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Positivity and Monotonicity Properties of Transport Equations with Spatially Dependent Cross Sections*

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Abstract

We investigate the transport equation

$$\mu \frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) = \frac{1}{2}\xi c(x)\int_{-1}^{+1}\psi(x,\mu')d\mu' + f(x,\mu)\,(0<x<\tau<\infty)$$

with suitable boundary conditions through an equivalent integral equation. Assuming the incoming fluxes, the internal source term $f(x,\mu)$, the cross section $c(x)$ and the parameter ξ to be nonnegative, we prove the existence of a unique dominant eigenvalue $\xi = \xi_0(\tau)$ for which the homogeneous problem has a positive solution (critical case), the existence of a unique positive solution for $\xi < \xi_0(\tau)$ (non-critical case), and the absence of positive solutions for $\xi > \xi_0(\tau)$ (supercritical case). We show $\xi_0(\tau)$ to decrease continuously from ∞ to some $\xi_0(\infty)>0$ whenever τ increases from 0 to ∞ (monotonicity). The results are obtained by studying an operator that leaves invariant the cone of nonnegative functions in $L_\infty(0,\tau)$.

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Introduction

The time-independent neutron transport equation in a homogeneous finite or semi-infinite slab medium with spatially dependent collision ratio $\xi c(x)$ reads as follows:

$$\mu \frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) = \frac{1}{2}\xi c(x)\int_{-1}^{+1} \psi(x,\mu')d\mu' + f(x,\mu)\,(0<x<\tau). \tag{1}$$

Studied on the finite slab $0<x<\tau<\infty$ one imposes the boundary conditions

$$\psi(0,\mu) = \phi(\mu)\,(0\leq\mu\leq 1), \quad \psi(\tau,\mu) = \phi(\mu)\,(-1\leq\mu<0); \tag{2}$$

on the half-line $0<x<\tau = \infty$ one imposes the conditions

$$\psi(0,\mu) = \phi(\mu)\,(0\leq\mu\leq 1), \quad \int_{-1}^{+1} |\psi(x,\mu)|^2 d\mu = O(1)\,(x\to\infty). \tag{3}$$

The popular method to study Eq. (1) is the method of singular eigenfunction expansion for which the present state of affairs has been reviewed by Larsen.[1] In the present article a different method is adopted that allows one to investigate the existence and uniqueness of nonnegative solutions ψ of Eqs. (1)-(2) and Eqs. (1)-(3), where we assume that $\xi>0$ and $\phi(\mu)$, $c(x)$ and $f(x,\mu)$ are suitable nonnegative functions. These positivity requirements all are prescribed by the physics behind the mathematical problem.

Let us introduce some notation. By $H = L_2[-1,1]$ we denote the Hilbert space of square integrable real-valued functions on $[-1,1]$ with inner product $\langle f,g\rangle = \int_{-1}^{+1} f(\mu)g(\mu)d\mu$. Further, let $\psi(x)$ and $f(x)$ be vectors in H, T and B the self-adjoint operators and Q_+ and Q_- the complementary orthogonal projections defined by

$$\psi(x)(\mu) = \psi(x,\mu), \quad f(x)(\mu) = f(x,\mu) \; ; \tag{4}$$

$$(Th)(\mu) = \mu h(\mu), \quad (Bh)(\mu) = \frac{1}{2}\int_{-1}^{+1} h(\mu')d\mu'; \tag{4b}$$

$$(Q_\pm h)(\mu) = h(\mu)\,(\mu\gtrless 0), \quad (Q_\pm h)(\mu) = 0\,(\mu\lessgtr 0). \tag{4c}$$

Then the ranges of $Q_\pm$ are the subspaces Ran $Q_+ = L_2[0,1]$ and Ran $Q_- = L_2[-1,0]$. The integro-differential equation (1) can now be written in the concise form

$$(T\psi)'(x) = -\psi(x) + \xi c(x)B\psi(x) + f(x)(0<x<\tau) \tag{5a}$$

with boundary conditions

$$\lim_{x\downarrow 0} \| Q_+\psi(x) - Q_+\phi \| = \lim_{x\uparrow\tau} \| Q_-\psi(x) - Q_-\phi \| = 0 \tag{5b}$$

for finite τ with $\phi \in H$, and with boundary conditions

$$\lim_{x\downarrow 0} \| Q_+\psi(x) - \phi \| = 0, \quad \| \psi(x) \| = O(1)(x\to\infty) \tag{5c}$$

for $\tau = \infty$ and $\phi \in$ Ran $Q_+ \subset H$. By a solution of the boundary value problem (5a)-(5b) (resp.(5a)-(5c)) we mean a function ψ from $(0,\tau)$ (resp.$(0,\infty)$) into H such that $T\psi$ is strongly differentiable on $(0,\tau)$ (resp.$(0,\infty)$) and satisfies (5a)-(5b)(resp.(5a)-(5c)).

