The One-Dimensional Inverse Scattering Problem for Nonhomogeneous Media with Discontinuous Wavespeed

TUNCAY AKTOSUN Department of Mathematics, North Dakota State University, Fargo, ND 58105

MARTIN KLAUS Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, VA 24061

CORNELIUS VAN DER MEE Department of Mathematics, University of Cagliari, Cagliari, Italy

ABSTRACT: We consider the inverse problem of reconstructing the wavespeed in a one-dimensional nonhomogeneous medium from appropriate scattering data. The wavespeed is allowed to have jump discontinuities and the medium may be subject to a nonhomogeneous external restoring force. In the frequency-domain this inverse problem leads to a Riemann-Hilbert problem and an associated singular integral equation. Under suitable conditions we prove that the singular integral equation is uniquely solvable and we discuss how its solution leads to the recovery of the wavespeed. We also show that certain characteristic properties of the wavespeed can be reconstructed more quickly, that is, without completely solving the inverse problem first. Some examples illustrating the reconstruction of the wavespeed are presented.

1 INTRODUCTION

In this lecture we consider an inverse scattering problem for the one-dimensional Schrödinger-type equation

$$\psi''(k,x)+k^2H(x)^2\psi(k,x)=Q(x)\,\psi(k,x),\quad -\infty<x<\infty, \tag{1.1}$$

which arises in electromagnetic and elastic wave scattering. For example, the wave equation

$$E_{xx} - \mu_0\,\epsilon(x)\,E_{tt} = 0, \quad -\infty < x < \infty, \tag{1.2}$$

where $E(x,t)$ is the electric field as a function of position and time, $\epsilon(x)$ is the permittivity, and μ_0 is the (constant) permeability, can be converted into an equation of the form (1.1) with $H(x) = \sqrt{\mu_0\,\epsilon(x)}$ and $Q(x) = 0$ by means of the substitution $E(x,t) = e^{ikt}\,\psi(k,x)$. Then $1/H(x)$ represents the position-dependent propagation speed. Similarly, the wave equation

$$[\rho(z)c(z)^2 u_z]_z = \rho(z)\,u_{tt}, \quad -\infty < z < \infty, \tag{1.3}$$

describing wave propagation in a one-dimensional nonhomogeneous elastic medium, can be transformed into (1.1) by means of the substitutions

$$u(z,t) = e^{ikt}\,\psi(k,x), \quad x = \int_0^z \frac{ds}{\rho(s)\,c(s)^2}. \tag{1.4}$$

Then $H(x) = \rho(z)\,c(z)$, where $z = z(x)$ is given by the second relation in (1.4), and again $Q(x) = 0$. In (1.3), $u(z,t)$ is the displacement at position z and time t, $\rho(z)$ is the density, and $c(z)$ is the wavespeed. For more information about (1.2) and (1.3) in the present context we refer the reader to Grinberg (1991a) and Krueger (1982). We have included the term with $Q(x)$ in (1.1) to account for possible external restoring forces acting on the medium and to achieve greater flexibility. The impact of $Q(x)$ on the inversion method is substantial, and it interesting to see how the special case $Q(x) = 0$ is embedded in the general case. The precise assumptions on $H(x)$ and $Q(x)$ will be stated below; we only mention at this point that $H(x)$ is a strictly positive function with at most a finite number of jump discontinuities and that the limits $\lim_{x\to\pm\infty} H(x) = H_\pm$ are assumed to exist and are strictly positive. The discontinuities of $H(x)$ represent abrupt changes in the material properties of the medium in which the wave propagates. The inverse scattering problem we are interested in here is that of recovering the function $H(x)$ from a suitable set of scattering data. The question of what constitutes appropriate scattering data is part of the problem and can only be answered after the procedure for solving the inverse problem has been set up. Certainly, as in the more familiar case of the inverse problem in quantum mechanics (Marchenko (1986)), in which $H(x) = 1$ and the potential $Q(x)$ is to be reconstructed from a reflection coefficient, and, if bound states are present, from the bound state energies and the norming constants, one also expects a reflection coefficient to be part of the scattering data here. This is true, but it turns out that knowing a reflection coefficient (assuming no bound states) is not always sufficient to reconstruct $H(x)$ uniquely. Sometimes it is also necessary to know the value of H_+ (or H_-); for example, this is the case when $Q(x) = 0$. Various authors have studied inverse scattering problems for differential equations with discontinuous coefficients; for example, Ware and Aki (1969), Razavy (1975), Krueger (1976, 1978, 1982), Sabatier (1988), and Grinberg (1991a,b). The work most directly related to ours is that of Grinberg, who, in the special case $Q(x) = 0$, developed a method to recover $H(x)$ using the solution of a singular integral equation. This will also be our strategy here, but when $Q(x) \neq 0$ the analysis of the problem becomes more involved and there are essential differences in the results as compared to the case $Q(x) = 0$.

In addition to solving the inverse problem we are also interested in questions regarding the practical applicability of the method. For example, it is natural to ask whether certain characteristic properties of $H(x)$ can be recovered more quickly, that is, without having to solve the inverse problem first. As it turns out, quantities that fall into this category include the number of discontinuities, certain ratios like $H(x_n - 0)/H(x_n + 0)$, where x_n is a point of discontinuity of $H(x)$, or integrals like $\int_0^{x_n} H(s)\,ds$, which, in the context of (1.2), represent the times (travel times) it takes the wave to travel from the origin to the discontinuity x_n. It turns out that information about such quantities can be extracted from the large-k asymptotics of the reflection and transmission coefficients.

This article is organized as follows. In Section 2 we summarize the properties of certain solutions of (1.1) that are relevant to the subsequent discussion, and of quantities related to them. In particular, we study the asymptotic behavior of these quantities for both small and large k. In Section 3 we formulate the inverse problem as a matrix Riemann-Hilbert problem and establish the unique solvability of the associated singular integral equation. The method presented here exploits the large-k asymptotics of the solutions to (1.1) and of the scattering data. There is an alternate approach using the small-k asymptotics which will only be reviewed briefly for comparison purposes. In Section 3 we also identify the scattering data and obtain a corresponding uniqueness result (Theorem 3.3). In Section 4 we present an algorithm to compute characteristic quantities associated with $H(x)$ directly from a reflection coefficient without solving the inverse problem first. Finally, in Section 5, we illustrate the inversion method by two examples. Throughout this lecture we shall assume that there are no bound states. The inverse problem with bound states has been studied in Aktosun et al. (1995a, Section 8); the inclusion of bound states here would take up too much space.

The lecture is based on recent work of ours (Aktosun et al., 1995a,b and 1996a,b), to which we will often refer for details, in particular for proofs and computations that are too long to be included here.

2. SCATTERING COEFFICIENTS AND THEIR ASYMPTOTICS

In this section we introduce the scattering coefficients and discuss their asymptotic properties in the limits $k \to 0$ and $k \to \infty$. We first list our assumptions on $H(x)$ and $Q(x)$:

(H1) $H(x)$ is strictly positive and piecewise continuous with jump discontinuities at x_n for $n = 1, \ldots, N$ such that $x_1 < \cdots < x_N$.

(H2) $H(x) \to H_\pm$ as $x \to \pm\infty$, where $H_\pm$ are positive constants.

(H3) $H(x)-H_\pm \in L^1(\mathbf{R}^\pm)$, where $\mathbf{R}^-=(-\infty,0)$ and $\mathbf{R}^+=(0,+\infty)$.
(H4) $H'(x)$ is absolutely continuous on (x_n,x_{n+1}) and $2H''(x)\,H(x)-3[H'(x)]^2 \in L_1^1(x_n,x_{n+1})$ for $n=0,\dots,N$, where $x_0=-\infty$ and $x_{N+1}=+\infty$.
(H5) $Q(x)$ is real-valued and belongs to $L_1^1(\mathbf{R})$.

Here $L_\beta^1(I)$ denotes the space of measurable functions $f(x)$ on I such that $\int_I (1+|x|)^\beta\,|f(x)|\,dx<\infty$.

We remark that (H4) will play a crucial role in our analysis of the large-k asymptotics of the scattering coefficients. Assumptions (H1)-(H5) are the same as those in Aktosun et al. (1996a). While (H5) is sufficient for the solution of the inverse problem, for technical reasons, some related results in Section 3 (e.g. Theorem 3.3) have so far only been established under the stronger condition $Q(x)\in L_2^1(\mathbf{R})$.

