SOLUTION METHODS FOR SEMI-INFINITE LINEAR SYSTEMS OF BLOCK TOEPLITZ TYPE AND THEIR PERTURBATIONS

CORNELIS V.M. VAN DER MEE[†§], SEBASTIANO SEATZU[†§], AND GIUSEPPE RODRIGUEZ[‡§]

Abstract. We discuss two methods to obtain the spectral factorizations of the inverse of a bi-infinite real block Toeplitz matrix, one for arbitrary banded and one for positive definite and possibly nonbanded block Toeplitz matrices. The results are then applied to solve semi-infinite linear systems of block Toeplitz type. Furthermore we propose a new method for solving semi-infinite linear systems whose matrices are "perturbations" of block Toeplitz matrices. Some applications are also considered.

1. Introduction. Let $\mathbb{Z}$ be the set of all integers and $\mathbb{Z}_+$ the set of non-negative integers. By a semi-infinite (resp., bi-infinite) matrix of order k we mean a matrix $A = (A_{ij})$, indexed by $\mathbb{Z}_+$ (resp., by $\mathbb{Z}$), whose elements are real $k \times k$ matrices, where $k = 1, 2, \ldots$ does not depend on (i, j). It is a block Toeplitz matrix with blocks of order k if its diagonals are constant, i.e.,

$$A_{ij} = A_{i-j},$$

where $i, j \in \mathbb{Z}_+$ (resp., $i, j \in \mathbb{Z}$). Further, let $A = (A_j)_{j\in\mathbb{Z}}$ be a bi-infinite sequence of either real $k \times k$ matrices or real column vectors of length k, where $k \in \mathbb{N}$ does not depend on j. In either case we call it a bi-infinite sequence of order k. Taking a bi-infinite sequence of positive numbers $\beta = (\beta_j)_{j\in\mathbb{Z}}$ satisfying $\beta_{i+j} \le \beta_i\beta_j$ for all $i, j \in \mathbb{Z}$, by $\ell^k_{1,\beta}(\mathbb{Z})$ we mean the space of all bi-infinite sequences A of order k for which

$$\|A\|_{1,\beta} := \sum_{j\in\mathbb{Z}} \beta_j \|A_j\| < \infty, \tag{1.1}$$

where the fixed $k \times k$ matrix norm $\|\cdot\|$ is arbitrary. The most common choices of β are $\beta_j = (1 + |j|)^\rho$ for $\rho > 0$ and $\beta_j = g^{|j|}$ for $g > 1$, which correspond to algebraic and exponential weights, respectively. We write $\ell^k_1(\mathbb{Z})$ if $\beta_i = 1$ for all $i \in \mathbb{Z}$. Analogous definitions hold for semi-infinite sequences.

Given a weight sequence β, we study semi-infinite block Toeplitz linear systems

$$\sum_{j\in\mathbb{Z}_+} T_{i-j}x_j = b_i, \quad i \in \mathbb{Z}_+, \tag{1.2}$$

with $\|(T_h)_{h\in\mathbb{Z}_+}\|_{1,\beta} < \infty$ and $\|(b_h)_{h\in\mathbb{Z}_+}\|_{1,\beta} < \infty$. We provide two Wiener-Hopf factorization methods, one based on the theory of matrix polynomials [4, 5, 6, 13] and one based on band completion [2], for their solution with the help of Krein's method [10]. Furthermore, we study semi-infinite linear systems of the type

$$\sum_{j\in\mathbb{Z}_+} A_{ij}x_j = b_i, \quad i \in \mathbb{Z}_+, \tag{1.3}$$

that are perturbations of the aforementioned block Toeplitz systems in the sense that $\|A_{ij} - T_{i-j}\| \le c\lambda^{i+j}$, $i, j \in \mathbb{Z}_+$, for certain $c > 0$ and $\lambda \in (0, 1)$, i.e., the semi-infinite matrices $A = (A_{ij})$ and $T = (T_{i-j})$ are equivalent in the sense of [7, 12]. For systems of the form (1.3) we provide a numerical method for their solution.

The paper is organized as follows. In §2 we review some results pertaining to the spectral factorization of bi-infinite block Toeplitz matrices, which is the focal point in the solution of Toeplitz systems. In §3, after recalling Krein's method for solving system (1.2), we prove some new results

[†]Dipartimento di Matematica, Università di Cagliari, viale Merello 92, 09123 Cagliari, Italy (cornelis@krein.unica.it, seatzu@unica.it)

[‡]Dipartimento di Matematica, Università di Roma *Tor Vergata*, via della Ricerca Scientifica, 00133 Roma, Italy (rodriguez@mat.uniroma2.it)

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on the behaviour of its solution. In §4 we discuss two methods to obtain the spectral factorization of the inverse of a block Toeplitz matrix. The first method, based on the theory of matrix polynomials, applies to banded block Toeplitz matrices, while the second, based on the extension of banded block Toeplitz matrices, applies to positive definite block Toeplitz matrices. In §5 we propose a new method for solving semi-infinite linear systems that are perturbations of block Toeplitz systems. Finally, in order to assess the effectiveness of the proposed methods, we consider some numerical applications in §6.

2. Spectral factorization: theoretical results. By a *spectral factorization* of T (with respect to the weight sequence β) we mean a representation of T in any of the two forms

$$T = LDM^T \quad \text{and} \quad T = UDV^T, \tag{2.1}$$

where the superscript T denotes matrix transposition and L, M, D, U and V are bi-infinite Toeplitz matrices of order k having the following properties:

1. $L_0 = M_0 = U_0 = V_0 = I_k$ (the $k \times k$ identity matrix);
2. $D_s = 0$ for $s \neq 0$ and $L_s = M_s = 0$ for $s < 0$ and $U_s = V_s = 0$ for $s > 0$;
3. the inverses L^{-1}, M^{-1}, U^{-1} and V^{-1} of L, M, U and V are bi-infinite Toeplitz matrices of order k satisfying $[L^{-1}]_s = [M^{-1}]_s = 0$ for $s < 0$ and $[U^{-1}]_s = [V^{-1}]_s = 0$ for $s > 0$;
4. the matrix sequences with entries L_s, M_s, U_s, V_s, $[L^{-1}]_s$, $[M^{-1}]_s$, $[U^{-1}]_s$ and $[V^{-1}]_s$ belong to $\ell^k_{1,\beta}(\mathbb{Z})$.

If such a factorization exists, it is unique.

If T is positive definite, setting $\mathcal{L}_s = L_s D_0^{1/2}$ or $\mathcal{U}_s = U_s D_0^{1/2}$, we obtain the *block Cholesky factorization* $T = \mathcal{L}\mathcal{L}^T$ or the *block Wiener-Hopf factorization* $T = \mathcal{U}\mathcal{U}^T$, where $\mathcal{L} = (\mathcal{L}_{i-j})_{i,j\in\mathbb{Z}}$ and $\mathcal{U} = (\mathcal{U}_{i-j})_{i,j\in\mathbb{Z}}$ are invertible on $\ell^k_{1,\beta}(\mathbb{Z})$ with inverses $\mathcal{L}^{-1}$ and $\mathcal{U}^{-1}$, the sequences $(\mathcal{L}_s)_{s\in\mathbb{Z}}$, $([\mathcal{L}^{-1}]_s)_{s\in\mathbb{Z}}$, $(\mathcal{U}_s)_{s\in\mathbb{Z}}$ and $([\mathcal{U}^{-1}]_s)_{s\in\mathbb{Z}}$ belong to $\ell^k_{1,\beta}(\mathbb{Z})$, and $\mathcal{L}_0$ and $\mathcal{U}_0$ are positive definite hermitian.

Several results on block Cholesky factorization of real bi-infinite and semi-infinite Toeplitz matrices were obtained in [11], where a method for its computation in the bi-infinite block tridiagonal case was established. For banded Toeplitz matrices of order k this method has been applied in [12]. For $k = 1$, an analysis of existing factorization algorithms has been given in [8].

It is known [2] that the class of matrix functions

$$\widehat{T}(z) = \sum_{j\in\mathbb{Z}} z^j T_j$$

on the unit circle $\mathcal{T}$, where the coefficients T_j satisfy (1.1), is a Banach algebra with respect to the norm

$$\|\widehat{T}\|_{1,\beta} := \|(T_h)_{h\in\mathbb{Z}}\|_{1,\beta},$$

denoted as $\mathcal{W}^k_\beta$. We write $\mathcal{W}^k$ if $\beta_i = 1$ for all $i \in \mathbb{Z}$. As usual we call $\widehat{T}(z)$ the *symbol* of the block Toeplitz matrix $T = (T_{i-j})_{i,j\in\mathbb{Z}}$.