In this article we are searching for nonnegative solutions of (5a)-(5b) and (5a)-(5c). Throughout the paper c(x) will be a bounded measurable nonnegative function on $(0,\tau)$ and K will denote the subset of functions $h \in H = L_2[-1,1]$ such that $h(\mu) \geq 0$ for almost every $-1\leq\mu\leq 1$. We assume throughout that $\xi > 0$, $\phi \in K$ and f is a bounded uniformly Hölder continuous function[2] from $[0,\tau]$(τ finite) or $[0,\infty)$ ($\tau=\infty$) into K. We shall prove the following results:

Theorem 1. Let τ be finite and $c(x) > 0$, except for a set in $(0,\tau)$ of measure zero. Then one of the following three situations occurs:

(a) non-critical case. For every $\phi \in H$ and bounded uniformly Hölder continuous $f:(0,\tau) \to H$ there exists a unique solution $\psi:(0,\tau) \to H$. Whenever $\phi \in K$ and $f(x) \in K$ for $0<x<\tau$, also the solution ψ assumes its values in K.

b. critical case. For $\phi = 0$ and $f \equiv 0$ there exists a nontrivial solution $\psi:(0,\tau) \to K$, which is unique up to multiplication by a positive constant. Whenever $\phi \in K$ and $f(x) \in K$ for $0<x<\tau$, and at least one of them is non-trivial, there does not exist any solution ψ taking its values in K.

c. supercritical case. For $\phi \in K$ and $f(x) \in K$ for $0<x<\tau$, the existence of a solution $\psi:(0,\tau) \to K$ implies that $\phi = 0$, $f \equiv 0$ and $\psi \equiv 0$. Moreover, there exists a unique positive number $\xi_0(\tau)$, such that for $0<\xi<\xi_0(\tau)$, $\xi = \xi_0(\tau)$ and $\xi > \xi_0(\tau)$ the non-critical, critical and supercritical cases occur, respectively. The function $\tau \to \xi_0(\tau)$ is strictly monotonic and continuous with $\lim_{\tau\downarrow 0} \xi_0(\tau) = \infty$ and $\lim_{\tau\to\infty} \xi_0(\tau) \overset{\text{def.}}{=} \xi_0(\infty) > 0$.

Theorem 1 concerns the criticality problem of finding for fixed τ the unique parameter ξ for which (5a)-(5b) with $\phi = 0$ and $f(x) \equiv 0$ has a positive solution. The monotonicity of $\tau \mapsto \xi_0(\tau)$ implies that for every $\xi > \xi_0(\infty)$ there also is a unique τ for which this problem has a positive solution. The former problem one could call the dominant eigenvalue problem; the latter is the actual critical slab problem.

For $c_0(x)\equiv 1$ one has $\xi_0(\infty)=1$, which corresponds to the c=1 half-space problem.[3] For this case Busoni, Mangiarotti and Frosali[4] proved Theorem 1 partially.[5] In fact, they settled the existence of $\xi_0(\tau)$, the occurrence of the case (a) for $0<\xi<\xi_0(\tau)$ and the uniqueness of a solution $\psi:(0,\tau)\to K$ for $\xi=\xi_0(\tau)$, $\phi=0$ and $f\equiv 0$. In the inhomogeneous case and below criticality Busbridge[6] studied an integral form of Eq.(1). Kelley[6] proved part (b) and the existence of $\xi_0(\tau)$ in a different way for positive $c\in L_1(0,\tau)$. Teczan[7] recently showed interest in the monotonicity of the critical slab value τ_0 for the homogeneous case with a backward scattering term.

The critical half-space problem was dealt with for exponential media ($c(x) = e^{-x/s}$) by Pomraning,[8] who obtained numerical results. In the present article we prove analogues of Theorem 1, (a) and (c).

Theorem 2. Let $\tau = \infty$ and $c(x) > 0$, except on a set of measure zero. Then one of the following situations occurs:

(a) $0<\xi<\xi_0(\infty)$: non-critical case. For every $\phi \in H$ and bounded uniformly Hölder continuous[2] function $f:(0,\infty) \to H$ there exists a unique solution $\psi:(0,\infty) \to H$. Whenever $\phi \in K$ and $f(x) \in K$ for $0<x<\infty$, the solution ψ also assumes its values in K.

(c) $\xi>\xi_0(\infty)$: supercritical case. For $\phi \in K$ and $f(x) \in K$ there does not exist a solution $\psi:(0,\infty) \to K$, unless $\phi = 0$, $f \equiv 0$ and $\psi \equiv 0$.

Both theorems will be proved by reducing the boundary value problems (5a)-(5b) and (5a)-(5c) to a scalar integral equation, which can be written in the form

$$(I - \xi\, \mathcal{R})\zeta = \tilde{\omega}\ , \tag{6}$$

on the Banach space $BC(0,\tau)$ of bounded continuous functions $\zeta:(0,\tau) \to \mathbb{R}$ with norm $\|\zeta\| = \sup\{|\zeta(x)|: 0<x<\tau\}$. In this space we consider the cone (see Section 2 for the definition)

$$K_c = \{\zeta \in BC(0,\tau): \zeta(x) \geq 0 \text{ for } 0<x<\tau\}.$$

For $\phi \in K$ and $f(x) \in K$ $(0<x<\tau)$ it appears that $\tilde{\omega} \in K_c$ and $\mathcal{R}[K_c] \subseteq K_c$. We are searching for solutions ζ in K_c.

In order to investigate Eq. (6) we shall apply the theory of operators leaving invariant a cone in a Banach space. This theory made considerable progress by the pioneering work of Krein and Rutman,[9] and Krasnoselškii.[10] Theorems 1 and 2 will appear as consequences of this theory.

