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7

LINEARIZED BGK AND NEUTRON TRANSPORT EQUATIONS IN FINITE DOMAINS

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ABSTRACT

A study is made of the integro-differential equation

$$\mu \frac{d\psi}{dx}(x,\mu) + \psi(x,\mu) = \int_{-\infty}^{+\infty} \psi(x,\nu)\Psi(\nu)d\nu + f(x,\mu) \qquad (0<x<\tau<+\infty).$$

For $\Psi(\mu) = \pi^{-\frac{1}{2}}e^{-\mu^2}$ this equation arises in the linearized BGK model; for $\Psi(\mu) = \frac{1}{2}c(|\mu|\le 1)$, $\Psi(\mu) = 0$ $(|\mu|>1)$ it describes neutron transport in a slab with isotropic scattering. With the help of the existing theory of convolution equations solutions of various boundary value problems are obtained. The auxiliary func-

tions appearing in these solutions are shown to satisfy certain X- and Y-equations. A rational approximation method for the computation of these X- and Y-functions is explored.

INTRODUCTION

The analogy of the linearized BGK equation in the kinetic theory of gases and the neutron transport equation in a slab with isotropic scattering is well-known. Cercignani[1,2] and Kaper[3,4] have exploited this analogy to solve a half-space problem.

A mathematical replica of the one-speed neutron transport equation can be found in astrophysics. For finite optical layers with isotropic scattering this replica, the equation of transfer of unpolarized radiation, has been solved in terms of so-called X- and Y-functions by Ambartsumian[5] and Chandrasekhar[6]. By applying the existing theory of convolution equations Van der Mee[7] has given a rigorous derivation of this solution.

The linearized BGK equation in a finite domain, with boundary conditions as for the finite-slab problem in neutron transport theory, constitutes a well-posed problem[8]. Special cases of this problem, describing plane Couette and Poiseuille flows between parallel plates, have been solved by series expansion[9,10]. In this article we solve in a mathematically rigorous way a few boundary value problems in a finite domain in terms of X- and Y-functions, including the neutron transport equation with isotropic scattering and the linearized BGK equation for plane Couette and Poiseuille flows.

In radiative transfer the X- and Y-functions have been thoroughly investigated by Busbridge[11] and Mullikin[12,13]. Here in a general context stability properties of such functions are established and a rational approximation method for their computation is offered. This method has been inspired by work of Masson[14] on Chandrasekhar's H-equation.

In the first two sections the linearized BGK and neutron transport equations are stated and shown to be equivalent to a convolution equation. To make this paper self-contained, in Section 3 we review the existing theory of convolution equations. In Section 4 we apply this theory and reduce various boundary value problems to the calculation of X- and Y-functions. Generalizations of these functions are studied in Section 5. In the last two sections for these generalizations and for H-functions stability properties are established and their computation by rational approximation is explored.

1. Statement of the problem

The linearized BGK and neutron transport equations in a finite domain can be written in the form of the operator differential equation

(1.1) $\quad (T\psi)'(x) = -(I-B)\psi(x) + f(x) \quad (0<x<\tau<+\infty)$

on a suitable Hilbert space H. For the neutron transport equation one takes $H = L_2[-1,+1]$ and defines the vectors $\psi(x)$ and $f(x)$ and the operators T and B by

(1.2a) $\quad \psi(x)(\mu) = \psi(x,\mu), \quad f(x)(\mu) = f(x,\mu); \quad (-1\le\mu\le+1)$

(1.2b) $\quad (Th)(\mu) = \mu h(\mu), \quad (Bh)(\mu) = \tfrac{1}{2}c\int_{-1}^{+1} h(\nu)d\nu;$

we restrict ourselves to non-multiplying media where $0\le c\le 1$. For the linearized BGK equation one takes as H the Hilbert space $L_2(\mathbb{R})_\rho$ of all scalar functions on the real line that are square integrable with respect to the measure ρ with Radon-Nikodym derivative $d\rho/d\mu = \pi^{-\frac{1}{2}}e^{-\mu^2}$; this Hilbert space is endowed with the inner product

$$(1.3)\qquad \langle h_1,h_2\rangle = \pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty} h_1(\nu)\overline{h_2(\nu)}e^{-\nu^2}d\nu;\quad h_1,h_2\in L_2(\mathbb{R})_\rho .$$

One takes the vectors $\psi(x)$ and $f(x)$ in $L_2(\mathbb{R})_\rho$ as in (1.2a) and the operators T and B on $L_2(\mathbb{R})_\rho$ as follows:

$$(1.4a)\qquad D_T = \{h\in L_2(\mathbb{R})_\rho : \int_{-\infty}^{+\infty}\nu^2|h(\nu)|^2e^{-\nu^2}d\nu<+\infty\};$$

$$(1.4b)\qquad (Th)(\mu) = \mu h(\mu),\quad (Bh)(\mu) = \pi^{-\frac{1}{2}}\int_{-\infty}^{+\infty}h(\nu)e^{-\nu^2}d\nu.$$

Here D_T denotes the domain of the unbounded operator T. For both models by a solution of Eq. (1.1) we mean a vector-valued function $\psi:(0,\tau)\to D_T\subset H$ such that $T\psi$ is differentiable on $(0,\tau)$ in the strong sense and (1.1) holds.

To impose boundary conditions on Eq. (1.1), both in $L_2[-1,+1]$ and $L_2(\mathbb{R})_\rho$ one defines the orthogonal projections P_+ and P_- by the formulas

$$(1.5)\qquad (P_+h)(\mu) = \begin{cases} h(\mu), & \mu\ge 0;\\ 0, & \mu<0;\end{cases}\qquad (P_-h)(\mu) = \begin{cases} 0, & \mu\ge 0;\\ h(\mu), & \mu<0.\end{cases}$$

For given $\phi\in D_T\subset H$ we impose the boundary conditions

$$(1.6)\qquad \lim_{x\downarrow 0} P_+\psi(x) = P_+\phi,\qquad \lim_{x\uparrow\tau} P_-\psi(x) = P_-\phi,$$

where the limit is taken in the norm of H. For $H = L_2[-1,+1]$ one gets the finite-slab problem of neutron transport theory. For $H = L_2(\mathbb{R})_\rho$, $f(x)\equiv 0$ and $\phi(\mu) = -\mathrm{sgn}\,\mu$ one obtains the linearized BGK problem for a plane Couette flow between parallel plates[9]; for $H = L_2(\mathbb{R})_\rho$, $f(x,\mu)\equiv -\frac{1}{2}k$ and $\phi(\mu)\equiv 0$ one has the linearized BGK problem for a plane Poiseuille flow between parallel plates[10]. Here k is some dimensionless constant.

As an operator differential equation of the form (1.1) Kaper[8] has dealt with the linearized BGK equation. For a large class of inhomogeneous terms f and every function $\phi\in D_T\subset L_2(\mathbb{R})_\rho$ he showed (a slightly modified version of) the boundary value problem (1.1) - (1.6) to be well-posed. The analogous result for the neutron transport equation in a non-multiplying slab is due to Van der Mee[7]. For a non-conservative slab a related result has been announced by Hangelbroek[15].

2. Hermitian admissible pairs

The analogous form the linearized BGK and neutron transport equations have suggests an abstract treatment of the operator differential equation

$$(2.1)\qquad (T\psi)'(x) = -(I-B)\psi(x) + f(x),\qquad 0<x<\tau<+\infty,$$

where the operators T and B are defined on an abstract Hilbert space H and have specific abstract properties. More precisely, we require (T,B) to be a hermitian admissible pair on H, i.e., a pair of operators T (H → H) and B: H → H with the following properties:

(C1) T is a (possibly unbounded) self-adjoint operator on H with a trivial null space;

(C2) B is a compact and self-adjoint operator on H;

(C3) there exist $0<\alpha<1$ and a bounded operator D: H → H such that the range $\{Dx: x\in H\}$ of the operator D is contained in the domain of $|T|^\alpha$ and
$$B = |T|^\alpha D.$$

To formulate boundary conditions to Eq. (2.1) we define two projections P_+ and P_- on H. According to the Spectral Theorem there exists a unique resolution of the identity E of the self-adjoint operator T such that $T = \int \lambda E(d\lambda)$. Put $P_+ = E([0,+\infty))$, $P_- = E((-\infty,0])$. Then P_+ and P_- are the spectral projections of T corresponding to the parts of the spectrum of T on the positive and negative real line, respectively. Since T is assumed to have a trivial null space, the projections P_+ and P_- are complementary. With the help of these projections we now impose on Eq. (2.1) the boundary conditions

$$(2.2)\qquad \lim_{x\downarrow 0} P_+\psi(x) = P_+\phi, \qquad \lim_{x\uparrow \tau} P_-\psi(x) = P_-\phi.$$

Here ϕ is a given vector in the domain D_T of T. By a solution of Eq. (2.1) we mean a vector-valued function $\psi: (0,\tau) \to D_T \subset H$ such that $T\psi$ is differentiable on $(0,\tau)$ in the strong sense and (2.1) holds.