Passing to the symbols $\widehat{L}(z)$, $\widehat{D}(z) \equiv D_0$ and $\widehat{M}(z)$ of L, D and M, [resp., $\widehat{U}(z)$, $\widehat{D}(z) \equiv D_0$ and $\widehat{V}(z)$ of U, D and V], we have

$$\widehat{T}(z) = \widehat{L}(z) D_0 \widehat{M}(z)^T, \quad \left[\widehat{T}(z) = \widehat{U}(z) D_0 \widehat{V}(z)^T\right], \quad |z| = 1, \tag{2.2}$$

where $\widehat{L}(z)$ and $\widehat{M}(z)$ [$\widehat{U}(z)$ and $\widehat{V}(z)$] extend to matrix functions that are continuous on the closed unit disk $\{z \in \mathbb{C} : |z| \leq 1\}$ and analytic on the open unit disk. Further, $\widehat{L}(z)$ and $\widehat{M}(z)$ [resp., $\widehat{U}(z)$ and $\widehat{V}(z)$] are nonsingular matrices for $|z| \leq 1$ and $\widehat{L}(0) = \widehat{M}(0) = I_k$ [resp., $\widehat{U}(0) = \widehat{V}(0) = I_k$].

Defining $\beta_\pm = \lim_{j\to\pm\infty} \beta_j^{1/|j|}$ ([2], Ch. XXX), we can prove the factorization problems (2.1) and (2.2), for Toeplitz matrices T with symbols in $\mathcal{W}^k_\beta$, to be equivalent. In fact, when $\widehat{T}(z)$ is invertible for all z with $(1/\beta_-) \leq |z| \leq \beta_+$ and the factorization (2.2) exists in $\mathcal{W}^k$, the factors $\widehat{L}(z)$ and $\widehat{M}(z)$

and their inverses $\widehat{L}(z)^{-1}$ and $\widehat{M}(z)^{-1}$ belong to $\mathcal{W}_\beta^k$. In particular, if $\beta_j = g^{|j|}$ for some $g \geq 1$, we have $\beta_\pm = g$ and hence the condition for equivalence is that $\widehat{T}(z)$ is invertible for $(1/g) \leq |z| \leq g$.

Positive definite $k \times k$ matrix functions in $\mathcal{W}^k$ and $k \times k$ matrix functions $\widehat{T}(z)$ in $\mathcal{W}^k$ such that $\|I - \widehat{T}(z)\| < 1$ for all $z \in \mathcal{T}$ (the norm representing the largest singular value of $I - \widehat{T}(z)$) have factorizations of the type (2.2). When the Toeplitz matrix T of order k is banded (i.e., if $T_{i-j} = 0$ whenever $|i-j| \geq m$ for some integer m), necessary and sufficient conditions for the existence of the spectral factorization (2.2) as well as a numerical procedure for its calculation have been given (cf. [12]), based on matrix polynomial theory (e.g., [13]).

3. Semi-infinite block Toeplitz systems. Let us now recall Krein's method [10] for solving system (1.2). Denoting by $\widehat{T}(z)$ the symbol associated to the Toeplitz matrix T, the system (1.2) has a unique solution in $\ell_{1,\beta}^k(\mathbb{Z})$ if and only if $\widehat{T}(z) \neq 0$ for all z with $(1/\beta_-) \leq |z| \leq \beta_+$ and $\beta_\pm$ as above. Under such a hypothesis $\widehat{T}(z)^{-1}$ has the factorization

$$\widehat{T}(z)^{-1} = \Gamma_+(z)\Gamma_-(z) \tag{3.1}$$

in $\mathcal{W}_\beta^k$, where

$$\Gamma_+(z) = \sum_{j\in\mathbb{Z}_+} z^j \Gamma_j^{(1)}, \qquad \Gamma_-(z) = \sum_{j\in\mathbb{Z}_+} z^{-j} \Gamma_j^{(2)}, \tag{3.2}$$

with $\|\Gamma^{(1)}\|_{1,\beta} < \infty$ and $\|\Gamma^{(2)}\|_{1,\beta} < \infty$. One then has the resolvent formula

$$x_\ell = \sum_{s\in\mathbb{Z}_+} \Gamma_{\ell s} b_s, \quad \ell \in \mathbb{Z}_+, \tag{3.3}$$

where

$$\Gamma_{\ell s} = \sum_{h=0}^{\min(\ell,s)} \Gamma_{\ell-h}^{(1)} \Gamma_{s-h}^{(2)}. \tag{3.4}$$

The factorization (3.1), which is equivalent to the spectral factorization LDM^T of T^{-1}, is generally quite hard to obtain. However, if T is banded, the factors in (2.1) can be computed using a method based on matrix polynomial theory [13], whereas a band extension method [2] can be used if T is positive definite.

THEOREM 3.1. *Consider the semi-infinite Toeplitz system (1.2) of order k, where, for some fixed weight sequence β, $\|(T_h)_{h\in\mathbb{Z}_+}\|_{1,\beta} < \infty$ and $\left\|(b_i)_{i\in\mathbb{Z}_+}\right\|_{1,\beta} < \infty$. Suppose $\widehat{T}(z)$ is invertible for every z with $(1/\beta_-) \leq |z| \leq \beta_+$. Then each solution $(x_i)_{i\in\mathbb{Z}_+}$ of (1.2) with $\left\|(x_i)_{i\in\mathbb{Z}_+}\right\|_1 < \infty$ satisfies $\left\|(x_i)_{i\in\mathbb{Z}_+}\right\|_{1,\beta} < \infty$.*

Proof. Under the above assumptions on the invertibility of the symbol, the linear operator representing the left-hand side of (1.2) is Fredholm on both $\ell_1^k(\mathbb{Z})$ and $\ell_{1,\beta}^k(\mathbb{Z})$ (see [10, 2]). Since the latter space is continuously and densely imbedded in the former, its kernel, null index and defect index are the same on either space, which implies the theorem. □

Let us now examine the behaviour of the coefficient sequence $\{\Gamma_{\ell s}\}$ with respect to T. To do so, we recall a definition, already introduced in [12] and previously, in an essentially equivalent form, in [9]. Let A be a semi-infinite matrix of order k. It is called exponentially decaying if there exist $c > 0$ and $\lambda \in (0,1)$ such that

$$\|A_{ij}\| \leq c\lambda^{|i-j|}, \quad i,j \in \mathbb{Z}_+.$$

THEOREM 3.2. *Let A be a semi-infinite nonsingular Toeplitz matrix of order k which decays exponentially. Then A^{-1} also decays exponentially.*

Proof. Let us first assume that A is positive definite. Since there exist constants $c > 0$ and $\lambda \in (0,1)$ with $\|A_j\| \leq c\lambda^{|j|}$, the symbol $\widehat{A}(z)$ associated to A belongs to the Banach algebra $\mathcal{W}_\beta^k$

with $\beta_j = g^{|j|}$ for every $g \in (1, 1/\lambda)$ and is positive for $z \in \mathcal{T}$. Following the reasoning of the final paragraph of §2, the Wiener-Hopf factors of $\widehat{A}(z)$ and their inverses exist and belong to $\mathcal{W}_\beta^k$ with $\beta_j = h^{|j|}$, for some $h \in (1, g]$. Thus, the Cholesky factors of A and A^{-1} are exponentially decaying and hence so is A^{-1}. Now suppose A is not positive definite. As the product of two exponentially decaying block matrices also decays exponentially [12], recalling that $A^{-1} = A^T(AA^T)^{-1}$, we have that A^{-1} decays exponentially. □

We note that, as a result of the proof of Theorem 3.2, the semi-infinite lower triangular matrices $\Gamma^{(1)}$ and $\Gamma^{(2)}$ of order k in (3.2) decay exponentially if the semi-infinite matrix T in (3.1) decays exponentially. In that case, the semi-infinite matrix Γ defined in (3.4) also decays exponentially, which implies that the partial sums of the series (3.3) converge exponentially to the solution of system (1.2). More specifically, this result holds for any banded system.

4. Spectral factorization: numerical methods. In this section we present two methods to compute the Wiener-Hopf factors of a given bi-infinite matrix of order k and their inverses. The first method applies only to banded bi-infinite matrices and relies on matrix polynomial theory. The second method applies only to positive definite bi-infinite matrices, but these need not be banded.

4.1. A method based on matrix polynomial theory. Let $A = (A_{i-j})_{i,j\in\mathbb{Z}}$ be a bi-infinite banded block Toeplitz matrix, where the square matrices $A_{-m}, A_{-m+1}, \ldots, A_n$ have order k and $A_s = 0$ for $s < -m$ and $s > n$. We shall assume that A_n is nonsingular. Similar results hold if one assumes the invertibility of A_{-m}. Apart from the expressions for the inverses of the factors, the analogous results for LDU factorization were given in [12].

