Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

MATHEMATICAL CENTRE TRACTS 146

SEMIGROUP AND FACTORIZATION METHODS IN TRANSPORT THEORY

C.V.M. van der MEE

MATHEMATISCH CENTRUM AMSTERDAM 1981

1980 Mathematics subject classification: Primary: 85A25, 82A70, 45E10
Secondary: 47B35, 47D05, 47D45

ISBN 90 6196 233 1

TABLE OF CONTENTS

ACKNOWLEDGEMENTS

As to the completion of a printed version of this monograph in Aug 25, 1981, I am grately indebted to Prof.dr. M.A. Kaashoek, Prof.dr. I. G and Dr. H. Bart, because their comments, critical remarks and advice du many discussions about Transport Theory, linearization and other topics Operator Theory have been a major stimulus to my investigations. I am v grateful to the social psychologist G.A.C. Warning, because his friends and professional help proved to be indispensable for writing this monog

The present work has been made possible by a stipendium granted by Netherlands Organization for the Advancement of Pure Research (Z.W.O.) January 1978 up to August 1981. Prof.dr. C.G. Lekkerkerker I thank for time he spent reading the manuscript and for the way he commented upon contents. I am grately indebted to all those people at the Mathematical Centre who have contributed in a positive sense to the technical realiz of this monograph.

Chapter 0

INTRODUCTION

1. The Transport Equation appears in several branches of physics. has been and still is a subject of intensive studies [10,65,9,12]. In t monograph we consider the one-dimensional, linear Transport Equation, w is an integro-differential equation of the following form

$$\mu\frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) = \int_{-1}^{+1} g(\mu,\mu')\psi(x,\mu')d\mu' + f(x,\mu), \quad 0 < x < \tau, \tag{0.1}$$

where for $-1 \le \mu,\mu' \le +1$

$$g(\mu,\mu') = (2\pi)^{-1}\int_0^{2\pi} \hat{g}(\mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos\alpha)d\alpha. \tag{0.2}$$

Here $\hat{g}$ is the so-called scattering indicatrix or phase function, which assumed to be given as is the inhomogeneous term $f(x,\mu)$. The problem is determine the unknown function ψ under certain boundary conditions.

If τ is finite, one speaks about the *finite-slab problem*, which is sidered under the following boundary conditions:

$$\psi(0,\mu) = \phi_+(\mu) \quad (0 \le \mu \le +1); \quad \psi(\tau,\mu) = \phi_-(\mu) \quad (-1 \le \mu \le 0) \tag{0.3}$$

Here ϕ_+ and ϕ_- are given functions on $[0,1]$ and $[-1,0]$, respectively.

In the *half-space problem*, where τ is infinite, one considers eith the boundary conditions

$$\psi(0,\mu) = \phi_+(\mu) \quad (0 \le \mu \le +1); \quad \lim_{x\to+\infty} \psi(x,\mu) = 0 \quad (-1 \le \mu \le \tag{0.4a}$$

or, for some (small) fixed positive constant k,

$$\psi(0,\mu) = \phi_+(\mu) \quad (0 \le \mu \le +1); \quad \psi(x,\mu) = O(e^{kx}) \quad (-1 \le \mu \le \tag{0.4b}$$

2. To write Eq. (0.1) with its boundary conditions as a vector equation one introduces the Hilbert space $H := L_2[-1,+1]$ of square integrable functions on the closed interval $[-1,+1]$, endowed with the inner product

$$\langle h_1,h_2\rangle = \int_{-1}^{+1} h_1(\mu)\overline{h_2(\mu)}d\mu; \quad h_1,h_2 \in L_2[-1,+1].$$

Further, one introduces the vectors $\psi(x) \in H$ and $f(x) \in H$, and the operators $T: H \to H$ and $B: H \to H$ by

(0.5a) $$\psi(x)(\mu) = \psi(x,\mu), \quad f(x)(\mu) = f(x,\mu);$$

(0.5b) $$(Th)(\mu) = \mu h(\mu), \quad (Bh)(\mu) = \int_{-1}^{+1} g(\mu,\mu')h(\mu')d\mu'. \qquad (0 < x < \tau;\ -1 \le \mu \le +1)$$

Now Eq. (0.1) can be written as an operator differential equation of the form

(0.6) $$(T\psi)'(x) = -(I-B)\psi(x) + f(x), \quad 0 < x < \tau,$$

where the differentiation is performed with respect to the variable x.

To rewrite the boundary conditions one uses two projections P_+ and P_- on $H = L_2[-1,+1]$, namely

(0.7) $$(P_+h)(\mu) = \begin{cases} h(\mu), & 0 \le \mu \le 1; \\ 0, & -1 \le \mu < 0; \end{cases} \qquad (P_-h)(\mu) = \begin{cases} 0, & 0 < \mu \le +1; \\ h(\mu), & -1 \le \mu \le 0. \end{cases}$$

With the help of the vectors $\phi_+ \in H_+ := L_2[0,1]$ and $\phi_- \in H_- := L_2[-1,0]$ the boundary conditions (0.3) of the finite-slab problem can be written concisely as

(0.8) $$\lim_{x\downarrow 0} P_+\psi(x) = \phi_+, \quad \lim_{x\uparrow\tau} P_-\psi(x) = \phi_-.$$

The boundary conditions (0.4a) or (0.4b) of the half-space problem can be rephrased as either

(0.9a) $$\lim_{x\downarrow 0} P_+\psi(x) = \phi_+, \quad \lim_{x\to+\infty} P_-\psi(x) = 0,$$

or, with the norm taken in the space $H = L_2[-1,+1]$, as

(0.9b) $$\lim_{x\downarrow 0} P_+\psi(x) = \phi_+, \quad \|P_-\psi(x)\| = O(e^{kx}) \quad (x \to +\infty).$$

3. In this monograph we employ two approaches to deal with the half space problem. One approach, a semigroup method, deals with the operator differential equation (0.6) directly. With the help of semigroup theory obtain formulas for the solutions of the operator differential equation (0.6) under the boundary conditions (0.9a) and (0.9b).

The second approach, a Wiener-Hopf method, deals with an equivalent form of Eq. (0.6) (including its boundary conditions), which is a convol tion equation with an operator-valued kernel. To present this convolutio equation we define the so-called propagator function $H(x)$ by

(0.10) $$(H(x)h)(\mu) = \begin{cases} +\mu^{-1} e^{-x/\mu} h(\mu); & 0 < x < +\infty,\ 0 < \mu \le +1; \\ -\mu^{-1} e^{-x/\mu} h(\mu); & -\infty < x < 0,\ -1 \le \mu < 0; \\ 0 \quad ; & x\mu < 0. \end{cases}$$

The half-space problem in the form of the operator differential equation (0.6) with boundary conditions (0.9a) can be shown to be equivalent with the Wiener-Hopf operator integral equation

(0.11) $$\psi(x) - \int_0^{+\infty} H(x-y)B\psi(y)dy = e^{-xT^{-1}}\phi_+ + \int_0^{+\infty} H(x-y)f(y)dy$$

$(0 < x < +\infty)$. We apply a factorization method developed in [2,3], constr a so-called Wiener-Hopf factorization of the symbol of Eq. (0.11) and fi formulas for the solution of the above Wiener-Hopf equation.

4. To deal with the finite-slab problem we also use two approaches. Again, the first approach is a semigroup method, with the help of which w formulas for the solutions of the operator differential equation (0.6) with boundary conditions (0.8). The second approach deals with an equiva form of Eq. (0.6) with boundary conditions (0.8), which is a finite-sect convolution equation of the type (0.11), where the integration is perfor over the finite interval $(0,\tau)$ and for $f = 0$ the right-hand side has the : $\omega(x) = e^{-xT^{-1}}\phi_+ + e^{(\tau-x)T^{-1}}\phi_-$ $(0 < x < \tau)$. Using a method of GOHBERG & HEINIG [22] we represent the solution ψ in the form

$$\psi(x) = \omega(x) + \int_0^{\tau} \gamma(x,y)\omega(y)dy, \quad 0 < x < \tau,$$

and we derive a formula for the resolvent kernel $\gamma(x,y)$.

5. We are now at a stage to acknowledge to the work of others, whose methods have influenced and stimulated the present study. The Wiener-Hopf approach has been influenced by the work of FELDMAN on the asymptotics of the solutions of the Transport Equation (cf. [15] to [19]), the work of GOHBERG & HEINIG on finite-section matrix Wiener-Hopf equations (cf. [22]) and a factorization method developed by BART, GOHBERG, KAASHOEK & VAN DOOREN (cf. [2,3]). Of course, the semigroup approach hinges upon the theory of strongly continuous operator semigroups (cf. [13], for instance). In the factorization of the symbol a projection operator plays an important role, which as a projection operator first appeared in [35] and was the object of study by HANGELBROEK & LEKKERKERKER in [40].

6. In the present monograph we disregard the specific form the operators T and B have in the case of the Transport Equation and offer an abstract theory of *hermitian admissible pairs* (T,B) on an abstract Hilbert space H. By this we mean a pair of bounded operators $T: H \to H$ and $B: H \to H$ such that

(C.1) T is self-adjoint and 0 is not an eigenvalue of T;
(C.2) B is compact and self-adjoint;
(C.3) there exist $0 < \alpha < 1$ and a bounded operator $D: H \to H$ such that $B = |T|^{\alpha} D$.

If, in addition, $A = I - B$ is a (non-strictly) positive operator, the term "semi-definite admissible pair" is employed. Under the assumption that the scattering indicatrix $\hat{g}$ appearing in (0.2) is real-valued and satisfies $\int_{-1}^{+1} |\hat{g}(t)|^r dt < +\infty$ for some $r > 1$ the pair (T,B) defined by (0.5b) is, indeed, a hermitian admissible pair on the Hilbert space $H = L_2[-1,+1]$. Often we shall choose the indicatrix $\hat{g}$ in such a way that the operator $A = I - B$ is positive, but not necessarily invertible (i.e., that $\langle Ah,h\rangle \geq 0$, $h \in H$).

In this monograph a general study is made of the operator differential equation (0.6) with boundary conditions (0.8), (0.9a) or (0.9b) and of convolution equations like (0.11), where the operators T and B appearing in these equations form an arbitrary hermitian admissible pair. This generalization allows us to deal with different versions of the Transport Equation from one point of view. With these abstract operators T and B the half-space and finite-slab versions of Eqs (0.6) and (0.11) are studie[d] using the approaches described earlier. As in general in Transport The[ory] the equivalence of the two approaches is more or less taken for grante[d] a mathematical justification of their equivalence is provided. Finally, the theory of hermitian admissible pairs is applied to obtain concrete results in Transport Theory.

7. We now give a description of the contents of this monograph. Th[e] first two chapters have an auxiliary character. The first chapter deal[s] with linearization of transfer functions, i.e., of operator-valued func[tions] of the form

$$W(\lambda) = I + C(\lambda - A)^{-1}B. \tag{0.12}$$

It appears that (on the resolvent set of the operator A) the spectral properties of the operator function W are characterized by the spectral properties of the linear operator polynomial $\lambda - (A - BC)$, and therefore $A - BC$ is called a linearization of W on the resolvent set of A (cf. Sec[tion] I.3; see also [23]). In view of transfer functions we remark that, up t[o a] trivial change of variable, the symbol of the Wiener-Hopf equation (0.1[1]) is a transfer function like (0.12).

In the second chapter some terminology on Wiener-Hopf factorizati[on] and Wiener-Hopf equations is introduced. We describe a method taken fr[om] [2,3] to factorize transfer functions of the form (0.12). In this meth[od] projections play a prominent role.

In the third chapter we establish the basic ingredients to deal wi[th] Eqs (0.6) and (0.11). First we introduce hermitian admissible pairs (T,[B]) on an abstract Hilbert space H. Putting $A = I - B$ a detailed study is ma[de] of the spectral properties of the operator $T^{-1}A$, which has its spectrum [on] the real line. For the operator $T^{-1}A$ the spectral subspaces H_p, H_m and [H_0] corresponding to the parts of its spectrum on $(0,+\infty)$, $(-\infty,0)$ and the po[int] $\lambda = 0$, are constructed. Similarly for the self-adjoint operator T one h[as] spectral subspaces H_+ and H_-, corresponding to the parts of its spectru[m] on $(0,+\infty)$ and $(-\infty,0)$, respectively; for the special pair (T,B) defined [by] (0.5b) we have already specified H_+ and H_- by the sentence following fo[r]mulas (0.7). The following decomposition theorem is derived:

THEOREM 0.1. *Let* (T,B) *be a semi-definite admissible pair on a Hilbert space* H. *If the operator* $A = I - B$ *is strictly positive, then*

$$H_p \oplus H_- = H_m \oplus H_+ = H. \tag{0.13}$$

If we only assume the operator $A = I - B$ *to be (non-strictly) positive, then*

$$H_p \oplus [(H_m \oplus H_+) \cap H_0] \oplus H_- = H_m \oplus [(H_p \oplus H_-) \cap H_0] \oplus H_+ = H. \tag{0.14}$$

If P(Q) *denotes the projection of* H *onto* $H_p \oplus [(H_m \oplus H_+) \cap H_0]$ *(resp.* $H_m \oplus [(H_p \oplus H_-) \cap H_0]$*) along* H_- (H_+), *then*

$$TP = (I - Q^*)T, \quad TQ = (I - P^*)T. \tag{0.15}$$

For the isotropic and the degenerate anisotropic case of the Transport Equation HANGELBROEK has found the decomposition (0.13) using a different method (see [35,36]). The decomposition (0.14) is connected to the Milne problem and has not been studied by Hangelbroek. For a case when the operator $A = I - B$ is positive definite, the intertwining properties (0.15) are due to Hangelbroek too.

In the fourth chapter Theorem 0.1 and the theory of strongly continuous semigroups are employed to get solutions of the operator differential equation (0.6). Here we state the two main solution theorems for the homogeneous version of Eq. (0.6), in which $f(x) \equiv 0$.

THEOREM 0.2. *Let* (T,B) *be a semi-definite admissible pair on* H. *Then for every* $0 < \tau < +\infty$ *and every* $\phi \in H$ *the operator differential equation*

$$(T\psi)'(x) = (I-B)\psi(x), \quad 0 < x < \tau, \tag{0.16}$$

with boundary conditions (0.8), *where* $\phi_+ = P_+\phi$ *and* $\phi_- = P_-\phi$, *has a unique solution* ψ, *namely*

$$\psi(x) = e^{-xT^{-1}A}P_pV_\tau^{-1}\phi + e^{(\tau-x)T^{-1}A}P_mV_\tau^{-1}\phi + (I-xT^{-1}A)P_0V_\tau^{-1}\phi.$$

In this expression V_τ *is the invertible operator, defined by*

$$V_\tau = P_+\left[P_p + e^{+\tau T^{-1}A}P_m\right] + P_-\left[P_m + e^{-\tau T^{-1}A}P_p\right] + P_0 - \tau P_-T^{-1}AP_0. \tag{0.17}$$

Here P_p, P_m *and* P_0 *are the spectral projections of* $T^{-1}A$ *corresponding t the parts of its spectrum on* $(0,+\infty)$, $(-\infty,0)$ *and the point* $\lambda = 0$, *respec*

Parallel to the present work Hangelbroek has proved the invertibil of the operator V_τ for a case when the pair (T,B) is positive definite.

THEOREM 0.3. *Let* (T,B) *be a positive definite admissible pair on* H. *The for every* $\phi_+ \in H_+$ *the operator differential equation*

$$(T\psi)'(x) = -(I-B)\psi(x), \quad 0 < x < +\infty,$$

with boundary conditions (0.9a) *has a unique bounded solution* ψ, *which given by the formula*

$$\psi(x) = e^{-xT^{-1}A}P\phi_+, \quad 0 < x < +\infty. \tag{0.18}$$

Here P *is the projection of* H *onto* H_p *along* H_-, *which exists according to Theorem* 0.1.

Theorems 0.2 and 0.3 are existence and uniqueness theorems for the finite-slab and half-space problem, respectively.

In the fifth chapter we first prove the equivalence of the semigro approach and the Wiener-Hopf approach. For a hermitian admissible pair (T,B) on an abstract Hilbert space H we define a propagator function *H*(: which is a generalization of (0.10). A typical equivalence theorem is t: following

THEOREM 0.4. *Let* (T,B) *be a hermitian admissible pair on* H. *Then for every* $0 < \tau < +\infty$ *and every* $\phi \in H$ *a bounded vector function* $\psi: (0,\tau) \to H$ *a solution of the operator differential equation* (0.16) *with boundary c ditions* (0.8), *where* $\phi_+ = P_+\phi$ *and* $\phi_- = P_-\phi$, *if and only if the finite-s tion Wiener-Hopf operator integral equation*

$$\psi(x) - \int_0^\tau H(x-y)B\psi(y)\,dy = e^{-xT^{-1}}\phi_+ + e^{(\tau-x)T^{-1}}\phi_-, \quad 0 < x < \tau, \tag{0.19}$$

has the function ψ *as a solution.*

With the help of Theorem 0.2 it is clear that the convolution equa (0.19) has a unique solution. Using results of GOHBERG & HEINIG [22] we solve Eq. (0.19) for an arbitrary bounded right-hand side $\omega(x)$.

The half-space problem is solved in three ways. First we exploit the equivalence of the semigroup and Wiener-Hopf approach, and obtain the solution from Theorem 0.3. Secondly we use the so-called projection method (cf. [24,17]) and obtain the solution of the half-space problem as a limit of solutions of the finite-slab problem as $\tau \to +\infty$. The third method amounts to solving the convolution equation

$$(0.20) \qquad \psi(x) - \int_0^{+\infty} H(x-y)B\psi(y)\,dy = \omega(x), \qquad 0 < x < +\infty,$$

by canonical factorization of its symbol.