Associated with (1.1) are the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ which satisfy the boundary conditions

$$f_l(k,x)=\begin{cases} e^{ikH_+x}+o(1), & x\to+\infty,\\ \dfrac{1}{T_l(k)}e^{ikH_-x}+\dfrac{L(k)}{T_l(k)}e^{-ikH_-x}+o(1), & x\to-\infty,\end{cases} \tag{2.1}$$

$$f_r(k,x)=\begin{cases} \dfrac{1}{T_r(k)}e^{-ikH_+x}+\dfrac{R(k)}{T_r(k)}e^{ikH_+x}+o(1), & x\to+\infty,\\ e^{-ikH_-x}+o(1), & x\to-\infty.\end{cases} \tag{2.2}$$

In (2.1) and (2.2), $T_l(k)$ and $T_r(k)$ are the transmission coefficients from the left and from the right, respectively, and $L(k)$ and $R(k)$ are the reflection coefficients from the left and from the right, respectively. These coefficients will collectively be referred to as scattering coefficients. The physical picture behind (2.1) is this: Multiply (2.1) by $T_l(k)$ so that

$$T_l(k)\,f_l(k,x)=\begin{cases} T_l(k)\,e^{ikH_+x}+o(1), & x\to+\infty,\\ e^{ikH_-x}+L(k)\,e^{-ikH_-x}+o(1), & x\to-\infty.\end{cases} \tag{2.3}$$

Then (2.3) represents a wave e^{ikH_-x} sent in from $-\infty$ which, because it interacts with the nonhomogeneous medium, produces a reflected wave $L(k)\,e^{-ikH_-x}$ and a transmitted wave $T_l(k)\,e^{ikH_+x}$. The scattering coefficients are given in terms of the Jost solutions by

$$[f_l(k,x);f_r(k,x)]=-2ik\frac{H_+}{T_r(k)}=-2ik\frac{H_-}{T_l(k)},$$

$$[f_l(k,x);f_r(-k,x)]=2ikH_-\frac{L(k)}{T_l(k)}=-2ikH_+\frac{R(-k)}{T_r(-k)},$$

where $[f;g] = fg' - f'g$ denotes the Wronskian of f and g. Therefore $T_l(k)$ and $T_r(k)$ are related by

$$H_+ T_l(k) = H_- T_r(k). \tag{2.4}$$

Moreover, for $k \in \mathbf{R}$, using (2.1) and (2.2) we have

$$f_l(-k,x) = \overline{f_l(k,x)}, \quad f_r(-k,x) = \overline{f_r(k,x)},$$

$$T_r(k)L(-k) + R(k)T_r(-k) = 0, \tag{2.5}$$

$$T_r(k)\,T_l(-k) + L(k)\,L(-k) = T_r(-k)\,T_l(k) + R(k)\,R(-k) = 1. \tag{2.6}$$

In the following it will often be necessary to distinguish between two cases:
(i) The generic case: $f_l(0,x)$ and $f_r(0,x)$ are linearly independent.
(ii) The exceptional case: $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent and hence

$$f_l(0,x) = \gamma\, f_r(0,x) \tag{2.7}$$

for some nonzero real constant γ.

This division into a generic and exceptional case is governed solely by the properties of $Q(x)$ because $H(x)$ does not affect the solutions of (1.1) at $k = 0$. For example, if $Q(x) = 0$, then $f_l(0,x) = f_r(0,x) = 1$, and we are in the exceptional case. If $Q(x) \geq 0$ but $Q(x) \not\equiv 0$, we have the generic case, since then $f_l(0,x)$ grows linearly as $x \to -\infty$. This follows from the integral equation for $f_l(0,x)$ (see (3.27) below).

For reasons that will become clear later we introduce the "reduced scattering coefficients"

(2.8)
$$\tau(k) = \sqrt{\frac{H_+}{H_-}}\, T_l(k)e^{ikA} = \sqrt{\frac{H_-}{H_+}}\, T_r(k)e^{ikA},\ \rho(k) = R(k)e^{2ikA_+},\ \ell(k) = L(k)e^{2ikA_-},$$

where

$$A_\pm = \pm \int_0^{\pm\infty} [H_\pm - H(s)]\, ds, \quad A = A_- + A_+. \tag{2.9}$$

Then, by (2.4)-(2.6), the matrix

$$\sigma(k) = \begin{bmatrix} \tau(k) & \rho(k) \\ \ell(k) & \tau(k) \end{bmatrix}, \tag{2.10}$$

is unitary, and it will be called the "reduced scattering matrix".

In our first theorem we collect some results about the reduced scattering coefficients, in particular their small-k behavior. We let $\mathbf{C}^+$ denote the upper-half complex plane and $\overline{\mathbf{C}^+}$ its closure, $\overline{\mathbf{C}^+} = \mathbf{C}^+ \cup \mathbf{R}$. Similarly, $\mathbf{C}^-$ denotes the lower-half complex plane and $\overline{\mathbf{C}^-} = \mathbf{C}^- \cup \mathbf{R}$.

THEOREM 2.1 Suppose that (H1)-H(5) hold and that there are no bound states. Then:

(i) $\tau(k)$ is analytic in $\mathbf{C}^+$ and continuous on $\overline{\mathbf{C}^+}$.

(ii) In the generic case

$$\tau(k) = ick + o(k), \quad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{2.11}$$

where c is a nonzero real constant, and

$$\rho(k) = -1 + o(1), \quad \ell(k) = -1 + o(1), \quad k \to 0 \text{ in } \mathbf{R}.$$

Furthermore, $\rho(k)$ and $\ell(k)$ are continuous for $k \in \mathbf{R}$, and $|\rho(k)| = |\ell(k)| < 1$ for $k \neq 0$.

(iii) In the exceptional case

$$\tau(k) = \frac{2\sqrt{H_- H_+}\,\gamma}{H_-\gamma^2 + H_+} + o(1), \quad k \to 0 \text{ in } \overline{\mathbf{C}^+},$$

$$\rho(k) = \frac{H_+ - H_-\gamma^2}{H_-\gamma^2 + H_+} + o(1), \quad k \to 0 \text{ in } \mathbf{R}, \tag{2.12}$$

$$\ell(k) = \frac{H_-\gamma^2 - H_+}{H_-\gamma^2 + H_+} + o(1), \quad k \to 0 \text{ in } \mathbf{R},$$

where γ is the constant defined in (2.7). Furthermore, $|\rho(k)| = |\ell(k)| < 1$ for all $k \in \mathbf{R}$.

PROOF: See Theorems 4.1 and 4.2 in Aktosun et al. (1995a), putting $\alpha = 0$ in Theorem 4.2. ∎

Note that (i) holds because we have excluded bound states. If bound states are present, then each bound state gives rise to a simple pole of $\tau(k)$ lying on the positive imaginary axis.

The constant c in (2.11) is given by

$$c = -\frac{2\sqrt{H_+ H_-}}{[f_l(0,x); f_r(0,x)]},$$

so that

$$H_+ = \frac{c^2\,[f_l(0,x); f_r(0,x)]^2}{4H_-}.$$

Since $c = \lim_{k\to 0} \frac{\tau(k)}{ik}$ and $|\tau(k)|^2 + |\rho(k)|^2 = 1$, we also have

$$c^2 = \lim_{k\to 0} \frac{1 - |\rho(k)|^2}{k^2} = \lim_{k\to 0} \frac{1 - |R(k)|^2}{k^2},$$

from which we see that in the generic case H_- is determined by H_+, $Q(x)$, and $R(k)$. In the exceptional case we infer from (2.8) and (2.12) that

$$H_+ = \frac{\gamma^2[1 + R(0)]}{1 - R(0)}\,H_-,$$

so that also in this case H_- is determined by H_+, $Q(x)$, and $R(k)$.

Next we consider the asymptotic behavior of the reduced scattering coefficients as $k \to \infty$. To analyze this limit we perform the Liouville transformation

$$\psi(k,x) = \frac{1}{\sqrt{H(x)}}\phi(k,y), \quad y = y(x) = \int_0^x H(s)\,ds, \tag{2.13}$$

on (1.1). As a result, we obtain

$$\frac{d^2\phi(k,y)}{dy^2} + k^2\phi(k,y) = V(y)\,\phi(k,y), \quad y \in \mathbf{R} \setminus \{y_1, \ldots, y_N\}, \tag{2.14}$$

where $y_j = y(x_j)$ and

$$V(y) = \frac{H''(x)}{2H(x)^3} - \frac{3}{4}\frac{H'(x)^2}{H(x)^4} + \frac{Q(x)}{H(x)^2}. \tag{2.15}$$

Since $H(x)$ has jump discontinuities at $x_1, \dots, x_N$, the function $V(y)$ is undefined at y_j. From the continuity of $\psi(k,x)$ and $\psi'(k,x)$ at each x_j it follows that $\phi(k,y)$ and $d\phi(k,y)/dy$ satisfy the following (internal) boundary conditions at y_j :

$$\begin{aligned} \phi(k,y_j-0) &= \sqrt{q_j}\,\phi(k,y_j+0), \\ \frac{d\phi(k,y_j-0)}{dy} &= \nu_j\,\phi(k,y_j+0) + \frac{1}{\sqrt{q_j}}\,\frac{d\phi(k,y_j+0)}{dy}, \end{aligned} \tag{2.16}$$

where we have defined

$$q_j = \frac{H(x_j-0)}{H(x_j+0)}, \quad \nu_j = \frac{1}{2\sqrt{H(x_j-0)\,H(x_j+0)}}\left[\frac{H'(x_j-0)}{H(x_j-0)} - \frac{H'(x_j+0)}{H(x_j+0)}\right]. \tag{2.17}$$

It is straightforward to check that the boundary conditions (2.16) are self-adjoint. So we can think of (2.14) as a Schrödinger equation with potential $V(y)$ given by (2.15) for $y \in \mathbf{R} \setminus \{y_1, \dots, y_N\}$ and supplemented by the boundary conditions (2.16) at the points y_j. In Aktosun et al. (1996b, Prop. 2.2) it was shown that the scattering matrix associated with the potential $V(y)$ is just the reduced scattering matrix $\sigma(k)$ defined in (2.10). For $j = 0, \dots, N$, let

$$V_{j,j+1}(y) = \begin{cases} V(y), & y \in (y_j, y_{j+1}), \\ 0, & \text{elsewhere}, \end{cases}$$

where we have defined $y_0 = -\infty$ and $y_{N+1} = +\infty$. Because of hypotheses (H4) and (H5) it follows that $V_{j,j+1}(y) \in L^1_1(\mathbf{R})$, where the finiteness of the first moment is effective only on the semi-infinite intervals (y_0, y_1) and (y_N, y_{N+1}).