We are interested in obtaining a UDL factorization of A, that is a factorization of the form

$$A = L^T D R, \tag{4.1}$$

where $L = (L_{i-j})_{i,j\in\mathbb{Z}}$, $R = (R_{i-j})_{i,j\in\mathbb{Z}}$ and $D = (D_{i-j})_{i,j\in\mathbb{Z}}$ are banded block Toeplitz matrices with $L_0 = R_0 = I_k$ (the $k \times k$ identity matrix), $L_s = -\mathcal{L}_s$ $(s = 1, \ldots, m)$ and $R_s = -\mathcal{R}_s$ $(s = 1, 2, \ldots, n)$, $D_0 = \mathcal{D}$, $D_s = 0$ for $s \neq 0$, $L_s = 0$ for $s \neq 0, 1, \ldots, m$, and $R_s = 0$ for $s \neq 0, 1, \ldots, n$. In other words, L and R are lower block triangular matrices and D is a block diagonal matrix.

Now let the matrix function

$$\Sigma(z) = \sum_{j=-m}^{n} z^j A_j$$

be the *symbol* associated to the matrix A. Then $\mathcal{P}(z) = z^m A_n^{-1}\Sigma(z)$ is a monic matrix polynomial of degree $m + n$, that is a matrix polynomial whose leading coefficient is the identity matrix I_k.

Let us consider a simple closed positively oriented rectifiable Jordan curve Γ, with $0 \notin \Gamma$, dividing the complex plane into an *interior* bounded domain Ω_+ with $0 \in \Omega_+$ and an *exterior* domain Ω_-, and assume that $\det(z^m\Sigma(z))$ does not vanish for $z \in \Gamma$. Then [13] the matrix factorization (4.1) is equivalent to the factorization

$$\Sigma(z) = \left(I_k - \sum_{i=1}^{m} z^{-i}\mathcal{L}_i\right)^T \mathcal{D} \left(I_k - \sum_{j=1}^{n} z^j \mathcal{R}_j\right), \quad z \in \Gamma, \tag{4.2}$$

of the symbol of A. The nonsingularity of A_n implies that both $\mathcal{R}_n$ and $\mathcal{D}$ are nonsingular.

We recall some basic properties of matrix polynomials [4, 6, 13]. Consider the monic matrix polynomial

$$\mathcal{P}(z) = z^\ell I_k + z^{\ell-1}A_{\ell-1} + \cdots + zA_1 + A_0,$$

of degree ℓ and order k, meaning that its coefficients are $k \times k$ matrices.

We call $z_j \in \mathbb{C}$ an *eigenvalue* of $\mathcal{P}$ if $\det \mathcal{P}(z_j) = 0$. The corresponding *eigenvector* $\mathbf{x}_{j1}$ is a non-null vector in $\mathbb{C}^k$ satisfying $\mathcal{P}(z_j)\mathbf{x}_{j1} = 0$. It is immediate to observe that $\mathcal{P}$ has a *spectrum* of exactly $k\ell$ eigenvalues, taking into account their multiplicities as zeros of $\det \mathcal{P}(z)$, which we consider ordered by decreasing modulus. The complex vectors $\{\mathbf{x}_{j1}, \mathbf{x}_{j2}, \dots, \mathbf{x}_{jr}\}$ constitute a *Jordan chain* at z_j of length r, if $\mathbf{x}_{j1} \neq 0$ and the lower triangular linear system of equations

$$\sum_{v=1}^{u} \frac{\mathcal{P}^{(u-v)}(z_j)}{(u-v)!} \mathbf{x}_{jv} = 0, \quad u = 1, 2, \dots, r,$$

is valid. The lengths of the Jordan chains at the eigenvalue z_j in a system of maximal Jordan chains are called *partial multiplicities* of $\mathcal{P}$ at z_j. The sum of the partial multiplicities at z_j coincides with the order of z_j as a zero of $\det \mathcal{P}(z)$, that is to its *algebraic multiplicity*; the dimension of the kernel of $\mathcal{P}(z_j)$ is called its *geometric multiplicity*, and is equal to the number of linearly independent maximal Jordan chains at z_j.

A pair of matrices (X, T), where X is of size $k \times k\ell$ and T is of size $k\ell \times k\ell$, is called a *right spectral pair* for the polynomial $\mathcal{P}(z)$ if the matrix

$$\operatorname{col}[XT^j]_{j=0}^{\ell-1} := \begin{bmatrix} X \\ XT \\ \vdots \\ XT^{\ell-1} \end{bmatrix}$$

is invertible and the following equality holds

$$XT^\ell + \sum_{j=0}^{\ell-1} A_j X T^j = 0.$$

By a *left spectral pair* for the polynomial $\mathcal{P}(z)$ we mean a right spectral pair for $\mathcal{P}(z)^T$. The *right canonical form* for $\mathcal{P}(z)$ is given by

$$\mathcal{P}(z) = z^\ell I_k - XT^\ell (V_1 + zV_2 + \cdots + z^{\ell-1} V_\ell),$$

where $V_1, \dots, V_\ell$ are the $k\ell \times k$ matrices defined by

$$\begin{bmatrix} V_1 & V_2 & \cdots & V_\ell \end{bmatrix} = \left(\operatorname{col}[XT^j]_{j=0}^{\ell-1}\right)^{-1}. \tag{4.3}$$

A similar definition holds for the *left canonical form*.

Denoting by $z_1, \dots, z_p$ the *distinct* eigenvalues of $\mathcal{P}(z)$, each of algebraic multiplicity m_j, it can be shown [13] that a right spectral pair (X, T) for $\mathcal{P}(z)$ is given by

$$X = \begin{bmatrix} X_1 & X_2 & \cdots & X_p \end{bmatrix}, \quad T = T_1 \oplus T_2 \oplus \cdots \oplus T_p, \tag{4.4}$$

where the matrices X_j, of size $k \times m_j$, and T_j, of size $m_j \times m_j$, are given by

$$\begin{aligned} X_j &= \begin{bmatrix} \mathbf{x}_{11}^{(j)} & \cdots & \mathbf{x}_{1r_{j1}}^{(j)} & \mathbf{x}_{21}^{(j)} & \cdots & \mathbf{x}_{2r_{j2}}^{(j)} & \cdots & \mathbf{x}_{q_j 1}^{(j)} & \cdots & \mathbf{x}_{q_j r_{jq_j}}^{(j)} \end{bmatrix}, \\ T_j &= J_{r_{j1}}(z_j) \oplus J_{r_{j2}}(z_j) \oplus \cdots \oplus J_{r_{jq_j}}(z_j). \end{aligned}$$

Here $\mathbf{x}_{s1}^{(j)}, \mathbf{x}_{s2}^{(j)}, \dots, \mathbf{x}_{sr_{js}}^{(j)}$, $s = 1, \dots, q_j$, are the maximal Jordan chains for $\mathcal{P}(z)$ corresponding to z_j, $J_{r_s}(z_j)$ is the $r_s \times r_s$ upper triangular Jordan block with eigenvalue z_j, and $r_{j1} + r_{j2} + \cdots + r_{jq_j} = m_j$.

To obtain the factorization the following theorem is crucial.

THEOREM 4.1. *Let $z_1, \dots, z_s$ be the distinct zeros of $\det(z^m \Sigma(z))$ in Ω_- and let $z_{s+1}, \dots, z_{s+t}$ be its distinct zeros in Ω_+. Moreover, let (X, T) be a right spectral pair of the monic matrix polynomial $\mathcal{P}(z) = z^m A_n^{-1} \Sigma(z)$. Then there exists a factorization of $\Sigma(z)$ of the type (4.2) where $\det(I_k - \sum_{i=1}^m z^{-i} \mathcal{L}_i) \neq 0$ for $z \in \Omega_-$ and $\det(I_k - \sum_{j=1}^n z^j \mathcal{R}_j) \neq 0$ for $z \in \Omega_+$, if and only if*

$$m_1 + \cdots + m_s = nk, \quad m_{s+1} + \cdots + m_{s+t} = mk,$$

and the restriction of $\operatorname{col}[XT^j]_{j=0}^{n-1}$ to the linear span of the eigenvectors and generalized eigenvectors of T in Ω_+ is invertible. This factorization is unique and is called the spectral factorization of A.

The proof of this result can be easily adapted from Theorem 2.1 of [12], where a result essentially equivalent to Theorem 4.1 was proved.