In an earlier work[7] for bounded T hermitian admissible pairs were introduced and for these pairs the boundary value problem (2.1) - (2.2) was studied. In this work[7] and for bounded T this problem was proved to be equivalent to a vector-valued convolution equation. To formulate this equivalence theorem (and to state it for unbounded and bounded T as well) we need the notion of a propagator function. By the <u>propagator function</u> of a hermitian admissible pair (T,B) on H we mean the function H from the non-zero part of the real line into the algebra of bounded linear operators on H defined by

$$(2.3)\qquad H(x) = \begin{cases} +T^{-1}e^{-xT^{-1}}P_+ = +\int_0^{+\infty} t^{-1}e^{-x/t}E(dt), & x>0; \\ -T^{-1}e^{-xT^{-1}}P_- = -\int_{-\infty}^{0} t^{-1}e^{-x/t}E(dt), & x<0, \end{cases}$$

where E denotes the resolution of the identity of the self-adjoint operator T.

<u>THEOREM 2.1</u>. *Let $0<\tau<+\infty$, and let (T,B) be a hermitian admissible pair on H. Let $\omega: [0,\tau] \to D_T \subset H$ be*

a continuous function such that $T\omega$ is differentiable on $(0,\tau)$. Then an essentially bounded (strongly measurable[16]) vector-valued function $\psi: (0,\tau) \to D_T \subset H$ is a solution of the convolution equation

$$(2.4)\qquad \psi(x) - {}_0\int^{\tau} H(x-y)B\psi(y)dy = \omega(x), \qquad 0<x<\tau,$$

if and only if $T\psi$ is strongly differentiable and satisfies the equation

$$(2.5a)\qquad (T\psi)'(x) = -(I-B)\psi(x) + (T\omega)'(x) + \omega(x)\,(0<x<\tau)$$

with boundary conditions

$$(2.5b)\qquad \lim_{x\downarrow 0} P_+\psi(x) = P_+\omega(0), \qquad \lim_{x\uparrow \tau} P_-\psi(x) = P_-\omega(\tau).$$

The proof of Theorem 2.1 is the same as the proof in the case when T is bounded[17]. In the special case when $f \equiv 0$ and $\phi \in D_T$ the boundary value problem (2.1) - (2.2) is equivalent to the convolution equation (2.4) with right-hand side

$$(2.6a)\qquad \omega(x) = e^{-xT^{-1}}P_+\phi + e^{(\tau-x)T^{-1}}P_-\phi, \qquad 0\le x\le\tau.$$

If $\phi = 0$ and $f(x) = |T|^{\gamma}g(x)$ $(0<x<\tau)$ for some $0<\gamma<1$ and bounded continuous function g, then the boundary value problem (2.1) - (2.2) is equivalent to the convolution equation (2.4) with right-hand side

$$(2.6b)\qquad \omega(x) = {}_0\int^{\tau} H(x-y)f(y)dy, \qquad 0\le x\le\tau.$$

If $\phi = 0$ and $f(x) = \chi \in H$ is constant, then the problem (2.1) - (2.2) is equivalent to the convolution equation (2.4) with right-hand side

$$(2.6c)\qquad \omega(x) = \left[I-e^{-xT^{-1}}P_+ - e^{(\tau-x)T^{-1}}P_-\right]\chi, \qquad 0\le x\le\tau.$$

The operators T and B in (1.2b) (resp. (1.4)) form a hermitian admissible pair on $L_2[-1,+1]$ (resp. $L_2(\mathbb{R})_\rho$) with bounded (resp. unbounded) T. By Theorem 2.1 the concrete version (1.1) - (1.6) of the abstract boundary value problem (2.1) - (2.2) is equivalent to a convolution equation.

3. Convolution equations on the finite interval $(0,\tau)$

In the previous section the linearized BGK equation and the neutron transport equation with isotropic scattering were stated to be equivalent to an equation of the form

$$(3.1)\qquad \psi(x) - \int_0^\tau H(x-y)B\psi(y)dy = \omega(x), \qquad 0<x<\tau<+\infty,$$

where $H(.)$ is the propagator function of a hermitian admissible pair (T,B). In both instances the operator B has the special form $B = c\langle .,e\rangle e$, where e is a vector of unit norm, $\langle .,.\rangle$ denotes the inner product and $0\le c\le 1$[18]. In fact, for the linearized BGK (resp. neutron transport) equation one has $e(\mu) \equiv 1$ (resp. $e(\mu) \equiv \tfrac12\sqrt{2}$) and $c=1$ (resp. $c \in [0,1]$). In view of (3.1) it is clear that

$$(3.2)\qquad \psi(x) = \omega(x) + c\int_0^\tau \langle\psi(y),e\rangle H(x-y)e\,dy, \qquad 0<x<\tau,$$

and hence Eq. (3.1) is easily reduced to the scalar convolution equation

$$(3.3)\qquad \langle\psi(x),e\rangle - c\int_0^\tau \langle H(x-y)e,e\rangle\langle\psi(y),e\rangle\,dy = \langle\omega(x),e\rangle, \qquad 0<x<\tau.$$

It appears that the kernel $c\langle H(.)e,e\rangle$ of Eq. (3.3) is an even function. In particular, for $0 \ne x \in \mathbb{R}$ one has

$$(3.4)\qquad \langle H(x)e,e\rangle = \begin{cases} \tfrac12\int_1^{+\infty} z^{-1}e^{-z|x|}dz\ ^{19}, & \text{for the neutron transport equation;} \\ \pi^{-\frac12}\int_0^{+\infty} z^{-1}e^{-z|x|}e^{-1/z^2}dz, & \text{for the BGK model.}\end{cases}$$

So in this section we review the existing mathematical theory[20,21] of convolution equations of the form

$$(3.5)\qquad \chi(x) - \int_0^\tau k(x-y)\chi(y)dy = \zeta(x), \qquad 0<x<\tau,$$

where $k\colon (-\tau,+\tau) \to \mathbb{C}$ is an even function such that $\int_{-\tau}^{+\tau}|k(t)|dt < +\infty$.

THEOREM 3.1. *Let* $0<\tau<+\infty$, *and let* $k \in L_1(-\tau,+\tau)$ *be an even function. Suppose that the convolution equation*

$$(3.6)\qquad \xi(x) - \int_0^\tau k(x-y)\xi(y)dy = k(x) \qquad (0<x<\tau)$$

has a solution $\xi \in L_1(0,\tau)$. *Then this solution is unique. Further, for every* $1\le p\le+\infty$ *and every* $\zeta \in L_p(0,\tau)$ *the convolution equation* (3.5) *has a unique solution* χ *in* $L_p(0,\tau)$, *namely*

$$(3.7a)\qquad \chi(x) = \zeta(x) + \int_0^\tau \delta(x,y)\zeta(y)dy, \qquad 0<x<\tau.$$

Here the resolvent kernel $\delta(x,y)$ *is given in terms of the function* ξ *by*

$$(3.7b)\qquad \delta(x,y) = \begin{cases} \xi(|x-y|) + \int_0^{\min(x,y)} \{\xi(x-z)\xi(y-z)-\xi(\tau+z-x)\xi(\tau+z-y)\}dz; \\ \qquad\qquad (0\le x\ne y\le\tau) \\ \xi(|x-y|) + \int_{\max(x,y)}^{\tau} \{\xi(z-x)\xi(z-y)-\xi(\tau-z+x)\xi(\tau-z+y)\}dz. \end{cases}$$

In the present form this theorem is due to Gohberg and Semençul[21].