In Section 1 the problems (5a)-(5b) and (5a)-(5c) are proved to be equivalent to a scalar integral equation, which will be put into the form (6). Section 2 reviews some of the theory of cones in a Banach space. In Section 3 the proofs of Theorems 1 and 2 will be given, and at the end of the section we prove a statement of Tezcan[7] rigorously.

1. Statement as a vector equation

Consider the vector $e(\mu) \equiv 1$ in H and define the propagator function $\mathcal{H}(x)$ $(0 \neq x \in \mathbb{R})$ by

$$(\mathcal{H}(x)h)(\mu) = \begin{cases} |\mu|^{-1} e^{-x/\mu}\, h(\mu), & x\mu>0\ ; \\ 0\ , & x\mu<0\ . \end{cases}$$

One easily computes that

$$Ei(x) = \langle \mathcal{H}(x)e,\ e\rangle = \int_1^\infty z^{-1} e^{-z|x|} dz\ ,^{11} \quad 0 \neq x \in \mathbb{R}. \tag{7}$$

<u>LEMMA</u> 1. For finite (resp. infinite) τ let f be a bounded

uniformly Hölder continuous[2] function from $[0,\tau]$ (resp. $[0,\infty)$) into H. Then

$$F(x) = [I-e^{-xT^{-1}}Q_{+}-e^{(\tau-x)T^{-1}}Q_{-}]f(x) + \int_0^\tau H(x-y)\cdot\{f(y)-f(x)\}dy,$$

with the second semigroup taken zero for $\tau = \infty$, is a bounded continuous function from $[0,\tau]$(resp. $0,\infty)$) into H. Further, $TF(x)$ is strongly differentiable on $(0,\tau)$ and

$$(TF)'(x) = -F(x) + f(x) \ , \quad 0<x<\tau \ . \tag{8}$$

Proof. Choose $0<\alpha\leq 1$ and M such that $\| f(y) - f(x) \| \leq M|y-x|^\alpha$. Then

$$\int_0^\tau \| H(x-y)\{f(y)-f(x)\} \| \, dy \leq M \int_{-x}^{\tau-x} |z|^{\alpha-1} \| H(z) \| \, dz < \infty \ .$$[12]

As Ref. 12 shows that $N \overset{\text{def.}}{=} \int_{-\infty}^{\infty} |z|^{\alpha-1} \| H(z) \| \, dz < \infty$, the boundedness of F and its vanishing at ∞ are clear. Now define $G(x,y)$ such that

$$f(x) - f(y) = |x-y|^\alpha \, G(x,y).$$

Then G is bounded on $[0,\tau]^2$ (resp. $[0,\infty)^2$ for $\tau = \infty$) and, if necessary, by lowering α, one could choose G to be continuous. Fix $\varepsilon > 0$. If τ is finite, there exists a two-variable H-valued polynomial $p(x,y)$ such that the difference $\| G(x,y) - p(x,y) \| \leq (\varepsilon/4N)(0\leq x,y\leq\tau)$. At the same time the coefficients of the polynomial $p(x,y)$ could be taken to be polynomials in $H = L_2[-1,1]$. Without any trouble one establishes that $q(x) = \int_0^\tau |x-y|^\alpha H(x-y)p(x,y)dy$ is continuous in x on $[0,\infty)$ with $\| q(x) \| \to 0$ as $x \to \infty$. Put $r(x) = \int_0^\tau H(x-y)\{f(y)-f(x)\}dy$. For $0\leq x_1, x_2<\infty$ we have the estimate

$$\| r(x_1) - r(x_2) \| \leq \| q(x_1) - q(x_2) \| + \varepsilon \leq d|x_1-x_2| + \tfrac{1}{2}\varepsilon \ ,$$

which, for finite τ, proves the continuity of r and thus of F. For infinite τ one proceeds differently. Singling out a compact interval $[a,b] \subseteq [0,\infty)$, one chooses finite σ such that

$\| \int_\sigma^\infty H(x-y)\{f(y) - f(x)\}dy \| < \frac{1}{4}\varepsilon$ for $a\leq x\leq b$. Then one proves the continuity of $s(x) = \int_0^\sigma H(x-y)\{f(y) - f(x)\}dy$ on $[0,\infty)$, analogously to the continuity proof for finite τ, and establishes the continuity of F on $[0,\infty)$.

Let us derive (8). Premultiplying (8) by $Q_\pm$ one gets the Cauchy problems

$$(TF_+)'(x) = -F_+(x) + Q_+f(x)(0<x<\tau),\ (TF_+)(0) = 0; \tag{9a}$$

$$(TF_-)'(x) = -F_-(x) + Q_-f(x)(0<x<\tau),\ (TF_-)(\tau) = 0. \tag{9b}$$

where $F_\pm(x) = Q_\pm F(x)$. Let us construct the solutions. Because in $\mathrm{Ran}\,Q_\pm$ the self-adjoint operator $\mp T^{-1}Q_\pm$ generates a bounded analytic semigroup[13] and $Q_\pm f:(0,\tau) \to H_\pm$ are bounded and uniformly Hölder continuous, the above Cauchy problems both have unique solutions TF_+ and TF_-[14] and these solutions have the form

$$TF_+(x) = \int_0^x e^{-(x-y)T^{-1}}Q_+f(y)dy,\ TF_-(x) = -\int_x^\tau e^{-(x-y)T^{-1}}Q_-f(y)dy.$$

Define $G(x) = TF_+(x) + TF_-(x)$; then one can write

$$G(x) = T[I-e^{-xT^{-1}}Q_+-e^{(\tau-x)T^{-1}}Q_-]f(x) + \int_0^\tau TH(x-y)\{f(y)-f(x)\}dy,$$

and thus $G(x) = TF(x)$ with $F(x)$ as in the statement of the lemma. Hence, TF is strongly differentiable on $(0,\tau)$ and satisfies Eq. (8).