Following the proof of Theorem 2.1 of [12] we can give the explicit construction of the factorization (4.2). We first rewrite (4.2) as the factorization of a monic matrix polynomial

$$\mathcal{P}(z) = z^m A_n^{-1}\Sigma(z) = \mathcal{L}(z)^T \mathcal{D} \mathcal{R}(z),$$

where

$$\mathcal{L}(z) = \left(z^m I_k - \sum_{i=1}^{m} z^{m-i}\mathcal{L}_i \right) (A_n^T)^{-1}$$

and

$$\mathcal{R}(z) = I_k - \sum_{j=1}^{n} z^j \mathcal{R}_j.$$

Then, as specified in (4.4), we construct the right spectral pair (X_R, T_R) of $\mathcal{P}(z)$ by using its Jordan chains, which coincide with the Jordan chains of $z^m\Sigma(z)$, and its left spectral pair (X_L, T_L) in a similar fashion. Partitioning the $k \times (m+n)k$ matrices X_L and X_R into a $k \times nk$ block and a $k \times mk$ block, we have

$$X_L = [\ ?\quad V\ ], \quad X_R = [\ W \quad ?\ ],$$

$$T_L = T_R = \bigoplus_{j=1}^{s+t} \left(J_{r_{j1}}(z_j) \oplus \cdots \oplus J_{r_{jq_j}}(z_j) \right),$$

where the matrices at the question marks are irrelevant and $r_{j1} + \cdots + r_{jq_j} = m_j$ $(j = 1, \ldots, s+t)$. We now set

$$\Lambda_L = \bigoplus_{j=s+1}^{s+t} \left(J_{r_{j1}}(z_j) \oplus \cdots \oplus J_{r_{jq_j}}(z_j) \right),$$

$$\Lambda_R = \bigoplus_{j=1}^{s} \left(J_{r_{j1}}(z_j) \oplus \cdots \oplus J_{r_{jq_j}}(z_j) \right),$$

and construct the $mk \times k$ matrices $W_1, \ldots, W_m$ and the $nk \times k$ matrices $V_1, \ldots, V_n$ by putting

$$[\ W_1 \quad W_2 \quad \cdots \quad W_m\ ] = \left(\mathrm{col}[V\Lambda_L^j]_{j=0}^{m-1} \right)^{-1},$$

$$[\ V_1 \quad V_2 \quad \cdots \quad V_n\ ] = \left(\mathrm{col}[W\Lambda_R^j]_{j=0}^{n-1} \right)^{-1}.$$

Using the spectral pairs of $\mathcal{P}(z)$ it is possible to compute directly the inverses of its spectral factors, obtaining a LDU factorization of A^{-1} without actually computing the factors themselves. Indeed, for a right spectral pair (X, T) there holds the inversion formula

$$\mathcal{P}(z)^{-1} = X(zI_{k\ell} - T)^{-1}V_\ell,$$

where V_ℓ is given by (4.3) [5].

This formula allows one to state the following equalities for the inverses of the spectral factors of $\mathcal{P}(z) = z^m A_n^{-1}\Sigma(z)$:

$$A_n^T \left(z^m I_k - \sum_{i=1}^{m} z^{m-i}\mathcal{L}_i \right)^{-1} (A_n^T)^{-1} = V(z - \Lambda_L)^{-1}W_m,$$

$$-\left(I_k - \sum_{j=1}^{n} z^j \mathcal{R}_j \right)^{-1} \mathcal{R}_n = W(z - \Lambda_R)^{-1}V_n.$$

Now suppose $\Gamma = \{z \in \mathbb{C} : |z| = \rho\}$, $\Omega_+ = \{z \in \mathbb{C} : |z| < \rho\}$ and $\Omega_- = \{z \in \mathbb{C} : |z| > \rho\}$. Then $\Sigma(z)$ has the factorization (4.2) for $z \in \Gamma$, and the inverses of its factors are given by

$$\begin{aligned}\left(I_k - \sum_{i=1}^{m} z^{-i}\mathcal{L}_i\right)^{-1} &= z^m (A_n^T)^{-1} V (zI_{km} - \Lambda_L)^{-1} W_m A_n^T \\ &= I_k + \sum_{\mu=1}^{\infty} z^{-\mu} (A_n^T)^{-1} V \Lambda_L^{\mu+m-1} W_m A_n^T\end{aligned} \tag{4.5}$$

for $z \in \Omega_-$ and

$$\begin{aligned}\left(I_k - \sum_{j=1}^{n} z^{j}\mathcal{R}_j\right)^{-1} &= W (I_{kn} - z\Lambda_R^{-1})^{-1} \Lambda_R^{-1} V_n \mathcal{R}_n^{-1} \\ &= I_k + \sum_{\mu=1}^{\infty} z^{\mu} W \Lambda_R^{-(\mu+1)} V_n W \Lambda_R^n V_1\end{aligned} \tag{4.6}$$

for $z \in \Omega_+$. To obtain (4.5) we make use of the *biorthogonality conditions* [5]

$$V\Lambda_L^\mu W_m = \begin{cases} 0, & \mu = 0, 1, \dots, m-2, \\ I_k, & \mu = m-1. \end{cases}$$

We note that the series expansion in (4.6) implies that

$$\mathcal{R}_n = W\Lambda_R^{-1} V_n,$$

while (4.2) implies that

$$\mathcal{D} = -A_n \mathcal{R}_n^{-1}.$$

Hence, the inverse of the diagonal factor $\mathcal{D}$ can be effectively computed as follows

$$\mathcal{D}^{-1} = -W\Lambda_R^{-1} V_n\, A_n^{-1}.$$

For its intrinsic interest, we remark that the coefficients $\mathcal{L}_i$, $\mathcal{R}_j$ and $\mathcal{D}$ of the spectral factors of $\mathcal{P}(z)$ are given by

$$\begin{aligned}&\mathcal{L}_i = (A_n^T)^{-1}\, V\Lambda_L^m W_{m-i+1}\, A_n^T, && i = 1, \dots, m, \\ &\mathcal{R}_j = -W\Lambda_R^{-1} V_n\, W\Lambda_R^n V_{j+1}, && j = 1, \dots, n-1, \\ &\mathcal{R}_n = W\Lambda_R^{-1} V_n,\end{aligned}$$

and

$$\begin{aligned}\mathcal{D} &= -(\mathcal{L}_m^T)^{-1} A_{-m} = -A_n \left(V\Lambda_L^{-1} W_m\right)^T A_n^{-1} A_{-m} \\ &= -A_n \mathcal{R}_n^{-1} = -A_n\, W\Lambda_R^n\, V_1.\end{aligned} \tag{4.7}$$

This corresponds to taking right spectral pairs (V, Λ_L) and (W, Λ_R) for the matrix polynomials $\mathcal{L}(z)(A_n^T)^{-1}$ and $\mathcal{R}(z)$, respectively, and recovering the spectral factors from their canonical forms. In writing these formulae we make use of the identity

$$\mathcal{R}_n = (W\Lambda_R^n V_1)^{-1} = W\Lambda_R^{-1} V_n.$$

Finally, formula (4.7) implies that

$$\mathcal{D}^{-1} = -A_{-m}^{-1} A_n \left(V\Lambda_L^m W_1\right)^T A_n^{-1},$$

which gives an alternative expression for the inverse of the block diagonal spectral factor.

From the computational point of view, it is useful to remark that, once the spectral pairs are obtained, (4.5) and (4.6) can be readily and efficiently computed as $z^{-1}\Lambda_L$ and $z\Lambda_R^{-1}$ are block diagonal matrices whose powers decay rapidly when the eigenvalues of $\mathcal{P}(z)$ are well separated from the curve Γ.

When $\Sigma(z)$ is the symbol corresponding to a real matrix A which is symmetric ($m = n$, $A_{-j} = A_j^T$), positive definite, block Toeplitz and banded, we can assume the curve Γ to be the unit circle, so that Theorem 4.1 assures that there exists only one factorization of A of the form

$$A = UU^T,$$

where $U = L^T D^{1/2}$. This is often referred to as the *Wiener-Hopf factorization* of A. In this case the matrix D in (4.1) is real positive definite, as well as A_0 and $\mathcal{D}$. Moreover, $\mathcal{L}_j = \mathcal{R}_j$ and the symbol

$$\mathcal{U}(z) := \sum_{j=0}^{m} z^{-j}\mathcal{U}_j$$

of the matrix U has coefficients defined by

$$\begin{aligned} \mathcal{U}_0 &= \mathcal{D}^{1/2}, \\ \mathcal{U}_j &= -\mathcal{R}_j^T\mathcal{D}^{1/2} = -\mathcal{L}_j^T\mathcal{D}^{1/2}, \quad j = 1, \dots, m. \end{aligned} \tag{4.8}$$

We note that the precision of the results essentially depends on the evaluation of the eigenvalues of $\mathcal{P}(z)$. The accuracy of this computation is influenced both by the dimension of the blocks and by the bandwidth of A.