Assume the conditions of Theorem 3.1 to be fulfilled. Let $\eta_{\tau,\mu}$ be the unique solution in $L_1(0,\tau)$ (and thus in $L_\infty(0,\tau)$) of the convolution equation

$$(3.8a)\qquad \eta_{\tau,\mu}(x) - \int_0^\tau k(x-y)\eta_{\tau,\mu}(y)dy = e^{-x/\mu}; \qquad 0<x<\tau, \quad 0\ne\mu\in\mathbb{C}.$$

As $k \in L_1(-\tau,+\tau)$ and $\eta_{\tau,\mu} \in L_\infty(0,\tau)$, it follows that $\eta_{\tau,\mu}$ can be extended to a continuous function on $[0,\tau]$. Put

$$(3.8b)\qquad X(\mu) = \eta_{\tau,\mu}(0), \qquad Y(\mu) = \eta_{\tau,\mu}(\tau); \qquad 0\ne\mu\in\mathbb{C}.$$

Then a straightforward application of Theorem 3.1 (especially (3.7b)) shows that

$$X(\mu) = 1 + {}_0\int^\tau e^{-y/\mu}\xi(y)dy;$$

$$Y(\mu) = e^{-\tau/\mu} + {}_0\int^\tau e^{-(\tau-y)/\mu}\xi(y)dy.$$

So X and Y are analytic on $\mathbb{C}/\{0\}$ and at infinity and

(3.8c) $\quad Y(\mu) = e^{-\tau/\mu}X(-\mu), \quad 0\neq\mu\in\mathbb{C}.$

A useful identity[22] relating the resolvent kernel to the functions X and Y is

(3.9a) $$ {}_0\int^\tau e^{-x/\nu}\left[e^{-x/\mu} + {}_0\int^\tau e^{-y/\mu}\delta(x,y)dy\right]dx = \frac{\mu\nu}{\mu+\nu}\{X(\mu)X(\nu) - Y(\mu)Y(\nu)\}.$$

With the help of (3.8c) one easily derives from it the identity

(3.9b) $$ {}_0\int^\tau e^{-(\tau-x)/\nu}\left[e^{-x/\mu} + {}_0\int^\tau e^{-y/\mu}\delta(x,y)dy\right]dx = \frac{\mu\nu}{\nu-\mu}\{Y(\nu)X(\mu) - X(\nu)Y(\mu)\}.$$

Note that $X(\infty) = Y(\infty) = 1 + {}_0\int^\tau\xi(y)dy$. Taking $\nu\to\infty$ in (3.9a) one gets

(3.9c) $$ {}_0\int^\tau\left[e^{-x/\mu} + {}_0\int^\tau e^{-y/\mu}\delta(x,y)dy\right]dx = \mu X(\infty)\{X(\mu) - Y(\mu)\}.$$

4. Solutions of boundary value problems

In this section we apply Theorem 3.1 to the convolution equation (3.3). Certain boundary value problems that are equivalent to Eq. (3.1) will be solved here in terms of X- and Y-functions using the connection (3.2) between solutions of Eqs. (3.1) and (3.3).

First we consider the scalar equation (3.3) and show that Theorem 3.1 applies to it. Recall that the kernel $k(x) \overset{\text{def}}{=} c\langle H(x)e,e\rangle$ (e is a vector of unit norm and $0\le c\le 1$) is an even nonnegative function (see (3.4)). Further, the norm on $L_1(0,\tau)$ (and also on $L_p(0,\tau)$) of the integral operator $\chi \mapsto {}_0\int^\tau k(\cdot-y)\chi(y)dy$ is estimated above by

$$ {}_{-\tau}\int^{+\tau}|k(y)|dy = 2\,{}_0\int^\tau k(y)dy = \begin{cases} c\,{}_1\int^{+\infty} z^{-2}(1-e^{-z\tau})dz, & \text{for neutron transport;}\\ 2\pi^{-\frac12}\,{}_0\int^{+\infty} e^{-w^2}(1-e^{-\tau/w})dw, & \text{for the BGK model.}\end{cases}$$

Since $c\,{}_1\int^{+\infty}z^{-2}dz = c \le 1$[18] and $2\pi^{-\frac12}\,{}_0\int^{+\infty}e^{-w^2}dw = 1$, for $0<\tau<+\infty$ both expressions at the right-hand side are strictly less than 1. So for $k(x) = c\langle H(x)e,e\rangle$ Eq. (3.6) has a unique solution ξ in $L_1(0,\tau)$. Clearly ξ is a nonnegative function.

Using (3.2) one concludes that for every $1\le p\le+\infty$ and every strongly measurable[16] right-hand side ω of Eq. (3.1) with $\|\omega(.)\| \in L_p(0,\tau)$ there is a unique[23] solution ψ of Eq. (3.1) with the property that ψ is strongly measurable and $\|\psi(.)\| \in L_p(0,\tau)$. By (3.2) this solution is given by

(4.1) $$\psi(x,\mu) = \begin{cases}\omega(x,\mu)+c\mu^{-1}\,{}_0\int^x e^{-(x-y)/\mu}\langle\omega(y),e\rangle dy, & \mu>0;\\ \omega(x,\mu)-c\mu^{-1}\,{}_x\int^\tau e^{-(x-y)/\mu}\langle\omega(y),e\rangle dy, & \mu>0.\end{cases}$$

In neutron physics $\sqrt{2}\langle\psi(x),e\rangle = {}_{-1}\int^{+1}\psi(x,\mu)d\mu$ represents the (angular-averaged) neutron density; in kinetic theory $\langle\psi(x),e\rangle = \pi^{-\frac12}\,{}_{-\infty}\int^{+\infty}\psi(x,\mu)e^{-\mu^2}d\mu$ represents the mass velocity. The function $\langle\psi(x),e\rangle$ can be calculated by solving Eq. (3.3) with the help of Theorem 3.1. One gets the solution ψ in the form

(4.2) $$\psi(x) = \omega(x) + c\,{}_0\int^\tau \langle\omega(y),e\rangle\left\{H(x-y)e + {}_0\int^\tau \delta(z,y)H(x-z)e\right\}dy.$$

Formula (4.2) will be our starting point for computing solutions of various boundary value problems; in all cases we calculate $\psi(0,\mu)$ and $\psi(\tau,\mu)$. To shorten notations we define a function Ψ by

$$\Psi(\mu) = \pi^{-\frac{1}{2}}e^{-\mu^2}, \quad \mu \in \mathbb{R}; \qquad \text{(BGK model)}$$

$$\text{(4.3)} \quad \Psi(\mu) = \begin{cases} \frac{1}{2}c, & -1\le\mu\le+1; \\ 0, & \mu<-1 \text{ or } \mu>+1. \end{cases} \qquad \text{(neutron transport)}$$

First we consider the boundary value problem (2.1) - (2.2) with $\phi \in D_T \subset H$ and $f \equiv 0$. Then the right-hand side ω of Eq. (3.1) has the form

$$\omega(x,\mu) = e^{-x/\mu}\phi(\mu) \qquad (\mu\ge0),$$
$$\omega(x,\mu) = e^{(\tau-x)/\mu}\phi(\mu) \qquad (\mu<0)$$

(see (2.6a)). Using (4.2) - (4.3) and the identities (3.9a) - (3.9b), and applying Theorem 3.1 one gets $\psi(0,\mu) = \phi(\mu)$ $(\mu\ge0)$, $\psi(\tau,\mu) = \phi(\mu)$ $(\mu<0)$ and

$$\psi(0,\mu) = e^{\tau/\mu}\phi(\mu) +$$
$$\text{(4.4a)} \quad +\int_{-\infty}^{0} \nu(\nu-\mu)^{-1}\Psi(\nu)\{X(-\mu)Y(-\nu)-X(-\nu)Y(-\mu)\}\phi(\nu)d\nu + \qquad (\mu<0)$$
$$+\int_{0}^{+\infty} \nu(\nu-\mu)^{-1}\Psi(\nu)\{X(-\mu)X(\nu)-Y(-\mu)Y(\nu)\}\phi(\nu)d\nu;$$

$$\psi(\tau,\mu) = e^{-\tau/\mu}\phi(\mu) +$$
$$\text{(4.4b)} \quad +\int_{0}^{+\infty} \nu(\nu-\mu)^{-1}\Psi(\nu)\{X(\mu)Y(\nu)-X(\nu)Y(\mu)\}\phi(\nu)d\nu + \qquad (\mu\ge0)$$
$$+\int_{-\infty}^{0} \nu(\nu-\mu)^{-1}\Psi(\nu)\{X(\mu)X(-\nu)-Y(\mu)Y(-\nu)\}\phi(\nu)d\nu.$$

For neutron transport these formulas have been derived in this way before by Van der Mee[7]; they agree with related formulas found by astrophysicists[5,6].