Our next goal is to express the reduced scattering coefficients in terms of the scattering coefficients for the potentials $V_{j,j+1}(y)$ and quantities associated with the boundary conditions (2.16). Let $t_{j,j+1}(k)$, $r_{j,j+1}(k)$, and $l_{j,j+1}(k)$ denote the scattering coefficients for the potential $V_{j,j+1}(y)$ and define

$$\Lambda(k) = \begin{bmatrix} \dfrac{1}{\tau(k)} & -\dfrac{\rho(k)}{\tau(k)} \\ \dfrac{\ell(k)}{\tau(k)} & \dfrac{1}{\tau(-k)} \end{bmatrix}, \quad \Lambda_{j,j+1}(k) = \begin{bmatrix} \dfrac{1}{t_{j,j+1}(k)} & -\dfrac{r_{j,j+1}(k)}{t_{j,j+1}(k)} \\ \dfrac{l_{j,j+1}(k)}{t_{j,j+1}(k)} & \dfrac{1}{t_{j,j+1}(-k)} \end{bmatrix}, \quad j = 0, \dots, N, \tag{2.18}$$

$$F_j(k) = \begin{bmatrix} \alpha_j + \dfrac{\nu_j}{2ik} & \left(\beta_j + \dfrac{\nu_j}{2ik}\right) e^{-2iky_j} \\ \left(\beta_j - \dfrac{\nu_j}{2ik}\right) e^{2iky_j} & \alpha_j - \dfrac{\nu_j}{2ik} \end{bmatrix}, \quad j = 1, \dots, N, \tag{2.19}$$

where

$$(2.20) \qquad \alpha_j = \frac{1}{2}\left(\sqrt{q_j} + \frac{1}{\sqrt{q_j}}\right), \quad \beta_j = \frac{1}{2}\left(\sqrt{q_j} - \frac{1}{\sqrt{q_j}}\right).$$

Note that $\beta_j = 0$ if and only if $H(x)$ is continuous at x_j, and that $\nu_j = 0$ if and only if $H'(x)/H(x)$ is continuous at x_j. In Aktosun et al. (1996a, Theorem 4.1) it was shown that $\Lambda(k)$ can be factored as

$$(2.21) \qquad \Lambda = \Lambda_{0,1} F_1 \Lambda_{1,2} F_2 \Lambda_{2,3} \cdots F_N \Lambda_{N,N+1}.$$

For large k we have (Deift and Trubowitz, 1979)

$$(2.22) \qquad t_{j,j+1}(k) = 1 + O(1/k), \quad k \to \infty \text{ in } \overline{\mathbf{C}^+},$$

$$(2.23) \qquad r_{j,j+1}(k) = o(1/k), \quad l_{j,j+1}(k) = o(1/k), \quad k \to \pm\infty,$$

and thus, by (2.18) and (2.21)-(2.23),

$$(2.24) \qquad \Lambda(k) = F_1 \cdots F_N + O(1/k), \quad k \to \pm\infty.$$

Let

$$(2.25) \qquad E(k, x_j) = \begin{bmatrix} \alpha_j & \beta_j e^{-2iky_j} \\ \beta_j e^{2iky_j} & \alpha_j \end{bmatrix},$$

$$(2.26) \qquad \begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix} = \prod_{j=1}^{N} E(k, x_j).$$

Note that $\det E(k, x_j) = 1$ and hence

$$(2.27) \qquad |a(k)|^2 - |b(k)|^2 = 1, \quad k \in \mathbf{R}.$$

From (2.19) and (2.25) we conclude that $F_j(k) = E(k, x_j) + O(1/k)$ as $k \to \pm\infty$, and hence, by (2.24)-(2.26), we obtain

$$\Lambda(k) = \begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix} + O(1/k). \tag{2.28}$$

Let AP^W denote the algebra of all complex-valued functions $f(k)$ on $\mathbf{R}$ which are of the form $f(k) = \sum_{j=-\infty}^{\infty} f_j e^{ik\lambda_j}$, where $f_j \in \mathbf{C}$, $\lambda_j \in \mathbf{R}$ for all j, and $\sum_j |f_j| < \infty$. Note that the closure of AP^W in $L^\infty(\mathbf{R})$ is the algebra of almost periodic functions.

THEOREM 2.2 Suppose that (H1)-(H5) hold and that there are no bound states. Then:

(i) $\dfrac{1}{\tau(k)} = a(k) + O(1/k), \quad k \to \infty \text{ in } \overline{\mathbf{C}^+}.$

(ii) $\rho(k) = \dfrac{-b(k)}{a(k)} + O(1/k), \quad k \to \pm\infty.$

(iii) $a(k)$, $b(k)$, $\dfrac{1}{a(k)}$, and $\dfrac{b(k)}{a(k)}$ belong to AP^W.

(iv) $|\tau(k)| \leq 1$ and $\limsup_{k \to \pm\infty} |\rho(k)| < 1$.

This theorem is a combination of results from Aktosun et al. (1995a, Theorems 4.4 and 4.5) and Aktosun et al. (1996a, Prop. 2.1). Here we only comment briefly on the various statements. First, (i) and (ii) follow directly from (2.18), (2.24), and (2.28). From (2.25) and (2.26) we also see that $a(k)$ and $b(k)$ are exponential polynomials. They have certain characteristics that will be exploited in Section 4. The proof of (iii) relies on the fact that $a(k)$ is bounded away from zero, in particular, $|a(k)| \geq 1$ in $\overline{\mathbf{C}^+}$ by Prop. 4.3 in Aktosun et al. (1995a). The latter property is a consequence of the fact that $1/a(k)$ can be interpreted as the transmission coefficient for an equation of the form (1.1) with $Q(x) = 0$ and $H(x)$ piecewise constant. Note that $a(k)$ and $b(k)$ only contain information about $H(x)$ through the constants q_j defined in (2.17) and the values y_j, and are completely independent of $Q(x)$. Since the invertible elements of AP^W are exactly those elements of AP^W that are invertible in $L^\infty(\mathbf{R})$, $a(k)$ is invertible in AP^W. The first inequality in (iv) follows from $|a(k)| \geq 1$, (i), and the maximum modulus principle. The second inequality is a consequence of (2.27), the boundedness of $a(k)$, i.e. $|a(k)| \leq C$, which together imply $|b(k)|/|a(k)| \leq \sqrt{1 - C^{-2}} < 1$, and (ii).

3. RIEMANN-HILBERT PROBLEM AND A SINGULAR INTEGRAL EQUATION

In this section we formulate the inverse problem as a Riemann-Hilbert problem which we then convert into a singular integral equation. The solution of the integral equation will lead to an implicit equation for $y(x)$ so that $H(x)$ can be obtained by differentiation. The method employed here follows Aktosun et al. (1996a) and uses in a crucial way the large-k asymptotics of the scattering coefficients. We also establish the connection with an alternate method based on the small-k asymptotics (Aktosun et al., 1995a and 1996b) and, as a result, are able to identify the scattering data needed to recover $H(x)$ uniquely.

We introduce two functions $Z_l(k,y)$ and $Z_r(k,y)$ by

(3.1)
$$f_l(k,x)=\sqrt{\frac{H_+}{H(x)}}e^{iky+ikA_+}Z_l(k,y),\quad f_r(k,x)=\sqrt{\frac{H_-}{H(x)}}e^{-iky+ikA_-}Z_r(k,y),$$

where $y=y(x)$ was defined in (2.13), and $A_\pm$ are the constants defined in (2.9). Note that $A_\pm$ also appear in the expansions

$$\begin{aligned} y&=H_+x-A_++o(1),\quad x\to+\infty,\\ y&=H_-x+A_-+o(1),\quad x\to-\infty. \end{aligned} \tag{3.2}$$

The functions $e^{iky}Z_l(k,y)$ and $e^{-iky}Z_r(k,y)$ are the Jost solutions from the left and from the right, respectively, associated with (2.14). It was shown in Aktosun et al. (1996a) that for each fixed $y\in\mathbf{R}\setminus\{y_1,\dots,y_N\}$

$$Z_l(k,y)=J_l(k,y)+O(1/k),\quad Z_r(k,y)=J_r(k,y)+O(1/k),\quad k\to\infty \text{ in } \overline{\mathbf{C}^+}, \tag{3.3}$$

where

(3.4)
$$J_l(k,y)=\begin{cases}\begin{bmatrix}1 & e^{-2iky}\end{bmatrix}\left(\prod_{n=j+1}^{N}E(k,x_n)\right)\begin{bmatrix}1\\0\end{bmatrix}, & y\in(y_j,y_{j+1}),\ j=0,\dots,N-1,\\ 1,\quad y\in(y_N,+\infty),\end{cases}$$

$$J_r(k,y)=\begin{cases}\begin{bmatrix}e^{2iky} & -1\end{bmatrix}\left(\prod_{n=j}^{1}E(k,x_n)\right)\begin{bmatrix}0\\-1\end{bmatrix}, & y\in(y_j,y_{j+1}),\ j=1,\dots,N,\\ 1,\quad y\in(-\infty,y_1).\end{cases}$$

Here by $\prod_{n=j}^{1}E(k,x_n)$ we mean the matrix product $E(k,x_j)\cdots E(k,x_1)$.