As mentioned above, the symbol of Eq. (0.11) has the form of a transfer function. If (T,B) is a positive definite admissible pair on H, we put $C = A^{-1} - I$, apply a factorization method taken from [2,3] and derive the following

<u>THEOREM 0.5</u>. *For a positive definite admissible pair* (T,B) *on* H, *where* $C = A^{-1} - I$, *the symbol of the Wiener-Hopf operator integral equation* (0.19) *has a canonical factorization of the form*

$$(0.21) \qquad (T-\lambda)^{-1}(T-\lambda A) = W_+(\lambda)W_-(\lambda), \qquad \mathrm{Re}\,\lambda = 0,$$

where W_+ (W_-) *is an analytic operator-valued function on the open right (left) half-plane, which has a continuous extension up to the extended imaginary line; further, for* $\mathrm{Re}\,\lambda \geq 0$ ($\mathrm{Re}\,\lambda \leq 0$) *the operator* $W_+(\lambda)$ ($W_-(\lambda)$) *is invertible. In fact,*

$$W_+(\lambda) = I + T(T-\lambda)^{-1}(I-P)C;$$

$$W_+(\lambda)^{-1} = I - T(I-P)(A^{-1}T-\lambda)^{-1}C;$$

$$W_-(\lambda) = [I + TP(T-\lambda)^{-1}C]A;$$

$$W_-(\lambda)^{-1} = A^{-1}[I - T(A^{-1}T-\lambda)^{-1}PC].$$

Here P *is the projection of* H *onto* H_p *along* H_-, *which exists according to Theorem* 0.1.

With the help of the factorization (0.21) we construct a formula f the unique solution of Eq. (0.20). In particular, if $\omega(x) = e^{-xT^{-1}}\phi_+$ ($0 < x < +\infty$) for some $\phi_+ \in H_+$, we obtain (0.18) as the solution, which in full agreement with the equivalence of the semigroup and the Wiener- approach.

In the last section of this chapter we construct unbounded solutic of Eq. (0.20) using an equivalence theorem. By this we recover the asyn tics of the solutions, which have been found before by FELDMAN (cf. [15

In the sixth and final chapter the formal theory of hermitian admi sible pairs is applied in Transport Theory. First we prove a theorem th allows us to apply this theory to the one-speed Transport Equation in a multiplying medium.

<u>THEOREM 0.6</u>. *Let the indicatrix* $\hat{g}$ *be real-valued, and let* $\int_{-1}^{+1} |\hat{g}(t)|^r dt$ *for some* $r > 1$. *On* $L_2[-1,+1]$ *we define the operators* T *and* B *by*

$$(Th)(\mu) = \mu h(\mu), \qquad (Bh)(\mu) = \int_{-1}^{+1} g(\mu,\mu')h(\mu')\,d\mu'; \qquad -1 \leq \mu \leq +$$

where g *is given by* (0.2). *Then* (T,B) *is a hermitian admissible pair on* $L_2[-1,+1]$. *Moreover, if* $\hat{g}$ *is nonnegative and* $c = \int_{-1}^{+1} \hat{g}(t)\,dt \leq 1$, *then pair* (T,B) *is semi-definite.*

By Theorem 0.6 the theory of hermitian admissible pairs applies to huge class of non-degenerate scattering indicatrices. For the major rest we refer to Theorems 0.1 to 0.5.

Up to now we have only been able to show the existence and uniquen of the solution of the half-space and the finite-slab problem. For the practical purpose of finding analytic expressions for the solutions of Eq. (0.1) we need an explicit expression for the projection P appearing Theorems 0.3 and 0.5 and the operator V_τ^{-1} (cf. (0.17)). For the isotrop case of the Transport Equation we provide analytic expressions for the solution. In the derivation of these expressions Wiener-Hopf methods ar heavily used (in fact, we use Theorem 0.5 and the Gohberg-Heinig method We also derive an analytic solution of the Milne problem with isotropic scattering, thereby supplementing results obtained in [42,47]. Also the corresponding anisotropic transport problems are analyzed. In particula the factorization of the dispersion function in Transport Theory is show to be a quite trivial corollary of a result of MUSKHELISHVILI [59].

Further, for the degenerate anisotropic case we specify the projection P and establish its connection to the scattering function introduced by CHANDRASEKHAR [10]. Using this connection a formula derived previously in a not completely rigorous way by BUSBRIDGE [8] is recovered. The expression for P recovered here improves upon a result of HANGELBROEK & LEKKERKERKER (published in [48]). Because in Transport Theory the image of the operator B contains a cyclic vector of T and B is an operator of finite rank, it is possible to derive such an expression, indeed. Further applications concern the symmetric multigroup Transport Equation, which has been analyzed by other means by GREENBERG [34]. For this equation the analogue of Theorem 0.6 is derived.

8. Let us now make a few remarks about the physical background of the Transport Equation

$$(0.22) \qquad \mu\frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) = \int_{-1}^{+1} g(\mu,\mu')\psi(x,\mu')d\mu' + f(x,\mu), \qquad 0 < x < \tau.$$

This integro-differential equation plays a prominent role in astrophysics and in neutron physics. In astrophysics it describes the stationary transfer of unpolarized electromagnetic radiation through a homogeneous stellar or planetary atmosphere. In neutron physics this equation gives a description of the stationary transport of undelayed, mutually non-interacting neutrons through a homogeneous fuel plate of a nuclear reactor. In both cases the medium (atmosphere or fuel plate) is taken to be a plane-parallel layer.

If Eq. (0.22) concerns radiative transfer, then the unknown function ψ is the intensity of the radiation, $x \in (0,\tau)$ is the optical depth (really a position coordinate weighted by the absorption coefficient), τ the optical thickness of the atmosphere, μ the cosine of the scattering angle and $f(x,\mu)$ is an inhomogeneous term accounting for radiative sources. The function $g(\mu,\mu')$ is given by (0.2), where $\hat{g}$ is the scattering indicatrix or phase function, which accounts for the scattering properties of the atmosphere. In this monograph the phase function $\hat{g}$ includes the albedo (or fraction of the radiation scattered) $c \in [0,1]$ of the atmosphere as a factor. For more details we refer to [10,65].

If Eq. (0.22) involves neutron transport, then the unknown function ψ is the angular density of the neutrons, $x \in (0,\tau)$ is the distance from the surface of the fuel plate (measured by the mean free path of the neutrons as the unit of length), τ the thickness of the plate (measured by the sa unit), μ the cosine of the scattering angle, $f(x,\mu)$ an inhomogeneous ter accounting for neutron sources and $g(\mu,\mu')$ is the scattering function, w describes the scattering properties of the medium. In this monograph thi function includes the average number $c \geq 0$ of neutrons produced by colli (with uranium atoms) as a factor. If $c > 1$, the medium is called multipl (a case we hardly consider here). If $c = 1$, we call the medium conservat for $0 \leq c < 1$ the medium is called non-conservative. For more details we refer to [12,9].

In astrophysics Eq. (0.22) applies to unpolarized radiation. In neu physics it is the so-called one-speed approximation of a more general ec tion also involving the neutron speed, which is assumed here to be const If one assumes instead that the neutrons can be divided into a finite nu ber of groups of particles of constant speed, then Eq. (0.22) is replace by a vector-valued version called the multigroup Transport Equation.

In astrophysics the half-space problem (where $\tau = +\infty$) involves an atmosphere of infinite optical depth: a stellar atmosphere with radiatic coming from the interior of the star, or a very opaque planetary atmosphere. In the former case boundary conditions like (0.4b) are imposed; i the latter case boundary conditions like (0.4a). Here the function ϕ_+ ap ing in the boundary conditions is given and signifies the inward intensi of the radiation on the surface. The finite-slab problem applies to a planetary atmosphere of finite optical depth. In this case we impose the boundary conditions (0.3), which specify the inward intensity of the radiation on the surface (usually we have $\phi_- = 0$).

In neutron physics the boundary conditions specify the inward angul density of the neutrons at the surface(s) of the fuel plate. By the fini size of the reactor the half-space problem is a less realistic descripti of neutron transport phenomena than the finite-slab problem is. For prac tical purposes the multigroup Transport Equation, where the neutron spee is taken into account as an additional (discrete) variable, plays an eve more important role.

9. We conclude the introduction by some terminological remarks. Throughout this monograph all Banach spaces will be complex and their norm is denoted by $\|\cdot\|$. Similarly, all Hilbert spaces are complex and th inner product is denoted by $<\cdot,\cdot>$. Contrary to the bracket notation comm in quantum mechanics inner products have the property $<if,g> = i<f,g> = -<$

If Y is a Banach space, the Banach algebra of bounded linear operators on Y is denoted by $L(Y)$ and the identity operator on Y by I_Y. If no confusion is possible, we shall write I rather than I_Y. The direct sum of two Banach spaces Y and Z is denoted by $Y \oplus Z$. By Y^{ℓ} we mean the direct sum of ℓ copies of Y. The spectrum and the resolvent set of an operator T are denoted by $\sigma(T)$ and $\rho(T)$, respectively. If T is a linear operator on a Banach space Y, then the domain of T is denoted by D(T), the kernel or null space of T by Ker T and the image or range of T by Im T. For the Riemann sphere we write $\mathbb{C}_\infty$. The restriction of an operator T to a T-invariant subspace M is denoted by $T|_M$. The orthogonal complement of a subset M of a Hilbert space is denoted by $M^{\perp}$.

CHAPTER I

LINEARIZATION OF TRANSFER FUNCTIONS

In this chapter we recall the basic facts concerning linearization and realization. For a transfer function a general linearization theorem is proved. In the final section we consider the realization problem for meromorphic operator functions and for such functions a kind of linearization i constructed.

1. Preliminaries about equivalence and linearization

Let Ω be an open subset of the Riemann sphere $\mathbb{C}_\infty$, and let Y_1 and Y_2 be complex Banach spaces. Two holomorphic operator functions $W_1: \Omega \to L(Y_1)$ and $W_2: \Omega \to L(Y_2)$ are called *equivalent on* Ω (cf. [23]) if

$$W_2(\lambda) = E(\lambda)W_1(\lambda)F(\lambda), \quad \lambda \in \Omega,$$

where $E(\lambda): Y_1 \to Y_2$ and $F(\lambda): Y_2 \to Y_1$ are invertible operators which depend holomorphically on λ in Ω. The functions E and F will be called *equivalence functions*.

Given a holomorphic operator function $W: \Omega \to L(Y)$ and a complex Banach space Z, we define the *Z-extension* of W to be the operator function on Ω whose value at $\lambda \in \Omega$ is equal to the operator $W(\lambda)\oplus I_Z \in L(Y\oplus Z)$. A linear operator $T \in L(X)$ is called a *linearization* of W on Ω if the linear pencil $\lambda I_X - T$ is equivalent on Ω to some extension of W. If Ω is a bounded open set, then W always admits a linearization on Ω (see [23,56,5]).

Let $W: \Omega \to L(Y)$ be a holomorphic operator function on the open set Ω. The *spectrum* $\Sigma(W)$ of W is the set of all $\lambda \in \Omega$ such that $W(\lambda)$ is not invertible. If T is a linearization of W on Ω, then

$$\sigma(T) \cap \Omega = \Sigma(W). \tag{1.1}$$

For the case when Ω is bounded one can construct a linearization T of W on Ω such that in addition to (1.1) one has the following: (1) $\sigma(T) \subset \bar{\Omega}$, (2) all limit points of the boundary of the closure of Ω are contained in $\sigma(T)$, (3) an isolated boundary point λ_0 of Ω does not belong to $\sigma(T)$ if and only if λ_0 has a deleted neighbourhood on which $W(\lambda)$ is invertible, while both W and W^{-1} have a removable singularity at λ_0 (cf. [52], Theorem II 4.2; for bounded Cauchy domains see also [23], Theorem 2.3).

We shall employ the term linearization also in a somewhat other context. Let $W: \Omega \to L(Y)$ be a holomorphic operator function, and let S and T be bounded linear operators on the complex Banach space X. We call the pencil $\lambda S-T$ a *linearization* of W on Ω if some extension of W is equivalent on Ω to $\lambda S-T$. (Thus the operator T is a linearization of W on Ω if and only if the pencil $\lambda I-T$ is a linearization of W on Ω.) Any holomorphic operator function (it does not matter whether Ω is bounded or not) admits a linearization in this more general sense (see [5]).

2. Preliminaries about transfer functions and nodes

In this monograph we shall often work with an operator function W of the form

$$W(\lambda) = D + C(\lambda-A)^{-1} B. \tag{2.1}$$

Here $A: X \to X$, $B: Y \to X$, $C: X \to Y$ and $D: Y \to Y$ are bounded linear operators acting between complex Banach spaces X and Y. The function (2.1) is defined and holomorphic on the resolvent set $\rho(A)$ of A and at infinity.

A system $\theta = (A,B,C,D;X,Y)$, where A,B,C and D are as in (2.1), is called an *(operator) node*, and the function (2.1) is called the *(monic) transfer function* of the node θ (cf. [2], Section 1.1). With the operator node $\theta = (A,B,C,D;X,Y)$ we also associate the *comonic transfer function*, namely the function

$$Z(\lambda) = D + \lambda C(I_X-\lambda A)^{-1} B,$$

which is defined and analytic on the set of all $\lambda \in \mathbb{C}$ such that $I_X - \lambda A$ is invertible.

Let $\theta = (A,B,C,D;X,Y)$ be a node. The operators D and A are called, respectively, the *external operator* and the *main operator* of the node θ.

The space X is referred to as the *state space* of θ. If the external oper D of θ is invertible, one defines $A^\times = A-BD^{-1}C$. The operator $A^\times$ is calle the *associate operator* (cf. [2], Section 5.5); it appears as the main op tor of the *associate node* $\theta^\times$, which is defined by

$$\theta^\times = (A^\times, BD^{-1}, -D^{-1}C, D^{-1}; X,Y).$$

Let W_θ be the transfer function of the node $\theta = (A,B,C,D;X,Y)$, and assume that the external operator D is invertible. Then the X-extension W_θ is equivalent on $\rho(A)$ to the Y-extension of $\lambda-A^\times$. In fact, one has (c [2], Theorem 4.5):

$$\begin{pmatrix} W_\theta(\lambda) & 0 \\ 0 & I_X \end{pmatrix} = E(\lambda)\begin{pmatrix} I_Y & 0 \\ 0 & \lambda-A^\times \end{pmatrix}F(\lambda), \quad \lambda \in \rho(A), \tag{2.2}$$

where

$$E(\lambda) = \begin{pmatrix} W_\theta(\lambda) & C(\lambda-A)^{-1} \\ (\lambda-A)^{-1}B & (\lambda-A)^{-1} \end{pmatrix}, \quad F(\lambda) = \begin{pmatrix} D^{-1}W_\theta(\lambda) & -D^{-1}C \\ (A-\lambda)^{-1}B & I_X \end{pmatrix}.$$

Note that this implies that the spectrum $\Sigma(W_\theta)$ of the transfer function ' is equal to $\sigma(A^\times) \cap \rho(A)$. Further, one easily computes that

$$W_\theta(\lambda)^{-1} = W_{\theta^\times}(\lambda), \quad \lambda \in \rho(A^\times) \cap \rho(A), \tag{2.3}$$

where $W_{\theta^\times}$ is the transfer function of the associate node $\theta^\times$. By direct computation one shows that for $\lambda \in \rho(A^\times) \cap \rho(A)$ the identity

$$(\lambda-A^\times)^{-1} = (\lambda-A)^{-1} - (\lambda-A)^{-1}BW_\theta(\lambda)^{-1}C(\lambda-A)^{-1} \tag{2.4}$$

holds true.

Formula (2.2) already suggests that $A^\times$ may appear as a linearizatio of W_θ on $\rho(A)$. Under certain invertibility conditions this is, indeed, tl case, as we shall see in the next section (cf. [2], Section 2.4).

3. Linearization of transfer functions

THEOREM 3.1. *Let* W *be the transfer function of the node* $\theta = (A,B,C,D;X,Y)$, *and assume that* B *has a left inverse* $B^\dagger$. *Let* $Z = \operatorname{Ker} B^\dagger$. *For* $y \in Y$, $z \in Z$ *and* $\lambda \in \rho(A)$ *we define*

$$E(\lambda)(y,z) = By + z + BC(\lambda-A)^{-1}z,$$

$$F(\lambda)(y,z) = (\lambda-A)^{-1}(By+z).$$

Then $E(\lambda), F(\lambda): Y\oplus Z \to X$ *are invertible operators which depend analytically on* λ *in* $\rho(A)$. *Further, for* $\lambda \in \rho(A)$

$$E(\lambda)[W(\lambda)\oplus I_Z] = [\lambda(I+B(D-I)B^\dagger) - (A-BC+B(D-I)B^\dagger A)]F(\lambda).$$

In particular, $\lambda(I+B(D-I)B^\dagger) - (A-BC+B(D-I)B^\dagger A)$ *is a linearization of* W *on* $\rho(A)$.