Secondly we consider the boundary value problem (2.1) - (2.2) with $\phi \equiv 0$ and $f(x) \equiv \chi$. Then the right-hand side ω of Eq. (2.1) has the form

$$\omega(x,\mu) = \{1-e^{-x/\mu}\}\chi(\mu) \qquad (\mu\ge0),$$
$$\omega(x,\mu) = \{1-e^{(\tau-x)/\mu}\}\chi(\mu) \qquad (\mu<0)$$

(see (2.6c)). Using (4.2) - (4.3) and the identities (3.9a) - (3.9c) one gets

$$\psi(0,\mu) = (1-e^{\tau/\mu})\chi(\mu) -$$
$$-\int_{-\infty}^{0} \nu(\nu-\mu)^{-1}\Psi(\nu)\{X(-\mu)Y(-\nu)-X(-\nu)Y(-\mu)\}\chi(\nu)d\nu +$$
$$\text{(4.5a)} \quad -X(\infty)\{X(-\mu)-Y(-\mu)\}\int_{-1}^{+1}\Psi(\nu)\chi(\nu)d\nu - \qquad (\mu<0)$$
$$-\int_{0}^{+\infty} \nu(\nu-\mu)^{-1}\Psi(\nu)\{X(-\mu)X(\nu)-Y(-\mu)Y(\nu)\}\chi(\nu)d\nu;$$

$$\psi(\tau,\mu) = (1-e^{-\tau/\mu})\chi(\mu) -$$
$$-\int_{0}^{+\infty} \nu(\nu-\mu)^{-1}\Psi(\nu)\{X(\mu)Y(\nu)-X(\nu)Y(\mu)\}\chi(\nu)d\nu +$$
$$\text{(4.5b)} \quad -X(\infty)\{X(\mu)-Y(\mu)\}\int_{-1}^{+1}\Psi(\nu)\chi(\nu)d\nu - \qquad (\mu\ge0)$$
$$-\int_{-\infty}^{0} \nu(\nu-\mu)^{-1}\Psi(\nu)\{X(\mu)X(-\nu)-Y(\mu)Y(-\nu)\}\chi(\nu)d\nu.$$

Within the linearized BGK model a plane Couette flow[9] is described by Eqs. (2.1) - (2.2) with T and B as in (1.2b), $f(x) \equiv 0$ and $\phi(\mu) = -\text{sgn}\,\mu$; a plane Poiseuille flow[10] is described by the same equations, but with $\phi(\mu) \equiv 0$ and $f(x,\mu) \equiv -\frac{1}{2}k$. Here k is some dimensionless constant. To obtain $\psi(0,\mu)$ and $\psi(\tau,\mu)$ for a plane Couette flow one substitutes $\phi(\mu) = -\text{sgn}\,\mu$ into (4.4), writes $\nu(\nu-\mu)^{-1} = 1 + \mu(\nu-\mu)^{-1}$, employs (5.12a) - (5.12b), (5.16) - (5.17) (to be derived later in Section 5) and obtains

$$\psi(0,\mu) = -1 + \{1+X(\infty)^{-1}\}[X(-\mu)+Y(-\mu)] \qquad (\mu<0),$$
$$\psi(0,\mu) = -1 \qquad (\mu\ge0);$$
$$\psi(\tau,\mu) = 1 - \{1+X(\infty)^{-1}\}[X(\mu)+Y(\mu)] \qquad (\mu\ge0),$$
$$\psi(\tau,\mu) = +1 \qquad (\mu<0).$$

To compute $\psi(0,\mu)$ and $\psi(\tau,\mu)$ for a plane Poiseuille flow one substitutes $\chi(\mu) = -1$ into (4.5), writes $\nu(\nu-\mu)^{-1} = 1 + \mu(\nu-\mu)^{-1}$, employs (5.12a) - (5.12b), (5.16) - (5.17) and obtains[24]

$$\psi(0,\mu) = \{1+(2c-1)X(\infty)\}[X(-\mu)-Y(-\mu)] \quad (\mu<0),$$
$$\psi(0,\mu) = 0 \quad (\mu\geq 0);$$
$$\psi(\tau,\mu) = \{1+(2c-1)X(\infty)\}[X(\mu)-Y(\mu)] \quad (\mu\geq 0),$$
$$\psi(\tau,\mu) = 0 \quad (\mu<0).$$

5. X- and Y-functions and the equations they satisfy

To generalize the X- and Y-functions of the previous section we consider the Banach space NBV$[0,+\infty]$[25] of all functions $f:[0,+\infty]\to\mathbb{C}$ of bounded variation that are continuous on the right and satisfy $f(0)=0$. This space is endowed with the norm

$$V(f) = \sup\{\sum_{k=1}^{n}|f(b_k)-f(a_k)| : 0\leq a_1<b_1\leq a_2<b_2\leq\ldots<b_n\leq+\infty; n\in\mathbb{N}\};$$

the number $V(f)$ is usually called the total variation of f. To every $f\in$ NBV$[0,+\infty]$ one associates a dispersion function Λ_f and an exponential integral function Ei_f, defined by

$$\Lambda_f(\lambda) = 1 - 2\lambda^2\int_0^{+\infty}\frac{df(t)}{\lambda^2-t^2};$$

(5.1)

$$Ei_f(z) = \int_0^{+\infty} t^{-1}e^{-|z|/t}df(t), \quad 0\neq z\in\mathbb{R}.$$

Let S_f denote the part of $[-\infty,+\infty]$ where $f(t)$ or $f(-t)$ is non-constant. Then Λ_f is analytic on the Riemann-sphere cut along S_f. Further, Λ_f is continuous on the extended imaginary line with $\Lambda_f(0)=1$ and $\Lambda_f(\infty) = 1-2f(+\infty)$. Also

(5.2) $$\Lambda_f(\lambda) = \Lambda_f(-\lambda); \quad \overline{\Lambda_f(\bar\lambda)} = \Lambda_{\bar f}(\lambda).$$

So for real-valued f the function Λ_f is real-valued on the imaginary line and on $\mathbb{R}\backslash S_f$. If $F(t)$ denotes the total variation of f on $[0,t]$ (so that $V(F)=V(f)$), then

(5.3a) $$\int_{-\infty}^{+\infty}|Ei_f(z)|dz \leq 2\int_0^{+\infty}\int_0^{+\infty}t^{-1}e^{-z/t}dzdF(t) = 2V(F) = 2V(f)<+\infty;$$

(5.3b) $$\int_{-\tau}^{+\tau}|Ei_f(z)|dz \leq 2\int_0^{+\infty}\int_0^{\tau}t^{-1}e^{-z/t}dzdF(t) = 2\int_0^{+\infty}(1-e^{-\tau/t})dF(t) \leq 2V(f) < +\infty.$$

The connection between Λ_f and Ei_f is given by

(5.4) $$\Lambda_f(\lambda) = 1 - \int_{-\infty}^{+\infty}e^{z/\lambda}Ei_f(z)dz, \quad \text{Re}\,\lambda = 0.$$

THEOREM 5.1. *Let $f\in$ NBV$[0,+\infty]$. Then the following four statements are equivalent:*

(1) *the dispersion function Λ_f does not vanish on the extended imaginary line;*

(2) *there exists a (unique) continuous function H_f on the closed right half-plane that is analytic on the open right half-plane, does not vanish, has the property $H_f(0)=1$ and satisfies the identity*

(5.5) $$\Lambda_f(\lambda) = H_f(\lambda)^{-1}H_f(-\lambda)^{-1}, \quad \text{Re}\,\lambda = 0;$$

(3) *there exists a (unique) solution ξ_f in $L_1(0,+\infty)$ of the Wiener-Hopf equation*

(5.6) $$\xi_f(x) - \int_0^{+\infty}Ei_f(x-y)\xi_f(y)dy = Ei_f(x) \quad (0<x<+\infty);$$

(4) *for every $1\leq p\leq+\infty$ and $\zeta\in L_p(0,+\infty)$ there is a unique solution χ in $L_p(0,+\infty)$ of the Wiener-Hopf equation*

(5.7) $$\chi(x) - \int_0^{+\infty}Ei_f(x-y)\chi(y)dy = \zeta(x) \quad (0<x<+\infty).$$

Proof. Because $\int_{-\infty}^{+\infty}|Ei_f(z)|dz < +\infty$, this theorem is basically known and can be derived from the theory of Wiener-Hopf equations developed by Krein[26,20]. Therefore, we sketch the proof only. If (3) is fulfilled, then Condition (4) is fulfilled too. In fact,

$$\chi(x) = \zeta(x) + \int_0^{+\infty}\delta_f(x,y)\zeta(y)dy \quad (0<x<+\infty),$$

where the resolvent kernel $\delta_f(x,y)$ is given by

$$\delta_f(x,y) = \begin{cases} \xi_f(|x-y|) + \int_0^{\min(x,y)} \xi_f(x-z)\xi_f(y-z)dz; & \\ & (0\le x\ne y<+\infty) \\ \xi_f(|x-y|) + \int_{\max(x,y)}^{+\infty} \xi_f(z-x)\xi_f(z-y)dz. & \end{cases}$$

The converse implication (4) ⇒ (3) is clear.