Recall that the Hardy spaces $\mathbf{H}^p_\pm(\mathbf{R})$ $(1 < p < \infty)$ are defined as the spaces of all functions $f(k)$ that are analytic in $\mathbf{C}^\pm$ and satisfy $\sup_{\epsilon>0} \int_{-\infty}^{\infty} |f(k \pm i\epsilon)|^p \, dk < \infty$.

THEOREM 3.1 For each fixed $y \in \mathbf{R} \setminus \{y_1, \ldots, y_N\}$, the functions $Z_l(k,y) - J_l(k,y)$ and $Z_r(k,y) - J_r(k,y)$ belong to the Hardy space $\mathbf{H}^2_+(\mathbf{R})$.
PROOF: The Jost solutions $f_l(k,x)$ and $f_r(k,x)$ are analytic in $\mathbf{C}^+$, and hence, by (3.1), this also holds for the functions $Z_l(k,y)$ and $Z_r(k,y)$. Then the assertion follows from (3.3). ∎

From (2.1), (2.2), and (3.1) it follows that for each fixed $y \in \mathbf{R} \setminus \{y_1, \ldots, y_N\}$,

$$\begin{bmatrix} Z_l(-k,y) \\ Z_r(-k,y) \end{bmatrix} = \begin{bmatrix} \tau(k) & -\rho(k)e^{2iky} \\ -\ell(k)e^{-2iky} & \tau(k) \end{bmatrix} \begin{bmatrix} Z_r(k,y) \\ Z_l(k,y) \end{bmatrix}, \qquad k \in \mathbf{R}. \tag{3.5}$$

This equation constitutes a problem of Riemann-Hilbert type for the vector $\mathbf{Z}(k,y) = [\,Z_l(k,y) \quad Z_r(k,y)\,]^T$, where the superscript T denotes the transpose. Note that (3.5) relates the boundary values on the real axis of two vector functions, $\mathbf{Z}(-k,y)$ being analytic in $\mathbf{C}^-$ and $\mathbf{Z}(k,y)$ being analytic in $\mathbf{C}^+$. However, in view of (3.3) and in contrast to more familiar problems of this type, the functions $Z_l(k,y)$ and $Z_r(k,y)$ do not approach constant limits as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Our next step is to recast (3.5) as a singular integral equation. First, by using Theorem 2.2 and (3.3), we obtain

$$\begin{bmatrix} J_l(-k,y) \\ J_r(-k,y) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a(k)} & \dfrac{b(k)}{a(k)}e^{2iky} \\ -\dfrac{b(-k)}{a(k)}e^{-2iky} & \dfrac{1}{a(k)} \end{bmatrix} \begin{bmatrix} J_r(k,y) \\ J_l(k,y) \end{bmatrix}, \qquad k \in \mathbf{R}. \tag{3.6}$$

Subtracting (3.6) from (3.5) gives

(3.7)
$$\begin{aligned} Z_l(-k,y) - J_l(-k,y) = &\left[\tau(k) - \frac{1}{a(k)}\right] Z_r(k,y) + \frac{1}{a(k)}\left[Z_r(k,y) - J_r(k,y)\right] \\ &- \rho(k)e^{2iky}\left[Z_l(k,y) - J_l(k,y)\right] - \left[\rho(k) + \frac{b(k)}{a(k)}\right] e^{2iky} J_l(k,y). \end{aligned}$$

There is a similar equation for $Z_r(-k,y) - J_r(-k,y)$ which could be used in place of (3.7). From now on we will only work with the solutions from the left. We see that the left side of (3.7) belongs to $\mathbf{H}^2_-(\mathbf{R})$, the first two terms on the right belong to $\mathbf{H}^2_+(\mathbf{R})$, and the last two terms belong to $L^2(\mathbf{R})$. Let $\Pi_\pm$ denote the orthogonal projections from $L^2(\mathbf{R})$ onto $\mathbf{H}^2_\pm(\mathbf{R})$, i.e.

$$(\Pi_\pm f)(k) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(s)}{s-k\mp i0}\,ds,$$

and define

$$X_l(k,y) = Z_l(-k,y) - J_l(-k,y). \tag{3.8}$$

Applying the projection Π_- to both sides of (3.7) we obtain

$$X_l(\cdot,y) + \Pi_-\left(\rho(\cdot)\,e^{2i(\cdot)y}\mathcal{J}X_l(\cdot,y)\right) = -\Pi_-\left(\left[\rho(\cdot)+\frac{b(\cdot)}{a(\cdot)}\right]e^{2i(\cdot)y}J_l(\cdot,y)\right), \tag{3.9}$$

where $(\mathcal{J}f)(k) = f(-k)$. Defining the singular integral operator $\mathcal{O}_l$ by

$$(\mathcal{O}_l X)(k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{\rho(-s)\,e^{-2isy}}{s+k-i0}\,X(s)\,ds, \tag{3.10}$$

we can write (3.9) as

$$X_l(k,y) + (\mathcal{O}_l X_l)(k,y) = P_l(k,y), \quad k\in\mathbf{R}, \tag{3.11}$$

where

$$P_l(k,y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{e^{2isy}}{s-k+i0}\left[\rho(s)+\frac{b(s)}{a(s)}\right]J_l(s,y)\,ds. \tag{3.12}$$

THEOREM 3.2 The singular integral equation (3.11) has a unique solution $X_l(\cdot,y)\in\mathbf{H}^2_-(\mathbf{R})$ which can be obtained by iteration.
PROOF: In Aktosun et al. (1995a, Theorem 7.1) it is shown that $\|\mathcal{O}_l\| < 1$ on $\mathbf{H}^2_-(\mathbf{R})$. ∎

We remark that in the exceptional case we have $||\rho||_\infty = \sup_{k\in\mathbf{R}} |\rho(k)| < 1$ which immediately implies $||\mathcal{O}_l|| < 1$ in $\mathbf{H}^2_-(\mathbf{R})$. In the generic case the same conclusion holds true, but the argument is more involved, since $\rho(0) = -1$ and hence $||\rho||_\infty = 1$.

In practice it is convenient to transform (3.11) into a Marchenko-type integral equation by means of a Fourier transform. We indicate the main steps of this procedure. First, since $b(k)/a(k) \in \mathrm{AP}^W$, we can write

$$\frac{b(k)}{a(k)} = -\sum_s \gamma_s e^{ikb_s}, \tag{3.13}$$

where the b_s are distinct real numbers and the γ_s are real constants satisfying $\sum_s |\gamma_s| < \infty$. Then, since $\rho(k) + b(k)/a(k) \in L^2(\mathbf{R})$ by Theorem 2.2 (ii), we have that

$$\rho(k) = \sum_s \gamma_s e^{ikb_s} + \int_{-\infty}^{\infty} e^{ikz} \varrho(z)\, dz, \tag{3.14}$$

where $\varrho(z) \in L^2(\mathbf{R})$. The symmetry relation $F(-k) = \overline{F(k)}$ $(k \in \mathbf{R})$ which is valid for $\rho(k)$, $a(k)$, and $b(k)$, implies that $\varrho(z)$ is real-valued. We now define the Fourier transform $\mathcal{F}$ and its inverse by

$$(\mathcal{F}g)(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikt}\, g(k)\, dk, \quad (\mathcal{F}^{-1}h)(k) = \int_{-\infty}^{\infty} e^{-ikt}\, h(t)\, dt.$$

Since $X_l(\cdot, y)$ and $P_l(\cdot, y)$ belong to $\mathbf{H}^2_-(\mathbf{R})$, their Fourier transforms are supported on the positive half line and so we put

$$X_l(k, y) = \int_0^{\infty} e^{-ikt}\, h_l(t, y)\, dt, \quad P_l(k, y) = \int_0^{\infty} e^{-ikt}\, h_{l,0}(t, y)\, dt, \tag{3.15}$$

where $h_l(\cdot, y), h_{l,0}(\cdot, y) \in L^2(\mathbf{R}^+)$. By (3.4), $J_l(k, y)$ is of the form

$$J_l(k, y) = \sum_s \omega_s(y) e^{ik\varsigma_s(y)}, \tag{3.16}$$

where, in each interval (y_j, y_{j+1}), $\omega_s(y)$ is a constant and $\varsigma_s(y)$ is a linear function of y. Thus, from (3.12) and (3.15) we obtain

$$h_{l,0}(t,y) = -\sum_s \omega_s(y)\, \varrho(-t-2y-\varsigma_s(y)), \quad t \geq 0. \tag{3.17}$$

Taking the Fourier transform of (3.11) and using (3.14) and (3.17) yields

$$\begin{aligned} h_l(t,y) &+ \sum_{\{s:b_s<-t-2y\}} \gamma_s\, h_l(-t-2y-b_s,y) \\ &+ \int_0^\infty \varrho(-s-t-2y)\, h_l(s,y)\, ds = h_{l,0}(t,y), \quad t \geq 0. \end{aligned} \tag{3.18}$$