Theorem 3. For finite (resp. infinite) τ let f be a bounded uniformly Hölder continuous function from $[0,\tau]$ (resp. $[0,\infty)$) into H. Then an essentially bounded (strongly measurable[15]) function $\psi:(0,\tau)\to H$ is a solution of the boundary value problem (5a)-(5b) (for finite τ) or (5a)-(5c) (for $\tau=\infty$), if and only if it is a solution of the vector-valued integral equation

$$\psi(x) - \xi\int_0^\tau c(y)H(x-y)B\psi(y)dy = \omega(x) \quad (0<x<\tau), \tag{10}$$

where the right-hand side is given by

$$\omega(x) = [e^{-xT^{-1}}Q_+ + e^{(\tau-x)T^{-1}}Q_-](\phi-f(x)) + f(x) + \\ + \int_0^\tau H(x-y)\{f(y) - f(x)\}dy. \tag{11}$$

Every such solution is continuous on $[0,\tau]$ for finite τ (resp. $[0,\infty)$ for $\tau = \infty$).

Proof. Note that

$$\omega(x) = [e^{-xT^{-1}}Q_+ + e^{(\tau-x)T^{-1}}Q_-]\phi + F(x),\ 0<x<\tau,$$

which implies the differential equation

$$(T\omega)'(x) = -\omega(x) + f(x)\ ,\quad 0<x<\tau.$$

The theorem can now be proved as is done for homogeneous media (i.e., $c(x) \equiv 1$).[16] The boundedness of $\xi c(y)$ on $(0,\tau)$ assures the validity of all estimates made in the proof.

Using $e(\mu) \equiv 1$ and putting $\zeta(x) = \langle\psi(x), e\rangle$ one immediately derives from (10) and (7) the scalar integral equation

$$\zeta(x) - \tfrac{1}{2}\xi \int_0^\tau c(y)Ei(x-y)\zeta(y)dy = \langle\omega(x), e\rangle (0<x<\tau). \qquad (12)$$

Conversely, once Eq. (12) is solved for $\zeta \in L_\infty(0,\tau)$, one puts

$$\psi(x) = \omega(x) + \tfrac{1}{2}\xi \int_0^\tau c(y)\zeta(y)H(x-y)e\ dy \quad (0<x<\tau) \qquad (13)$$

and gets an essentially bounded (strongly measurable) solution of Eq. (10). Hence, the boundary value problems (5a) - (5b) and and (5a) - (5c) have been reduced to the scalar integral equation (12). For homogeneous media ($c(x) \equiv 1$) Eq. (12) was discovered in radiative transfer applications;[17] inhomogeneous media versions appeared much later.[18]

Let us study Eq. (12) on the Banach space $BC(0,\tau)$. On this space the operator

$$(R\zeta)(x) = \frac{1}{2}\int_0^\tau c(y)Ei(x-y)\zeta(y)dy \qquad (14)$$

is bounded with norm

$$\|R\| \leq \frac{1}{2} \operatorname*{ess\ sup}_{0<y<\tau} |c(y)| . \int_{-\tau}^{\tau} Ei(z)dz, \qquad (15)$$

and Eq. (12) is written in the form

$$(I-\xi R)\zeta = \tilde{\omega}, \tag{16}$$

where $\tilde{\omega}(x) = \langle\omega(x), e\rangle (0<x<\tau)$. We shall come back to Eq. (16) in Section 3.

2. Operators leaving invariant a cone in a Banach space[9,10]

Let X be a real Banach space. A non-empty closed subset K in X is called a cone in X if it has the following properties: (1) $x,y \in K \Rightarrow x+y \in K$, (2) $x \in K$, $\lambda\geq 0 \Rightarrow \lambda x \in K$, (3) $x,-x \in K \Rightarrow x = 0$. A cone K in X is called reproducing if every $z \in X$ can be written in the form $z = x-y$ with $x,y \in K$. A cone K in X is called normal if there exists a constant M such that $x \in K$ and $y-x \in K$ imply that $0 \leq \|x\| \leq M \|y\|$.[19] The set of nonnegative functions in $L_p[-1,1]$ $(1\leq p\leq\infty)$ is a normal and reproducing cone. The set of nonnegative functions in $BC(0,\tau)$ is a normal and reproducing cone also.

A bounded linear operator R on X is called positive (with respect to K) if R leaves invariant the cone K. For $0 \neq e \in K$ one defines R to be e-bounded below (resp. above) if for all $0\neq x\in K$ there are $m \in \mathbb{N}$ and $\alpha >0$ (resp. $m \in \mathbb{N}$ and $\beta>0$) such that $R^m x-\alpha e \in K$ (resp. $\beta e- R^m x \in K$). If R is both e-bounded above and e-bounded below for the same $0 \neq e \in K$, for every $0 \neq x \in K$ there exist $m \in N$ and $\alpha, \beta>0$ such that $\{R^m x-\alpha e, \beta e- R^m x\} \subseteq K$, and in this case R is said to be e-positive.