If (1) is fulfilled, there exist continuous functions H_f and K_f on the closed right half-plane that are analytic on the open right half-plane, do not vanish and satisfy $H_f(0) = K_f(0) = 1$, and there exists a unique integer κ such that

$$\Lambda_f(\lambda) = H_f(\lambda)^{-1}((1+\lambda)/(1-\lambda))^{\kappa}K_f(-\lambda)^{-1}, \quad \mathrm{Re}\ \lambda = 0.$$

The index κ is uniquely determined by Λ_f. So using (5.2) one has

$$\Lambda_f(\lambda) = K_f(\lambda)^{-1}((1+\lambda)/(1-\lambda))^{-\kappa}H_f(-\lambda)^{-1}, \quad \mathrm{Re}\ \lambda = 0,$$

and thus $\kappa = -\kappa = 0$. But then $K_f(\lambda)H_f(\lambda)^{-1} = K_f(-\lambda)H_f(-\lambda)^{-1}$, Re $\lambda = 0$. By Liouville's theorem and $H_f(0) = K_f(0) = 1$ one has $K_f(\lambda)/H_f(\lambda) \equiv 1$. So (2) is clear. The converse implication (2) ⇒ (1) is trivially established.

If (3) is fulfilled, then Condition (2) is fulfilled for

$$(5.8)\qquad H_f(\lambda) = 1 + \int_0^{+\infty} e^{-t/\lambda}\xi_f(t)dt, \qquad \mathrm{Re}\ \lambda \ge 0.$$

The implication (2) ⇒ (3) (or (2) ⇒ (4)) is a standard argument that involves the reduction of Eq. (5.6) (or Eq. (5.7)) to a uniquely solvable Riemann-Hilbert problem[20]. □

The function H_f appearing in (5.5) will be called the <u>H-function</u>. As it will be clear from Eqs. (5.12) (for $\tau \to +\infty$), it satisfies the <u>H-equation</u>

$$(5.9)\qquad H_f(\lambda)^{-1} = 1 - \lambda\int_0^{+\infty} (t+\lambda)^{-1}H_f(t)df(t), \qquad \lambda \notin [-\infty,0] \cap S_f.$$

In astrophysics a special form of this equation is studied and applied intensively[6,11]. With the H-function in the present form (with $f \in NBV[0,+\infty]$) Krein[27] has derived Eq. (5.9) for functions $f \in NBV[0,+\infty]$ satisfying Condition (1) of Theorem 5.1.

<u>THEOREM 5.2</u>. *Let* $0<\tau<+\infty$ *and* $f \in NBV[0,+\infty]$. *Then the following two statements are equivalent:*

(1) *there exists a (unique) solution* ξ_f^τ *in* $L_1(0,\tau)$ *of the convolution equation*

$$\xi_f^\tau(x) - \int_0^\tau Ei_f(x-y)\xi_f^\tau(y)dy = Ei_f(x) \quad (0<x<\tau);$$

(2) *for every* $1\le p\le+\infty$ *and* $\zeta \in L_p(0,+\infty)$ *there is a unique solution* χ *in* $L_p(0,\tau)$ *of the convolution equation*

$$(5.10)\qquad \chi(x) - \int_0^\tau Ei_f(x-y)\chi(y)dy = \zeta(x) \quad (0<x<\tau).$$

By (5.3b) and the fact that Ei_f is an even function, this theorem is immediate from Theorem 3.1. The solution χ of Eq. (5.10) can be obtained using Theorem 3.1.

With $f \in NBV[0,+\infty]$ we associate X- and Y-functions. Suppose that for $0<\tau<+\infty$ the first (and thus the second) statement of Theorem 5.2 holds true. Put

$$(5.11a)\qquad \begin{aligned} X_{\tau,f}(\mu) &= 1 + \int_0^\tau e^{-t/\mu}\xi_f^\tau(t)dt, \\ Y_{\tau,f}(\mu) &= e^{-\tau/\mu} + \int_0^\tau e^{-(\tau-t)/\mu}\xi_f^\tau(t)dt; \end{aligned} \qquad 0\ne\mu\in\mathbb{C}.$$

It follows from the contents of Section 3 that $X_{\tau,f}(\mu) = \eta_{\tau,\mu}^f(0)$ and $Y_{\tau,f}(\mu) = \eta_{\tau,\mu}^f(\tau)$, where $\eta_{\tau,\mu}^f$ is the unique solution in $L_1(0,\tau)$ of the convolution equation

$$\text{(5.11b)}\quad \eta_{\tau,\mu}^{f}(x) - \int_0^\tau Ei_f(x-y)\eta_{\tau,\mu}^{f}(y)dy = e^{-x/\mu} \qquad (0<x<\tau;\ 0\neq\mu\in\mathbb{C}).$$

THEOREM 5.3. *Let* $0<\tau<+\infty$ *and* $f \in NBV[0,+\infty]$. *Suppose that the conditions of Theorem* 5.2 *are fulfilled. Then the functions* $X_{\tau,f}$ *and* $Y_{\tau,f}$ *satisfy the non-linear equations*

$$\text{(5.12a)}\quad X_{\tau,f}(\mu) = 1+\mu\int_0^{+\infty}\frac{X_{\tau,f}(\mu)X_{\tau,f}(\nu)-Y_{\tau,f}(\mu)Y_{\tau,f}(\nu)}{\nu+\mu}df(\nu);$$

$$(\mu\notin[-\infty,0]\cap S_f)$$

$$\text{(5.12b)}\quad Y_{\tau,f}(\mu) = e^{-\tau/\mu}+\mu\int_0^{+\infty}\frac{X_{\tau,f}(\mu)Y_{\tau,f}(\nu)-Y_{\tau,f}(\mu)X_{\tau,f}(\nu)}{\nu-\mu}df(\nu),$$

and the linear equations

$$\text{(5.13a)}\quad \Lambda_f(\mu)X_{\tau,f}(\mu) = 1+\mu\int_0^{+\infty}\frac{X_{\tau,f}(\nu)}{\nu-\mu}df(\nu) - e^{-\tau/\mu}\mu\int_0^{+\infty}\frac{Y_{\tau,f}(\nu)}{\nu+\mu}df(\nu);$$

$$\text{(5.13b)}\quad \Lambda_f(\mu)Y_{\tau,f}(\mu) = e^{-\tau/\mu} - e^{-\tau/\mu}\mu\int_0^{+\infty}\frac{X_{\tau,f}(\nu)}{\nu+\mu}df(\nu) + \mu\int_0^{+\infty}\frac{Y_{\tau,f}(\nu)}{\nu-\mu}df(\nu). \qquad (\mu\notin S_f)$$

Proof. To derive (5.12a) one premultiplies (3.9a) (with $k(x) = Ei_f(x)$) by ν^{-1} and integrates with respect to the (complex-valued) measure on $[0,+\infty]$ induced by $f \in NBV[0,+\infty]$. After this the second part of (5.1) is employed to yield

$$\text{(5.14a)}\quad \int_0^\tau Ei_f(x)\{e^{-x/\mu}+\int_0^\tau e^{-y/\mu}\delta(x,y)dy\}dx = \mu\int_0^{+\infty}\frac{X_{\tau,f}(\mu)X_{\tau,f}(\nu)-Y_{\tau,f}(\mu)Y_{\tau,f}(\nu)}{\nu+\mu}df(\nu),$$

where $\delta(x,y)$ is the resolvent kernel of Eq. (5.10). Next one applies Theorem 3.1 to Eq. (5.11b) and gets

$$\text{(5.14b)}\quad \eta_{\tau,\mu}^{f}(x) = e^{-x/\mu}+\int_0^\tau e^{-y/\mu}\delta(x,y)dy, \qquad 0<x<\tau.$$

From (5.11b) it is clear that $\int_0^\tau Ei_f(x)\eta_{\tau,\mu}^{f}(x)dx = \eta_{\tau,\mu}^{f}(0) - 1$. Together with (5.14a) and (5.14b) this yields (5.12a). To derive (5.12b) one proceeds similarly, but starts from (3.9b).