4.2. A method based on band extension. The factorization method discussed above has the disadvantage that it can only be used for banded block Toeplitz matrices, where either the bandwidth $m + n + 1$ or the matrix order k is relatively small. We now present a method based on band extension (cf. Section XXXV.3 of [2]) which applies to positive definite bi-infinite block Toeplitz matrices, not necessarily banded.

Let $A = (A_{i-j})_{i,j\in\mathbb{Z}}$ be a bi-infinite block Toeplitz matrix of order k. Fixing a positive integer p, which we call the *extension parameter*, we associate to the matrix A the bi-infinite banded matrix $\Psi = (\Psi_{i-j})_{i,j\in\mathbb{Z}}$ defined by

$$\Psi_j = \begin{cases} A_j, & |j| \le p, \\ 0, & |j| > p. \end{cases} \tag{4.9}$$

Given the real $k \times k$ matrices $A_{-p}, A_{-p+1}, \dots, A_p$, and fixing a sequence of positive numbers β_j, $j \in \mathbb{Z}$, with $\beta_i\beta_j \le \beta_{i+j}$, a *Carathéodory-Toeplitz extension* of Ψ is a bi-infinite block Toeplitz matrix $\Phi = (\Phi_{i-j})_{i,j\in\mathbb{Z}}$ which is positive definite, satisfies $\Phi_j = A_j$ for $|j| \le p$, and has the so-called Wiener property $\sum_{j\in\mathbb{Z}} \beta_j\|\Phi_j\| < +\infty$. In other words, we seek, in $\ell^k_{1,\beta}(\mathbb{Z})$, a positive definite *band extension* Φ of the banded bi-infinite block Toeplitz matrix Ψ.

The following theorem indicates when such a *Carathéodory-Toeplitz extension* exists [2]. Furthermore, it gives an algorithm for constructing one particular such extension as well as all such extensions.

THEOREM 4.2. *Given the real $k \times k$ matrices $A_{-p}, A_{-p+1}, \cdots, A_p$, there exists a Carathéodory-Toeplitz extension Φ of the banded bi-infinite block Toeplitz matrix Ψ (4.9) if and only if the $k(p+1) \times k(p+1)$ matrix*

$$\widetilde{A} = \begin{bmatrix} A_0 & A_{-1} & \cdots & A_{-p} \\ A_1 & A_0 & \cdots & A_{-p+1} \\ \vdots & \vdots & & \vdots \\ A_p & A_{p-1} & \cdots & A_0 \end{bmatrix} \tag{4.10}$$

is positive definite. In that case there exists a unique solution Φ_b with the additional property that its inverse is banded. Defining the $k(p+1)\times k$ matrices

$$\begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_p \end{bmatrix} = \widetilde{A}^{-1}\begin{bmatrix} I_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \begin{bmatrix} Y_{-p} \\ \vdots \\ Y_{-1} \\ Y_0 \end{bmatrix} = \widetilde{A}^{-1}\begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_k \end{bmatrix},$$

and using $X_j = 0$ for $j<0$ and $j>p$ and $Y_j = 0$ for $j<-p$ and $j>0$, the bi-infinite block Toeplitz matrices

$$L = (X_{i-j}X_0^{-1/2})_{i,j\in\mathbb{Z}_+}, \quad U = (Y_{i-j}Y_0^{-1/2})_{i,j\in\mathbb{Z}_+}$$

satisfy the following factorization relations

$$\Phi_b^{-1} = LL^T = UU^T. \tag{4.11}$$

Moreover, each real solution Φ of the Carathéodory-Toeplitz extension problem has the form

$$\Phi^{-1} = (UG+L)(I-G^TG)^{-1}(G^TU^T+L^T),$$

where $G = (G_{i-j})_{i,j\in\mathbb{Z}_+}$ is a bi-infinite real block Toeplitz matrix satisfying $G_j = 0$ for $j \le p$ which is a strict contraction on $\ell_2(\mathbb{Z}_+)$. This formula yields a 1,1-correspondence between real solutions Φ and such matrices G.

The above theorem leads to the following factorization of a positive definite bi-infinite block Toeplitz matrix A. For $p\in\mathbb{Z}_+$ large enough, the bi-infinite block Toeplitz matrix with symbol (4.9) has a positive definite banded extension Φ such that the bi-infinite inverse Φ^{-1} is banded. The factors of the inverse Φ^{-1}, as given by (4.11), are approximations of the factors of the inverse of the original positive definite block Toeplitz matrix. This follows from well-known convergence results on the bilateral projection method (cf. [1], Theorem 3.4; [3], Theorems 2.1 and 3.1).

5. Perturbed semi-infinite block Toeplitz systems. As in [7] and [12], we say that two semi-infinite block matrices $A = (A_{ij})_{i,j\in\mathbb{Z}_+}$ and $B = (B_{ij})_{i,j\in\mathbb{Z}_+}$ of order k are called equivalent ($A\sim B$) if there exist $c>0$ and $\lambda\in(0,1)$ such that

$$\|A_{ij}-B_{ij}\| \le c\lambda^{i+j}, \quad i,j\in\mathbb{Z}_+.$$

THEOREM 5.1. *Let A and B be nonsingular semi-infinite matrices of order k that decay exponentially, with $A\sim B$ and A of Toeplitz type. Then A^{-1} and B^{-1} also decay exponentially and $A^{-1}\sim B^{-1}$.*

Proof. Let us first assume that A and B are positive definite. Further let L and M be the Cholesky factors of A and B, respectively. As A is of Toeplitz type, as explained in the proof of Theorem 3.2, L and L^{-1} decay exponentially. Furthermore M and M^{-1} also decay exponentially and $M\sim L$ and $M^{-1}\sim L^{-1}$ ([12], Theorem 3.10). Therefore, as the product of two exponentially decaying matrices also decays exponentially ([12], Lemma 3.3) and $A_1B_1\sim A_2B_2$ if $A_1\sim A_2$ and $B_1\sim B_2$ ([12], Lemma 3.5), A^{-1} and B^{-1} decay exponentially and $A^{-1}\sim B^{-1}$. Hence, in view of Theorem 3.2, this result holds true even if A and B are not positive definite. □

Let us now consider a semi-infinite linear system

$$A\mathbf{x} = \mathbf{b},$$

where A is nonsingular and bounded on $\ell_1^k(\mathbb{Z}_+)$ with $\mathbf{b}\in\ell_1^k(\mathbb{Z}_+)$. Further, let $A\sim T$, where T is a nonsingular semi-infinite Toeplitz matrix of order k.

The method we propose consists of two steps. First, choose $N\in\mathbb{N}$ and let

$$(T_N^{-1})_{ij} = (T^{-1})_{ij} \quad\text{and}\quad (A_N^{-1})_{ij} = (A^{-1})_{ij} \tag{5.1}$$

for $i = N+1, N+2, \dots$ and $j \in \mathbb{Z}_+$. Then take

$$\widehat{\mathbf{x}}_2^N = T_N^{-1}\mathbf{b}$$

as an approximant of $\mathbf{x}_2^N = A_N^{-1}\mathbf{b}$. We call N the *projection parameter*. Note that $T_N^{-1}\mathbf{b}$ can be obtained by omitting the first $N+1$ components from the solution of system $T\mathbf{x} = \mathbf{b}$, which could be solved as explained in §3.

Next, take the subsystem $A_1^N\mathbf{x} = \mathbf{b}_1^N$ consisting of the first $N+1$ equations of the system $A\mathbf{x} = \mathbf{b}$ and partition it as follows

$$\begin{bmatrix} A_{11}^N & A_{12}^N \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^N \\ \mathbf{x}_2^N \end{bmatrix} = \mathbf{b}_1^N,$$

where $(A_{11}^N)_{ij} = A_{ij}$ for $i,j = 0,1,\dots,N$ and $(A_{12}^N)_{ij} = A_{ij}$ for $i = 0,1,\dots,N$ and $j = N+1, N+2, \dots$. Then, assuming A_{11}^N to be nonsingular, the second step consists of approximating $\mathbf{x}_1^N$ by

$$\widehat{\mathbf{x}}_1^N = (A_{11}^N)^{-1}\left(\mathbf{b}_1^N - A_{12}^N\widehat{\mathbf{x}}_2^N\right).$$

As regards the error estimate, we have the following result.

