1} & \Lambda_{r2} \\ \Lambda_{r3} & \Lambda_{r4} \end{bmatrix},$$

we now easily find the expressions

$$\begin{aligned} \mathcal{Q} &= \begin{bmatrix} \Lambda_{l1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1} & -\Lambda_{l1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} \\ \Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1} & -\Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} \end{bmatrix}, \\ I_{2p} - \Pi &= \begin{bmatrix} -\Lambda_{r2}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y} & \Lambda_{r2}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1} \\ -\Lambda_{r4}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y} & \Lambda_{r4}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1} \end{bmatrix}, \end{aligned}$$

where the inverses of $\Lambda_{l1} - \mathcal{X}\Lambda_{l3}$ and $\Lambda_{r4} - \mathcal{Y}\Lambda_{r2}$ exist. We obtain

$$\begin{aligned} T_l(k)^{-1} &= I_n - i\,\mathcal{C}\Lambda_{l1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}(k - i\mathcal{E}_l)^{-1}\mathcal{X}\mathcal{C}^\dagger, \\ \left[T_l(k^*)^\dagger\right]^{-1} &= I_n + i\,\mathcal{C}\mathcal{X}(k + i\mathcal{E}_l^\dagger)^{-1}\mathcal{J}_1\mathcal{C}^\dagger, \\ T_r(k)^{-1} &= I_n - i\,\mathcal{B}^\dagger\mathcal{Y}(k - i\mathcal{E}_r)^{-1}\mathcal{J}_2\mathcal{B}, \\ \left[T_r(k^*)^\dagger\right]^{-1} &= I_n + i\,\mathcal{B}^\dagger\Lambda_{r4}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}(k + i\mathcal{E}_r^\dagger)^{-1}\mathcal{Y}\mathcal{B}, \end{aligned}$$

where

$$\begin{aligned} \mathcal{J}_1 &= (I_p - \mathcal{Y}\mathcal{X})^{-1}\left[I_p + (\Lambda_{l3} - \mathcal{Y}\Lambda_{l1})(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X}\right], \\ \mathcal{J}_2 &= (I_p - \mathcal{X}\mathcal{Y})^{-1}\left[I_p + (\Lambda_{r2} - \mathcal{X}\Lambda_{r4})(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y}\right]. \end{aligned}$$

Let us partition $\Phi_l$ and $\Phi_r$ into $p \times p$ blocks as

$$\Phi_l = \begin{bmatrix} \Phi_{l1} & \Phi_{l2} \\ \Phi_{l3} & \Phi_{l4} \end{bmatrix}, \qquad \Phi_r = \begin{bmatrix} \Phi_{r1} & \Phi_{r2} \\ \Phi_{r3} & \Phi_{r4} \end{bmatrix}.$$

Then

$$\begin{aligned} T_l(k) &= I_n + i\,\mathcal{C}\Lambda_{l1}(k - i\Omega_3)^{-1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X}\mathcal{C}^\dagger, \\ T_l(k^*)^\dagger &= I_n - i\,\mathcal{C}\mathcal{J}_3(k + i\Omega_3^\dagger)^{-1}\Phi_{l4}\mathcal{C}^\dagger, \\ T_r(k) &= I_n + i\,\mathcal{B}^\dagger\mathcal{J}_4(k - i\Omega_4)^{-1}\Phi_{r1}\mathcal{B}, \\ T_r(k^*)^\dagger &= I_n - i\,\mathcal{B}^\dagger\Lambda_{r4}(k + i\Omega_4^\dagger)^{-1}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}\mathcal{Y}\mathcal{B}, \end{aligned}$$



where

$$\begin{aligned} \mathcal{J}_3 &= \Lambda_{l2} - \Lambda_{l1}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}(\Lambda_{l2} - \mathcal{X}\Lambda_{l4}), \\ \mathcal{J}_4 &= \Lambda_{r3} - \Lambda_{r4}(\Lambda_{r4} - \mathcal{Y}\Lambda_{r2})^{-1}(\Lambda_{r3} - \mathcal{Y}\Lambda_{r1}), \end{aligned} \qquad \Omega_4 = \begin{bmatrix} \mathcal{A}_- & 0 \\ \mathcal{C}_+^\dagger\mathcal{C}_- & -\mathcal{A}_+^\dagger \end{bmatrix}.$$

We finally obtain

$$L(k) = i\,\Omega_8(k - i\Omega_7)^{-1}\Omega_9,$$

where

$$\Omega_7 = \begin{bmatrix} \Omega_4 & -\Phi_{r1}\mathcal{B}\mathcal{B}^\dagger & \Phi_{r1}\mathcal{B}\mathcal{B}^\dagger \\ 0 & -\mathcal{A}^\dagger & 0 \\ 0 & 0 & -\mathcal{E}_l^\dagger \end{bmatrix},$$

$$\Omega_8 = \mathcal{B}^\dagger\left[\,\mathcal{J}_4 \quad -I_p \quad I_p\,\right], \qquad \Omega_9 = \begin{bmatrix} 0 \\ \Lambda_{l3}(\Lambda_{l1} - \mathcal{X}\Lambda_{l3})^{-1}\mathcal{X} \\ \mathcal{J}_1 \end{bmatrix}\mathcal{C}^\dagger.$$