To deduce (5.13) one rewrites Eqs. (5.12) as follows:

$$\text{(5.15a)}\quad \left\{1-\mu\int_0^{+\infty}\frac{X_{\tau,f}(\nu)}{\nu+\mu}df(\nu)\right\}X_{\tau,f}(\mu) + \left\{\mu\int_0^{+\infty}\frac{Y_{\tau,f}(\nu)}{\nu+\mu}df(\nu)\right\}Y_{\tau,f}(\mu) = 1;$$

$$(\mu\notin S_f)$$

$$\text{(5.15b)}\quad \left\{-\mu\int_0^{+\infty}\frac{Y_{\tau,f}(\nu)}{\nu-\mu}df(\nu)\right\}X_{\tau,f}(\mu) + \left\{1+\mu\int_0^{+\infty}\frac{X_{\tau,f}(\nu)}{\nu-\mu}df(\nu)\right\}Y_{\tau,f}(\mu) = e^{-\tau/\mu}.$$

With the help of (5.12a) one computes the determinant of this linear system of equations for the unknown $X_{\tau,f}(\mu)$ and $Y_{\tau,f}(\mu)$ and gets $\Lambda_f(\mu)$. The solution of the system (5.15) has the form (5.13). □

From (5.11a) it is clear that $X_{\tau,f}(\infty) = Y_{\tau,f}(\infty) = 1 + \int_0^\tau \xi_f^\tau(t)dt$. Using (5.12a) (for $\mu\to\infty$) and (5.13a) (for $\mu\to\infty$) one gets

$$X_{\tau,f}(\infty) = 1 + X_{\tau,f}(\infty)\left\{\int_0^{+\infty}X_{\tau,f}(\nu)df(\nu) - \int_0^{+\infty}Y_{\tau,f}(\nu)df(\nu)\right\};$$

$$\Lambda_f(\infty)X_{\tau,f}(\infty) = 1 - \left\{\int_0^{+\infty}X_{\tau,f}(\nu)df(\nu) + \int_0^{+\infty}Y_{\tau,f}(\nu)df(\nu)\right\}.$$

Thus $X_{\tau,f}(\infty) = Y_{\tau,f}(\infty) \neq 0$, and for $x_\infty = \int_0^{+\infty}X_{\tau,f}(\nu)df(\nu)$

and for $y_\infty = \int_0^{+\infty} Y_{\tau,f}(\nu)df(\nu)$ one has

$$x_\infty = 1 - \tfrac{1}{2}X_{\tau,f}(\infty)^{-1} - \tfrac{1}{2}\Lambda_f(\infty)X_{\tau,f}(\infty), \tag{5.16}$$
$$y_\infty = 1 + \tfrac{1}{2}X_{\tau,f}(\infty)^{-1} - \tfrac{1}{2}\Lambda_f(\infty)X_{\tau,f}(\infty).$$

In astrophysics the functions $f \in NBV[0,+\infty]$ appearing in (5.12) - (5.13) are of a special type. They have the form

$$f(t) = \int_0^{\min(t,1)} \Psi(u)du, \quad 0\le t\le +\infty, \tag{5.17}$$

where Ψ: $[0,1] \to \mathbb{R}$ is a continuous function called the <u>characteristic function</u>. For this case the functions H_f, $X_{\tau,f}$ and $Y_{\tau,f}$ have been applied frequently[6]. A systematic study of these functions has been made by Busbridge[11] and Mullikin[12] for the case when Ψ is non-negative and satisfies a Hölder condition. For this case Busbridge[11] derived (5.13) from (5.12). Mullikin[12,13] found constraints on the equations (5.12) and (5.13) such that the functions in (5.11a) are the unique solutions of the (non-linear) equations (5.12) and the (linear) equations (5.13) that satisfy these constraints.

For the linearized BGK model one has $f(t) = \pi^{-\frac{1}{2}}\int_0^t e^{-u^2}du$, $0\le t\le +\infty$. It is known[3,4] that in this case Λ_f is analytic on $\mathbb{C}\setminus\mathbb{R}$, has a cut on the full real line, does not vanish on $\mathbb{C}\setminus\mathbb{R}$ and is continuous on the extended imaginary line with a double zero at infinity. For this case the derivation[28] of the constraints for the X- and Y-equations in the conservative case of the equation of radiative transfer can be repeated. It appears that the functions $X_{\tau,f}$ and $Y_{\tau,f}$ in (5.11a) are the unique solutions of Eqs. (5.12) and of Eqs. (5.13) that satisfy the constraints

$$\pi^{-\frac{1}{2}}\int_0^{+\infty} e^{-u^2}\left\{X_{\tau,f}(u) + Y_{\tau,f}(u)\right\}du = 1;$$
$$\int_0^{+\infty} ue^{-u^2}\left\{X_{\tau,f}(u) + Y_{\tau,f}(u)\right\}du = \tau\int_0^{+\infty} e^{-u^2}Y_{\tau,f}(u)du.$$

6. <u>Stability and rational approximation of H-functions</u>

In this section we prove the following stability theorem and apply it to get rational approximations of H-functions.

<u>Theorem 6.1</u>. *Let* $(f_n)_{n=1}^{+\infty}$ *be a sequence in* $NBV[0,+\infty]$ *such that* $V(f_n-f) \to 0$ $(n\to+\infty)$. *Suppose that the dispersion function does not vanish on the extended imaginary line. Then for* $n\ge n_0$ *the dispersion function* Λ_{f_n} *has the same property and for the* H-*functions one has*

$$\lim_{n\to+\infty} \max_{\mathrm{Re}\lambda\ge 0} |H_{f_n}(\lambda) - H_f(\lambda)| = 0.$$

<u>Proof</u>. On the Banach space $L_1(0,+\infty)$ one considers the integral operators K_1, K_2, ... and K, defined by

$$(K_nh)(x) = \int_0^{+\infty} Ei_{f_n}(x-y)h(y)dy,$$
$$(Kh)(x) = \int_0^{+\infty} Ei_f(x-y)h(y)dy.$$

Using the identity $Ei_{f_n} - Ei_f = Ei_{f_n-f}$ and (5.3a) one gets the estimate

$$\|K_n - K\| \le \int_{-\infty}^{+\infty} |Ei_{f_n}(z) - Ei_f(z)|dz =$$
$$= \int_{-\infty}^{+\infty} |Ei_{f_n-f}(z)|dz \le 2V(f_n - f).$$

By Theorem 5.1 the operator I-K is invertible. So for $n\ge n_0$ the operator $I-K_n$ is invertible and for the function $\xi_{f_n} = (I-K_n)^{-1}Ei_{f_n}$ one has

$$\lim_{n\to+\infty} \int_0^{+\infty} |\xi_{f_n}(x) - \xi_f(x)|\,dx = 0.$$

Using (5.8) (and its analogue for f_n) the theorem is clear. □

Now let $g = g(t_1,\ldots,t_n;\alpha_1,\ldots,\alpha_n)$ be the following step function:

$$(6.1)\qquad \begin{aligned} &g(t) = 0 \ (0\le t<t_1),\ g(t) = \alpha_1 \ (t_1\le t<t_2),\ \ldots, \\ &g(t) = \alpha_1 + \ldots + \alpha_n \ (t_n \le t \le +\infty). \end{aligned}$$

Here the jump points $0<t_1<t_2<\ldots<t_n\le+\infty$ and the jumps $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ are given. For such g the functions Λ_g and Ei_g are of a simple type, namely

$$(6.2)\qquad \Lambda_g(\lambda) = 1-2\lambda^2 \sum_{m=1}^{n} \frac{\alpha_m}{\lambda^2-t_m^2}, \quad Ei_g(z) = \sum_{m=1}^{n} \frac{\alpha_m}{t_m} e^{-|z|/t_m}.$$

If Λ_g does not vanish on the extended imaginary line, then

$$(6.3)\qquad \begin{aligned} &\Lambda_g(\infty) = 1-2\sum_{m=1}^{n}\alpha_m \neq 0, \\ &\Lambda_g(\lambda) = (1-2\sum_{m=1}^{n}\alpha_m)\frac{(\lambda^2-x_1^2)\ldots(\lambda^2-x_n^2)}{(\lambda^2-t_1^2)\ldots(\lambda^2-t_n^2)}, \end{aligned}$$

where $x_1,\ldots,x_n$ are certain points in the open right half-plane. Using that $H_g(0) = 1$, one gets

$$(6.4)\qquad H_g(\lambda) = \frac{1}{\sqrt{1-2(\alpha_1+\ldots+\alpha_n)}}\,\frac{\lambda+t_1}{\lambda+x_1}\cdots\frac{\lambda+t_n}{\lambda+x_n}, \quad \mathrm{Re}\ \lambda \ge 0,$$

where the square root is taken in the open right half-plane.