This is the analog of the Marchenko integral equation for the ordinary Schrödinger equation (see Marchenko (1986)). It allows us to obtain $X_l(k,y)$ uniquely from $\rho(k)$. Of course, eventually we want to use $R(k)$ as input rather than $\rho(k)$. The necessary modifications will be discussed later. For the moment we assume that $\rho(k)$ is known and proceed with the construction of $H(x)$ from $\rho(k)$. Assuming $X_l(k,y)$ and $J_l(k,y)$ are known, from (3.8) we obtain

$$Z_l(0,y) = X_l(0,y) + J_l(0,y),$$

and then, using (3.1) and (2.13), we deduce that

$$\frac{dy}{Z_l(0,y)^2} = H_+ \frac{dx}{f_l(0,x)^2}, \tag{3.19}$$

so that the equation for $y(x)$ becomes

$$\int_0^y \frac{dz}{Z_l(0,z)^2} = H_+ \int_0^x \frac{dz}{f_l(0,z)^2}. \tag{3.20}$$

From $y(x)$ we obtain $H(x)$ by differentiation. We remark that $J_l(0,y)$ is piecewise constant; explicitly

$$J_l(0,y) = \sqrt{q_{j+1}} \cdots \sqrt{q_N}, \quad y \in (y_j, y_{j+1}), \quad j = 0, \ldots, N-1,$$

and $J_l(0,y) = 1$ for $y > y_N$. This follows from (3.4), (2.20), (2.25), and the fact that when $k = 0$ the matrices $E(0,x_j)$ have eigenvectors $[1 \;\; 1]^T$ and $[1 \;\; -1]^T$

with corresponding eigenvalues $\sqrt{q_j}$ and $1/\sqrt{q_j}$, respectively. In the generic case, $f_l(0,x)^2$ grows proportional to x^2 as $x \to -\infty$, and so, taking $x \to -\infty$ in (3.20) gives

$$H_+ = \frac{\int_{-\infty}^0 dz/Z_l(0,z)^2}{\int_{-\infty}^0 dx/f_l(0,x)^2}. \tag{3.21}$$

Therefore, in the generic case, H_+ is determined by $\rho(k)$. In the exceptional case, H_+ is a free parameter. The precise meaning of this will be explained below when we discuss the recovery of $H(x)$ from $R(k)$.

We now describe the reconstruction of $H(x)$ from $R(k)$. First, put $\rho(k) = R(k)\, e^{2ikA_+}$, regarding A_+ as a parameter. Let $X_{l,1}(k,y)$ and $J_{l,1}(k,y)$ denote the solution of (3.11) and the function given by (3.4), respectively, if in (3.10) and (3.12) we replace $\rho(k)$ by $R(k)$ and change $a(k)$ and $b(k)$ accordingly. Then $X_l(k,y)$ and $J_l(k,y)$ corresponding to $\rho(k) = R(k)\, e^{2ikA_+}$ are given by $X_l(k,y) = X_{l,1}(k,y+A_+)$ and $J_l(k,y) = J_{l,1}(k,y+A_+)$. In analogy to (3.8) let

$$Z_{l,1}(-k,y) = X_{l,1}(k,y) + J_{l,1}(-k,y) = Z_l(-k,y-A_+). \tag{3.22}$$

Inserting (3.22) in (3.20) yields

$$\int_{A_+}^{y+A_+} \frac{dz}{Z_{l,1}(0,z)^2} = H_+ \int_0^x \frac{dz}{f_l(0,z)^2}. \tag{3.23}$$

In the generic case we can let $x \to -\infty$ and $y \to -\infty$, so that

$$\int_{-\infty}^{A_+} \frac{dz}{Z_{l,1}(0,z)^2} = H_+ \int_{-\infty}^0 \frac{dz}{f_l(0,z)^2}. \tag{3.24}$$

This equation gives a relation between A_+ and H_+ which, in view of (3.22), is equivalent to (3.21). In the generic case, by adding (3.23) and (3.24), we obtain

$$\int_{-\infty}^{y+A_+} \frac{dz}{Z_{l,1}(0,z)^2} = H_+ \int_{-\infty}^x \frac{dz}{f_l(0,z)^2}. \tag{3.25}$$

In the exceptional case the relation between A_+ and H_+ is less direct. In fact, there are difficulties in finding this relation if $Q(x)$ is only assumed to be in $L^1_1(\mathbf{R})$. To see this, let us try to take the limit as $x \to +\infty$ in (3.23). First we write (3.23) as

$$y + \int_{A_+}^{y+A_+} \frac{1 - Z_{l,1}(0,z)^2}{Z_{l,1}(0,z)^2}\, dz = H_+ x + H_+ \int_0^x \frac{1 - f_l(0,z)^2}{f_l(0,z)^2}\, dz.$$

Substituting the first relation of (3.2) for y in the above equation yields

(3.26)
$$A_+ = -H_+ \int_0^x \frac{1 - f_l(0,z)^2}{f_l(0,z)^2}\, dz + \int_{A_+}^{y+A_+} \frac{1 - Z_{l,1}(0,z)^2}{Z_{l,1}(0,z)^2}\, dz + o(1), \quad x \to +\infty.$$

From this we see that the right-hand side must converge as $x \to +\infty$. However, we cannot conclude that each integral on the right must converge separately. In fact, this conclusion is generally false. For example, if $Q(x) = (1 + |x|)^{-2-\epsilon}$ with $0 < \epsilon < 1$, then $Q(x) \in L^1_1(\mathbf{R})$ but $Q(x) \notin L^1_2(\mathbf{R})$, and using the integral equation

$$f_l(0,x) = 1 + \int_x^\infty (z - x)\, Q(z)\, f_l(0,z)\, dz, \tag{3.27}$$

we deduce that $f_l(0,x) \geq 1 + cx^{-\epsilon}$ with some $c > 0$. Therefore the first integral on the right-hand side of (3.26) diverges as $x \to +\infty$. Hence the second integral must also diverge but in such a way that it compensates for the growth of the first term. In order to ensure that both integrals in (3.26) have separate limits as $x \to +\infty$, we now make the assumption that

$$Q(x) \in L^1_2(\mathbf{R}). \tag{3.28}$$

Then (3.27) implies that $1 - f_l(0,\cdot)^2 \in L^1(\mathbf{R})$ and, by using (3.1) and (H3), we conclude that $1 - Z_{l,1}(0,\cdot)^2 \in L^1(\mathbf{R})$. Then, taking $x \to +\infty$ and $y \to +\infty$ in (3.26), we obtain

$$\int_0^{A_+} \frac{dz}{Z_{l,1}(0,z)^2} = -H_+ \int_0^\infty \frac{1 - f_l(0,z)^2}{f_l(0,z)^2}\, dz + \int_0^\infty \frac{1 - Z_{l,1}(0,z)^2}{Z_{l,1}(0,z)^2}\, dz, \tag{3.29}$$

and this relation holds in both the generic and exceptional case. Defining

$$G_1(x) = -\int_0^\infty \frac{1 - f_l(0,z)^2}{f_l(0,z)^2}\,dz + \int_0^x \frac{dz}{f_l(0,z)^2}, \tag{3.30}$$

$$\widehat{G}_1(y) = -\int_0^\infty \frac{1 - Z_{l,1}(0,z)^2}{Z_{l,1}(0,z)^2}\,dz + \int_0^y \frac{dz}{Z_{l,1}(0,z)^2}, \tag{3.31}$$

we can write (3.29) as

$$\int_0^{A_+} \frac{dz}{Z_{l,1}(0,z)^2} = H_+\,G_1(0) - \widehat{G}_1(0). \tag{3.32}$$

In the generic case, subtracting (3.29) from (3.24) and using (3.30) and (3.31), we obtain

$$\widehat{G}_1(-\infty) = H_+\,G_1(-\infty). \tag{3.33}$$

So, if $G_1(-\infty) \neq 0$, then

$$H_+ = \frac{\widehat{G}_1(-\infty)}{G_1(-\infty)}, \tag{3.34}$$

meaning that in the generic case with $G_1(-\infty) \neq 0$, H_+ is determined by $R(k)$. Then A_+ can be found from (3.32), and thus both H_+ and A_+ are determined by $R(k)$ alone. If $G_1(-\infty) = 0$, then $\widehat{G}_1(-\infty) = 0$, and so (3.34) does not allow us to determine H_+. To see what happens when $G_1(-\infty) = 0$, it is convenient to write (3.23) in a different form. At the same time this will establish the connection between (3.23) and a corresponding equation in Aktosun et al. (1996b) obtained there by using the small-k asymptotics. First, adding (3.23) and (3.29), and letting $x \to -\infty$, we get

$$\widehat{G}_1(y + A_+) = H_+\,G_1(x). \tag{3.35}$$

Note that (3.35) can also be verified by differentiating both sides with respect to x and using (3.22), which shows that (3.19) is satisfied. Moreover, at $x = 0$, the two sides of (3.35) agree on account of (3.29)-(3.31). By means of the substitution

$$\tilde{X}_1(0,y) = -y + \widehat{G}_1(y), \tag{3.36}$$

we can write (3.35) in the form

$$y + A_+ + \tilde{X}_1(0, y + A_+) = H_+ G_1(x). \tag{3.37}$$

Then (3.37) agrees with (3.38) in Aktosun et al. (1996b). There the function $\tilde{X}_1(0, y + A_+)$ arose as the value at $k = 0$ of a function $\tilde{X}(k,y)$ given by $\tilde{X}(k,y) = \tilde{X}_1(k, y + A_+)$, where

$$\tilde{X}(k,y) = i\,\frac{Z_l(-k,y) - Z_l(0,y)}{k\, Z_l(0,y)}. \tag{3.38}$$

Moreover, $\tilde{X}(k,y)$ satisfies the singular integral equation

$$\tilde{X}(k,y) - (\mathcal{O}_l \tilde{X})(k,y) = \tilde{X}_0(k,y), \tag{3.39}$$

where

$$\tilde{X}_0(k,y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\rho(s)\, e^{2isy} - \rho(0)}{s\,(s - k + i0)}\, ds. \tag{3.40}$$