PROOF. An easy computation yields that $E(\lambda)$ and $F(\lambda)$ are invertible for all $\lambda \in \rho(A)$ and that their inverses are given by

$$E(\lambda)^{-1}x = [B^\dagger x - C(\lambda-A)^{-1}(I-BB^\dagger)x] \oplus (I-BB^\dagger)x,$$

$$F(\lambda)^{-1}x = B^\dagger(\lambda-A)x \oplus (I-BB^\dagger)(\lambda-A)x.$$

We now compute $E(\lambda)[W(\lambda)\oplus I_Z]F(\lambda)^{-1}x = [BW(\lambda)B^\dagger + (I-BB^\dagger) + BC(\lambda-A)^{-1}(I-BB^\dagger)](\lambda-A)x$. Using that

$$W(\lambda) = D + C(\lambda-A)^{-1}B, \quad \lambda \in \rho(A),$$

we get

$$E(\lambda)[W(\lambda)\oplus I_Z]F(\lambda)^{-1}x =$$

$$[B(D-I)B^\dagger + BB^\dagger + BC(\lambda-A)^{-1}BB^\dagger + (I-BB^\dagger) + BC(\lambda-A)^{-1}(I-BB^\dagger)](\lambda-A)x =$$

$$[B(D-I)B^\dagger + I + BC(\lambda-A)^{-1}](\lambda-A)x.$$

From this we easily get the theorem. □

For the case when $D = I$ the previous theorem shows that the operat[or] $A-BC$ is a linearization of $I + C(\lambda-A)^{-1}B$ on $\rho(A)$. More generally, if D [is] invertible, we can apply Theorem 3.1 to the function $I + D^{-1}C(\lambda-A)^{-1}B$ an[d] deduce that $A^\times = A - BD^{-1}C$ is a linearization of $W(\lambda) = D + C(\lambda-A)^{-1}B$ on [$\rho(A)$] (cf. [2], Theorem 2.6). Note that Theorem 3.1 may be viewed as a slight[ly] more general version of [2], Theorem 2.6.

Under the extra condition

$$B^\dagger(\lambda-A)^{-1}AB = 0, \quad \lambda \in \rho(A), \tag{3.2}$$

one can prove that the pencil

$$\lambda[I+B(D-I)B^\dagger] - (A-BC) \tag{3.3}$$

is a linearization of $W(\lambda) = D + C(\lambda-A)^{-1}B$ on $\rho(A)$. To see this, first [note] that (3.2) implies that $E_1(\lambda) = I + B(D-I)B^\dagger(\lambda-A)^{-1}A$ is invertible for [all] $\lambda \in \rho(A)$. But then it is clear from (3.1) that

$$E_1(\lambda)E(\lambda)[W(\lambda)\oplus I_Z]F(\lambda)^{-1} = \lambda[I+B(D-I)B^\dagger] - (A-BC),$$

which proves that the pencil (3.3) is a linearization of W on $\rho(A)$ (cf. second part of Theorem 1.2 in [41]).

By substituting $1/\lambda$ for λ in the statement of Theorem 3.1 one easil[y] shows that the pencil

$$[I+B(D-I)B^\dagger] - \lambda[A-BC+B(D-I)B^\dagger A]$$

is a linearization for the comonic transfer function $Z(\lambda) = D + \lambda C(I-\lambda A)^{-1}B$ on the open set

$$\Omega = \{\lambda \in \mathbb{C}: I - \lambda A \text{ is invertible}\}.$$

If, in addition, one assumes that

$$B^\dagger(I-\lambda A)^{-1}AB = 0, \quad \lambda \in \Omega,$$

then one can show that $[I+B(D-I)B^\dagger] - \lambda(A-BC)$ is a linearization of $D + \lambda C(I-\lambda A)^{-1}B$ on Ω (cf. the first part of Theorem 1.2 in [41]).

Without proof we state the dual of Theorem 3.1 (cf. [2], Section 2.4).

THEOREM 3.2. *Let* W *be the transfer function of the node* $\theta = (A,B,C,D;X,Y)$, *and assume that* C *has a right inverse* $C^{\dagger}$. *Put* $Z = \operatorname{Ker} C$. *For* $y \in Y$, $z \in Z$ *and* $\lambda \in \rho(A)$ *we define*

$$E(\lambda)(y,z) = (\lambda-A)C^{\dagger}y + (\lambda-A)z,$$

$$F(\lambda)(y,z) = C^{\dagger}y - (I-C^{\dagger}C)(\lambda-A)^{-1}By + z.$$

Then $E(\lambda), F(\lambda): Y\oplus Z \to X$ *are invertible and depend analytically on* λ *in* $\rho(A)$. *Further, for* $\lambda \in \rho(A)$

$$E(\lambda)[W(\lambda)\oplus I_Z] = [\lambda(I+C^{\dagger}(D-I)C) - (A-BC+AC^{\dagger}(D-I)C)]F(\lambda).$$

In particular, $\lambda(I+C^{\dagger}(D-I)C) - (A-BC+AC^{\dagger}(D-I)C)$ *is a linearization of* W *on* $\rho(A)$.

4. Realization of meromorphic operator functions

The problem to determine all operator functions that can be written in the form (2.1) is an important one. In Mathematical Systems Theory this problem is known as the realization problem (cf. [43]) and concerns rational matrix functions. In this section we consider the realization problem for meromorphic operator functions in general.

Let Ω be an open subset of $\mathbb{C}_\infty$, and let W be an operator function that is meromorphic on Ω and has values in $L(Y)$. The set of poles of W in Ω is denoted by Σ. By a *realization* of W on Ω we mean a representation of W in the form

$$W(\lambda) = I + C(\lambda-A)^{-1}B, \quad \lambda \in \Omega\setminus\Sigma_0,$$

where $\sigma(A) \cap \Omega$ is a discrete subset Σ_0 of Ω. Obviously, the set Σ of poles of W is contained in Σ_0. The main problem of this section is to construct a realization for a given meromorphic operator function. We shall show that at least on a bounded domain this is always possible. Together with the solution of the realization problem we obtain a kind of linearization of a meromorphic operator function.

In the proof we use the fact that for holomorphic operator functio
a bounded domain the realization problem has been solved (cf. [2], Sect
2.3; [52], Section II.4; [56], Lemma 3.3).

THEOREM 4.1. *Suppose* Ω *is a bounded open set, and let* W *be a meromorphi operator function on* Ω *with values in* $L(Y)$. *Then* W *admits a realization the form*

$$W(\lambda) = I + C(\lambda-A)^{-1}B, \quad \lambda \in \Omega\setminus\Sigma,$$

where $\theta = (A,B,C,I;X,Y)$ *is a node,* Σ *is the set of poles of* W *and* C *is invertible. Further, for* $Z = \operatorname{Ker} C$ *there exist invertible operators* $F_1(\lambda), F_2(\lambda): Y\oplus Z \to X$, *depending analytically on* λ *in all of* Ω, *such tha*

$$F_2(\lambda)[W(\lambda)\oplus I_Z]F_1(\lambda)^{-1} = (\lambda-A)^{-1}(\lambda-A+BC), \quad \lambda \in \Omega\setminus\Sigma. \tag{4.1}$$

THEOREM 4.2. *Suppose* Ω *is a bounded open set and* W *a meromorphic operat function on* Ω *with values in* $L(Y)$. *Then* W *admits a realization of the f*

$$W(\lambda) = I + C(\lambda-A)^{-1}B, \quad \lambda \in \Omega\setminus\Sigma,$$

where $\theta = (A,B,C,I;X,Y)$ *is a node,* Σ *is the set of poles of* W *and* B *is invertible. Further, if* $B^{\dagger}B = I_Y$ *and* $Z = \operatorname{Ker} B^{\dagger}$, *there exist invertible operators* $E_1(\lambda), E_2(\lambda): Y\oplus Z \to X$, *depending analytically on* λ *in* Ω, *such*

$$E_1(\lambda)[W(\lambda)\oplus I_Z]E_2(\lambda)^{-1} = (\lambda-A+BC)(\lambda-A)^{-1}, \quad \lambda \in \Omega\setminus\Sigma. \tag{4.2}$$

The fractional linear operator functions appearing at the right-ha sides of (4.1) and (4.2) are to some extent "linearizations" of W on Ω. W is holomorphic on Ω, then the set of poles Σ of W is empty and $\lambda-A$ is invertible on all of Ω. So in that case $A-BC$ is a linearization of W on (in the sense of Sections I.1 and I.3).

We now give the proof of Theorem 4.1 and mention how to modify it get a proof of Theorem 4.2.

PROOF OF THEOREM 4.1. Suppose Ω is a bounded open set and W a meromorphi operator function on Ω with values in $L(Y)$. By Σ we denote the set of pc of W in Ω. Then it is clear from the Weierstrass product theorem that th exist two holomorphic operator functions $W_1, W_2: \Omega \to L(Y)$ such that the spectrum of W_2 coincides with Σ and $W(\lambda) = W_2(\lambda)^{-1} W_1(\lambda)$, $\lambda \in \Omega\backslash\Sigma$. According to a realization theorem for holomorphic operator functions on bounded open sets (cf. [52], formula (II4.4)), there exist nodes $\theta_i = (A_i, B_i, C_i, I; X_i, Y)$ with $\Omega \subset \rho(A_i)$ and C_i right invertible, such that

$$W_i(\lambda) = I + C_i(\lambda - A_i)^{-1} B_i; \quad \lambda \in \Omega,\ i = 1,2. \tag{4.3}$$

However, the state spaces X_1 and X_2 appear to be weighted Banach spaces of Y-valued functions in which only the weighted norm depends on W_1 and W_2, respectively. On the respective spaces the operators A_1 and A_2, and C_1 and C_2 have the same form, whereas the forms of the operators B_1 and B_2 differ considerably. An elementary modification of the weighted norms enables us to get the realizations (4.3) of W_1 and W_2 in such a way that these realizations have the same state space X, the same main operator A, while $C_1 = C_2 = C$ is still right invertible (cf. Section III.2 of [52] for a more detailed treatment). More precisely, there exist nodes $\tilde\theta_i = (A, B_i, C, I; X, Y)$ with $\Omega \subset \rho(A)$ and C right invertible, such that

$$W_i(\lambda) = I + C(\lambda - A)^{-1} B_i; \quad \lambda \in \Omega,\ i = 1,2.$$

A straightforward computation, using that $B_2 C = (\lambda - A_2^\times) - (\lambda - A)$, with $A_2^\times = A - B_2 C$ as the associate operator of the node $\tilde\theta_2$, yields

$$W(\lambda) = [I - C(\lambda - A_2^\times)^{-1} B_2][I + C(\lambda - A)^{-1} B_1] = I + C(\lambda - A_2^\times)^{-1}(B_1 - B_2). \tag{4.4}$$

This settles the first part of Theorem 4.1, because $\sigma(A_2^\times) \cap \Omega = \Sigma(W_2) = \Sigma$ is the discrete subset of Ω of poles of W.

From Theorem 3.2 (with D = I) we know that the formulas

$$F_1(\lambda)(y,z) = C^\dagger y + z - (I - C^\dagger C)(\lambda - A)^{-1} B_1 y,$$

$$F_2(\lambda)(y,z) = C^\dagger y + z - (I - C^\dagger C)(\lambda - A)^{-1} B_2 y,$$

$$E(\lambda)(y,z) = (\lambda - A)C^\dagger y + (\lambda - A)z,$$

define invertible operators that depend analytically on λ in Ω and act from $Y \oplus \operatorname{Ker} C$ onto X, where $CC^\dagger = I_Y$. Further, by this same Theorem 3.2 (with D = I), we have

$$E(\lambda)[W_i(\lambda) \oplus I_{\operatorname{Ker} C}] = (\lambda - A_i^\times) F_i(\lambda); \quad \lambda \in \Omega,\ i = 1,2.$$

Here $A_1^\times = A - B_1 C$ is the associate operator of the node $\tilde\theta_1$. From this id tity we directly infer that

$$F_2(\lambda)[W(\lambda) \oplus I_{\operatorname{Ker} C}] F_1(\lambda)^{-1} = (\lambda - A_2^\times)^{-1}(\lambda - A_1^\times), \quad \lambda \in \Omega\backslash\Sigma.$$

Finally, observe that $A_1^\times = A_2^\times - (B_1 - B_2)C$. □

With the help of Theorem 2.5 of [2] one can prove that Theorem 4.1 also applies to meromorphic operator functions W on a neighbourhood of infinity with $W(\infty) = I$.

The proof of Theorem 4.2 is a modification of the one of Theorem 4 First we apply the Weierstrass product theorem and factorize the given meromorphic operator function W as the quotient $W(\lambda) = W_1(\lambda) W_2(\lambda)^{-1}$ of holomorphic operator functions W_1 and W_2. Instead of the realization in [52], Section II.4, we now use the realizations

$$W_i(\lambda) = I + C_i(\lambda - A)^{-1} B; \quad \lambda \in \Omega,\ i = 1,2$$

(cf. [2], Lemma 3.3). The proof now proceeds along the same lines as the one of Theorem 4.1 (cf. [52], Section III.2).

For rational matrix functions the realization problem has been sol in many different ways (cf. [43] and [2], for instance). For such functi the identities (4.1) and (4.2) can be replaced by more explicit expressi as the next theorem shows.

<u>THEOREM 4.3</u>. *Let W be an operator function with values in L(Y) that can written as the quotient* $W(\lambda) = Q(\lambda)^{-1} P(\lambda)$ *of two monic operator polynomi P and Q of degree* ℓ, *while the spectrum of Q is a finite set. Write* $P(\lambda) = \lambda^\ell I + \sum_{i=0}^{\ell-1} \lambda^i P_i$ *and* $Q(\lambda) = \lambda^\ell I + \sum_{i=0}^{\ell-1} \lambda^i Q_i$. *Then there exist inve ible operators* $F_P(\lambda), F_Q(\lambda): Y^\ell \to Y^\ell$, *depending analytically on* λ *in* $\mathbb{C}$, *s that*

$$W(\lambda) \oplus I_{Y^{\ell-1}} = F_Q(\lambda)^{-1}(\lambda - C_Q)^{-1}(\lambda - C_P) F_P(\lambda), \quad \lambda \notin \Sigma(Q). \tag{4.5}$$

Here C_P (C_Q) *is the second companion operator of the monic polynomial* $P(Q)$, *and the operators* $F_P(\lambda)$ *and* $F_Q(\lambda)$ *have the form*

$$F_P(\lambda) = \begin{bmatrix} B^P_{\ell-1}(\lambda) - I & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ \vdots & \bigcirc & & \ddots & -I \\ B^P_0(\lambda) & 0 & \cdots & \cdots & 0 \end{bmatrix},$$

$$F_Q(\lambda) = \begin{bmatrix} B^Q_{\ell-1}(\lambda) - I & 0 & \cdots & \cdots & 0 \\ \vdots & 0 & \ddots & & \vdots \\ & & \ddots & \ddots & 0 \\ \vdots & \bigcirc & & \ddots & -I \\ B^Q_0(\lambda) & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

Here $B^P_0 = B^Q_0(\lambda) = I$, *while for* $n = 1,2,\ldots,\ell-1$

$$B^P_0(\lambda) = \lambda^n I + \lambda^{n-1}P_{\ell-1} + \ldots + P_{\ell-n},$$

$$B^Q_n(\lambda) = \lambda^n I + \lambda^{n-1}Q_{\ell-1} + \ldots + Q_{\ell-n}.$$

<u>PROOF</u>. Put

$$E_P(\lambda) = E_Q(\lambda) = \begin{bmatrix} I & & & \bigcirc \\ -\lambda I & \ddots & & \\ & \ddots & \ddots & \\ \bigcirc & & -\lambda I & I \end{bmatrix} : Y^\ell \to Y^\ell, \quad \lambda \in \mathbb{C}.$$

As this operator has the form $I - \lambda S$ with S nilpotent, this operator is invertible and depends analytically on λ in $\mathbb{C}$. Further,

$$\left.\begin{aligned} F_P(\lambda)[P(\lambda)\oplus I_{Y^{\ell-1}}] &= (\lambda - C_P)E_P(\lambda); \\ F_Q(\lambda)[Q(\lambda)\oplus I_{Y^{\ell-1}}] &= (\lambda - C_Q)E_Q(\lambda) \end{aligned}\right\} \quad (\lambda \in \mathbb{C})$$

(cf. [26]; see also [2] at pages 60 and 61). From these identities form (4.5) is clear. □

CHAPTER II

WIENER-HOPF FACTORIZATION OF TRANSFER FUNCTIONS

In this chapter we review the main elements of the theory of Wiene Hopf operator integral equations and Wiener-Hopf factorization. Particu attention is given to the case when the symbol of the Wiener-Hopf opera integral equation can be written in the form of a transfer function.

1. Factorization of transfer functions

In this section we describe the basic factorization theorem for fu tions of the form $W(\lambda) = D + C(\lambda - A)^{-1}B$ with D invertible (see [2,3]). W begin with the notion of the product of nodes (cf. [2], Section 1.1).

Consider two operator nodes $\theta_i = (A_i, B_i, C_i, D_i; X_i, Y)$ $(i = 1,2)$. By definition the product $\theta = \theta_1\theta_2$ of the nodes θ_1 and θ_2 is the node $\theta = (A,B,C,D;X,Y)$ in which $X = X_1 \oplus X_2$ is the new state space, $D = D_1D_2$, while the operators $A \in L(X_1 \oplus X_2)$, $B\colon Y \to X_1 \oplus X_2$ and $C\colon X_1 \oplus X_2 \to Y$ are giv by

$$(1.1)\qquad A = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, \quad C = [C_1 \;\; D_1C_2].$$

Note that $\sigma(A_1) \cup \sigma(A_2) \supset \sigma(A)$ (or equivalently, that $\rho(A_1) \cap \rho(A_2) \subset \rho($ An easy computation shows that

$$W_\theta(\lambda) = W_{\theta_1}(\lambda)W_{\theta_2}(\lambda), \qquad \lambda \in \rho(A_1) \cap \rho(A_2),$$

where W_θ, W_{θ_1} and W_{θ_2} are the transfer functions of θ, θ_1 and θ_2. Hence, the transfer function of the product node $\theta = \theta_1\theta_2$ is the product of the transfer functions of θ_1 and θ_2.