Let $g \in NBV[0,+\infty]$ such that Λ_f does not vanish on the extended imaginary line. If $(g_n)_{n=1}^{+\infty}$ is a sequence of step functions converging to f in the norm of $NBV[0,+\infty]$, then, according to Theorem 6.1, for $n \ge n_0$ the sequence of rational functions $(H_{g_n})_n$ tends to H_f uniformly on the closed right half-plane. In this way a sequence of rational approximants of H_g is constructed.

Masson[14] has considered the H-function H_f in the case when $f \in NBV[0,+\infty]$ is monotonically non-decreasing, constant on $[1,+\infty]$ (the case needed for many astrophysical applications) and continuous at $t=1$. Assuming $f(1) = f(+\infty) < \frac{4}{9}$ Masson has proved the convergence to H_f of a sequence of special rational approximants of the form (6.4). The method has been stated by him for $f(1) = f(+\infty) \le \frac{1}{2}$ (but the convergence proof has been given for $f(1) = f(+\infty) < \frac{4}{9}$). Here we extend Massons method for $f(1) = f(+\infty) < \frac{1}{2}$, but allow a more extensive class of functions f and rational approximants to f.

7. Stability and rational approximation of X- and Y-functions

In this section the following stability theorem is derived and applied to obtain rational approximations of X- and Y-functions.

THEOREM 7.1. *Let* $0<\tau<+\infty$, *and let* $(f_n)_{n=1}^{+\infty}$ *be a sequence in* $NBV[0,+\infty]$ *such that* $V(f_n-f) \to 0$ $(n\to+\infty)$. *Suppose that the convolution equation*

$$(7.1)\qquad \xi_f^\tau(x) - \int_0^\tau Ei_f(x-y)\xi_f^\tau(y)\,dy = Ei_f(x) \qquad (0<x<\tau)$$

has a (unique) solution ξ_f^τ *in* $L_1(0,\tau)$. *Then for* $n\ge n_0$ *this equation with* f_n *instead of* f *has a unique solution* $\xi_{f_n}^\tau$ *in* $L_1(0,\tau)$ *and for the corresponding* X- *and* Y-*functions one has*

$$\lim_{n\to+\infty} \max_{\mathrm{Re}\lambda\ge0} |X_{\tau,f_n}(\lambda) - X_{\tau,f}(\lambda)| = 0;$$

$$\lim_{n\to+\infty} \max_{\mathrm{Re}\lambda\ge0} |Y_{\tau,f_n}(\lambda) - Y_{\tau,f}(\lambda)| = 0.$$

As the proof of this theorem is analogous to the one of Theorem 6.1, it is omitted.

Now let $g = g(t_1,\dots,t_n;\alpha_1,\dots,\alpha_n)$ be the step function in (6.1), and suppose that Eq. (7.1) with f replaced by g has a (unique) solution ξ_g^τ in $L_1(0,\tau)$. Then $X_{\tau,g}$ and $Y_{\tau,g}$ are analytic on $\mathbb{C}\setminus\{0\}$ and satisfy Eqs. (5.13) with f replaced by g. So

$$\Lambda_g(\mu)X_{\tau,g}(\mu) = 1 + \mu\sum_{j=1}^{n}\frac{\alpha_j X_{\tau,g}(t_j)}{t_j-\mu} - e^{-\tau/\mu}\mu\sum_{j=1}^{n}\frac{\alpha_j Y_{\tau,g}(t_j)}{t_j+\mu}; \tag{7.2a}$$

$$(\mu\notin\{-t_n,\dots,-t_1,0,t_1,\dots t_n\})$$

$$\Lambda_g(\mu)Y_{\tau,g}(\mu) = e^{-\tau/\mu} - e^{-\tau/\mu}\mu\sum_{j=1}^{n}\frac{\alpha_j X_{\tau,g}(t_j)}{t_j+\mu} + \mu\sum_{j=1}^{n}\frac{\alpha_j X_{\tau,g}(t_j)}{t_j+\mu}. \tag{7.2b}$$

Hence, for $Z_{\tau,g}^{\pm} = X_{\tau,g} \pm Y_{\tau,g}$ and $c_\pm(\mu) = 1 \pm e^{-\tau/\mu}$ one gets

$$\Lambda_g(\mu)Z_{\tau,g}^{\pm}(\mu) = c_\pm(\mu) + \mu\sum_{j=1}^{n}\frac{\alpha_j Z_{\tau,g}^{\pm}(t_j)}{t_j-\mu} \mp e^{-\tau/\mu}\mu\sum_{j=1}^{n}\frac{\alpha_j Z_{\tau,g}^{\pm}(t_j)}{t_j-\mu}. \tag{7.3}$$

As Λ_g is explicitly given by (6.2), it suffices to compute the numbers $X_{\tau,g}(t_j)$ and $Y_{\tau,g}(t_j)$ $(j=1,\dots,n)$, or the numbers $Z_{\tau,g}^{\pm}(t_j)$ $(j=1,\dots,n)$.

By (5.2) one has

$$Z_{\tau,g}^{\pm}(-\mu) = \pm e^{+\tau/\mu}Z_{\tau,g}^{\pm}(\mu).$$

With the help of (6.2) one gets

$$Z_{\tau,g}^{\pm}(\mu) = c_\pm(\mu) + \mu\sum_{j=1}^{n}\alpha_j\frac{Z_{\tau,g}^{\pm}(\mu)-Z_{\tau,g}^{\pm}(t_j)}{\mu-t_j} \pm e^{-\tau/\mu}\mu\sum_{j=1}^{n}\frac{Z_{\tau,g}^{\pm}(-\mu)-Z_{\tau,g}^{\pm}(t_j)}{\mu+t_j}. \tag{7.4}$$

From (7.3) and the analyticity of Λ_g and Λ_g^{-1} in a neighbourhood of $\mu=0$ one sees that $Z_{\tau,g}^{\pm}(1/\mu)$ is an

entire function of order $\le \tau$. An easy Liouville argument based on (7.4) yields that $Z_{\tau,g}^{\pm}$ is the only solution of Eq. (7.3) that is analytic on $\mathbb{C}_\infty\setminus\{0\}$ and has the property that $Z_{\tau,g}^{\pm}(1/\mu)$ is entire of order $\le \tau$.

Assume that $\Lambda_g(\infty) = 1-2(\alpha_1+\dots+\alpha_n) \neq 0$. Then Λ_g has the form (6.3) for certain numbers $x_1,\dots,x_n$ with $-\frac{1}{2}\pi<\arg x_m\le\frac{1}{2}\pi$ $(m = 1,2,\dots,n)$. Then $\Lambda_g(x_m) = 0$ for $m = 1,\dots,n$. For $m = 1,\dots,n$ one substitutes $\mu = x_m$ into (7.3) and gets

$$\sum_{j=1}^{n}\left(\frac{1}{x_m-t_j} \pm \frac{e^{-\tau/x_m}}{x_m+t_j}\right)\alpha_j Z_{\tau,g}^{\pm}(t_j) = \frac{c_\pm(x_m)}{x_m} \tag{7.5}$$

$$(m = 1,\dots,n).$$

If the determinant of the matrix

$$V_{\tau,g}^{\pm} = \left[(x_m-t_j)^{-1} \pm e^{-\tau/x_m}(x_m+t_j)^{-1}\right]_{m,j=1}^{n} \tag{7.6}$$

would vanish, then the homogeneous version of the linear system of equations (7.5) for the unknown $\alpha_j Z_{\tau,g}^{\pm}(t_j)$ $(j = 1,\dots,n)$ would have a non-trivial solution, and thus there would exist a solution $W^\pm$ of the homogeneous version of Eq. (7.3) that is non-trivial, analytic on $\mathbb{C}_\infty\setminus\{0\}$ and has the property that $W^\pm(1/\mu)$ is an entire function of order $\le \tau$. Contradiction. Hence, the matrices $V_{\tau,g}^{\pm}$ are invertible.