When referring to spectral properties of R on real Banach spaces X, one complexifies X by embedding X into the complex space $\tilde{X} = X+iX$ and defining $\tilde{R}$ by $\tilde{R}(x+iy) = Rx + iRy$. The complex space $\tilde{X}$ will be a Banach space with respect to the norm

$$\|x+iy\| = \sup_{0\leq\theta<2\pi} \{\|x\| \cos\theta + \|y\| \sin\theta\} \quad (x,y\in X),$$

and in this way X is isometrically embedded in $\tilde{X}$ and $\|\tilde{R}\| = \|R\|$. The spectral properties of R, such as eigenvalues and spectral radius, are simply defined as the ones of $\tilde{R}$. In particular, for the spectral radius one has $r(R)\overset{\text{def.}}{=\!=}r(\tilde{R}) = \lim_{n\to\infty} \|R^n\|^{1/n}$.

Proposition 1.[20] Let K be a normal and reproducing cone in X and R a positive operator on X. Then the spectral radius $r(R)$ of R belongs to the spectrum of R. In particular, R does not have eigenvalues λ such that $|\lambda| > r(R)$.

Proposition 2.[9,10] Let K be a reproducing cone in X, and let R be a compact positive operator. Then either $r(R) = \{0\}$ or there exist $\lambda > 0$ and $0 \neq x \in K$ with $Rx = \lambda x$.

Proposition 3.[9,10] Let K be a reproducing cone in X, and let R be e-positive for some $0 \neq e \in K$. Assume that $\tilde{R}x = \lambda x$ for some $\lambda \in \mathbb{C}$ and $0 \neq x \in K$. Then

(1) the operator R is x-positive;

(2) λ is a positive eigenvalue of R of geometric multiplicity one;

(3) any other eigenvalue of $\tilde{R}$ does not correspond to eigenvectors in K;

(4) all eigenvalues of $\tilde{R}$ are contained in $\{\lambda\} \cup \{z \in C : 0 \; |z| < \lambda\}$.

Proposition 4.[10] Let K be a reproducing cone in X, and let R be e-positive for some $0 \neq e \in K$. If λ is a positive eigenvalue of R corresponding to an eigenvector in K, then the equation

$$(\mu - R)x = y \tag{17}$$

does not have any solutions x in K whenever $0 \neq y \in K$ and $\mu \leq \lambda$.

Under the conditions of Proposition 4, certainly $\lambda \leq r(R)$. For $\mu > r(R)$ the equation (17) has a unique solution x for $y \in X$, and

$$x = (\mu - R)^{-1}y = \sum_{n=0}^{\infty} \mu^{-(n+1)} R^n y.$$

Thus $x \in K$ whenever $y \in K$.

3. Proofs of Theorems 1 and 2.

In Section 1 we have written Eq. (12) in the form (16), where R is given by (14). On the real Banach space $BC(0,\tau)$ we consider the cone K_c of nonnegative functions. Directly from the definitions one shows K_c to be a reproducing and normal cone invariant under R. Since one could factorize $R = CM$, where $(C\zeta)(x) = \frac{1}{2}\int_0^\tau Ei(x-y)\zeta(y)dy$ is a convolution operator[21] and $(M\zeta)(x) = c(x)\zeta(x)$ is a multiplication by an L_∞-function, the operator $L = CM$ is bounded on $L_\infty(0,\tau)$ with range contained in $BC(0,\tau)$; for finite τ, C is a compact operator from $L_\infty(0,\tau)$ into $BC(0,\tau)$, and thus $L = CM$ is compact on $BC(0,\tau)$.

LEMMA 2. Let $c(x)>0$, except on a subset of $(0,\tau)$ of measure zero. Then, for $e(\mu) \equiv 1$ in K_c and finite τ, the operator R is e-positive.

Proof. Certainly, if $\zeta \in K_c$, then for $0<x<\tau$

$$0 \leq (R\zeta)(x) \leq \sup\{|(R\zeta)(z)|: 0<z<\tau\} \leq \| R \| \; \| \zeta \|,$$

and thus $\| R \| . \; \| \zeta \| \; e-\zeta \in K_c$, thereby proving that R is e-bounded above.

Take $0 \neq \zeta \in K_c$, and choose $\varepsilon > 0$ and $(a,b) \subseteq (0,\tau)$ such that $\zeta(x) \geq \varepsilon$ on (a,b). Because $\{x \in (0,\tau): c(x) = 0\}$ has measure zero, there exist $E \subset (a,b)$ with positive measure and $\delta > 0$ such that $c(x) \geq \delta$ almost everywhere on E. So $c(x)\zeta(x) \geq \delta \; \varepsilon > 0$ almost everywhere on E. Thus

$$(R\zeta)(x) \geq \frac{1}{2} \; \delta \; \varepsilon \int_E \; \mathrm{Ei}(x-y)dy = \frac{1}{2} \; \delta \; \varepsilon \int_{E-x} \mathrm{Ei}(z)dz \geq M > 0,$$

which for finite τ implies that R is e-bounded below.