Suppose, in addition, that the external operators D_1 and D_2 of the nodes θ_1 and θ_2 are invertible. For $i = 1,2$, put $A_i^\times = A_i - B_iD_i^{-1}C_i$, and l $\theta_i^\times = (A_i^\times, B_iD_i^{-1}, C_i, D_i^{-1}; X_i, Y)$ be the associate node of θ_i. Up to the natura identification of $X_1 \oplus X_2$ and $X_2 \oplus X_1$, one has $(\theta_1\theta_2)^\times = \theta_2^\times\theta_1^\times$. Further, the associate operator of the node $\theta = \theta_1\theta_2$ has the form

$$A^\times = \begin{bmatrix} A_1^\times & 0 \\ -B_2D^{-1}C_1 & A_2^\times \end{bmatrix}.$$

Therefore, $A[X_1] \subset X_1$ and $A^\times[X_2] \subset X_2$. In other words, the state space X_1 (X_2) of the node θ_1 (θ_2) appears to be invariant under the main (associate) operator of the product node $\theta = \theta_1\theta_2$.

Conversely, let $\theta = (A,B,C,D;X,Y)$ be a node with an invertible external operator D and with associate operator $A^\times = A - BD^{-1}C$. Assume we have a decomposition $X = X_1 \oplus X_2$ of X into two closed linear subspaces X_1 and X_2 such that

$$A[X_1] \subset X_1, \quad A^\times[X_2] \subset X_2. \tag{1.2}$$

Then the node θ can be written as a product $\theta = \theta_1\theta_2$ of two nodes θ_1 and θ_2 which have the form

$$\theta_1 = (A_1,B_1,C_1,D_1;X_1,Y), \quad \theta_2 = (A_2,B_2,C_2,D_2;X_2,Y).$$

Here D_1 and D_2 are invertible operators on Y such that $D = D_1D_2$ and

$$A_1 = A|_{X_1}, \quad B_1 = (I-\pi)BD_2^{-1}, \quad C_1 = C|_{X_1},$$

$$A_2 = \pi A|_{X_2}, \quad B_2 = \pi B, \quad C_2 = D_1^{-1}\,C|_{X},$$

where π denotes the projection of X along X_1 onto X_2. Note that according to (1.2) the kernel of the projection π is invariant under A and its image is invariant under $A^\times$. A projection π with these two properties is called a *supporting projection* for the node θ. As the transfer function of the product of two nodes θ_1 and θ_2 is the product of the transfer functions of θ_1 and θ_2, any supporting projection π of the node $\theta = (A,B,C,D;X,Y)$ yields a factorization of the transfer function $W(\lambda) = D + C(\lambda-A)^{-1}B$. From this we easily deduce the next theorem (cf. [2], Section 1.1).

<u>THEOREM 1.1</u>. *Let* $W(\lambda) = D + C(\lambda-A)^{-1}B$ *be a transfer function with invertible operator* D. *Let* $A^\times = A - BD^{-1}C$, *and let* π *be a projection on the state space such that*

$$A[\operatorname{Ker}\pi] \subset \operatorname{Ker}\pi, \quad A^\times[\operatorname{Im}\pi] \subset \operatorname{Im}\pi.$$

Let $D = D_1D_2$, *where* D_1 *and* D_2 *are invertible operators. Then*

$$W(\lambda) = W_1(\lambda)W_2(\lambda), \quad \lambda \in \rho(A),$$

where

$$W_1(\lambda) = D_1 + C(\lambda-A)^{-1}(I-\pi)BD_2^{-1},$$

$$W_2(\lambda) = D_2 + D_1^{-1}C\pi(\lambda-A)^{-1}B,$$

$$W_1(\lambda)^{-1} = D_1^{-1} - D_1^{-1}C(I-\pi)(\lambda-A^\times)^{-1}BD^{-1},$$

$$W_2(\lambda)^{-1} = D_2^{-1} - D^{-1}C(\lambda-A^\times)^{-1}\pi BD_2^{-1}.$$

For a given transfer function W the previous theorem can be used tc obtain all possible factorizations of W on a neighbourhood of infinity ($\mathbb{C}_\infty$). However, for that purpose one cannot do with a single realization c but one has to consider all possible realizations of W. For this and oth aspects of the factorization theory of transfer functions we refer to [2

2. <u>Wiener-Hopf factorization</u>

Let Γ be a simple closed rectifiable Jordan curve on the Riemann sphere $\mathbb{C}_\infty$. We assume Γ to be oriented. The interior domain of Γ will be denoted by F_+ and the exterior domain by F_-. Often we shall take Γ to be the extended imaginary axis and then (unless stated otherwise) the orien tion of Γ is chosen in such a way that F_+ is the open right half-plane.

Let Y be a complex Banach space, and let $W: \Gamma \to L(Y)$ be a continuou operator function whose values are invertible operators on Y. By a *left Wiener-Hopf factorization* of W with respect to the curve Γ we mean a fac rization of W of the form

$$W(\lambda) = W_+(\lambda)D(\lambda)W_-(\lambda), \quad \lambda \in \Gamma, \tag{2.1}$$

where $W_\pm: \Gamma \to L(Y)$ is a continuous operator function whose values are invertible operators on Y and the function $W_\pm$ and its inverse $W_\pm^{-1}$ have a continuous extension to $F_{\pm} \cup \Gamma$ which is analytic on $F_{\pm}$. Further, the so-called diagonal factor D has the form

$$(2.2) \qquad D(\lambda) = P_0 + \sum_{j=1}^{r} \left(\frac{\lambda-\lambda_+}{\lambda-\lambda_-}\right)^{\kappa_j} P_j, \qquad \lambda \in \Gamma.$$

Here $\lambda_+ \in F_+$ and $\lambda_- \in F_-$ are given points, $P_1,\dots,P_r$ are mutually disjoint one-dimensional projections and P_0 is the projection given by $P_0 = I - P_1 - \dots - P_r$. The numbers $\kappa_1,\dots,\kappa_r$, which are non-zero integers such that $\kappa_1 \geq \dots \geq \kappa_r$, are called (the) *left indices* of W with respect to the curve Γ.

It can be proved (cf. [20]) that the left indices of an operator function W with respect to the curve Γ do not depend on the choice of the factors W_+ and W_- in (2.1) nor on the choice of the projections $P_0,P_1,\dots,P_r$ in (2.2) and the points λ_+ and λ_-. Therefore, the indices $\kappa_1,\dots,\kappa_r$ are called *the left indices* of W with respect to Γ. The sum of the left indices of W is called the *left sum index* of W with respect to Γ.

If all left indices of W vanish, then (2.1) takes the form

$$(2.3) \qquad W(\lambda) = W_+(\lambda)W_-(\lambda), \qquad \lambda \in \Gamma.$$

The factorization is then referred to as a *left canonical (Wiener-Hopf) factorization* of W with respect to Γ. The factor W_+ (W_-) in (2.3) is uniquely determined up to a constant invertible factor at the right (left).

If the roles of the factors W_+ and W_- in (2.1) and (2.3) are interchanged, one gets a so-called right Wiener-Hopf factorization. Right indices, right sum index and right canonical (Wiener-Hopf) factorization of the operator function W with respect to Γ may be defined in the same way as for left Wiener-Hopf factorization. We remark that the right indices of the operator function W with respect to Γ may be different from its left indices.

From (2.1) and (2.2) we see that a necessary condition for the existence of a left Wiener-Hopf factorization of an operator function $W: \Gamma \to L(Y)$ is that W is continuous and all its values are invertible operators. But this is not a sufficient condition: already for scalar continuous functions and Γ the unit circle a counterexample can be provided (cf. [21], Section I.5).

The general definition of a Wiener-Hopf factorization for operator functions first appeared in [20] and is a generalization of similar definitions given in the finite-dimensional case (cf. [70,21,20]). In the finite-dimensional case a left (and right) Wiener-Hopf factorization of $W: \Gamma \to L$ always exists, provided $\det W(\lambda) \neq 0$ ($\lambda \in \Gamma$) and, for example, W is Höl continuous (see [20,21,29,61] for further results in this direction). F operator functions that are compact perturbations of the identity a tho study of the existence of Wiener-Hopf factorizations has been made by GOHBERG & LEITERER (cf. [29,30,31,46]). In [30] it is proved that such ator functions admit a left and right canonical factorization whenever, addition, the number

$$\max_{\lambda\in\Gamma} \|W(\lambda)-I\|$$

is sufficiently small.

3. Wiener-Hopf operator integral equations

Let Y be a complex Banach space. For each $1 \leq p \leq +\infty$ and each subir val (a,b) of $\mathbb{R}$ we denote by $L_p((a,b);Y)$ the Banach space of strongly me able vector functions $\psi: (a,b) \to Y$, endowed with the norm

$$\|\psi\|_p = \begin{cases} [\int_a^b \|\psi(t)\|^p \, dt]^{1/p}, & 1 \leq p < +\infty, \\ \operatorname{ess\,sup}_{0<t<+\infty} \|\psi(t)\| , & p = +\infty. \end{cases}$$

In this monograph by strong measurability we mean measurability with res to the Lebesgue measure as exposed in [71], and (unless stated otherwise integrals of vector and operator functions will always be Bochner integr with respect to the Lebesgue measure (cf. [71], Section VI 31).

By $E_+(Y)$, $E_-(Y)$ and $E(Y)$ we mean one of the Banach spaces $L_p((a,b);$ where (a,b) denotes the interval $(0,+\infty)$, $(-\infty,0)$ and $(-\infty,+\infty)$, respectively

Suppose that $k \in L_1((-\infty,+\infty);L(Y))$. Then the operator K, defined by

$$(3.1) \qquad (K\psi)(t) = \int_0^{+\infty} k(t-s)\psi(s)ds \qquad (0 < t < +\infty),$$

is a well-defined bounded linear operator on $E_+(Y)$ whose norm is bounded above by $\|k\|_1 = \int_{-\infty}^{+\infty} \|k(t)\| dt$. By the *Wiener-Hopf operator integral equat* with *kernel* k and right-hand side ω we mean the convolution equation

$$(3.2) \qquad \psi(t) - \int_0^{+\infty} k(t-s)\psi(s)ds = \omega(t), \qquad 0 < t < +\infty,$$

where ω belongs to one of the spaces $E_+(Y)$. If $\omega \in E_+(Y)$, then we shall solve this equation in the same space $E_+(Y)$. I.e., we ask for a vector function $\psi \in E_+(Y)$ such that

$$(3.3) \qquad (I-K)\psi = \omega.$$

In the theory of Wiener-Hopf operator integral equations of the form (3.2) an essential role is played by an operator function $W: i\mathbb{R} \to L(Y)$, which is defined on the imaginary axis by the formula

$$(3.4) \qquad W(\lambda) = I - \int_{-\infty}^{+\infty} e^{\lambda t}\, k(t)dt.$$

We call this operator function the *symbol* of Eq. (3.2). Since $k \in L_1((-\infty,+\infty);Y)$, it is clear that W is continuous in the norm on the extended imaginary line; its value at infinity is the identity operator. Note that we are deviating from the common practice in the theory of Wiener-Hopf operator integral equations by using the Laplace rather than the Fourier transform.

For the one-dimensional (or scalar) and finite-dimensional (or matrix) case the method to solve Eq. (3.2) with the help of a (left) Wiener-Hopf factorization of its symbol (3.4) has been discovered by WIENER & HOPF (cf. [70]). However, they imposed rather heavy assumptions on the kernel k. These assumptions have been alleviated subsequently (for a concise historical account see the end of the introduction of [24]). The final results for the finite-dimensional case are due to GOHBERG & KREIN (see [24], where it is only assumed that $k \in L_1((-\infty,+\infty);L(Y))$). FELDMAN [17] studied Eq. (3.2) and the asymptotics of its solutions in the infinite dimensional case under the hypothesis that the kernel k is compact operator-valued and belongs to $L_1((-\infty,+\infty);L(Y))$. In [18] FELDMAN also gave an account of Wiener-Hopf integral equations on a separable Hilbert space with a weakly integrable compact operator-valued kernel.

In its present, infinite dimensional form, the next theorem is due to FELDMAN (cf. [17]).

<u>THEOREM 3.1</u>. *Let the kernel* k *take its values in the closure in* $L(Y)$ *of the operators of finite rank, and let* $k \in L_1((-\infty,+\infty);L(Y))$. *Then the operator* $I-K$ *is a Fredholm operator on the Banach space* $E_+(Y)$ *if and only if the symbol of the Wiener-Hopf operator integral equation* (3.2) *admits a left Wiener-Hopf factorization with respect to the imaginary axis of the for*[m] (2.1), *where for some function* $x_\pm \in L_1((0,+\infty);L(Y))$

$$(3.5) \qquad W_\pm(\lambda) = I + \int_0^{+\infty} e^{\mp\lambda t}\, x_\pm(t)dt, \qquad \mathrm{Re}\,\lambda = 0.$$

If this is the case and $\kappa_1,\ldots,\kappa_n$ *are the left indices of the symbol, t*[hen]

$$\dim \mathrm{Ker}(I-K) = -\sum_{\kappa_i<0} \kappa_i, \quad \mathrm{codim}\, \mathrm{Im}(I-K) = \sum_{\kappa_i>0} \kappa_i.$$

Hence, the operator $I-K$ *is invertible on the Banach space* $E_+(Y)$ *if and only if the symbol has a left canonical factorization with respect to t*[he] *imaginary line with factors of the form* (3.5).

In the rest of this section we suppose that the symbol (3.4) of Eq (3.2) has a left canonical factorization with respect to the imaginary [axis]. Then the inverse of the operator $I-K$ (or equivalently, a formula for t[he] unique solution of Eq. (3.3)) can be deduced. In fact, we have

<u>THEOREM 3.2</u>. *Let the symbol of* Eq. (3.2) *admit a left canonical factori*[za]*tion with respect to the imaginary line of the form*

$$(3.6) \qquad W(\lambda) = W_+(\lambda)W_-(\lambda), \qquad \mathrm{Re}\,\lambda = 0,$$

where $W_+(\infty) = W_-(\infty) = I$. *Suppose that for certain operator functions* $x \in L_1((0,+\infty);L(Y))$ *and* $y \in L_1((-\infty,0);L(Y))$ *we have*

$$(3.7) \qquad W_-(\lambda)^{-1} = I + \int_0^{+\infty} e^{\lambda t}x(t)dt, \qquad W_+(\lambda)^{-1} = I + \int_{-\infty}^{0} e^{\lambda t}y(t)dt, \quad \mathrm{Re}\,\lambda$$

Then on the Banach space $E_+(Y)$ *the operator* $I-K$ *is invertible and its inverse is given by*

$$(3.8) \qquad ((I-K)^{-1}\omega)(t) = \omega(t) + \int_0^{+\infty} \gamma(t,s)\omega(s)ds \qquad (0 < t < +\infty).$$

In this expression the resolvent kernel $\gamma(t,s)$ *has the form*

$$(3.9) \qquad \gamma(t,s) = \begin{cases} x(t-s) + \int_0^s x(t-r)y(r-s)dr, & 0 < s < t < +\infty, \\ y(t-s) + \int_0^t x(t-r)y(r-s)dr, & 0 < t < s < +\infty. \end{cases}$$

PROOF. Let $\psi: (0,+\infty) \to Y$ be a solution of Eq. (3.2) in the Banach space $E_+(Y)$, and let the symbol W have a left canonical factorization with respect to the imaginary axis of the form (3.6), where the factors W_+ and W_- are given by (3.7). For the sake of convenience we consider the case $E_+(Y) = L_1((0,+\infty);Y)$ only, because for other spaces $E_+(Y)$ the proof is similar. Put

$$\psi(t) = \int_0^{+\infty} k(t-s)\psi(s)ds, \quad -\infty < t < 0. \tag{3.10}$$

Since formula (3.10) defines a bounded linear operator from $E_+(Y)$ into $E_-(Y)$ whose norm is bounded above by $\|k\|_1 = \int_{-\infty}^{+\infty} \|k(t)\| dt$, it is clear that this extension of ψ to $(-\infty,0)$ belongs to $E_-(Y)$.

For imaginary λ we put

$$\begin{aligned} \hat{\psi}_+(\lambda) &= \int_{-\infty}^{0} e^{\lambda t}\psi(t)dt, \quad & \hat{\psi}_-(\lambda) &= \int_0^{+\infty} e^{\lambda t}\psi(t)dt; \\ \hat{\omega}(\lambda) &= \int_0^{+\infty} e^{\lambda t}\omega(t)dt, \quad & \hat{k}(\lambda) &= \int_{-\infty}^{+\infty} e^{\lambda t}k(t)dt, \end{aligned} \tag{3.11}$$

where $\omega \in E_+(Y)$ is the right-hand side of Eq. (3.2). Since $k \in L_1((-\infty,+\infty);L(Y))$, the function $\hat{k}$ is continuous. The functions $\hat{\psi}_+$, $\hat{\psi}_-$ and $\hat{\omega}$ are analytic and continuous up to the boundary of the open right, left and left half-plane, respectively, and vanish at infinity. From (3.2) and (3.10) we obtain the equation

$$W(\lambda)\hat{\psi}_-(\lambda) + \hat{\psi}_+(\lambda) = \hat{\omega}(\lambda), \quad \text{Re } \lambda = 0.$$

Inserting (3.6) we get the Riemann-Hilbert problem

$$W_-(\lambda)\hat{\psi}_-(\lambda) + W_+(\lambda)^{-1}\hat{\psi}_+(\lambda) = W_+(\lambda)^{-1}\hat{\omega}(\lambda), \quad \text{Re } \lambda = 0. \tag{3.12}$$

With the help of (3.7) and (3.11) we get

$$W_+(\lambda)^{-1}\hat{\omega}(\lambda) = \int_{-\infty}^{0} e^{\lambda t} \int_0^{+\infty} y(t-s)\omega(s)ds + \int_0^{+\infty} e^{\lambda t}\left\{\omega(t) + \int_t^{+\infty} y(t-s)\omega(s)ds\right\}dt, \quad \text{Re } \lambda = 0.$$

With the help of the above identity we obtain the unique solution of the Riemann-Hilbert problem (3.12), namely

$$W_-(\lambda)\hat{\psi}_-(\lambda) = \int_0^{+\infty} e^{\lambda t}\left\{\omega(t) + \int_t^{+\infty} y(t-s)\omega(s)ds\right\}dt, \quad \text{Re } \lambda = 0.$$

Using (3.7) we eventually get

$$\begin{aligned} \hat{\psi}_-(\lambda) &= \int_0^{+\infty} e^{\lambda t}\left\{\omega(t) + \int_t^{+\infty} y(t-s)\omega(s)ds\right\}dt + \\ &+ \int_0^{+\infty} e^{\lambda t} \int_0^{t} x(t-r)\left\{\omega(r) + \int_r^{+\infty} y(r-s)\omega(s)ds\right\}dr\, dt = \\ &= \int_0^{+\infty} e^{\lambda t}\omega(t)dt + \int_0^{+\infty} e^{\lambda t}\left\{\int_0^{t} x(t-s)\omega(s)ds + \int_t^{+\infty} y(t-s)\omega(s)ds\right. \\ &+ \int_0^{+\infty} e^{\lambda t}\left\{\int_0^{\min(t,s)} x(t-r)y(r-s)dr\right\}\omega(t)dt, \quad \text{Re } \lambda = 0. \end{aligned}$$

Hence, for $0 < t < +\infty$ the vector function $\psi(t)$ is given by the right-h side of (3.8), where the resolvent $\gamma(t,s)$ has the form (3.9). So we es lished the theorem in case $E_+(Y) = L_1((0,+\infty);L(Y))$. □

For the finite-dimensional case this theorem has appeared in [24] also [21]); an infinite dimensional result has been stated by FELDMAN who assumed that, in addition, the kernel takes its values in the clos in $L(Y)$ of the operators of finite rank. The proof given above follows method exposed in [24].