THEOREM 5.2. *Let A and T be semi-infinite matrices of order k that decay exponentially, with $A \sim T$. Then for any $\epsilon > 0$ and for large enough N we have*

$$\max\left\{\|\mathbf{x}_1^N - \widehat{\mathbf{x}}_1^N\|_1, \|\mathbf{x}_2^N - \widehat{\mathbf{x}}_2^N\|_1\right\} < \epsilon.$$

Proof. By Theorem 5.1 we can say that $A^{-1} \sim T^{-1}$, so that there exist constants $c > 0$ and $\lambda \in (0,1)$ such that

$$\|(A^{-1} - T^{-1})_{ij}\| \le c\lambda^{i+j}, \quad i,j \in \mathbb{Z}_+.$$

As a consequence

$$\left\|\left((A_N^{-1} - T_N^{-1})\mathbf{b}\right)_{N+1-i}\right\| \le c\lambda^{N+1+i}\|\mathbf{b}\|_{1,\lambda}, \quad i \in \mathbb{Z}_+,$$

where $\|\mathbf{b}\|_{1,\lambda} < \|\mathbf{b}\|_1$, since $\lambda_j = \lambda^j$ for $j \in \mathbb{Z}_+$ and $\lambda \in (0,1)$. Therefore, letting $\delta\mathbf{x}_2^N = \mathbf{x}_2^N - \widehat{\mathbf{x}}_2^N = (A_N^{-1} - T_N^{-1})\mathbf{b}$, we have

$$\|\delta\mathbf{x}_2^N\|_1 \le c\|\mathbf{b}\|_{1,\lambda}(1-\lambda)^{-1}\lambda^{N+1},$$

which can be made arbitrarily small by adequately increasing N.

Given $\mathbf{x}_2^N$ and $\widehat{\mathbf{x}}_2^N$, the vectors $\mathbf{x}_1^N$ and $\widehat{\mathbf{x}}_1^N$ are the respective solutions of

$$\begin{cases} A_{11}^N\mathbf{x}_1^N = \mathbf{b}_1^N - A_{12}^N\mathbf{x}_2^N \\ A_{11}^N\widehat{\mathbf{x}}_1^N = \mathbf{b}_1^N - A_{12}^N\widehat{\mathbf{x}}_2^N, \end{cases}$$

both being systems of order $N+1$. Setting $\delta\mathbf{x}_1^N = \mathbf{x}_1^N - \widehat{\mathbf{x}}_1^N$ and noting that

$$\|(A_{12}^N\,\delta\mathbf{x}_2^N)_i\| \le c\lambda^{N+1-i}\,\|\delta\mathbf{x}_2^N\|_{1,\lambda}, \quad i = 0,1,\dots,N,$$

where $\|\delta\mathbf{x}_2^N\|_{1,\lambda} < \|\delta\mathbf{x}_2^N\|_1$, it is immediate to derive the result from the estimate

$$\|\delta\mathbf{x}_1^N\|_1 \le c\lambda\frac{1-\lambda^{N+1}}{1-\lambda}\,\|(A_{11}^N)^{-1}\|_1\,\|\delta\mathbf{x}_2^N\|_{1,\lambda},$$

which completes the proof. $\square$

6. Applications.

6.1. Semi-infinite block Toeplitz systems. As an application of the previous method, let us consider the numerical solution of the Poisson equation on a semi-infinite strip. More precisely, setting $\Omega = \{(x,y) : 0 < x < \infty,\ 0 < y < 1\}$ and denoting by $\partial\Omega$ the boundary of Ω, we consider the numerical solution, by a finite differences method, of the differential boundary value problem

$$\begin{cases} \Delta u(x,y) = -f(x,y), & (x,y) \in \Omega, \\ u(x,y) = 0, & (x,y) \in \partial\Omega, \end{cases} \tag{6.1}$$

where Δ is the Laplace operator.

To be specific, take

$$f(x,y) = \left[2\pi\cos\pi x + (2\pi^2 - 1)\sin\pi x\right] \mathrm{e}^{-x}\sin\pi y,$$

so that the exact solution of (6.1) is

$$u(x,y) = \mathrm{e}^{-x}\sin\pi x\ \sin\pi y. \tag{6.2}$$

Discretizing the differential problem (6.1) by a 5-points scheme on the mesh points

$$(x_i, y_j) = (ih, jh), \quad j = 0,1,\dots,n+1, \quad i = 0,1,\dots$$

with the stepsize $h = \frac{1}{n+1}$, by choosing the usual order for the unknowns u_{ij} we obtain a semi-infinite linear system of block Toeplitz type, whose matrix is of the form

$$T = \begin{bmatrix} T_0 & -I & & \\ -I & T_0 & \ddots & \\ & \ddots & \ddots & \ddots \\ & & \ddots & \ddots \end{bmatrix}, \tag{6.3}$$

where I is the identity matrix of order n and T_0 is the $n \times n$ tridiagonal Toeplitz matrix

$$\begin{bmatrix} 4 & -1 & & \\ -1 & 4 & \ddots & \\ & \ddots & \ddots & -1 \\ & & -1 & 4 \end{bmatrix}.$$

The symbol associated to the matrix T is the Laurent matrix polynomial

$$\widehat{T}(z) = -z^{-1}I + T_0 - zI.$$

In this case the eigenvalues $\{\lambda_j\}$ of the corresponding monic matrix polynomial $\mathcal{P}(z) = -z\widehat{T}(z)$ can be obtained analytically. Indeed, it is straightforward to prove that

$$\lambda_j = \begin{cases} \dfrac{1}{2}\left(\mu_j + \sqrt{\mu_j^2 - 4}\right), & j = 1,\dots,n \\ \dfrac{1}{\lambda_{2n-j+1}}, & j = n+1,\dots,2n \end{cases}$$

where

$$\mu_j = 4 + 2\cos\frac{j\pi}{n+1}, \quad j = 1,\dots,n.$$

The first n eigenvalues lie outside the unit circle and the last n inside, as shown in Figure 1 for $n = 20$. As a consequence, a very accurate spectral factorization of $\mathcal{P}(z)^{-1}$ can be obtained using the first method, even for moderately high values of n.

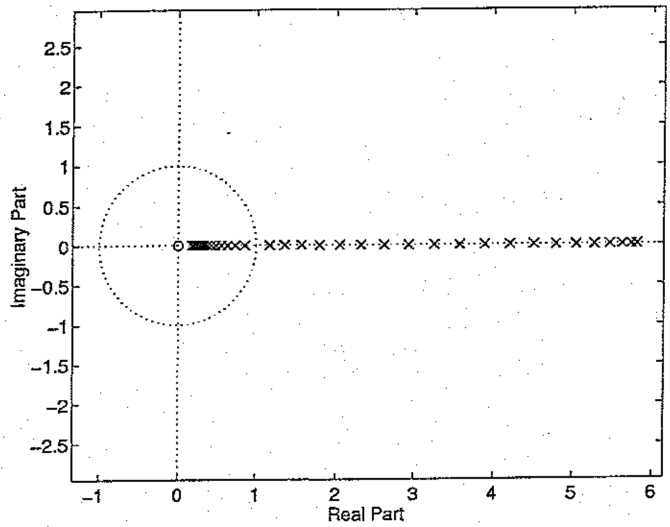

Fig. 1

Denoting by u_{ij}, for $i = 0, 1, \dots$ and $j = 0, 1, \dots, n+1$, the solution of the discretized problem, we assess the accuracy of the results, with respect to n, by the two following error estimates:

$$E_n^{(k)} = \max\left\{ |u(x_i, y_j) - u_{ij}^{(k)}|,\ i \in \mathcal{I}_{10n},\ j \in \mathcal{I}_n \right\} \tag{6.4}$$

$$E_{r,n}^{(k)} = \max\left\{ \frac{|u(x_i, y_j) - u_{ij}^{(k)}|}{|u(x_i, y_j)|},\ i \in \mathcal{I}_{10n},\ j \in \mathcal{I}_n,\ |u(x_i, y_j)| > 10^{-16} \right\} \tag{6.5}$$

where $\mathcal{I}_n = \{1, \dots, n\}$ and $k = 1, 2$ specifies if the spectral factorization has been carried out by the first or the second method, respectively. Figure 2 shows $E_n^{(1)}$, $E_{r,n}^{(1)}$ and $1/n^2$ in the range $5 \le n \le 40$. Note that, as we expect solving the Poisson equation by the 5-points discretization method [14], $E_n^{(1)} = O(n^{-2})$.