If there were an interval (a,b) in $(0,\tau)$ on which $c(x) = 0$, then $R\zeta = 0$ for all ζ which have their support on $[a,b]$. So in this case R cannot be e-positive.

Proof of Theorem 1. Define N by $(N\zeta)(x) = \sqrt{c(x)} \; \zeta(x)$. Then $N^2 = M$ and $R = CN^2$. The compact operators R on $BC(0,\tau)$, R on $L_2(0,\tau)$ and NCN on $L_2(0,\tau)$ have the same non-zero eigenvalues.[22] As NCN is self-adjoint and non-zero, the spectral radius of NCN must be positive. Thus $r(R) > 0$. By Proposition 2, there exist $\lambda > 0$ and $0 \neq \eta \in K_c$ such that $R\eta = \lambda\eta$.

Because R is e-positive, Proposition 3 implies that λ is a simple eigenvalue of R and that R does not have any other eigenvectors in K_c than positive scalar multiples of η. Further, $\lambda = r(R) = r(NCN) = \| NCN \|$. Given τ, put $\xi_0(\tau) = \lambda^{-1}$.

First let $0<\xi<\xi_0(\tau)$ (τ fixed). Then $|\xi|<r(R)^{-1}$, and thus Eq. (16) has a unique solution ζ for every $\tilde{\omega}$. Proposition 4 implies that $\zeta \in K_c$ whenever $\tilde{\omega} \in K_c$. However, if in the boundary value problem (5a)-(5b) the incoming flux $\phi \in K$ and the internal source term $f(x) \in K(0<x<\tau)$, then (11) implies that the right-hand side $\omega(x)$ of the equivalent integral equation (10) assumes its values in K. By virtue of (12), $\tilde{\omega}(.) = \langle\omega(.), e\rangle = \int_{-1}^{+1} \omega(.,\mu)d\mu \in K_c$, and therefore $\zeta \in K_c$. Now (13) implies that ψ assumes its values in K. So we have established Part (a) of the theorem.

Let $\xi > \xi_0(\tau)$, and take $\phi \in K$ and $f(x) \in K(0<x<\tau)$. Then, by (11), ω assumes its values in K. If there would be a solution $\psi:(0,\tau) \to K$ of (5a)-(5b), then $\zeta(.) = <\psi(.), e> = \int_{-1}^{+1} \psi(.,\mu)d\mu \in K_c$. Now Proposition 4 implies that $\zeta \equiv 0$, $\hat{\omega} = <\omega(.), e> \equiv 0$. Eq. (13) implies $\psi = \omega$. Because of (11) and the fact that f and ω assume their values in K, one has

$$\int_0^1 e^{-x/\mu} \phi(\mu)d\mu = 0, \quad \int_{-1}^{0} e^{(\tau-x)/\mu} \phi(\mu)d\mu = 0, \int_{-1}^{+1} F(x,\mu)d\mu = 0;$$

from this one easily derives $\phi = 0$ and $f \equiv 0$. Then $\omega \equiv 0$, and thus $\psi \equiv 0$. This proves Part (c) of the theorem.

Let $\xi = \xi_0(\tau)$. Then η satisfies Eq. (12) with zero right-hand side. Therefore, according to (13),

$$\psi(x) = \frac{1}{2} \xi_0(\tau) \int_0^{\tau} c(y)\eta(y)H(x-y)edy \quad (0<x<\tau)$$

is a non-trivial solution of Eq. (10) with right-hand side $\omega \equiv 0$, which assumes its values in K. However, Theorem 3 implies that $\psi:(0,\tau) \to K$ is a non-trivial solution of Eqs. (5a)-(5b) with $\phi = 0$ and $f \equiv 0$. Now suppose, there is another non-trivial solution $\tilde{\psi}$, of Eqs. (5a)-(5b)(with $\phi = 0$, $f \equiv 0$); then this solution $\tilde{\psi}$ also satisfies Eqs. (10)(with $\omega \equiv 0$), and therefore, after proper multiplication by a positive constant, $\eta(x) = <\tilde{\psi}(x), e> = \int_{-1}^{+1} \tilde{\psi}(x,\mu)d\mu$ $(0<x<\tau)$. Thus (13) implies that $\tilde{\psi} = \psi$. Hence, $\psi:(0,\tau) \to K$ is unique up to multiplication by a positive constant. Further, for $\phi \in K$ and $f(x) \in K$ $(0<x<\tau)$ there cannot be a solution $\psi:(0,\tau) \to K$ of (5a)-(5b), unless $\phi = 0$, $f \equiv 0$ and $\psi \equiv 0$. The latter is proved in exactly the same way as for $\xi > \xi_0(\tau)$. Thus Part (b) has been proved.