<u>THEOREM 7.2</u>. *Let $0<\tau<+\infty$, $\alpha_1,\dots,\alpha_n$ non-zero and $g = g(t_1,\dots,t_n;\alpha_1,\dots,\alpha_n)$ the step function in (6.1). Suppose that $\Lambda_g(\infty) = 1-2(\alpha_1+\dots+\alpha_n) \neq 0$. Then the convolution equation*

$$\xi_g^\tau(x) - \int_0^\tau Ei_g(x-y)\xi_g^\tau(y)dy = Ei_g(x) \quad (0<x<\tau) \tag{7.7}$$

has a (unique) solution ξ_g^τ in $L_1(0,\tau)$ if and only if for the zeros $x_1,\dots,x_n$ of the dispersion function Λ_g with $-\frac{1}{2}\pi<\arg x_m\le\frac{1}{2}\pi$ $(m = 1,\dots,n)$ the matrices $V_{\tau,g}^{\pm}$ in (7.6) are invertible. In that case the X- and Y-functions have the form

$$(7.8a)\quad X_{\tau,g}(\mu) = \Lambda_g(\mu)^{-1}\left\{1+\mu\sum_{j=1}^{n}\frac{\xi_j}{t_j-\mu} - e^{-\tau/\mu}\mu\sum_{j=1}^{n}\frac{\eta_j}{t_j+\mu}\right\};$$

$$(7.8b)\quad Y_{\tau,g}(\mu) = \Lambda_g(\mu)^{-1}\left\{e^{-\tau/\mu}-e^{-\tau/\mu}\mu\sum_{j=1}^{n}\frac{\xi_j}{t_j+\mu}+\mu\sum_{j=1}^{n}\frac{\eta_j}{t_j-\mu}\right\},$$

where $\xi = \frac{1}{2}(\zeta^+ + \zeta^-)$, $\eta = \frac{1}{2}(\zeta^+ - \zeta^-)$ *and*

$\zeta^\pm = (V^\pm_{\tau,g})^{-1}\left((1\pm e^{-\tau/x_j})/x_j\right)_{j=1}^{n}$.

Proof. One part of the theorem has been proved already and from this part the formulas (7.8) are immediate. To prove the converse part we assume that the matrices (7.8) are invertible. Then the functions (7.8) satisfy Eqs. (5.13) and are analytic on $\mathbb{C}_\infty\setminus\{0\}$. There exists $\varepsilon>0$ such that for $0<|\zeta|<\varepsilon$ the convolution equation (7.7) with g replaced by $(1+\zeta)g$ has a unique solution $\xi^\tau_{(1+\zeta)g}$ in $L_1(0,\tau)$ and such that the matrices $V^\pm_{\tau,(1+\zeta)g}$ (i.e., the matrices (7.6) with g replaced by $(1+\zeta)g$) are invertible. This is a consequence of the compactness of the integral operator $\xi \mapsto {}_0\int^\tau Ei_g(\cdot-y)\xi(y)dy$ on $L_1(0,\tau)$. Observe that $\|V^\pm_{\tau,(1+\zeta)g} - V^\pm_{\tau,g}\| \to 0$ as $\zeta \to 0$. Hence,

$$\lim_{\zeta\to 0}\ \max_{\mathrm{Re}\,\mu\geq 0}|X_{\tau,g}(\mu) - X_{\tau,(1+\zeta)g}(\mu)| = \lim_{\zeta\to 0}\ \max_{\mathrm{Re}\,\mu\geq 0}|Y_{\tau,g}(\mu) - Y_{\tau,(1+\zeta)g}(\mu)| = 0.$$

However, since $Ei_{(1+\zeta)g} \in L_2(0,\tau)$ $(0\leq|\zeta|<\varepsilon)$, it is clear that $\xi^\tau_{(1+\zeta)g} \in L_2(0,\tau)$ $(0<|\zeta|<\varepsilon)$. Using the representation (6.3) for Λ_g it follows that $X_{\tau,g}$ has the form

$$X_{\tau,g}(\mu) = 1 + \int_0^\tau e^{-x/\mu}\xi^\tau_g(x)dx, \qquad \mathrm{Re}\,\mu \geq 0,$$

where $\xi^\tau_g \in L_2(0,\tau)$. But at the same time one has

$$\lim_{\zeta\to 0}\int_{-\infty}^{+\infty}|X_{\tau,g}(-i/k) - X_{\tau,(1+\zeta)g}(-i/k)|^2dk = 0,$$

and therefore $\|\xi^\tau_{(1+\zeta)g} - \xi^\tau_g\|_2 \to 0$ as $\zeta \to 0$. From the latter identity it follows that $\xi^\tau_g \in L_1(0,\tau)$ is a solution of Eq. (7.7). □

Theorem 7.2 enables us to apply the stability theorem 7.1 to find rational approximations of the X- and Y-functions. To be more precise, let $f \in NBV[0,+\infty]$ fulfill the conditions of Theorem 7.1, and let $\Lambda_f(\infty) = 1-2f(+\infty) \neq 0$. Choose a sequence $(f_n)_{n=0}^{+\infty}$ of step functions such that $V(f_n-f) \to 0$ as $n \to +\infty$. Then for $n \geq n_0$ one has $\Lambda_{f_n}(\infty) = 1-2f_n(+\infty) \neq 0$, while the convolution equation (7.7) with g replaced by f_n has a solution in $L_1(0,\tau)$. Thus the conclusion of Theorem 7.1 holds true and the approximants X_{τ,f_n} and Y_{τ,f_n} of $X_{\tau,f}$ and $Y_{\tau,f}$ can be computed on the basis of Theorem 7.2. If one would drop the hypothesis $\Lambda_g(\infty) = 1-2(\alpha_1+\ldots+\alpha_n) \neq 0$, then Theorem 7.2 has to be modified, but in principle Theorem 7.1 is applicable.

If $g = g(t_1,\ldots,t_n;\alpha_1,\ldots,\alpha_n)$ is the step function in (6.1), then Eq. (7.7) has the form

$$\xi^\tau_g(x) - \sum_{j=1}^{n}\frac{\alpha_j}{t_j}\int_0^\tau e^{-|x-y|/t_j}\xi^\tau_g(y)dy = \sum_{j=1}^{n}\frac{\alpha_j}{t_j}e^{-x/t_j}$$
$$(0<x<\tau).$$

For such equations, whose symbols (up to a trivial change of variable, the dispersion functions) are rational functions, a mathematical theory exists[29,30] that deals with convolution equations of the above type and their connection to linear systems.

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(15) R.J. Hangelbroek announced this result during the Vienna Workshop on Transport Theory, June 23, 1980.
(16) The vector-valued integrals appearing in this article are Bochner integrals. By strong measurability we mean measurability with respect to Lebesgue measure as defined in Section VI.31 of A.C. Zaanen, "Integration", North-Holland, Amster-

dam (1967). This book also contains general information about Bochner integrals.
(17) For finite τ only the singularity of the kernel $H(.)B$ at $x=0$ and not the one at $x=+\infty$ has to be accounted for. Therefore, the equivalence proof does not change essentially if T is taken to be unbounded.
(18) We restrict ourselves to neutron transport in non-multiplying media.
(19) The function $Ei(x) = \int_1^{+\infty} z^{-1}e^{-z|x|}dz$ is known as the exponential integral function.
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(23) The uniqueness of the solution ψ of Eq. (3.1) follows from its existence, because on the spaces of strongly measurable L_p-functions on $(0,\tau)$ with values in a Hilbert space the integral operator $\psi \mapsto \int_0^\tau H(.-y)B\psi(y)dy$ is compact $(1 \le p \le +\infty)$. The compactness of this operator follows from (the infinite dimensional analogue of) Lemma 1.1 in: I.C. Gohberg and G. Heinig, Rev. Roum. Math. Pures et Appl., 20, 55 (1975).
(24) Of course, for the BGK model one has $c=1$. The constant c is still included to have available a

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