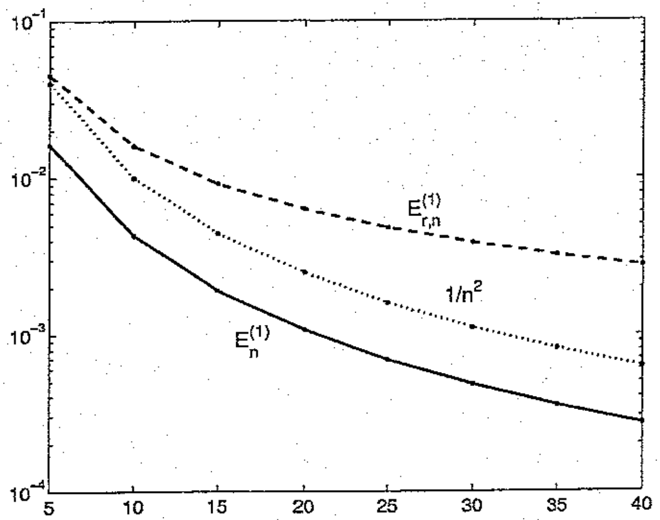

Fig. 2

The high level of accuracy of the results essentially depends on the exact knowledge of the eigenvalues of the matrix polynomial $\mathcal{P}(z) = -z\widehat{T}(z)$. Indeed, our experience suggests that the precision attainable by the first method primarily depends on the accurate evaluation of the eigenvalues, on their separation and on their distance from the mentioned curve Γ.

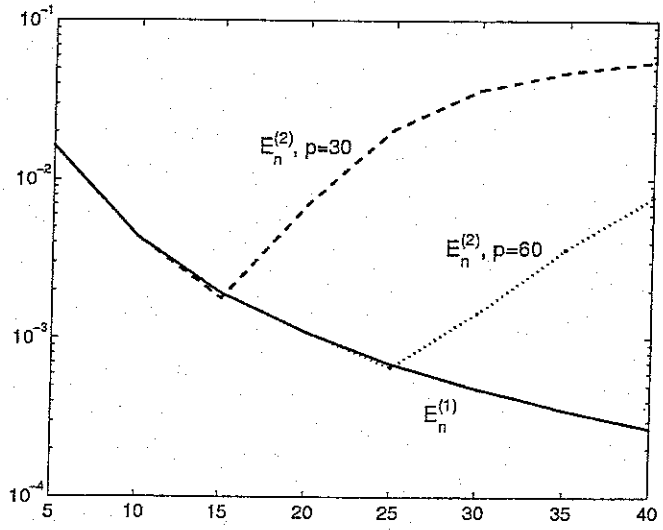

Fig. 3

Figure 3 reports the error $E_n^{(k)}$, $k = 1, 2$, for several values of n and for two values of the extension parameter p. It shows, in particular, that for small values of n both methods are equally effective. Furthermore, it highlights that the extension parameter depends on the size of the blocks n. In fact, as n gets larger it is necessary to increase the value of p to obtain, by the band extension method, the same accuracy as in the first factorization method. For example, when $n = 25$ and $p = 60$ we need to solve a block Toeplitz linear system of dimension $n(p+1) = 1525$. This fact poses no particular problem from the point of view of computational complexity, the matrix being banded, but implies a larger propagation of roundoff errors, with respect to the first method. However, as we will see in the next example, the first algorithm is not always more accurate than the second one.

Now, consider the following two B-splines, already introduced in [12]:

$$\begin{aligned}
B_1(x) &= -\frac{11}{36}\,(-x)_+^3 - \frac{1}{2}\,(-x)_+^2 + \frac{1}{2}\,(1-x)_+^3 \\
&\quad -\frac{1}{4}\,(2-x)_+^3 + \frac{1}{18}\,(3-x)_+^3\,, \\
B_2(x) &= \frac{1}{12}\,(3-x)_+^4 - 2\,(2-x)_+^3 + \frac{3}{4}\,(2-x)_+^4 \\
&\quad -\frac{3}{4}\,(1-x)_+^4 - 2\,(1-x)_+^3 - \frac{1}{12}\,(-x)_+^4\,.
\end{aligned}$$

Moreover, assuming $B_{ki}(x) := B_k(x-i)$, $k = 1, 2$, let T be the bi-infinite block Toeplitz matrix defined by

$$T_{ij} = \begin{bmatrix} (B_{1i}, B_{1j}) & (B_{1i}, B_{2j}) \\ (B_{2i}, B_{1j}) & (B_{2i}, B_{2j}) \end{bmatrix}, \quad i, j \in \mathbb{Z} \tag{6.6}$$

where the symbol $(\cdot,\cdot)$ denotes the usual inner product in $L^2(\mathbb{R})$.

The matrix T is a 5-diagonal block Toeplitz matrix of order 2, whose nonzero blocks $(T)_{ij} = T_{i-j}$, $|i-j| \le 2$, are

$$T_0 = \frac{1}{\alpha}\begin{bmatrix} 13176 & 10179 \\ 10179 & 11304 \end{bmatrix}, \quad T_1 = \frac{1}{\alpha}\begin{bmatrix} 4634 & 6573 \\ 1275 & 1688 \end{bmatrix}, \quad T_2 = \frac{1}{\alpha}\begin{bmatrix} 124 & 111 \\ 6 & 4 \end{bmatrix}$$

with $\alpha = 362880$, $T_{-2} = T_2^T$ and $T_{-1} = T_1^T$.

Now consider the semi-infinite matrix T_+ given by $(T_+)_{ij} = T_{ij}$, $i, j \in \mathbb{Z}_+$, set $\mathbf{b} = T_+\mathbf{x}$, with $x_n = \left(\frac{8}{9}\right)^n$, $n \in \mathbb{Z}_+$, and solve the semi-infinite block Toeplitz linear system

$$T_+\mathbf{x} = \mathbf{b}$$

by the two spectral factorization methods illustrated above.

Denoting by $x^{(1)}$ and $x^{(2)}$ the solution vectors obtained by using the two spectral factorization methods, let

$$e_n^{(k)} = \frac{|x_n - x_n^{(k)}|}{|x_n|}, \tag{6.7}$$

with $k = 1, 2$ and $n \in \mathbb{Z}_+$, be the corresponding componentwise relative error. Figure 4 gives the

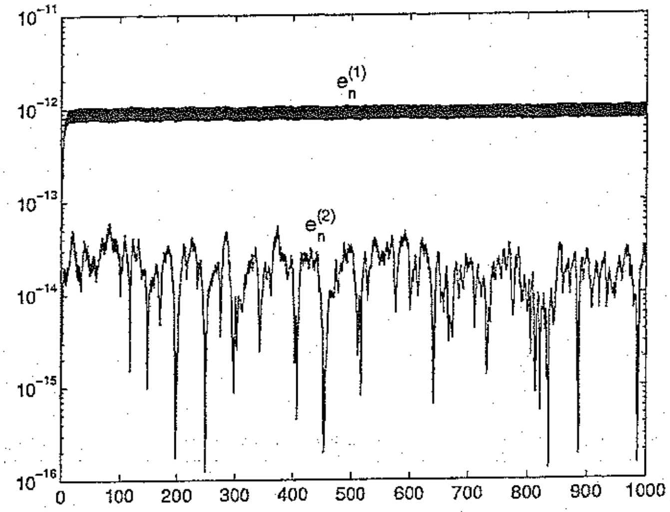

Fig. 4

values of $e_n^{(k)}$ in the range $0 \le n \le 1000$. Note that the thick black line representing $e_n^{(1)}$ is due to a small oscillation between even and odd components of the error.

Figure 4 clearly shows that in both cases the results are very accurate and that the second method gives better results than the first one for all values of n. We note that in this example the application of the band extension method is computationally less expensive than in the previous one. Indeed, for $p = 60$, which is a good extension parameter value, we have to solve a linear system of dimension 240. On the contrary, using the first method we must estimate the 8 eigenvalues of the matrix polynomial $z^2\widehat{T}(z)$, a task which requires a greater computational effort and does not guarantee the same precision of the results.

6.2. Perturbed semi-infinite block Toeplitz systems. As an application of the method proposed in §5 for the solution of perturbed block Toeplitz linear systems, let us consider the numerical solution of the following partial differential equation

$$\begin{cases} \Delta u(x,y) + 10\, \chi_{[0,2]}(x)\, u(x,y) = -f(x,y), & (x,y) \in \Omega \\ u(x,y) = 0, & (x,y) \in \partial\Omega, \end{cases} \tag{6.8}$$

where Ω and $\partial\Omega$ are defined as in (6.1) and $\chi_{[0,2]}(x)$ is the characteristic function of the interval $[0, 2]$. Equation (6.8) can then be considered as a perturbation of (6.1). As an example we take

$$f(x,y) = -\Delta u(x,y) - 10\chi_{[0,2]}(x)u(x,y),$$

where $u(x,y)$ is defined in (6.2).

Discretizing equation (6.8) as in §6.1, we obtain a semi-infinite linear system whose matrix A is a perturbation of the block Toeplitz matrix T given by (6.3). More precisely, A agrees with T off its first $n(N+1)$ rows, where n is the dimension of each block.