CHAPTER III

SPECTRAL THEORY OF HERMITIAN ADMISSIBLE PAIRS

In this chapter we consider transfer functions of the form

$$W(\lambda) = (I-B) - T(\lambda-T)^{-1}B.$$

Here T and B are self-adjoint operators on an abstract Hilbert space H, which are related in a special way. Transfer functions of this type ap[in a natural way in Transport Theory as symbols of Wiener-Hopf integral equations. The pair (T,B) is called a hermitian admissible pair on H. I this chapter we develop a little theory for hermitian admissible pairs. particular, the spectral subspaces of T and the associate operator (I-B are described and in terms of these subspaces canonical decompositions are constructed. Special attention is paid to the case when the transfe function W is symmetric with respect to inversion.

1. Preliminaries about self-adjoint operators

Let H be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Given a self-adjoint operator $T \in L(H)$ we denote by m(T) and M(T), respectively, the infimum and supremum of the numerical range of T, i.e.,

$$m(T) = \inf_{\|x\|\le 1} \langle Tx,x\rangle, \quad M(T) = \sup_{\|x\|\le 1} \langle Tx,x\rangle.$$

The interval [m(T),M(T)] is the smallest convex set containing $\sigma(T)$. For every continuous function $f\colon \sigma(T) \to \mathbb{C}$ with only a possible jump discont uity at 0, we define

$$f(T) = \int_{\sigma(T)} f(t)E(dt), \tag{1.1}$$

where E is the resolution of the identity of T (see [13] for the defini of this integral). As known (cf. [13]),

$$(1.2)\qquad \|f(T)\| = \sup_{t\in\sigma(T)} |f(T)|.$$

We now derive four propositions that will play an important role in what follows.

PROPOSITION 1.1. *For* $0 < \omega < \frac{1}{2}\pi$, *let* $\Omega_\omega = \{0 \neq \lambda \in \mathbb{C}\colon |\pm\frac{1}{2}\pi - \arg\lambda| \le \omega\}$. *For every self-adjoint operator* T *with resolution of the identity* E *we have*

$$(1.3)\qquad \lim_{\lambda\to 0,\lambda\in\Omega_\omega} T(T-\lambda)^{-1}x = [I - E(\{0\})]x, \quad x \in H.$$

PROOF. By (1.1) we have

$$T(T-\lambda)^{-1}x = \int_{\sigma(T)} \frac{t}{t-\lambda}\, E(dt)x.$$

Because of the estimate

$$\left|\frac{t}{t-\lambda}\right| \le 1 + \frac{1}{\cos\omega} \qquad (\lambda \in \Omega_\omega,\ t \in \sigma(T) \subset \mathbb{R}),$$

we may apply the theorem of dominated convergence for vector-valued measures (cf. [13], Theorem IV 10.10) and obtain

$$\lim_{\lambda\to 0,\lambda\in\Omega_\omega} T(T-\lambda)^{-1}x = \int_{\sigma(T)} [1 - \chi(t)]E(dt)x, \quad x \in H.$$

Here χ denotes the characteristic function of the set {0}. From this, formula (1.3) is clear. □

PROPOSITION 1.2. *For every nonnegative (self-adjoint) operator* T *with resolution of the identity* E *we have*

$$(1.4)\qquad \lim_{\lambda\to 0,\ \mathrm{Re}\,\lambda\le 0} T(T-\lambda)^{-1}x = [I - E(\{0\})]x, \quad x \in H.$$

This proposition is proved in the same way as the first proposition.

PROPOSITION 1.3. *For* $0 < \omega < \frac{1}{2}\pi$, *let* $\Omega_\omega = \{0 \neq \lambda \in \mathbb{C}\colon |\pm\frac{1}{2}\pi - \arg\lambda| \le \omega\}$. *If* $0 < \alpha < 1$, *then*

$$(1.5)\qquad \| |T|^\alpha (T-\lambda)^{-1}\| \le \frac{(1+\cos\omega)^\alpha}{\cos\omega}\, |\lambda|^{\alpha-1}.$$

PROOF. For $y \in H$ and $0 < \alpha < 1$ it follows from Hölder's inequality th

$$\| |T|^\alpha y\|^2 = \int_{\sigma(T)} |t|^{2\alpha} \langle E(dt)y,y\rangle \le$$

$$(1.6)\qquad \le \left[\int_{\sigma(T)} |t|^2 \langle E(dt)y,y\rangle\right]^\alpha \cdot \left[\int_{\sigma(T)} \langle E(dt)y,y\rangle\right]^{1-\alpha} =$$

$$= \|Ty\|^{2\alpha}\cdot\|y\|^{2(1-\alpha)}.$$

Inserting $y = (T-\lambda)^{-1}x$, $\|T(T-\lambda)^{-1}\| \le 1 + (\cos\omega)^{-1}$ (see the proof of Pr tion 1.1) and $\|(T-\lambda)^{-1}\| \le |\mathrm{Im}\,\lambda|^{-1} \le (\cos\omega)^{-1}\,|\lambda|^{-1}$ we obtain

$$\| |T|^\alpha (T-\lambda)^{-1}x\| \le (1 + (\cos\omega)^{-1})^\alpha (\cos\omega)^{\alpha-1} |\lambda|^{\alpha-1}\|x\|, \qquad \lambda \in$$

From this estimate the proposition is clear. □

A curve Γ that contains the point $\lambda_0 \in \mathbb{R}$ is called *non-tangentia* λ_0 if there exist $\varepsilon > 0$ and $0 < \omega < \frac{1}{2}\pi$ such that

$$(1.7)\qquad \{\lambda \in \Gamma\colon 0 < |\lambda-\lambda_0| < \varepsilon\} \subset \{\lambda_0 \neq \lambda \in \mathbb{C}\colon |\pm\tfrac{1}{2}\pi - \arg(\lambda-\lambda_0)| \le \omega\}$$

and the left-hand side of (1.7) is either a circle arc or a straight l broken at λ_0. We remark that the requirement that near λ_0 the curve Γ special type is not essential.

PROPOSITION 1.4. *Let* T *be a self-adjoint operator with resolution of t identity* E, *and suppose that the real numbers* a *and* b *(with* a < b*) are eigenvalues of* T. *Let* Γ *be a positively oriented simple closed rectifi Jordan contour that is non-tangential at* a *and* b *and whose inner regio contains* $\sigma(T) \cap (a,b)$. *Assume* ϕ *is a scalar function that is analytic uniformly Hölder continuous up to the boundary of* F_+. *Then*

$$(1.8a)\qquad (2\pi i)^{-1} \int_\Gamma \phi(\lambda)(\lambda-T)^{-1}x\,d\lambda = \int_a^b \phi(t)E(dt)x, \quad x \in H.$$

In particular, taking $\phi(\lambda) \equiv 1$ *one has*

(1.8b) $$(2\pi i)^{-1}\int_\Gamma (\lambda-T)^{-1}x\,d\lambda = E((a,b))x, \quad x\in H.$$

PROOF. For $\varepsilon > 0$ let Γ_ε be the curve obtained from Γ by omitting all points $\lambda\in\Gamma$ for which either $|\lambda-a| < \varepsilon$ or $|\lambda-b| < \varepsilon$, and let Γ_ε inherit its orientation from Γ. Then, by Fubini's theorem,

$$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\phi(\lambda)(\lambda-T)^{-1}x\,d\lambda = \int_{\sigma(T)}\left[\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\phi(\lambda)}{\lambda-t}\,d\lambda\right]E(dt)x, \quad x\in H.$$

Since a and b do not belong to the eigenvalue spectrum of T, we have $E(\{a\}) = E(\{b\}) = 0$. If one would simply apply the theorem of dominated convergence for vector-valued measures ([13], Theorem IV 10.10), one would obtain

$$\frac{1}{2\pi i}\int_\Gamma \phi(\lambda)(\lambda-T)^{-1}x\,d\lambda = \int_a^b \phi(t)E(dt)x, \quad x\in H.$$

It remains to justify the application of the principle of dominated convergence. This would require a proof of the boundedness of the set of integrals

(1.9) $$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\phi(\lambda)}{\lambda-t}\,d\lambda \quad (t\in\mathbb{R}\backslash\{a,b\};\ \varepsilon>0).$$

By the non-tangentiality of Γ at a and b there exist $\varepsilon > 0$ and $\omega\in(0,\frac{1}{2}\pi)$ such that

(1.10) $$\{\lambda\in\Gamma:\ 0<|\lambda-\lambda_0|<2\varepsilon\}\subset\{0\neq\lambda\in\mathbb{C}:\ |\pm\tfrac{1}{2}\pi-\arg(\lambda-\lambda_0)|\le\omega\}\ (\lambda_0=a,b)$$

and for $\lambda_0 = a,b$ the left-hand side is either a circle arc or a straight line broken at λ_0. Now we choose points $\lambda_+(a)$, $\lambda_-(a)$, $\lambda_+(b)$ and $\lambda_-(b)$ on the curve Γ such that $|a-\lambda_\pm(a)| = |b-\lambda_\pm(b)| = \varepsilon$, $\lambda_+(a)$ and $\lambda_+(b)$ are in the upper half-plane and $\lambda_-(a)$ and $\lambda_-(b)$ are in the lower half-plane. Further, we suppose that $\varepsilon < (b-a)(4\sin\omega)^{-1}$ so that for $\lambda_0 = a,b$ the left-hand sides of (1.10) do not intersect.

By Ξ_ε we denote the oriented curve composed of the four oriented segments from $a+i\varepsilon\cos\omega$ to $\lambda_+(a)$, from $\lambda_-(a)$ to $a-i\varepsilon\cos\omega$, from $b-i\varepsilon\cos\omega$ to $\lambda_-(b)$ and from $\lambda_+(b)$ to $b+i\varepsilon\cos\omega$. The total length of the four segments marking up Ξ_ε does not exceed $4\varepsilon\sin\omega$. By Δ_ε we denote the oriented union of the oriented segments from $a-i\varepsilon\cos\omega$ to $b-i\varepsilon\cos\omega$ and from $b+i\varepsilon\cos\omega$ to $a+i\varepsilon\cos\omega$. A close inspection of the orientations of Γ_ε, Δ_ε and Ξ_ε reveals the identity

$$\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{d\lambda}{\lambda-t} = \frac{1}{2\pi i}\int_{\Delta_\varepsilon}\frac{d\lambda}{\lambda-t} + \frac{1}{2\pi i}\int_{\Xi_\varepsilon}\frac{d\lambda}{\lambda-t}, \quad t\in\mathbb{R}\backslash\{a,b\}.$$

For $t\in\mathbb{R}$ and $\lambda\in\Xi_\varepsilon$ one has $|\lambda-t|\ge\varepsilon\cos\omega$. Since the length of Ξ_ε d[oes] not exceed $4\varepsilon\sin\omega$, we get the estimate

(1.11a) $$\left|\frac{1}{2\pi i}\int_{\Xi_\varepsilon}\frac{d\lambda}{\lambda-t}\right| \le \frac{2}{\pi}\tan\omega.$$

Another useful estimate is

(1.11b) $$\left|\frac{1}{2\pi i}\int_{\Delta_\varepsilon}\frac{d\lambda}{\lambda-t}\right| = \frac{\varepsilon}{\pi}\int_a^b\frac{d\mu}{(\mu-t)^2+\varepsilon^2} = \frac{1}{\pi}\left\{\arctan\frac{b-t}{\varepsilon} - \arctan\frac{a-t}{\varepsilon}\right\}:$$

By the uniform Hölder continuity of ϕ on $\overline{F}_+$ there exist $0<\alpha<1$ and a constant K_1 such that

(1.11c) $$|\phi(t_1)-\phi(t_2)| \le K_1|t_1-t_2|^\alpha \quad (t_1,t_2\in\overline{F}_+).$$

Finally, for $t\neq a,b$ and $|\pm\frac{1}{2}\pi-\arg(\lambda-\lambda_0)|\le\omega$ $(\lambda_0=a,b)$ the sine rule applied to the triangle with vertices λ_0, λ and t yields

(1.11d) $$\left|\frac{\lambda-\lambda_0}{t-\lambda}\right| \le \frac{1}{\cos\psi}.$$

Next, we apply the estimates (1.11) to find an upper bound for the integrals (1.9). For $t\in(a,b)$ we have

$$\left|\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\phi(\lambda)}{\lambda-t}\,d\lambda\right| =$$

$$= \left|\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\phi(\lambda)-\phi(t)}{\lambda-t}\,d\lambda\right| + \left|\frac{1}{2\pi i}\int_{\Delta_\varepsilon}\frac{\phi(t)}{\lambda-t}\,d\lambda\right| + \left|\frac{1}{2\pi i}\int_{\Xi_\varepsilon}\frac{\phi(t)}{\lambda-t}\,d\lambda\right| \le$$

$$\le K_1\frac{1}{2\pi}\int_{\Gamma_\varepsilon}\frac{d|\lambda|}{|\lambda-t|^{1-\alpha}} + |\phi(t)|\cdot(1+\frac{2}{\pi}\tan\omega),$$

where we have used (1.11c), (1.11b) and (1.11a). Now we exploit the estimate (1.11d) and the boundedness of ϕ on (a,b) and construct an upper bo[und] of the integrals (1.9), which does not depend on $\varepsilon>0$ and $t\in(a,b)$.

Similarly, for $t < a$ we have

$$\left|\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\phi(\lambda)}{\lambda-t}\,d\lambda\right| =$$

$$= \left|\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{\phi(\lambda)-\phi(a)}{\lambda-t}\,d\lambda\right| + \left|\frac{1}{2\pi i}\int_{\Delta_\varepsilon}\frac{\phi(a)}{\lambda-t}\,d\lambda\right| + \left|\frac{1}{2\pi i}\int_{\Xi_\varepsilon}\frac{\phi(t)}{\lambda-t}\,d\lambda\right| \le$$

$$\le K_1\,\frac{1}{2\pi}\int_{\Gamma_\varepsilon}\left|\frac{\lambda-a}{\lambda-t}\right|\frac{d|\lambda|}{|\lambda-a|^{1-\alpha}} + |\phi(a)|\cdot(1+\frac{2}{\pi}\tan\omega),$$

where we have employed (1.11c), (1.11b) and (1.11a). Now we use the inequality (1.11d) and construct an upper bound of the integrals (1.9), which does not depend on $\varepsilon > 0$ and $t < a$. An analogous upper bound is derived for $t > b$. Hence, the set of integrals (1.9) is, indeed, uniformly bounded in $t \in \mathbb{R}\setminus\{a,b\}$ and $\varepsilon > 0$. In this way the non-tangentiality of Γ and the Hölder continuity of ϕ at the points a and b have been essentially used to justify the application of the principle of dominated convergence. □

For $\phi(\lambda) \equiv 1$ Proposition 1.4 yields an expression for the resolution of the identity of a self-adjoint operator related (but not identical) to a well-known formula of STONE (cf. [66]; also [13]). Formula (1.8a) is related to Stieltjes' inversion formula and has not been found in literature.

2. Hermitian admissible pairs and their symbol

Let H be a complex Hilbert space with inner product $<\cdot,\cdot>$. A pair (T,B) of bounded linear operators on H is called a *hermitian admissible pair* on H if

(C.1) T is a self-adjoint operator with Ker T = {0};

(C.2) B is a compact self-adjoint operator;

(C.3) there exist $0 < \alpha < 1$ and an operator $D \in L(H)$ such that

$$B = |T|^\alpha D.$$

As we shall see later (see Proposition 2.2), in Condition (C.3) one can always take D to be compact.

The operator I - B will be denoted by A. If A is invertible, we call the pair (T,B) *regular*; otherwise, the pair (T,B) is called *singular*. If the operator A is nonnegative (i.e., $<Ax,x> \ge 0$ for all $x \in H$), then we call the pair (T,B) *semi-definite*; otherwise, (T,B) is referred to as an *indefinite* pair. If A is strictly positive, then the pair (T,B) is cal
positive definite and the operator

$$C = A^{-1} - I = B(I-B)^{-1}$$

is compact. Observe that in [53] positive definite admissible pairs ha been introduced as self-adjoint admissible pairs.