Thus $\tilde{X}(k,y)$ obeys an integral equation similar to (3.11), but with a minus sign in front of the operator $\mathcal{O}_l$ and a different nonhomogeneous term. Similarly, as in the case of $X_{l,1}(k,y)$, it is the function $\tilde{X}_1(k,y)$ that is determined first by solving (3.39) with $\rho(k)$ replaced by $R(k)$ in (3.40) and (3.10) and with $a(k)$ and $b(k)$ replaced accordingly. From (3.22) and (3.38) we also see that

$$\tilde{X}_1(k,y) = i\,\frac{Z_{l,1}(-k,y) - Z_{l,1}(0,y)}{k\, Z_{l,1}(0,y)},$$

and thus

$$\tilde{X}_1(0,y) = -i\,\frac{\dot{Z}_{l,1}(0,y)}{Z_{l,1}(0,y)}, \tag{3.41}$$

where the dot denotes differentiation with respect to k. Furthermore, by (3.31) and (3.36),

$$\frac{d\tilde{X}_1(0,y)}{dy} = \frac{1}{Z_{l,1}(0,y)^2} - 1. \tag{3.42}$$

This establishes the connection between the two inversion methods. In the method presented here based on the large-k asymptotics we determine $Z_{l,1}(0,y)$ from $R(k)$, while in the method based on the small-k asymptotics we determine $-i\dot{Z}_{l,1}(0,y)/Z_{l,1}(0,y)$ from $R(k)$. From the latter we can find $Z_{l,1}(0,y)$ by differentiation using (3.41) and (3.42). Conversely, $Z_{l,1}(0,y)$ determines $-i\dot{Z}_{l,1}(0,y)/Z_{l,1}(0,y)$ via (3.36) and (3.41). We remark that because of the factor $1/k$ in (3.38), the integral equation (3.39) must be studied in the Hardy spaces $\mathbf{H}^p_-(\mathbf{R})$ with suitable p. Precisely, if $Q(x) \in L^1_{1+\alpha}(\mathbf{R})$ with $\alpha \in (0,1)$, then p must satisfy $1 < p < 1/(1-\alpha)$; if $\alpha = 1$, then any $p \in (1,\infty)$ is allowed. If $p = 2$, then the unique solvability of (3.39) in $\mathbf{H}^2_-(\mathbf{R})$ follows as in the case of (3.11) (cf. Theorem 3.2). If $p \neq 2$, then we refer the reader to Aktosun et al. (1995a, Theorem 7.1) for the details. There it is shown that the spectral radius of $\mathcal{O}_l$ in the appropriate $\mathbf{H}^p_-$-space is strictly less than 1. Therefore (3.39) is uniquely solvable in $\mathbf{H}^p_-(\mathbf{R})$ and the solution can be obtained by iteration.

Eq. (3.35) in the form (3.37) was analyzed in Aktosun et al. (1996b). The basic observation is that if (3.35) is to have a solution $y(x)$ corresponding to a function $H(x)$, then the ranges of $\widehat{G}_1(y + A_+)$ and $H_+ G_1(x)$ must be the same. Since both sides of (3.35) are strictly monotone increasing functions of their respective variables, any solution $y(x)$ of (3.35) is necessarily unique and monotone increasing. In the exceptional case, the range of $G_1(x)$ is all of $\mathbf{R}$ and hence does not change if we vary H_+. Hence it is plausible that H_+ is a free parameter in the sense that once H_+ has been fixed, then (3.35) has a unique solution $y(x)$ resulting in a unique function $H(x)$. The main task is to verify that the $H(x)$ thus constructed satisfies (H1)-(H4). This is not easy and we refer the reader to the aforementioned reference. In the generic case, the range of $G_1(x)$ is the semi-infinite interval $(G_1(-\infty), +\infty)$. Hence, if $G_1(-\infty) \neq 0$, then H_+ is given by (3.34) and is not a free parameter. However, if $G_1(-\infty) = 0$, then the situation is similar as in the exceptional case. We can vary H_+ without changing the range of the right-hand side of (3.35) and, as in the exceptional case, one can then show that H_+ is a free parameter giving rise to a one-parameter family of functions $H(x)$ all of which satisfy (H1)-(H4).

These observations suggest that the proper scattering data for a unique inversion are:

(i) In the exceptional case: $Q(x)$, $R(k)$, and H_+.

(ii) In the generic case if $G_1(-\infty) = 0$: $Q(x)$, $R(k)$, and H_+.
(iii) In the generic case if $G_1(-\infty) \neq 0$: $Q(x)$ and $R(k)$.

The following uniqueness theorem justifies this choice. For the proof, see Aktosun et al. (1996b).

THEOREM 3.3 Suppose that $Q(x) \in L_2^1(\mathbf{R})$. Then in all cases the solution of the inverse problem given the above scattering data is unique. Moreover, in the generic case with $G_1(-\infty) = 0$ and in the exceptional case, the constant H_+ is a free parameter in the sense that for any choice of $H_+ > 0$, the function $H(x)$ resulting from the solution of (3.35) (if a solution exists) corresponds to the same reflection coefficient $R(k)$.

We remark that Theorem 3.3 does not make any claims about the existence of $H(x)$ for a given set of scattering data. It may well be that a given reflection coefficient $R(k)$ and a given $Q(x)$ are incompatible, so that no function $H(x)$ exists. On the other hand, the same reflection coefficient and a different $Q(x)$ may allow the construction of an $H(x)$. We will give examples for the various possibilities in Section 5.

From (3.4) and (3.12) we see that, in order to carry out the inversion procedure, we must know the constants q_j defined in (2.17) and the function $\rho_{\mathrm{as}}(k) = -b(k)/a(k)$ (cf. Theorem 2.2 (ii)). Furthermore, if we want to use the integral equation (3.18), we must know the constants γ_s and b_s in (3.13) and the functions ϱ, ω_s, and ς_s in (3.16) and (3.17). Of course, all of these quantities are known "in principle" if $\rho(k)$ is known. We indicate here how $\rho_{\mathrm{as}}(k)$ and then $a(k)$ and $b(k)$ can be determined from $\rho(k)$. How one gets the q_j from $\rho_{\mathrm{as}}(k)$ will be the subject of the next section. Since $\rho_{\mathrm{as}}(k)$ is almost periodic, the coefficients γ_s and b_s in (3.13) are uniquely determined by $\rho_{\mathrm{as}}(k)$, and since the difference $\rho(k) - \rho_{\mathrm{as}}(k)$ is $O(1/k)$ as $k \to \pm\infty$, we conclude that the set $\{b_s\}$ coincides with the set S of Fourier exponents of $\rho_{\mathrm{as}}(k)$, and that

$$S = \{z \in \mathbf{R} : \lim_{L\to+\infty} \frac{1}{2L} \int_{-L}^{L} e^{-ikz} \rho(k)\, dk \neq 0\}.$$

Then, if $b_s \in S$, we have

$$\gamma_s = \lim_{L\to+\infty} \frac{1}{2L} \int_{-L}^{L} e^{-ikb_s} \rho(k)\, dk.$$

Whether or not S is easy to find depends on the form in which $\rho(k)$ is available. In Section 5 we will start from a $\rho(k)$ given as a simple analytic expression and the problem of finding S will be trivial. Once we have found $\rho_{\mathrm{as}}(k)$, using (2.27) and the fact that $a(k) = \overline{a(-k)}$ for $k \in \mathbf{R}$, we can write

$$a(k)\, a(-k) = \frac{1}{1 - |\rho_{\mathrm{as}}(k)|^2}, \quad k \in \mathbf{R},$$

which is a scalar Riemann-Hilbert problem for $a(k)$. Its solution was given in Aktosun et al. (1995b). Hence we know $a(k)$ and thus $b(k) = -\rho_{\mathrm{as}}(k)\,a(k)$. Then from $a(k)$ and $b(k)$ we can determine α_j and β_j using the algorithm of the next section, and hence we also know ϱ, ω_s, and ς_s.

4. AN ALGORITHM TO RECOVER CHARACTERISTIC PROPERTIES OF $H(x)$

In this section we present an algorithm to recover certain quantities related to the discontinuities of $H(x)$ that does not require the prior solution of the inverse problem.