It remains to prove the monotonicity and continuity of the map $\tau \mapsto \xi_0(\tau)$, and the existence of $\lim \xi_0(\tau)$ for $\tau \downarrow 0$ and $\tau \to \infty$. Fix $0<\tau<\infty$. Let $\psi:(0,\tau) \to K$ be a non-trivial solution of the vector-valued integral equation

$$\psi(x) - \xi_0(\tau)\int_0^{\tau} c(y)H(x-y) B\psi(y)dy = 0, \quad 0<x<\tau. \tag{18}$$

If $\psi(x_0) = 0$ for some $0<x_0<\tau$, then $c(y)H(x_0-y)B\psi(y) \in K$ implies that $c(y)H(x_0-y)B\psi(y) = 0$ for $0<y<\tau$. As $c(y) > 0$, except on a set of measure zero, one gets $<\psi(y), e> = 0$ for $0<y<\tau$ (see (4b) for the form of B). Then (13) implies that $\psi \equiv 0$, which is a contradiction. Thus $\psi(x)>0$, $0<x<\tau$. For $0<x<\tau$ one gets from (18):

$$\psi(x)-\xi_0(\tau)\int_0^\sigma c(y)H(x-y)B\psi(y)dy = \xi_0(\tau)\int_\sigma^\tau c(y)H(x-y)B\psi(y)dy, \tag{19}$$

where the right-hand side is non-zero for every $0<x<\sigma$.[23] As $\psi(x) \in K$ for $0<x<\sigma$, the critical and supercritical cases are excluded, and thus

$$\xi_0(\tau) < \xi_0(\sigma)$$

(non-critical case). This establishes the monotonicity of $\tau \mapsto \xi_0(\tau)$. If this function would not be continuous, there would be a jump discontinuity at τ_0. Put $\xi_+ = \lim_{\tau\downarrow\tau_0} \xi_0(\tau)$, $\xi_- = \lim_{\tau\uparrow\tau_0} \xi_0(\tau)$. Either $\xi_-\geq\xi_0(\tau)>\xi_+$ or $\xi_->\xi_0(\tau)\geq\xi_+$. In the former case take $\xi_-\geq\xi_0(\tau)>\xi>\xi_+$, in the latter case $\xi_->\xi>\xi_0(\tau)\geq\xi_+$. In both cases an argument as applied to prove monotonicity would lead to a contradiction. Thus $\tau \mapsto \xi_0(\tau)$ is continuous.

Finally, let us prove that $\lim_{\tau\to 0} \xi_0(\tau) = \infty$ and $\lim_{\tau\to\infty} \xi_0(\tau) = \xi_0(\infty)>0$. By virtue of (15), the norm of R, and thus its spectral radius $r(R)$ too, vanishes for $\tau \downarrow 0$, which shows that $\xi_0(\tau) = r(R)^{-1} \to \infty$ as $\tau \downarrow 0$. The same estimate (15) and the integrability of Ei on $(-\infty,\infty)$[24] show that for all finite τ

$$r(R) \leq \| R \| \leq \frac{1}{2}\| c \|_\infty \int_{-\infty}^{\infty} Ei(z)dz = \| c \|_\infty < \infty.$$

By the monotonicity of $\xi_0(\tau)^{-1} = r(R)$ as a function of τ, the existence of the limit $\lim_{\tau\to\infty} \xi_0(\tau) \overset{\text{def.}}{=} \xi_0(\infty)$ is clear, and $\xi_0(\infty) \geq \| c \|_\infty^{-1} > 0$.

<u>Proof of Theorem</u> 2. The non-conservative case $\xi<\xi_0(\infty)$ is immediate from $\xi_0(\infty) = r(R)^{-1}$. The supercritical case $\xi>\xi_0(\infty)$ can be dealt with as follows: Consider Eq. (10) with $\tau = \infty$, $\omega(x) \in K(0<x<\infty)$ and $\psi(x) \in K(0<x<\infty)$, and fix finite τ with $\xi>\xi_0(\tau)>\xi_0(\infty)$. Writing

$$\psi(x) - \xi \int_0^\tau c(y) H(x-y) B\psi(y)\,dy = \omega(x) \quad (0<x<\tau),$$

Theorem 1 implies that $\omega \equiv 0$ and $\psi \equiv 0$ on $(0,\tau)$. As this is the case for every finite τ with $\xi>\xi_0(\tau)>\xi_0(\infty)$, one has $\omega \equiv 0$ and $\psi \equiv 0$ on $(0,\infty)$, which establishes the theorem in the supercritical case.

The only situation not covered by Theorem 2 is the "conservative" half-space problem (5a)-(5c), where $\xi = \xi_0(\infty)$. This problem can be reduced to the integral equation

$$\zeta(x) - \frac{1}{2}\xi_0(\infty) \int_0^\infty c(y)\mathrm{Ei}(x-y)\zeta(y)\,dy = \tilde{\omega}(x), \quad 0<x<\infty, \tag{20}$$

where $\tilde{\omega}$, $\zeta \in BC(0,\infty)$. In homogeneous media ($c(y) \equiv 1$, $\xi_0(\infty) = 1$) and for $\tilde{\omega} = 0$ the above equation is the Schwarzschild-Milne integral equation,[25,17] which does not have non-trivial bounded solutions. For inhomogeneous media Eq. (20) with $\tilde{\omega} = 0$ has a solution $0 \neq \zeta \in BC(0,\infty)$ if and only if $\xi_0(\infty)$ is an eigenvalue of R.