With a hermitian admissible pair (T,B) on H we associate the oper function W: $\rho(T) \to L(H)$ given by

(2.1) $$W(\lambda) = (T-\lambda)^{-1}(T-\lambda A) = I - \lambda(\lambda-T)^{-1}B.$$

The function W is called the *symbol* of the pair (T,B). Observe that in the symbol has been defined for positive definite admissible pairs onl but in a slightly different way. Note that $W(\lambda)$ may also be written in form

(2.2) $$W(\lambda) = (I-B) - T(\lambda-T)^{-1}B, \quad \lambda \in \rho(T),$$

and hence the symbol $W(\lambda)$ may be viewed as the transfer function of the node (T,B,-T,A;H,H). Obviously, the symbol is continuous (even analytic infinity. The next proposition shows that the symbol is Hölder continuc on the imaginary axis.

PROPOSITION 2.1. *Let* W *be the symbol of the hermitian admissible pair* (*Then* W *is uniformly Hölder continuous on the imaginary axis. Further, i* $\Omega_\omega = \{0 \ne \lambda \in \mathbb{C}: |\pm\frac{1}{2}\pi - \arg\lambda| \le \omega\}$, *then*

(2.3) $$\lim_{\lambda\to 0,\,\lambda\in\Omega_\omega} W(\lambda) = I$$

in the norm of $L(H)$.

PROOF. Assume $B = |T|^\alpha D$, where $0 < \alpha < 1$. We shall prove that on the im inary line the symbol W is Hölder continuous of exponent $0 < \beta < \alpha$. Thi requires an investigation of $W(\lambda)$ for λ in a neighbourhood of the point 0. Let E be the resolution of the identity of the self-adjoint operator By (1.1) we have

$$|\lambda|^{1-\alpha}|T|^{\alpha}(T-\lambda)^{-1}x = \int_{\sigma(T)} \frac{|\lambda|^{1-\alpha}|t|^{\alpha}}{t-\lambda} E(dt)x, \quad x \in H.$$

With the help of (1.2) and (1.5) (with $\omega = 0$) we derive that for imaginary $\lambda \neq 0$

$$(2.4) \qquad \frac{|\lambda|^{1-\alpha}|t|^{\alpha}}{(t^2+|\lambda|^2)^{\frac{1}{2}}} \leq 2^{\alpha}, \quad t \in \sigma(T) \subset \mathbb{R},$$

and therefore the operator $|\lambda|^{1-\alpha}|T|^{\alpha}(T-\lambda)^{-1}$ is uniformly bounded in λ on the non-zero part of the imaginary line. So for all imaginary $\lambda \neq 0$ and $0 < \beta < \alpha$ we have

$$|\lambda|^{-\beta}\|W(\lambda)-I\| = |\lambda|^{\alpha-\beta}\cdot\| |\lambda|^{1-\alpha}|T|^{\alpha}(T-\lambda)^{-1}D\| \to 0 \quad (\lambda \to 0,\ \mathrm{Re}\,\lambda = 0).$$

Here we used that $W(\lambda) - I = -\lambda(\lambda-T)^{-1}B = \lambda|T|^{\alpha}(T-\lambda)^{-1}D$. Since, because of $\sigma(T) \subset \mathbb{R}$, the function W is analytic on the cone $\{0 \neq \lambda \in \mathbb{C}: |\pm\frac{1}{2}\pi - \arg\lambda| < \omega\}$ $(0 < \omega < \frac{1}{2}\pi)$ and at infinity, it follows that for every $0 < \beta < \alpha$ the symbol W is Hölder continuous of exponent β on the extended imaginary line.

Finally, we establish (2.3). It suffices to prove that

$$(2.5) \qquad \lim_{\lambda\to 0,\lambda\in\Omega_{\omega}} \|\lambda(\lambda-T)^{-1}B\| = 0.$$

According to Proposition 1.1 we have $\lambda(\lambda-T)^{-1}y = y - T(T-\lambda)^{-1}y \to 0$ $(\lambda \to 0,\ \lambda \in \Omega_{\omega};\ y \in H)$. But this identity holds true uniformly in y on compact subsets of $L(H)$. Since the image under the compact operator B of the closed unit ball in $L(H)$ is relatively compact in H, formula (2.5) is clear. This settles (2.3). □

<u>PROPOSITION 2.2</u>. *Let* (T,B) *be a hermitian admissible pair on* H. *Then* $B = |T|^{\alpha}D$ *for some* $0 < \alpha < 1$ *and some compact operator* D *on* H.

<u>PROOF</u>. According to Condition (C.3) there exist $0 < \alpha < 1$ and a bounded linear operator D_{α} on H such that $B = |T|^{\alpha}D_{\alpha}$. For $0 < \beta < \alpha$ put $D_{\beta} = |T|^{\alpha-\beta}D_{\alpha}$. Then $B = |T|^{\beta}D_{\beta}$. We shall prove that D_{β} is compact.

Let $(\lambda_n)_{n=0}^{+\infty}$ be the sequence of eigenvalues of the compact self-adjoint operator B counted according to multiplicity, $(\phi_n)_{n=0}^{+\infty}$ the corresponding orthonormal set of eigenvectors of B, and suppose that $|\lambda_0| \geq |\lambda_1| \geq \ldots$. Then

$$B = \sum_{n=0}^{+\infty} \lambda_n\langle\cdot,\phi_n\rangle\phi_n.$$

Put $P_k = \sum_{n=0}^{k} \langle\cdot,\phi_n\rangle\phi_n$. Then P_k is the orthogonal projection of H onto linear span of $\phi_0,\phi_1,\ldots,\phi_k$, while $\|B-BP_k\| = s_{k+1}(B)$ is the $(k+1)$-th approximation number of B (cf. [25], Chapter II). Then $\|B-BP_k\| \to 0$ as $k \to +\infty$.

Let E be the resolution of the identity of the self-adjoint operat T. For $0 < \beta < \alpha$ and $x \in H$ we apply Hölder's inequality and get the est mate

$$\|D_{\beta}x\|^2 = \int_{\sigma(T)} |t|^{2(\alpha-\beta)}\|E(dt)D_{\alpha}x\|^2 \leq$$

$$\leq \left[\int_{\sigma(T)} |t|^{2\alpha}\|E(dt)D_{\alpha}x\|^2\right]^{1-\frac{\beta}{\alpha}}\cdot\left[\int_{\sigma(T)} \|E(dt)D_{\alpha}x\|^2\right]^{\beta/\alpha},$$

and therefore

$$\|D_{\beta}x\| \leq \|Bx\|^{1-\frac{\beta}{\alpha}}\|D_{\alpha}x\|^{\beta/\alpha}; \quad x \in H,\ 0 < \beta < \alpha.$$

Substitute $x = (I-P_k)y$ and use the estimate $\|D_{\alpha}(I-P_k)x\| \leq \|D_{\alpha}\|\cdot\|x\|$. Th

$$\|D_{\beta}(I-P_k)\| \leq \|B-BP_k\|^{1-\frac{\beta}{\alpha}}\|D_{\alpha}\|^{\beta/\alpha}; \quad 0 < \beta < \alpha.$$

Now $D_{\beta}P_k$ is an operator of finite rank $(k = 0,1,\ldots)$. Since $\|B-BP_k\| \to 0$ $k \to +\infty$, we see that D_{β} is a compact operator. Putting $M = \max(1,\|D_{\alpha}\|)$ w get, in view of Theorem II 2.1 of [25],

$$s_{k+1}(D_{\beta}) \leq M\, s_{k+1}(B)^{1-\frac{\beta}{\alpha}}; \quad k = 0,1,2,\ldots,\ 0 < \beta < \alpha,$$

where $s_{k+1}(D_{\beta})$ is the $(k+1)$-th approximation number of D_{β}. □

Whenever for some $p \geq 1$ the operator B belongs to the p-th Von Neu Schatten class and $\varepsilon > 0$ is given, there exists $0 < \beta < 1$ such that D_{β} belongs to the $(p+\varepsilon)$-th Von Neumann Schatten class (cf. [62,25] for the theory of these classes).

Assume the hermitian admissible pair (T,B) to be positive definite Then the operator $A = I - B$ is invertible, and hence the symbol W of the pair (T,B) is invertible at ∞ (cf. (2.2)). In fact, for λ in a neighbou hood of ∞ one has

(2.6) $$W(\lambda)^{-1} = A^{-1} + A^{-1}T(\lambda - A^{-1}T)^{-1}BA^{-1}.$$

Thus in this case the node $(A^{-1}T, BA^{-1}, A^{-1}T, A^{-1}; H, H)$ is the associate node of the node $(T, B, -T, A; H, H)$. The next theorem shows that with respect to a suitable inner product on H the pair $(A^{-1}T, -BA^{-1})$ is again a positive definite (hermitian) admissible pair.

<u>THEOREM 2.3</u>. *Let* (T,B) *be a positive definite (hermitian) admissible pair on* H, *and let* W *be its symbol. Put* $A = I - B$. *Then*

(2.7) $$\langle x,y\rangle_A = \langle Ax,y\rangle$$

defines an equivalent inner product on H, *the pair* $(A^{-1}T, -BA^{-1})$ *is a positive definite (hermitian) admissible pair on the space* H *endowed with the inner product* $\langle\cdot,\cdot\rangle_A$ *and its symbol is equal to* W^{-1}.

<u>PROOF</u>. As A is strictly positive, it is clear that $\langle\cdot,\cdot\rangle_A$ is an equivalent inner product on H. Put $S = A^{-1}T$. From

$$\langle Sx,y\rangle_A = \langle Tx,y\rangle = \langle x,Ty\rangle = \langle A^{-1}Ax,Ty\rangle = \langle x,A^{-1}Ty\rangle_A = \langle x,Sy\rangle_A,$$

we conclude that S is self-adjoint on H endowed with $\langle\cdot,\cdot\rangle_A$. Obviously, Ker S = {0}. So Condition (C.1) holds true.

Since $A = I - B$, the operators A and B commute. It follows that $BA^{-1} = A^{-1}B$. Using this one easily sees that BA^{-1} is self-adjoint with respect to $\langle\cdot,\cdot\rangle_A$. The fact that B is compact, implies that BA^{-1} is compact too. This proves Condition (C.2).

From formula (2.6) it is clear that the symbol of the pair $(A^{-1}T, -BA^{-1})$ is equal to W^{-1}. Further, A^{-1} is strictly positive with respect to the inner product (2.7). So it remains to prove that Condition (C.3) holds true.

By Proposition 2.2 there exist $0 < \alpha < 1$ and a compact operator D such that $B = |T|^{\alpha}D$. Put $S = A^{-1}T$ and $C = BA^{-1}$. We have to show that $C = |S|^{\alpha}\tilde{D}$ for some bounded operator $\tilde{D}$. (Here the absolute value of S is defined in terms of the inner product (2.7) on H.) In fact, it suffices to show that

(2.8) $$|S|^{\alpha} - |T|^{\alpha} = |T|^{\alpha}K$$

for some compact operator K. Indeed, from (2.8) it follows that $|S|^{\alpha} = |T|^{\alpha}(I+K)$, where $I + K$ is a Fredholm operator of index 0 such t[…] $\mathrm{Ker}(I+K) \subset \mathrm{Ker}|S|^{\alpha}$. Since $|S|^{\alpha}$ has a trivial kernel, it follows that […] invertible. But then $C = |S|^{\alpha}\tilde{D}$, where $\tilde{D} = (I+K)^{-1}DA^{-1}$, and the proof […] be complete.

Let Γ_+ (Γ_-) be a simple closed rectifiable Jordan contour in the closed right (left) half-plane that is non-tangential at $\lambda = 0$ and wh[…] inner region contains the parts of the spectra of T and S on the posi[…] (negative) real line. We assume that Γ_+ and Γ_- are oriented in the po[…] sense. By Γ we denote the oriented curve composed of the contours Γ_+ a[…] Γ_-.

By Proposition 1.4 we have for all $x \in H$

(2.9) $$(|S|^{\alpha} - |T|^{\alpha})x = (2\pi i)^{-1}\Big(\int_{\Gamma_+} + \int_{\Gamma_-}\Big)\lambda^{\alpha}[(\lambda-S)^{-1} - (\lambda-T)^{-1}]x\,d\lambda.$$

Here λ^{α} is defined on the closed right and left half-plane separately analytic function that is positive on the real line. By (I 2.4) we hav[…]

(2.10) $$(\lambda-S)^{-1} - (\lambda-T)^{-1} = -(\lambda-T)^{-1}BW(\lambda)^{-1}T(T-\lambda)^{-1}, \quad 0 \neq \lambda \in \Gamma.$$

By Propositions 1.1 and 2.1 the function $DW(\lambda)^{-1}T(T-\lambda)^{-1}$ is bounded on $\Gamma\setminus\{0\}$. Put $K(\lambda) = -(\lambda-T)^{-1}DW(\lambda)^{-1}T(T-\lambda)^{-1}$, $0 \neq \lambda \in \Gamma$. Using that $B = |$[…] we get, in view of Proposition 1.1,

$$(\lambda-S)^{-1} - (\lambda-T)^{-1} = |T|^{\alpha}K(\lambda); \|K(\lambda)\| = O(|\lambda|^{-1}) \quad (\lambda\to 0, 0\neq\lambda\in\Gamma$$[…]

For $\varepsilon > 0$ let $\Gamma_{+\varepsilon}$ $(\Gamma_{-\varepsilon})$ be the curve obtained from Γ_+ (Γ_-) by omitting of its points λ for which $|\lambda| < \varepsilon$. Assume $\Gamma_{+\varepsilon}$ $(\Gamma_{-\varepsilon})$ inherits its orien[…] tion from Γ_+ (Γ_-). For $\varepsilon > 0$ let Γ_{ε} be the oriented curve, which is co[…] of the two parts $\Gamma_{+\varepsilon}$ and $\Gamma_{-\varepsilon}$. Put

(2.11) $$K_{\varepsilon} = (2\pi i)^{-1}\Big(\int_{\Gamma_{+\varepsilon}} + \int_{\Gamma_{-\varepsilon}}\Big)\lambda^{\alpha}K(\lambda)\,d\lambda, \quad \varepsilon > 0.$$

Since $\|\lambda^{\alpha}K(\lambda)\| = O(|\lambda|^{\alpha-1})$ $(\lambda\to 0,\ 0\neq\lambda\in\Gamma)$, it follows that

$$\|K_{\varepsilon} - K_{\delta}\| = O\Big(\Big(\int_{\Gamma_{\delta}} - \int_{\Gamma_{\varepsilon}}\Big)|\lambda|^{\alpha-1}d\lambda\Big) \quad (\varepsilon > \delta > 0,\ \varepsilon \downarrow 0).$$

Hence, the operator K_{ε} has a limit in the norm of $L(H)$ as $\varepsilon \downarrow 0$. Since

$|T|^{\alpha}K_{\varepsilon} = (2\pi i)^{-1} \int_{\Gamma_{\varepsilon}} \lambda^{\alpha}[(\lambda-S)^{-1}-(\lambda-T)^{-1}]d\lambda$, it is clear from (2.9) that this limit $K = \lim_{\varepsilon\downarrow 0} K_{\varepsilon}$ satisfies (2.8).

Next, recall that D is a compact operator. By $B = |T|^{\alpha}D$ and formula (2.10) it is clear that $K(\lambda) = -(\lambda-T)^{-1}DW(\lambda)^{-1}T(T-\lambda)^{-1}$ is compact too $(0 \neq \lambda \in \Gamma)$. But then the operator K_{ε}, which is defined by (2.11) as a Riemann integral in the norm of $L(H)$, is also a compact operator. Since $\|K_{\varepsilon}-K\| \to 0$ as $\varepsilon \downarrow 0$, we conclude that the operator K is compact. From the compactness of K we finish the proof as described earlier. □

3. Spectral projections

Throughout this section H is a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and (T,B) is a semi-definite admissible pair on H. As usual $A = I - B$. In this section we shall study three different spectral decompositions of H. First we write H as a direct sum of spectral subspaces of T. Next we construct spectral decompositions of H corresponding to the unbounded operators $T^{-1}A$ and AT^{-1}.

Let E be the resolution of the identity of the self-adjoint operator T. Put $P_+ = E((0,\infty))$ and $P_- = E((-\infty,0))$, and let H_+ and H_- denote the ranges of the orthogonal projections P_+ and P_-, respectively. Since Ker T = {0}, we have

$$(3.1) \qquad H = H_- \oplus H_+.$$

This is the first spectral decomposition of H.

To study the spectral properties of the possibly unbounded operators $T^{-1}A$ and AT^{-1} we first consider the linear operator polynomial

$$L(\lambda) = A - \lambda T, \quad \lambda \in \mathbb{C}.$$

Put $\Xi = \{\lambda \in \mathbb{C}: I-\lambda T \text{ is invertible}\}$. Note that Ξ is an open connected subset of $\mathbb{C}$, which contains the set $\mathbb{C}\setminus\mathbb{R}$. Let $\Sigma(L)$ denote the spectrum of L, i.e., the set $\Sigma(L)$ consists of all $\lambda \in \mathbb{C}$ such that $L(\lambda)$ is not invertible. First we shall prove that $\Xi \cap \Sigma(L)$ is a discrete subset of Ξ.

Since $L(\lambda) = (I-\lambda T) - B$ and B is compact, the operator $L(\lambda)$ is a Fredholm operator of index zero for each $\lambda \in \Xi$. Further

$$L(\lambda) = (I-\lambda T)W(\tfrac{1}{\lambda}), \quad 0 \neq \lambda \in \Xi,$$

where W is the symbol of the pair (T,B). By Proposition 1.2 the opera
$W(\frac{1}{\lambda})$ is invertible for Re $\lambda = 0$ and $|\lambda|$ sufficiently large. It follow
the same is true for $L(\lambda)$. Since Ξ is connected we may conclude (cf.
that $L(\lambda)$ is invertible for all $\lambda \in \Xi$ with the exception of a discret
set of Ξ. Hence, $\Xi \cap \Sigma(L)$ is a discrete subset of Ξ.