For $N = 1, 2$ the explicit expressions for $a(k)$ and $b(k)$ are (cf. (2.25) and (2.26))

$$a(k) = \alpha_1, \quad e^{2iky_1}b(k) = \beta_1, \quad \text{if } N = 1, \tag{4.1}$$

$$a(k) = \alpha_1\alpha_2 + \beta_1\beta_2 e^{2ik(y_2-y_1)}, \quad e^{2iky_2}b(k) = \alpha_1\beta_2 + \beta_1\alpha_2 e^{2ik(y_2-y_1)}, \quad \text{if } N = 2,$$

and if $N = 3$ we have

$$a(k) = \alpha_1\alpha_2\alpha_3 + \beta_1\beta_2\alpha_3 e^{2ik(y_2-y_1)} + \alpha_1\beta_2\beta_3 e^{2ik(y_3-y_2)} + \beta_1\alpha_2\beta_3 e^{2ik(y_3-y_1)},$$

$$e^{2iky_3}b(k) = \alpha_1\alpha_2\beta_3 + \beta_1\beta_2\beta_3 e^{2ik(y_2-y_1)} + \alpha_1\beta_2\alpha_3 e^{2ik(y_3-y_2)} + \beta_1\alpha_2\alpha_3 e^{2ik(y_3-y_1)}.$$

As these formulas suggest, and as one can prove by induction, in general, $a(k)$ and $b(k)$ are exponential polynomials of the form

$$a(k) = a_0 + \sum_n a_n e^{2ik\lambda_n}, \quad a_0 = \prod_{j=1}^{N} \alpha_j, \quad b(k) = \sum_n b_n e^{2ik\xi_n}, \tag{4.2}$$

where a_n, b_n, λ_n, and ξ_n are nonzero real constants, $\lambda_n > 0$, and the ξ_n are of the form $\lambda_n - y_N$. Moreover, the number of terms in each case is at most 2^{N-1}. Next we list the steps of the algorithm which allows us to recover N, y_j, and q_j from $a(k)$ and $b(k)$.

1. From $b(k)$, obtain y_N as $y_N = -\min_n \xi_n$. Note that the coefficient b_N in $b(k)$ is $\alpha_1\alpha_2\cdots\alpha_{N-1}\beta_N$.
2. From $a(k)$, obtain the coefficient $a_0 = \alpha_1\cdots\alpha_N$.
3. From the ratio of the coefficients in steps 1 and 2, obtain $\dfrac{\beta_N}{\alpha_N}$ and hence $q_N = \dfrac{1+\beta_N/\alpha_N}{1-\beta_N/\alpha_N}$.

4. Construct the matrix $E(k, x_N)$ defined in (2.25) by using y_N and q_N from above.
5. From (2.25), obtain the matrix $E(k, x_N)^{-1}$ and then define

$$\begin{bmatrix} a^{[N-1]}(k) & b^{[N-1]}(k) \\ b^{[N-1]}(-k) & a^{[N-1]}(-k) \end{bmatrix} := \begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix} E(k, x_N)^{-1}.$$

   Note that $a^{[N-1]}(k)$ has again the form (4.2) with $a_0 > 0$ and a_n, b_n nonzero real constants, but with fewer terms.
6. Replace $a(k)$ and $b(k)$ by $a^{[N-1]}(k)$ and $b^{[N-1]}(k)$, respectively, and repeat steps 1-5. This results in functions $a^{[N-2]}(k)$ and $b^{[N-2]}(k)$. Repeat the procedure until the matrix in step 5 no longer contains any exponential terms on the diagonal, i.e. until the matrix is $E(k, x_1)$. From it, find y_1 and q_1.

There are a few ways to speed up the algorithm and these are described in Aktosun et al. (1996b). Also, the algorithm can be modified and applied to $R(k)$ instead of $\rho(k)$. In that case one can determine N, q_j, and the differences $y_j - y_k$. The values y_j themselves can only be obtained to within a shift; namely, one can determine $y_j + A_+$. Again we refer to the aforementioned reference for details.

In general we cannot determine the points x_j $(j = 1, \ldots, N)$ from $a(k)$ and $b(k)$. However, if it is known, for example, that $H(x)$ is piecewise constant, and if H_+ is given, then we can determine the x_j. To see this, note that we can first determine $H(x)$ on each interval (y_j, y_{j+1}) starting with (y_N, ∞), i.e. $H_{N,N+1} = H_+$, $H_{N-1,N} = H_+ q_N$, etc., where we put $H(x) = H_{j,j+1}$ on (y_j, y_{j+1}). It now suffices to find one of the points x_j because the others can then be obtained by using the formula $x_{n+1} - x_n = (y_{n+1} - y_n)/H_{n,n+1}$. Now, if $y_j = 0$ for some j, then $x_j = 0$, and we are done. Supposing $y_j \neq 0$ for every j, we can find a y_{j_0} that is closest to zero. If $y_{j_0} < 0$, then $x_{j_0} = y_{j_0}/H_{j_0,j_0+1}$, and, if $x_{j_0} > 0$, then $x_{j_0} = y_{j_0}/H_{j_0-1,j_0}$, and we have found one of the points of discontinuity.

In Aktosun et al. (1996b) we have also characterized those functions $H(x)$ for which $\rho(k) = \rho_{\text{as}}(k)$, respectively $R(k) = R_{\text{as}}(k)$. We are then necessarily in the exceptional case, because, by (2.27), $|\tau(k)| = \sqrt{1 - |\rho(k)|^2} = \sqrt{1 - |\rho_{\text{as}}(k)|^2} = 1/|a(k)|$, and thus $a(0) = 1/\tau(0)$ is finite. It turns out that the functions $H(x)$ are such that $H(x)\, f_l(0, x)^2$ is piecewise constant. Then one can argue as above and successively determine the points $x_1, \ldots, x_N$.

In Aktosun et al. (1996b) the above algorithm has been extended to recover the constants ν_j defined in (2.17) from $a(k)$ and $b(k)$, and some additional information that is also available without solving the inverse problem first.

5. EXAMPLES

In this section we consider two examples illustrating the inversion procedure, one for the exceptional case and one for the generic case. Example 5.1 deals with the exceptional case and has already been discussed in Aktosun et al. (1995a), but the solution given here is considerably simpler. Example 5.2 illustrates the two possibilities that can occur in the generic case (see cases (ii) and (iii) above Theorem 3.3) and it is a generalization of Examples 8.3 and 3.4 in Aktosun et al. (1996a and 1996b, respectively).

EXAMPLE 5.1 The simplest example illustrating the exceptional case is when $R(k)$ is constant, i.e.

$$R(k) = R_0, \quad -1 < R_0 < 1.$$

Putting $\rho(k) = R_0\, e^{2ikA_+}$ and applying the algorithm of Section 4, in particular (4.1), we get

$$N = 1, \quad \alpha_1 = \frac{1}{\sqrt{1-R_0^2}}, \quad \beta_1 = \frac{-R_0}{\sqrt{1-R_0^2}}, \quad y_1 = -A_+, \quad q_1 = \frac{1-R_0}{1+R_0},$$

$$a(k) = \frac{1}{\sqrt{1-R_0^2}}, \quad b(k) = -\frac{R_0\, e^{2ikA_+}}{\sqrt{1-R_0^2}}.$$

Note that $\rho(k) = \rho_{\mathrm{as}}(k)$ and that $a(k)$ and $b(k)$ can easily be guessed. Therefore, (3.4) yields

$$J_l(0,y) = \begin{cases} 1, & y > y_1, \\ \sqrt{q_1}, & y < y_1. \end{cases}$$

Since $\rho(k) = \rho_{\mathrm{as}}(k)$, we have $P_l(k,y) = 0$ and thus $X_l(k,y) = 0$. Hence $Z_l(0,y) = J_l(0,y)$ and, since $Z_{l,1}(0,y) = Z_l(0,y-A_+) = Z_l(0,y+y_1)$, we have that

$$Z_{l,1}(0,y) = \begin{cases} 1, & y > 0, \\ \sqrt{q_1}, & y < 0. \end{cases}$$

Thus, by (3.29)-(3.31), $\widehat{G}_1(0) = 0$, and

$$A_+ = \begin{cases} H_+ G_1(0), & G_1(0) \geq 0, \\ q_1 H_+ G_1(0), & G_1(0) < 0. \end{cases}$$

Note that the sign of A_+ is determined by that of $G_1(0)$. Evaluating the left-hand side of (3.23), we obtain
If $G_1(0) \geq 0$:

$$(5.1) \qquad y(x) = \begin{cases} H_+ [G_1(x) - G_1(0)], & y > y_1, \\ H_+[q_1 G_1(x) - G_1(0)], & y < y_1. \end{cases}$$

If $G_1(0) < 0$:

$$(5.2) \qquad y(x) = \begin{cases} H_+ [G_1(x) - q_1 G_1(0)], & y > y_1, \\ H_+ q_1 [G_1(x) - G_1(0)], & y < y_1. \end{cases}$$

The point x_1 (such that $y(x_1) = y_1$) is obtained by equating the expressions on the right-hand sides of (5.1) and (5.2), respectively. This gives in both cases

$$(5.3) \qquad G_1(x_1) = 0$$

for the equation determining x_1. Since in the exceptional case the range of $G_1(x)$ is all of $\mathbf{R}$, (5.3) has a unique solution. By differentiating (5.1) and (5.2), we obtain

$$H(x) = \begin{cases} \dfrac{H_+}{f_l(0,x)^2}, & x > x_1, \\ \dfrac{q_1 H_+}{f_l(0,x)^2}, & x < x_1. \end{cases}$$

We see that in order for (H3) to be satisfied it is necessary and sufficient that $1 - f_l(0,\cdot)^2 \in L^1(\mathbf{R})$, and we know that condition (3.28) is sufficient to guarantee this. In other words, if $Q(x) \in L^1_1(\mathbf{R})$ but $1 - f_l(0,\cdot)^2 \notin L^1(\mathbf{R})$, then there is no function $H(x)$ that satisfies (H3). This illustrates the remark made below Theorem 3.3 about the possible incompatibility of $R(k)$ and $Q(x)$.