Using a convenient similarity (as in [1]), we may write

$$L(k) = i\,\tilde{\mathcal{C}}(k - i\tilde{\mathcal{A}})^{-1}\tilde{\mathcal{B}},$$

where

$$\tilde{\mathcal{A}} = \begin{bmatrix} \tilde{\mathcal{A}}_+ & 0 \\ 0 & \tilde{\mathcal{A}}_- \end{bmatrix}, \quad \tilde{\mathcal{B}} = \begin{bmatrix} \tilde{\mathcal{B}}_+ \\ \tilde{\mathcal{B}}_- \end{bmatrix}, \quad \tilde{\mathcal{C}} = [\,\tilde{\mathcal{C}}_+ \quad \tilde{\mathcal{C}}_-\,],$$

$$\tilde{\mathcal{A}}_+ = \begin{bmatrix} -\mathcal{A}_l^\dagger & 0 \\ 0 & -\mathcal{A}_-^\dagger \end{bmatrix}, \quad \tilde{\mathcal{B}}_+ = \begin{bmatrix} \mathcal{J}_1\mathcal{C}^\dagger \\ 0 \end{bmatrix}, \quad \tilde{\mathcal{C}}_+ = [\,\mathcal{B}^\dagger(I_p + \mathcal{J}_4P_3) \quad \mathcal{B}^\dagger\mathcal{J}_4P_4 - \mathcal{B}_-^\dagger\,],$$

$$P_3 = \int_0^\infty dt\, e^{t\Omega_4}\Phi_{r1}\mathcal{B}\mathcal{B}^\dagger e^{t\mathcal{E}_l^\dagger}.$$

Let us now compute $Q(x)$ for $x < 0$ from the Marchenko equation (1.6), where $\hat{S}_r(\cdot) = \hat{L}(\cdot)$ and

$$\hat{L}(t) = -\tilde{\mathcal{C}}E(t; -\tilde{\mathcal{A}})\tilde{\mathcal{B}} = -\mathcal{B}^\dagger(I_p + \mathcal{J}_4P_3)e^{t\mathcal{E}_l^\dagger}\mathcal{J}_1\mathcal{C}^\dagger, \qquad t > 0.$$

Then

$$B_r(x,y) = \mathcal{B}^\dagger(I_p + \mathcal{J}_4P_3)[I_n + e^{-2x\mathcal{E}_l^\dagger}\mathcal{D}]^{-1}e^{(y-2x)\mathcal{E}_l^\dagger}\mathcal{J}_1\mathcal{C}^\dagger.$$

The potential $Q(x)$ can then be obtained by differentiation. In fact, for $x < 0$ we have

$$Q(x) = 4\mathcal{B}^\dagger(I_p + \mathcal{J}_4P_3)[I_n + e^{-2x\mathcal{E}_l^\dagger}\mathcal{D}]^{-1}\mathcal{E}_l^\dagger[I_n + e^{-2x\mathcal{E}_l^\dagger}\mathcal{D}]^{-1}e^{-2x\mathcal{E}_l^\dagger}\mathcal{J}_1\mathcal{C}^\dagger.$$

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# THE RIEMANN-HILBERT PROBLEM FOR A REPRESENTATION OF A HIGHER DIMENSIONAL HOMOLOGY GROUP ON A REAL MANIFOLD


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**Abstract**

A concept of regular singularities for higher codimensional submanifolds of a $C^\infty$-real manifold is introduced and the Riemann-Hilbert problem for a homology representation is formulated. The problem is solved and the Fuchs relation is obtained. The Fuchs relation becomes nothing but a theorem of Gauss-Bonnet type. As an application we can obtain the original Gauss-Bonnet theorem by use of the method of the Riemann-Hilbert problem.


## 1. Introduction.

A concept of regular singularities for a divisor on a complex manifold is introduced by R. Gerard([6]), P. Deligne([5]) and the Riemann-Hilbert problem is formulated for the monodromy representation of the complement space of the divisor and is solved by R. Gerard([6]), M. N. Katz([7]), and O. Suzuki([10]). Also the Fuchs relations are obtained([10]). In this paper we give the counterpart of this theory to a real manifold. At first we introduce a concept of regular singularities for a disjoint union of $p$-codimensional submanifods, $Y$ of a manifold $M$. Here we also consider a concept of "a family of regular singularities along a submanifold" (in 3). Next we formulate the Riemann-Hilbert problem for a representation

$$\rho : H_{p-1}(M - Y, \mathbf{R}) \longmapsto \mathbf{R}$$

and also for a family of representations. At the same time we formulate the Cousin problem for one cochain of forms of regular singularities (in 4). In 4 we prove that the both problems can be solved. Next we proceed to the Fuchs relation. The obstruction to the Riemann-Hilbert problem can be described by the "characteristic mapping":

$$d: C^0(M, \Phi^{p-1}[(p-1)Y]) \mapsto H^p(M, \mathbf{R}).$$

As for the notations we refer to 5. We see that this map describes general formulas of Gauss-Bonnet type systematically. Next we proceed to the deduction of the original