In particular, there exists $\varepsilon > 0$ such that $\Sigma(L) \cap \{\lambda \in \mathbb{C}: 0<|\lambda|\leq\varepsilon$
empty. Define

$$(3.2) \qquad P_0 = \frac{-1}{2\pi i}\int_{|\lambda|=\varepsilon} L(\lambda)^{-1}T\,d\lambda, \quad P_0^{\dagger} = \frac{-1}{2\pi i}\int_{|\lambda|=\varepsilon} TL(\lambda)^{-1}d\lambda.$$

Here the circle $|\lambda| = \varepsilon$ is assumed to have a positive orientation. A s
argument (cf. [67], Section 1.3) shows that P_0 and $P_0^{\dagger}$ are projections
H_0 $(H_0^{\dagger})$ denote the range and H_1 $(H_1^{\dagger})$ the kernel of the projection P_0
Then

$$(3.3) \qquad H = H_0 \oplus H_1, \quad H = H_0^{\dagger} \oplus H_1^{\dagger}.$$

PROPOSITION 3.1. *We have* dim H_0 = dim $H_0^{\dagger} < +\infty$ *and*
(i) $TH_0 = H_0^{\dagger}$, $\overline{TH_1} = H_1^{\dagger}$;
(ii) Ker $A \subset H_0$, $AH_0 \subset H_0^{\dagger}$, $\overline{AH_1} = H_1^{\dagger}$;
(iii) $H_0^{\perp} = H_1^{\dagger}$, $H_1^{\perp} = H_0^{\dagger}$.

PROOF. Take $\varepsilon > 0$ as in the definition of P_0 and $P_0^{\dagger}$ (cf. formula (3.2)
Put

$$R = \frac{-1}{2\pi i}\int_{|\lambda|=\varepsilon} L(\lambda)^{-1}d\lambda.$$

Since L(0) is a Fredholm operator, the coefficients of the principal p
of the Laurent expansion of $L(\lambda)^{-1}$ at zero are operators of finite ra
(cf. [33]). In particular, the operator R is of finite rank. Note that
$P_0 = RT$ and $P_0^{\dagger} = TR$. So both P_0 and $P_0^{\dagger}$ have finite rank.

The operator T maps H_0 in a one-one manner onto $H_0^{\dagger}$. To see this c
observes that $TP_0 = TRT = P_0^{\dagger}T$. This implies that $TH_0 \subset H_0^{\dagger}$. Also,
$RP_0^{\dagger} = RTR = P_0R$, and hence $RH_0^{\dagger} \subset H_0$. Now one checks directly, that

$$(T|_{H_0})^{-1} = R|_{H_0^{\dagger}}.$$

The fact that T maps H_0 in a one-one manner onto $H_0^\dagger$, implies that $\dim H_0 = \dim H_0^\dagger$.

To prove the second equality in (i) note that Im T is dense in H. Since T maps H_0 into $H_0^\dagger$ and H_1 into $H_1^\dagger$ and one has the decompositions (3.3), one must have that $\overline{TH_1} = H_1^\dagger$.

Statement (ii) is an immediate consequence of the definitions of P_0 and $P_0^\dagger$ (cf. [67], Section 1.3). Statement (iii) follows from the fact that

$$P_0^* = (RT)^* = TR^* = TR = P_0^\dagger,$$

which is a consequence of the self-adjointness of A and T. □

The space H_0 will be called the *singular subspace* of the pair (T,B), and the space H_1 will be called the *regular subspace* of the pair (T,B). If (T,B) is a positive definite pair, then $H_0 = \{0\}$ and H_1 is equal to the full space H.

Since Ker T = {0}, the operator $T^{-1}A$ is a well-defined possibly unbounded operator on H, whose domain of definition is equal to the set

$$D(T^{-1}A) = \{x \in H\colon Ax \in \operatorname{Im} T\}.$$

The fact that $TH_0 = H_0^\dagger$ and $AH_0 \subset H_0^\dagger$ implies that $H_0 \subset D(T^{-1}A)$ and $(T^{-1}A)H_0 \subset H_0$. So the restriction of $T^{-1}A$ to H_0 is a bounded operator on the whole of H_0.

So far we have not used that the pair (T,B) is semi-definite, i.e., we have not yet used the fact that A is a positive operator. However, in the proof of the next proposition this will play an essential role.

<u>PROPOSITION 3.2</u>. *We have* $(T^{-1}A)^2x = 0, \quad x \in H_0$.

<u>PROOF</u>. From the definitions of H_0 and $H_0^\dagger$ it follows that for each $0 \neq \lambda \in \mathbb{C}$ the operator $A-\lambda T$ maps H_0 in a one-one manner onto $H_0^\dagger$. Hence, we may conclude that the spectrum of the restriction of $T^{-1}A$ to H_0 consists of the point zero only. Since H_0 is finite-dimensional, there exists a positive number n such that $(T^{-1}A)^nx = 0$ for each $x \in H$.

Let $x \in H_0$, and assume that $(T^{-1}A)^3x = 0$. Put $y = (T^{-1}A)x$ and $z = (T^{-1}A)y$. Then $y,z \in H_0$, $Ax = Ty$, $Ay = Tz$ and $Az = 0$. Note that

$$\langle Ay,y\rangle = \langle Tz,y\rangle = \langle z,Ty\rangle = \langle z,Ax\rangle = \langle Az,x\rangle.$$

Since $Az = 0$, this implies $\langle Ay,y\rangle = 0$. Now we use the fact that A is a positive operator. So $\|A^{\frac{1}{2}}y\|^2 = \langle Ay,y\rangle = 0$, and therefore $Tz = Ay = 0$, means that $(T^{-1}A)^2x = 0$.

Given $x \in H_0$, we know that $(T^{-1}A)^nx = 0$ for some $n \geq 4$. Repeating above argument n-2 times we derive that $(T^{-1}A)^2x = 0$. □

In terms of $L(\lambda) = A - \lambda T$ the previous proposition tells us that I has a pole at $\lambda = 0$ of order at most 2. Hence, there exist operators c finite rank R_1 and R_2 such that $L(\lambda)^{-1} - \lambda^{-1}R_1 - \lambda^{-2}R_2$ is analytic at λ

Next we consider $T^{-1}A$ on H_1. The fact that A maps H_1 in a one-one manner onto $H_1^\dagger$ and $TH_1 \subset H_1^\dagger$ implies that there exists a unique bounde linear operator $S\colon H_1 \to H_2$ such that

$$(3.4)\qquad ASx = Tx, \quad x \in H_1.$$

Obviously, Ker S = {0}. We shall prove that with respect to an equival inner product on H_1 the operator S is self-adjoint. We call S the *asso operator* of the pair (T,B).

Consider on H_1 the sesquilinear form

$$(3.5)\qquad \langle x,y\rangle_A = \langle Ax,y\rangle, \quad (x,y \in H_1).$$

Since A is a positive operator and A acts as an invertible operator fr H_1 onto $H_1^\dagger$, it is clear that $\langle\cdot,\cdot\rangle_A$ defines an inner product on H_1 whi equivalent to the original inner product on H_1. With respect to the in product $\langle\cdot,\cdot\rangle_A$ the operator S is self-adjoint. Indeed, for $x,y \in H_1$ we

$$\langle Sx,y\rangle_A = \langle ASx,y\rangle = \langle Tx,y\rangle = \langle x,Ty\rangle =$$

$$= \langle x,ASy\rangle = \langle Ax,Sy\rangle = \langle x,Sy\rangle_A.$$

We shall use the self-adjointness of S to make a further spectral decomposition of H_1. Let F be the resolution of the identity of S as a self-adjoint operator on $(H_1\langle\cdot,\cdot\rangle_A)$. Put $H_p = \operatorname{Im} F((0,+\infty))$ and $H_m = \operatorname{Im} F((-\infty,0))$. The fact that Ker S = {0} implies that $H_1 = H_m \oplus H_p$, a thus we obtain

(3.6) $$H = H_0 \oplus H_m \oplus H_p.$$

We define P_p to be the projection of H onto H_p along $H_0 \oplus H_m$ and P_m is the projection of H onto H_m along $H_0 \oplus H_p$. Note that

(3.7) $$\begin{cases} \langle Tx,x\rangle = \langle Sx,x\rangle_A \le 0, & x \in H_m, \\ \langle Tx,x\rangle = \langle Sx,x\rangle_A \ge 0, & x \in H_p. \end{cases}$$

<u>PROPOSITION 3.3</u>. *Let Γ_+ (Γ_-) be a simple closed rectifiable Jordan contour in the closed right (left) half-plane that is oriented in the positive sense, non-tangential of $\lambda = 0$ and whose inner region contains the parts of the spectra of T and S on the open positive (negative) real line. Then for every $x \in H$ we have*

$$P_p x = (-2\pi i)^{-1} \int_{\Gamma_+} (T-\lambda A)^{-1} Ax\, d\lambda, \quad P_m x = (-2\pi i)^{-1} \int_{\Gamma_-} (T-\lambda A)^{-1} Ax\, d\lambda.$$

<u>PROOF</u>. We restrict ourselves to the proof of the formula for P_p. If $x \in H_1$, then one employs the self-adjointness of S in the inner product (3.5) of H_1 and Proposition 1.4 and obtains

$$P_p x = (2\pi i)^{-1} \int_{\Gamma_+} (\lambda - S)^{-1} x\, d\lambda = (-2\pi i)^{-1} \int_{\Gamma_+} (T-\lambda A)^{-1} Ax\, d\lambda.$$

On the other hand, if $x \in H_0$, then $P_p x = 0$ and, in view of $\sigma(T^{-1}A_{|H_0}) = \{0\}$,

$$(-2\pi i)^{-1} \int_{\Gamma_+} (T-\lambda A)^{-1} Ax\, d\lambda = (-2\pi i)^{-1} \int_{\Gamma_+} (I-\lambda T^{-1}A)^{-1} T^{-1} Ax\, d\lambda = 0.$$

Since $H = H_1 \oplus H_0$, the proposition is clear. □

<u>PROPOSITION 3.4</u>. *The following intertwining formulas hold true:*

(3.8) $$AP_0 = P_0^* A, \quad AP_p = P_p^* A, \quad AP_m = P_m^* A;$$

(3.9) $$TP_0 = P_0^* T, \quad TP_p = P_p^* T, \quad TP_m = P_m^* T.$$

<u>PROOF</u>. Since $P_0^* = P_0^\dagger$, the first equalities in (3.8) and (3.9) have already been proved. Let us prove $AP_p = P_p^* A$. Take x,y in H. Note that we may write

$$P_p x = F((0,+\infty))(I-P_0)x, \quad P_m x = F((-\infty,0))(I-P_0)x,$$

where F is the resolution of the identity of S (as a self-adjoint oper on H_1). Further, recall that $AH_p \subset H_1^\dagger = H_0^\perp$ (by Proposition 3.1(iii)).

$$\langle AP_p x,y\rangle = \langle AP_p x,(I-P_0)y\rangle = \langle P_p x,(I-P_0)y\rangle_A =$$
$$= \langle F((0,+\infty))(I-P_0)x,(I-P_0)y\rangle_A =$$
$$= \langle (I-P_0)x, F((0,+\infty))(I-P_0)y\rangle_A =$$
$$= \langle A(I-P_0)x, P_p y\rangle = \langle P_p^* Ax, y\rangle.$$

Thus $AP_p = P_p^* A$. The third equality in (3.8) is proved in a similar way.
Next we prove that $TP_p = P_p^* T$. For $x,y \in H$ we have

$$\langle TP_p x,y\rangle = \langle TP_p x,(I-P_0)y\rangle =$$
$$= \langle SF((0,+\infty))(I-P_0)x,(I-P_0)y\rangle_A =$$
$$= \langle (I-P_0)x, F((0,+\infty))S(I-P_0)y\rangle_A,$$

because S and $F((0,\infty))$ are self-adjoint on $(H_1, \langle\cdot,\cdot\rangle_A)$. Since S and $F((0,+\infty))$ commute, we get

$$\langle TP_p x,y\rangle = \langle (I-P_0)x, SF((0,+\infty))(I-P_0)y\rangle_A =$$
$$= \langle A(I-P_0)x, SP_p y\rangle = \langle Ax, SP_p y\rangle$$
$$= \langle x, TP_p y\rangle = \langle P_p^* Tx, y\rangle.$$

So $TP_p = P_p^* T$. The third equality in (3.9) can be proved in a similar wa

<u>COROLLARY 3.5</u>. *We have*

(3.10) $$AH_p = (H_m \oplus H_0)^\perp, \quad AH_m = (H_p \oplus H_0)^\perp;$$

(3.11) $$\overline{TH_p} = (H_m \oplus H_0)^\perp, \quad \overline{TH_m} = (H_p \oplus H_0)^\perp.$$

<u>PROOF</u>. Since H_p is closed in H_1 and A maps H_1 in a one-one manner onto

the space AH_p is closed. Further,

$$AH_p = \operatorname{Im} AP_p = (\operatorname{Ker}(AP_p)^*)^{\perp} = (\operatorname{Ker} P_p^*A)^{\perp} =$$

$$= (\operatorname{Ker} AP_p)^{\perp} = (H_0 \oplus H_m)^{\perp}.$$

The second equality in (3.10) and the equalities (3.11) can be proved in a similar way. □

Now let us return to the possibly unbounded operators $T^{-1}A$ and AT^{-1}. We have already seen that $H_0 \subset D(T^{-1}A)$ and $(T^{-1}A)[H_0] \subset H_0$. Since $AH_1 \subset H_1^{\dagger}$ and $TH_1 \subset H_1^{\dagger}$, it is clear that

$$(T^{-1}A)[H_1 \cap D(T^{-1}A)] \subset H_1.$$

So $T^{-1}A$ is completely reduced by the decomposition $H = H_0 \oplus H_1$. From the definition of S one sees that $\operatorname{Im} S = H_1 \cap D(T^{-1}A)$ and

(3.12) $$S = (T^{-1}A_{|H_1})^{-1}, \quad S^{-1} = T^{-1}A_{|H_1}.$$

Since T maps H_0 in a one-one manner onto $H_0^{\dagger}$ and $AH_0 \subset H_0^{\dagger}$, it is clear that $H_0^{\dagger} \subset D(AT^{-1})$. From $AH_1^{\dagger} \subset H_1$ and $TH_1 \subset H_1^{\dagger}$ it follows that

$$(AT^{-1})[H_1^{\dagger} \cap D(AT^{-1})] \subset H_1^{\dagger}.$$

So AT^{-1} is completely reduced by the decomposition $H = H_0^{\dagger} \oplus H_1^{\dagger}$.

To study AT^{-1} on H_1 one considers the bounded linear operator $S^{\dagger}: H_1^{\dagger} \to H_1^{\dagger}$ defined by

(3.13) $$S^{\dagger}Ax = Tx, \quad x \in H_1^{\dagger}.$$

It is easy to see that $\operatorname{Im} S^{\dagger} = H_1^{\dagger} \cap D(AT^{-1})$ and

$$S^{\dagger} = (AT^{-1}{}_{|H_1^{\dagger}})^{-1}, \quad (S^{\dagger})^{-1} = AT^{-1}{}_{|H_1^{\dagger}}.$$

Note that $ASx = S^{\dagger}Ax$ and $TSx = S^{\dagger}Tx$ for each $x \in H_1$. In particular, since A maps H_1 in a one-one manner onto $H_1^{\dagger}$, one sees that S and $S^{\dagger}$ are similar, the similarity being given by the map A. Let F be the resolution of the identity of S (as a self-adjoint operator on $(H_1, \langle\cdot,\cdot\rangle_A)$). For Borel set τ in $\mathbb{C}$ define $F^{\dagger}(\tau)$ on $H_1^{\dagger}$ by

$$F^{\dagger}(\tau)Ax = AF(\tau)x.$$

Then $F^{\dagger}$ is a resolution of the identity for $S^{\dagger}$. From Proposition 3.4 deduces easily that

$$P_p^*x = F^{\dagger}((0,+\infty))(I-P_0^*)x, \quad P_m^*x = F^{\dagger}((-\infty,0))(I-P_0^*)x$$

for each $x \in H$. For every bounded measurable function ϕ on $\sigma(S) = \sigma(S$ have

(3.14) $$A\phi(S)x = \phi(S^{\dagger})Ax, \quad T\phi(S)x = \phi(S^{\dagger})Tx, \quad x \in H_1.$$

Here

$$\phi(S) = \int_{\sigma(S)} \phi(t)F(dt), \quad \phi(S^{\dagger}) = \int_{\sigma(S^{\dagger})} \phi(t)F^{\dagger}(dt).$$

For the conservative isotropic case of the Transport Equation th decomposition $H = H_1 \oplus H_0$, the inner product (2.7) on H_1 and the self-a operator S have been introduced in [47]. The decomposition $H = H_1 \oplus H_0$, has been constructed here with the help of [67], is an improved gener tion of its analogue in [47].

4. Auxiliary operator semigroups

Often the solutions of the operator differential equations and t operator integral equations we shall study in Chapters IV and V will expressed in terms of certain operator semigroups. Here we introduce semigroups and state some of their properties. We continue using the tions of the previous section.