As to be expected, the results are effectively influenced by the parameter n, which characterizes the discretization stepsize, and by the projection parameter N. To highlight their relevance for the accuracy of the results, the absolute and relative errors, already defined in (6.4) and in (6.5), are now denoted by $E_{n,N}^{(k)}$ and $E_{r,n,N}^{(k)}$.

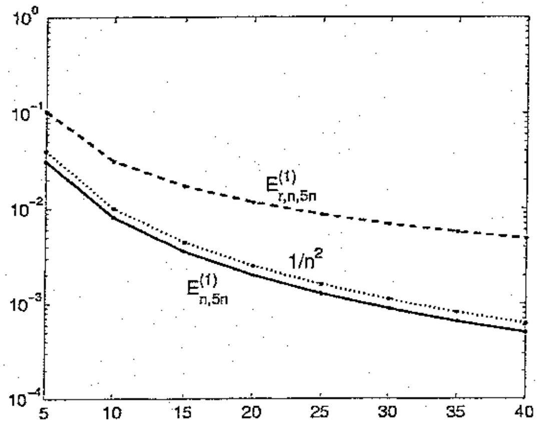

Fig. 5

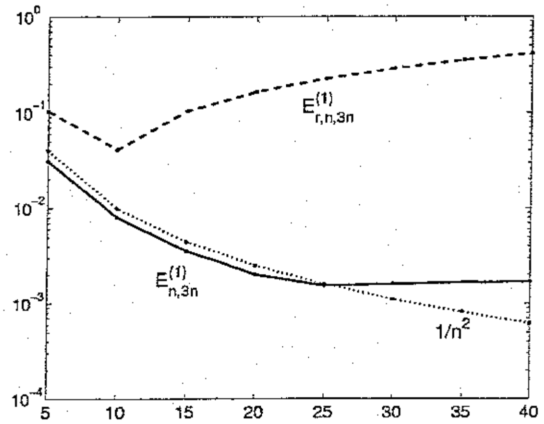

Fig. 6

In Figure 5 the values of $E_{n,N}^{(1)}$ and $E_{r,n,N}^{(1)}$ are depicted for $n = 5, 10, \dots, 40$ and $N = 5n$. This graph shows that the method of §5 is very effective if the spectral factorization of the associated block Toeplitz matrix is accurate, and if the value of the projection parameter N is large enough.

The effect of an insufficient value of N is illustrated in Figure 6, where the errors are plotted with the same range of variation of n, but with $N = 3n$. This loss of accuracy for moderately large values of n depends on the fact that in this case the matrices T_N and A_N defined in (5.1) are not *close enough* to guarantee the approximation result stated in Theorem 5.2.

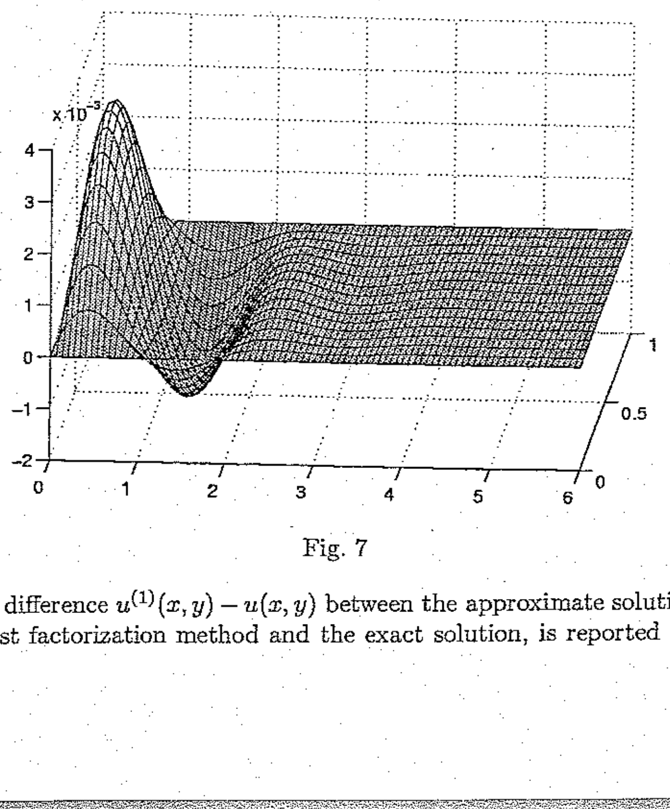

Fig. 7

A graph of the difference $u^{(1)}(x,y) - u(x,y)$ between the approximate solution of problem (6.8) obtained by the first factorization method and the exact solution, is reported in Figures 7 and 8.

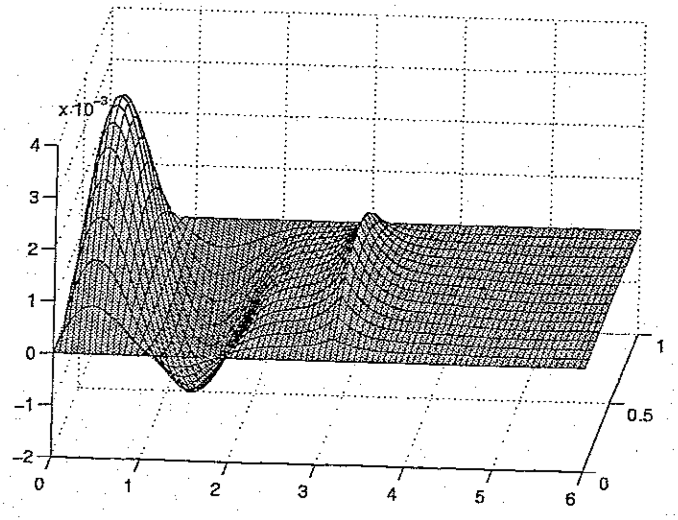

Fig. 8

In both figures $n = 15$, while we fixed $N = 75$ in Figure 7 and $N = 45$ in Figure 8. Again, these graphs show the effect of an under-estimation of the projection parameter that is a sharp increase of the error at the x-value corresponding to this under-estimation ($x = Nh$).

Now, as a second example, we consider the system

$$A\mathbf{x} = \mathbf{b},$$

where

$$A_{ij} = T_{ij} + \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \mathrm{e}^{-(i+j)}, \quad i, j \in \mathbb{Z}_+, \tag{6.9}$$

T_{ij} is the 2×2 matrix defined in (6.6), $x_n = \left(\frac{8}{9}\right)^n$, $n \in \mathbb{Z}_+$, and $\mathbf{b} := A\mathbf{x}$.

Let $e_{n,N}^{(k)}$ as in (6.7), where N specifies the value of the projection parameter. Figure 9 clearly

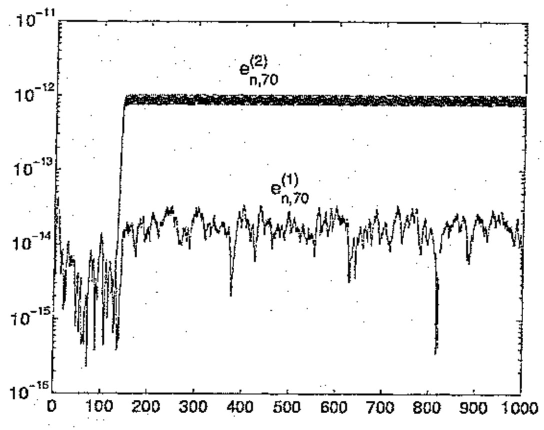

Fig. 9

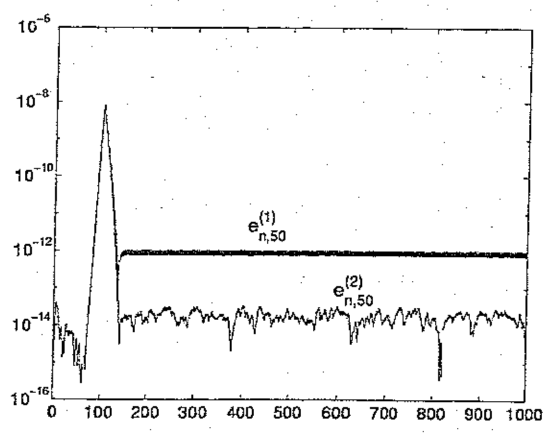

Fig. 10

exhibits that, for large enough N, the results obtained by the second spectral factorization method are very accurate. Similar results, even if not so accurate, are obtained using the first factorization method. Figure 10 shows the same error curves for $N = 50$. In both figures the extension parameter was fixed at $p = 60$. Finally, we note that, as our numerical experiments suggest, substituting the 2×2 perturbation matrix in (6.9) by any other non singular matrix does not change the accuracy of the results. Furthermore, using the damping factor $\lambda^{-(i+j)}$, with $1 < \lambda < e$, we can still get good results, provided that we increase the projection parameter N.

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