EXAMPLE 5.2 Let

$$R(k) = \frac{\epsilon k - i\gamma}{k + i\gamma}\, e^{-2ik\delta},$$

where $-1 < \epsilon < 1$, $\gamma > 0$, $\delta \in \mathbf{R}$. Thus $R(0) = -1$, and so we are in the generic case. Again, we put $\rho(k) = R(k)\, e^{2ikA_+}$ and note that $\rho_{\text{as}}(k) = \epsilon\, e^{2ik(A_+ - \delta)}$. Then the method of Section 4 gives

$$N=1,\quad \alpha_1=\frac{1}{\sqrt{1-\epsilon^2}},\quad \beta_1=\frac{-\epsilon}{\sqrt{1-\epsilon^2}},\quad y_1=\delta-A_+,\quad q_1=\frac{1-\epsilon}{1+\epsilon},$$

$$a(k)=\frac{1}{\sqrt{1-\epsilon^2}},\quad b(k)=-\frac{\epsilon\, e^{2ik(A_+-\delta)}}{\sqrt{1-\epsilon^2}}.$$

Also, $\gamma_1=\epsilon$, $b_1=2\,(A_+-\delta)$, and thus, from (3.4) and (3.14) we obtain

$$J_l(k,y)=\begin{cases}\dfrac{1-\epsilon\, e^{-2ik(y-y_1)}}{\sqrt{1-\epsilon^2}}, & y<y_1,\\ 1, & y>y_1,\end{cases}$$

$$\varrho(z)=\begin{cases}0, & z+2y_1<0,\\ -\gamma(1+\epsilon)e^{-\gamma(z+2y_1)}, & z+2y_1>0.\end{cases}$$

The Marchenko equation (3.18) takes the form

$$h_l(t,y)=0,\quad t>0,\quad t+2(y-y_1)>0,$$

$$h_l(t,y)+\epsilon\, h_l(-t-2(y-y_1),y)-\gamma\,(1+\epsilon)\,e^{\gamma[t+2(y-y_1)]}\int_0^{-[t+2(y-y_1)]}e^{\gamma s}\,h_l(s,y)\,ds$$
$$=\frac{\gamma(1+\epsilon)}{\sqrt{1-\epsilon^2}}e^{\gamma[t+2(y-y_1)]},\quad t>0,\quad t+2(y-y_1)<0,$$

and its solution is

$$h_l(t,y)=\begin{cases}0, & t>0,\quad t+2(y-y_1)>0,\\ \dfrac{\gamma}{\sqrt{1-\epsilon^2}}, & t>0,\quad t+2(y-y_1)<0.\end{cases}$$

By (3.15), we have $X_l(0,y)=\int_0^\infty h_l(t,y)\,dt$, and hence

$$Z_{l,1}(0,y)=\begin{cases}\dfrac{1-\epsilon-2\gamma(y-\delta)}{\sqrt{1-\epsilon^2}}, & y<\delta,\\ 1, & y>\delta.\end{cases}\tag{5.4}$$

Thus (3.31) gives

(5.5)
$$\widehat{G}_1(-\infty) = \delta - \int_{-\infty}^{\delta} \frac{dz}{Z_{l,1}(0,z)^2} = \frac{2\gamma\delta - 1 - \epsilon}{2\gamma}.$$

Therefore, $\widehat{G}_1(-\infty) = 0$ if and only if $2\gamma\delta - 1 - \epsilon = 0$, and, by (3.33), this happens if and only $G_1(-\infty) = 0$. Furthermore, by (3.34) and (5.5),

(5.6)
$$H_+ = \frac{2\gamma\delta - 1 - \epsilon}{2\gamma\, G_1(-\infty)} \qquad \text{if} \quad G_1(-\infty) \neq 0.$$

From (3.25), setting $x = x_1$, $y = y_1$, and using $y_1 = \delta - A_+$, we obtain

(5.7)
$$\int_{-\infty}^{\delta} \frac{dz}{Z_{l,1}(0,z)^2} = H_+ \int_{-\infty}^{x_1} \frac{dz}{f_l(0,z)^2}.$$

If $G_1(-\infty) \neq 0$, then using (5.5)-(5.7) yields

(5.8)
$$\int_{-\infty}^{x_1} \frac{dz}{f_l(0,z)^2} = \frac{(1+\epsilon)\, G_1(-\infty)}{2\gamma\delta - 1 - \epsilon}.$$

This is the equation for x_1, provided $G_1(-\infty) \neq 0$. Since the left-hand side of (5.8) is strictly positive, it is necessary that $G_1(-\infty)$ and $2\gamma\delta - 1 - \epsilon$ have the same sign. Otherwise, a solution of (5.8) does not exist, and there is no function $H(x)$ for the given reflection coefficient. Also, if $G_1(-\infty) = 0$ and $2\gamma\delta - 1 - \epsilon \neq 0$, or, if $G_1(-\infty) \neq 0$ and $2\gamma\delta - 1 - \epsilon = 0$, then a solution does not exist. This again illustrates the remark made below Theorem 3.3: $Q(x)$ and $R(k)$ may be incompatible. However, if both $G_1(-\infty) = 0$ and $2\gamma\delta - 1 - \epsilon = 0$, then H_+ is a free parameter and x_1 is determined in terms of H_+ by (5.7). This illustrates the case with $G_1(-\infty) = 0$ in Theorem 3.3. Moreover, in any case, if $H(x)$ exists, then, by using (3.25) and (5.4), we obtain

(5.9)
$$H(x) = \begin{cases} \dfrac{H_+}{f_l(0,x)^2}, & x > x_1, \\[2ex] \dfrac{1-\epsilon}{2\gamma f_l(0,x)^2} \dfrac{\int_{-\infty}^{x_1} dz/f_l(0,z)^2}{[\int_{-\infty}^{x} dz/f_l(0,z)^2]^2}, & x < x_1. \end{cases}$$

Using

$$f_r(0,x) = [f_l(0,\cdot); f_r(0,\cdot)]\, f_l(0,x) \int_{-\infty}^{x} \frac{dz}{f_l(0,z)^2},$$

we can write the second line of (5.9) as

$$H(x) = \frac{(1-\epsilon)[f_l(0,x); f_r(0,x)]^2}{2\gamma} \frac{\int_{-\infty}^{x_1} dz/f_l(0,z)^2}{f_r(0,x)^2}, \quad x < x_1.$$

We see that, as in Example 5.1, a condition like (3.28) is needed to ensure that $H(x)$ obeys hypothesis (H3).

REFERENCES

Aktosun, T., Klaus, M., and van der Mee, C. (1995a). Inverse wave scattering with discontinuous wave speed, J. Math. Phys., 36: 2880.

Aktosun, T., Klaus, M., and van der Mee, C. (1995b). "On the recovery of a discontinuous wavespeed in wave scattering in a nonhomogeneous medium." Proc. 20th Summer School "Applications of Mathematics in Engineering," Varna, Bulgaria, August 26-September 2, 1994. Ed. by D. Ivanchev and D. Mishev, pp. 9–27. Inst. Appl. Math. and Informatics, Techn. Univ., Sofia.

Aktosun, T., Klaus, M., and van der Mee, C. (1996a). Integral equation methods for the inverse problem with discontinuous wavespeed, J. Math. Phys., 37: 3218.

Aktosun, T., Klaus, M., and van der Mee, C. (1996b). Recovery of discontinuities in a nonhomogeneous medium, Inverse Problems, 12: 1.

Deift, P. and Trubowitz, E. (1979). Inverse scattering on the line, Comm. Pure Appl. Math., 32: 121.

Grinberg, N. I. (1991a). Inverse scattering problem for an elastic layered medium, Inverse Problems, 7: 567.

Grinberg, N. I. (1991b). The one-dimensional inverse scattering problem for the wave equation, Math. USSR Sb., 70: 557.

Krueger, R. J. (1976). An inverse problem for a dissipative hyperbolic equation with discontinuous coefficients, Quart. Appl. Math., 34: 129.

Krueger, R. J. (1978). An inverse problem for an absorbing medium with multiple discontinuities, Quart. Appl. Math., 36: 235.

Krueger, R. J. (1982). Inverse problems for nonabsorbing media with discontinuous material properties, J. Math. Phys., 23: 396.

Marchenko, V.A. (1986). Sturm-Liouville operators and applications, Birkhäuser, Basel.

Razavy, M. (1975). Determination of the wave velocity in an inhomogeneous medium from the reflection coefficient, J. Acoust. Soc. Am., 58: 956.

Sabatier, P. C. (1988). "For an impedance scattering theory," Proc. IVth Workshop on Nonlinear Evolution Equations and Dynamical Systems, Nonlinear Evolutions, Balaruc-les-Bains, France, June 11-25, 1987. Ed. by J. J. P. Leon, pp. 727–749. World Scientific, Singapore.

Ware, J. A. and Aki, K. (1969). Continuous and discrete inverse-scattering problems in a stratified elastic medium. I. Plane waves at normal incidence, J. Acoust. Soc. Am., 45: 911.