Tezcan[7] reduced a criticality problem containing an anisotropy parameter β to a transport equation of the form

$$\mu \frac{\partial\psi'}{\partial x'} + \psi'(x',\mu') = \frac{1}{2} c' \int_{-1}^{+1} \psi'(x',\mu')\,d\mu', \quad -b<x'<b,$$

with boundary conditions

$$\psi'(-b,\mu) = 0 \quad (0\leq\mu\leq 1), \quad \psi'(b,\mu) = 0 \quad (-1\leq\mu\leq 0);$$

transforming $x'' = x' + b$ and $\tau = 2b$ one gets the boundary value problem (5a)-(5b) with $\xi = c'$, $c(x) \equiv 1$, $f \equiv 0$, $\phi \equiv 0$. Tezcans[7] parameters are related to the original collision ratio c, anisotropy parameter β, position (in units of neutron mean free path) x by

$$c' = c(1-\beta)/(1-c\beta), \quad x' = qx, \quad q = \sqrt{1-c^2\beta^2}.$$

Tezcans[7] finding, which he explained physically and backed up numerically, is that the critical thickness vanishes for $\beta\uparrow(1/c)$. In our notation,

$$\lim_{\beta\uparrow(1/c)} \tau_0(c')/\sqrt{1-c^2\beta^2} = 0, \tag{21}$$

where $\xi_0(\tau_0(c)) = c$ for fixed $c > 1$. For homogeneous media ($c(x) \equiv 1$) one has to prove that

$$\lim_{\tau\downarrow 0} \tau^{-2} r(R_\tau)^{26} = \lim_{\tau\downarrow 0}(\tau^2 \xi_0(\tau))^{-1} = \infty, \tag{22}$$

because this would imply $\lim_{y\to\infty} \sqrt{y}\ \tau_0(y) = 0$, and thus (21). As R_τ is a self-adjoint operator on $L_2(0,\tau)$ and the spectra of the convolution operator R_τ on $L_2(0,\tau)$ and $BC(0,\tau)$ coincide,[21] one has $r(R_\tau) = \| R_\tau \|_{L_2(0,\tau)} \geq \tau^{-\frac{1}{2}} \| R_\tau e \|_2$, where $e(x) \equiv 1$ has L_2-norm $\sqrt{\tau}$. One easily computes that $(R_\tau e)(x) = \frac{1}{2}\{1-Ei_2(x)\} + \frac{1}{2}\{1-Ei_2(\tau-x)\}^{27} \geq \frac{1}{2}\{1-Ei_2(\tau)\}$, $0<x<\tau$, and thus

$$\tau^{-2} r(R_\tau) \geq \tau^{-5/2} \| R_\tau e \|_2 \geq \frac{1-Ei_2(\tau)}{2\tau^2} = \frac{1}{2}\int_\tau^\infty \frac{1-e^{-\omega}}{\omega^2}\, d\omega.$$

However, since the right-hand side tends to infinity as $\tau\downarrow 0$, Eq. (22) is clear. Herewith we have established (21).

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(11) We call $Ei(x) = \int_1^\infty z^{-1}e^{-z|x|}dz$ the exponential integral function.

(12) It is easy to compute that $\|H(x)\| = \sup\{\mu^{-1}e^{-|x|/\mu}: 0<\mu\leq 1\}$, which equals $1/(ex)$ for $|x|\leq 1$ and $e^{-|x|}$ for $|x|\geq 1$.

(13) Analytic semigroups are discussed in: T. Kato, "Perturbation Theory for Linear Operators", Springer Verlag (1966). The solutions of the Cauchy problems (9a) and (9b) are based on Theorem IX 1.27 of this book.

(14) We make abuse of notation. Here writing $TF_\pm$ does not mean that T could be splitted off.

(15) Strong measurability is defined with respect to Lebesque measure. See Section VI. 31 in: A. C. Zaanen, "Integration", North-Holland, Amsterdam (1967).

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The equation has been studied in Refs.6.

(19) In Refs. 9 and 10, K is called normal if there exists $\delta > 0$ such that $x,y \in K$ and $\|x\|=\|y\|= 1$ imply $\|x+y\| \geq \delta$. The equivalence of the two definitions is due to I. A. Bakhtin and the proof can be found in Ref. 10.

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(22) See, for instance, Theorem 4.5 in: I. Gohberg, M. A. Kaashoek, D. C. Lay, J. Func. Anal. 28, 102 (1978). In fact, $(I-\lambda NCN)^{-1} = I+\lambda N(I-\lambda CN^2)^{-1}CN$ and $(I-\lambda CN^2)^{-1} = I+\lambda CN(I-\lambda NCN)^{-1}N$ imply the statement. Further, as NCN is self-adjoint, R has real eigenvalues only.

(23) Note that $c(y)H(x-y)B\psi(y) \in K$, $0<y<\sigma$. So $\int_\sigma^\tau c(y)H(x-y)B\psi(y)dy=0$ would imply $B\psi(y) = \langle\psi(y),\ e\rangle = 0$ $(0<y<\tau)$, and thus, by (13), $\psi \equiv 0$, which is a contradiction.

(24) $\int_{-\infty}^{+\infty} Ei(z)dz = 2\int_1^\infty z^{-2}dz = 2$. See also Ref. (11).

(25) E. Hopf, "Mathematical Problems of Radiative Equilibrium", Cambridge University Press, Cambridge (1934).

(26) Here we use the subscript τ of R, which we have suppressed hitherto.

(27) The function $Ei_2(x) = \int_1^\infty z^{-2}e^{-|x|z}dz$ has been described in the first chapter of: H.C. van de Hulst, "Multiple Light Scattering", Pergamon Press (1980), as well as in the appendix of: S. Chandrasekhar, "Radiative Transfer", Second revised edition, Dover (1960).