First let T be a nonnegative (self-adjoint) operator with Ker T = By (1.2) the set

$$\{\|[T(T-\lambda)^{-1}]^n\| : n = 0,1,2,\ldots;\ -\infty < \lambda < 0\}$$

is contained in [0,1]. Therefore, the unbounded inverse of $-T$ is the infinitesimal generator of a strongly continuous contraction semigrou is clear from the Hille-Yosida-Phillips theorem ([13], Theorem VIII 1.13). Note that in this particular case

$$(4.1)\qquad e^{-tT^{-1}} = \int_0^{+\infty} e^{-t/\mu}\, E(d\mu), \quad 0 < t < +\infty,$$

where E denotes the resolution of the identity of T. One easily estimates that

$$(4.2)\qquad \| e^{-tT^{-1}} \| = \sup_{\mu\in\sigma(T)} |e^{-t/\mu}| = e^{-t/\|T\|}, \quad 0 < t < +\infty.$$

From (4.1) one easily shows that the semigroup (4.1) is analytic (cf. [45] for the definition and main properties of analytic semigroups), and therefore the function (4.1) is a C^∞-function in $\mathbb{R}\backslash\{0\}$ in the operator norm.

Now let (T,B) be a semi-definite admissible pair on a Hilbert space H, and let A = I − B. With T we associate two analytic contraction semigroups, namely

$$e^{-tT^{-1}}P_+ := \int_0^{+\infty} e^{-t/\mu}\, E(d\mu), \quad 0 \le t < +\infty;$$

$$e^{+tT^{-1}}P_- := \int_{-\infty}^{0} e^{+t/\mu}\, E(d\mu), \quad 0 \le t < +\infty.$$

Here E is the resolution of the identity of T.

Further we introduce four analytic semigroups related to $T^{-1}A$ and AT^{-1}, namely

$$(e^{-tT^{-1}A_{P_p}})x := \int_0^{+\infty} e^{-t/\mu}F(d\mu)(I-P_0)x, \quad 0 \le t < +\infty;$$

$$(e^{+tT^{-1}A_{P_m}})x := \int_{-\infty}^{0} e^{+t/\mu}F(d\mu)(I-P_0)x, \quad 0 \le t < +\infty;$$

$$(e^{-tAT^{-1}}P_p^*)x := \int_0^{+\infty} e^{-t/\mu}F^\dagger(d\mu)(I-P_0^*)x, \quad 0 \le t < +\infty;$$

$$(e^{+tAT^{-1}}P_m^*)x := \int_{-\infty}^{0} e^{+t/\mu}F^\dagger(d\mu)(I-P_0^*)x, \quad 0 \le t < +\infty.$$

Here x is an arbitrary vector in H and F and $F^\dagger$ are the resolutions of the identity of S and $S^\dagger$, respectively. From (3.14) it is clear that we h[...] intertwining between pairs of corresponding semigroups. For example, $0 \le t < +\infty$ we have

$$(4.3)\qquad Te^{-tT^{-1}A_{P_p}} = e^{-tAT^{-1}}P_p^*T, \qquad Te^{+tT^{-1}A_{P_m}} = e^{+tAT^{-1}}P_m^*T.$$

The operator $T^{-1}A$ is a bounded operator on H_0 and AT^{-1} is a boun[...] operator on $H_0^\dagger$. So the expressions

$$e^{-tT^{-1}A_{P_0}}, \quad e^{-tAT^{-1}}P_0^*$$

are well-defined for all $-\infty < t < +\infty$. From Proposition 3.2 it is clea[...]

$$(4.4)\qquad e^{-tT^{-1}A_{P_0}} = P_0 - tT^{-1}AP_0, \quad -\infty < t < +\infty.$$

Since $(AT^{-1})Tx = T(T^{-1}A)x$ for each $x \in H_0$, we also have

$$e^{-tAT^{-1}}P_0^* = P_0^* - tAT^{-1}P_0^*, \quad -\infty < t < +\infty.$$

5. Canonical decompositions

Let (T,B) be a semi-definite admissible pair on a Hilbert space [...] and let A = I − B. The aim of this section is to prove the following ba[...] decomposition theorems.

THEOREM 5.1. *Let* (T,B) *be a semi-definite admissible pair on* H. *Then*

$$(5.1a)\qquad H = H_p \oplus H_- \oplus \{[H_m \oplus H_+] \cap H_0\};$$

$$(5.1b)\qquad H = H_m \oplus H_+ \oplus \{[H_p \oplus H_-] \cap H_0\}.$$

Further, if P *denotes the projection of* H *onto* $H_p \oplus \{[H_m \oplus H_+] \cap H_0\}$ *along and* Q *denotes the projection of* H *onto* $H_m \oplus \{[H_p \oplus H_-] \cap H_0\}$ *along* H_+, *then* P *and* Q *are bounded and*

$$(5.2)\qquad TP = (I-Q^*)T, \quad TQ = (I-P^*)T.$$

THEOREM 5.2. *Let* (T,B) *be a semi-definite admissible pair on* H. *Put*

$$(5.3)\qquad V = P_+P_p + P_-P_m.$$

Then $I - V$ *is a compact operator*, $\operatorname{Ker} V = H_0$, $\operatorname{Im} V = [H_p \oplus H_-] \cap [H_m \oplus H_+]$ *and*

$$(5.4)\qquad H = \{[H_p \oplus H_-] \cap [H_m \oplus H_+]\} \oplus H_0.$$

The proofs of these two theorems hinge upon a basic lemma.

LEMMA 5.3. *Let* (T,B) *be a semi-definite admissible pair on* H. *Then the operators* $P_p - P_+$ *and* $P_m - P_-$ *are compact. More precisely, if* $B = |T|^\alpha D$ *for some* $0 < \alpha < 1$ *and some* $D \in J$, *where* J *is a symmetrically normed ideal in* $L(H)$, *then* $P_p - P_+$ *and* $P_m - P_-$ *belong to the ideal* J *too.*

PROOF. Let (T,B) be a semi-definite admissible pair on H. According to Proposition 2.2 there exist $0 < \alpha < 1$ and a compact operator D such that $B = |T|^\alpha D$. Now let J be a symmetrically normed ideal of compact operators on H (see [25] for the definition and main properties of such an ideal) such that $D \in J$. We shall prove that the operators $P_p - P_+$ and $P_m - P_-$ belong to the ideal J.

Let $\|\cdot\|_J$ be a symmetric norm of J. Since P_0 is an operator of finite rank, we have $P_0 \in J$. Also $B = |T|^\alpha D \in J$. Put $A = I - B$. Then $P_p - P_+ = (P_p - P_+A) - P_+B$ and $P_m - P_- = -(P_p - P_+) - P_0$. So it suffices to show that $P_p - P_+A \in J$.

Let Γ be a simple closed rectifiable Jordan contour in the closed right half-plane that is positively oriented, non-tangential at $\lambda = 0$ and whose inner region contains the parts of the spectra of $T \in L(H)$ and $S \in L(H_1)$ on the interval $(0,+\infty)$. From Propositions 3.3 and 1.4 it follows that

$$(5.5)\qquad (P_p - P_+A)x = -(2\pi i)^{-1} \int_\Gamma [(T-\lambda A)^{-1} - (T-\lambda)^{-1}]Ax\,d\lambda, \qquad x \in H.$$

By W we denote the symbol of the pair (T,B). Then for $0 \neq \lambda \in \Gamma$ we get

$$(T-\lambda A)^{-1} - (T-\lambda)^{-1} = [W(\lambda)^{-1} - I](T-\lambda)^{-1} = \lambda(\lambda-T)^{-1}BW(\lambda)^{-1}(T-\lambda)^{-1} =$$
$$= (T-\lambda)^{-1}BW(\lambda)^{-1}[I - T(T-\lambda)^{-1}].$$

First we exploit the symmetry of the norm $\|\cdot\|_J$ and obtain

$$\|(T-\lambda A)^{-1} - (T-\lambda)^{-1}\|_J \le \||T|^\alpha (T-\lambda)^{-1}\| \cdot \|D\|_J \cdot \|W(\lambda)^{-1}\| \cdot \|I - T(T-\lambda)^{-}$$

Secondly we employ Propositions 1.1 and 2.1 (i.e., $\|W(\lambda) - I\| \to 0$; $\lambda \to 0$, $\lambda \in \Gamma$) and infer that

$$\|(T-\lambda A)^{-1} - (T-\lambda)^{-1}\|_J = O(\||T|^\alpha (T-\lambda)^{-1}\|) \qquad (\lambda \to 0,\ \lambda \in \Gamma).$$

Finally, by Proposition 1.3, we get the estimate

$$(5.6)\qquad \|(T-\lambda A)^{-1} - (T-\lambda)^{-1}\|_J = O(|\lambda|^{\alpha-1}) \qquad (\lambda \to 0,\ \lambda \in \Gamma).$$

For $\varepsilon > 0$ let Γ_ε be the curve obtained from Γ by omitting all point $\lambda \in \Gamma$ for which $|\lambda| < \varepsilon$, and assume Γ_ε inherits its orientation from Γ. I

$$(5.7)\qquad K_\varepsilon = -(2\pi i)^{-1} \int_{\Gamma_\varepsilon} [(T-\lambda A)^{-1} - (T-\lambda)^{-1}]A\,d\lambda.$$

Since the ideal J endowed with the norm $\|\cdot\|_J$ is a Banach space and the integrand of the above integral is a continuous function from Γ_ε into J, it is clear that the operator K_ε belongs to the ideal J ($\varepsilon > 0$). Further, by (5.6),

$$(5.8a)\qquad \|K_\varepsilon - K_\delta\|_J \le M \int_{\Gamma_\delta \setminus \Gamma_\varepsilon} |\lambda|^{\alpha-1} d|\lambda| \qquad (0 < \delta < \varepsilon \le 1),$$

where M is some finite constant. By (5.5), (5.6) (applied for J being the ideal of all compact operators in $L(H)$) and (5.7) we have for all $x \in H$:

$$(5.8b)\qquad \|K_\varepsilon x - (P_p - P_+A)x\| \le N\|x\| \int_{\Gamma \setminus \Gamma_\varepsilon} |\lambda|^{\alpha-1} d|\lambda| \qquad (0 < \varepsilon \le 1),$$

where N is some finite constant. From the inequalities (5.8) one conclude that $P_p - P_+A \in J$ and $\|K_\varepsilon - [P_p - P_+A]\|_J \to 0$ as $\varepsilon \downarrow 0$. □

Important examples of symmetrically normed ideals are the ideals of trace class operators, Hilbert-Schmidt operators and all compact operator in $L(H)$. In particular, if B is an operator of finite rank, then $P_p - P_+$ a $P_m - P_-$ are trace class operators. Further, with the help of the remark following the proof of Proposition 2.2 one shows that for every $\varepsilon > 0$ the operators $P_p - P_+$ and $P_m - P_-$ belong to the $(p+\varepsilon)$-th Von Neumann-Schatten class whenever B belongs to the p-th Von Neumann-Schatten class $(1 \le p < +\infty)$ (cf. [62,25] for the theory of these specific symmetrically normed ideals).

Next, we prove Theorems 5.1 and 5.2. First we prove Theorem 5.2 except formula (5.4). Secondly, we derive two auxiliary propositions. Finally, we deduce the decompositions (5.1a), (5.1b) and (5.4), and the intertwining properties (5.2).

PROOF OF THEOREM 5.2 (except formula (5.4)). Let (T,B) be a semi-definite admissible pair on H, and put $V = P_+P_p + P_-P_m$. The compactness of $I - V$ is clear from Lemma 5.3 and the equality

$$I - V = P_-P_0 + (P_- - P_+)(P_p - P_+).$$

Next, we compute the kernel and range of V and get

(5.9a) $\quad \operatorname{Ker} V = [H_p \cap H_-] \oplus [H_m \cap H_+] \oplus H_0;$

(5.9b) $\quad \operatorname{Im} V = [H_p + H_-] \cap [H_m + H_+].$

Finally, we show that

(5.10) $\quad H_p \cap H_- = \{0\} = H_m \cap H_+.$

Take $x \in H_p \cap H_-$. On the one hand, $\langle Tx,x\rangle = \langle Sx,x\rangle_A \ge 0$; on the other hand, $\langle Tx,x\rangle \le 0$ (cf. (3.7)). So $Tx = 0$. Since $\operatorname{Ker} T = \{0\}$, we have $x = 0$, which settles $H_p \cap H_- = \{0\}$. Analogously one proves that $H_m \cap H_+ = \{0\}$. We substitute (5.10) into (5.9a) and (5.9b), and conclude that $\operatorname{Ker} V = H_0$ and $\operatorname{Im} V = [H_p \oplus H_-] \cap [H_m \oplus H_+]$. □

With the help of the second part of Lemma 5.3 one shows that $I - V$ belongs to the symmetrically normed ideal J in $L(H)$ whenever $B = |T|^{\alpha}D$ for some $0 < \alpha < 1$ and $D \in J$. In particular, if B is an operator of finite rank, then $I - V$ is a trace class operator.

Next, we derive the first auxiliary proposition.

PROPOSITION 5.4. *Let* (T,B) *be a semi-definite admissible pair on* H. *Then*

(5.11a) $\quad [H_p \oplus H_0] + H_- = [H_m \oplus H_0] + H_+ = H;$

(5.11b) $\quad \dim([H_p \oplus H_0] \cap H_-) = \operatorname{codim}(H_m + H_+) = \dim([H_p \oplus H_-] \cap H_0);$

(5.11c) $\quad \dim([H_m \oplus H_0] \cap H_+) = \operatorname{codim}(H_p + H_-) = \dim([H_m \oplus H_+] \cap H_0).$

PROOF. Put $V = P_+P_+ + P_-P_m$ and $V^\dagger = P_+P_p^* + P_-P_m^*$. Then $TV = V^\dagger T$ (cf. (3.9)), $I - V^\dagger = P_-P_0^* + (P_- - P_+)(P_p - P_+)^*$ is a compact operator (see Lemma 5.3) an therefore $V^\dagger$ is a Fredholm operator of index 0. From the intertwining property we conclude that $T[\operatorname{Ker} V] \subset \operatorname{Ker} V^\dagger$. By duality one also has $T[\operatorname{Ker}(V^\dagger)^*] \subset \operatorname{Ker} V^*$. Since T is one-one and V and $V^\dagger$ are both Fredholm operators of index 0, it follows that

$$\dim \operatorname{Ker} V^\dagger \ge \dim \operatorname{Ker} V = \dim \operatorname{Ker} V^* \ge \dim \operatorname{Ker}(V^\dagger)^* = \dim \operatorname{Ker} V^\dagger.$$

So $\dim \operatorname{Ker} V = \dim \operatorname{Ker} V^\dagger$, and thus

$$\operatorname{Ker} V^\dagger = T[\operatorname{Ker} V] = T[H_0] = H_0^\dagger.$$

Next, note that $\operatorname{Im} P_p^* = (\operatorname{Ker} P_p)^\perp = (H_m \oplus H_0)^\perp$, $\operatorname{Im} P_m^* = (H_p \oplus H_0)^\perp$ and $\operatorname{Im} P_0^* = (\operatorname{Ker} P_0)^\perp = H_1^\perp = H_0^\dagger$. In a way analogous to the proof of (5.9a) one shows that

$$\operatorname{Ker} V^\dagger = [(H_m \oplus H_0)^\perp \cap H_-] \oplus [(H_p \oplus H_0)^\perp \cap H_+] \oplus H_0^\dagger.$$

From the two expressions we have derived for $\operatorname{Ker} V^\dagger$ one sees that

$$(H_m \oplus H_0)^\perp \cap H_- = (H_p \oplus H_0)^\perp \cap H_+ = \{0\}.$$

Hence, $([H_p \oplus H_0] + H_-)^\perp = [H_p \oplus H_0]^\perp \cap H_-^\perp = [H_p \oplus H_0]^\perp \cap H_+ = \{0\}$. Observe that $\operatorname{Im} V = [H_p + H_-] \cap [H_m + H_+] \subset H_p + H_- \subset [H_p \oplus H_0] + H_-$, and hence $[H_p \oplus H_0] + H_-$ is closed in H. But then $[H_p \oplus H_0] + H_- = H$. In an analogous way one shows that $[H_m \oplus H_0] + H_+ = H$, and therefore the identities (5.11a) are establishec

By the compactness of P_+P_0 the operator $V_0 := P_+(P_p + P_0) + P_-P_m = V + P_+P_0$ is a Fredholm operator of index 0. Further, by (5.10) and (5.1 one has

$$\operatorname{Ker} V_0 = ([H_p \oplus H_0] \cap H_-) \oplus [H_m \cap H_+] = [H_p \oplus H_0] \cap H_-;$$

$$\operatorname{Im} V_0 = ([H_p \oplus H_0] + H_-) \cap [H_m + H_+] = H_m + H_+.$$

Therefore, $\dim([H_p \oplus H_0] \cap H_-) = \operatorname{codim}(H_m + H_+)$.

To show that $\dim([H_p \oplus H_0] \cap H_-) = \dim([H_p + H_-] \cap H_0)$ we prove that P_0 acts as an invertible operator from $[H_p \oplus H_0] \cap H_-$ onto $[H_p \oplus H_-] \cap H_0$. Indeed, if $x_- \in [H_p \oplus H_0] \cap H_-$, then $P_0 x_- = -P_p x_- + x_- \in [H_p + H_-] \cap H_0$, and therefore P_0 maps $[H_p \oplus H_0] \cap H_-$ into $[H_p + H_-] \cap H_0$. Further, if $P_0 x_- = 0$ for some $x_- \in [H_p \oplus H_0] \cap H_-$, then $P_p x_- = x_- \in H_p \cap H_- = \{0\}$ (cf. (5.10)); therefore, P_0 maps $[H_p \oplus H_0] \cap H_-$ in a one-one manner into $[H_p + H_-] \cap H_0$. Finally, if $x_0 \in [H_p + H_-] \cap H_0$, then there exist vectors $x_p \in H_p$ and $x_- \in H_-$ such that $x_p + x_- = x_0$. Then $x_- = -x_p + x_0 \in [H_p \oplus H_0] \cap H_-$ and $P_0 x_- = x_0$, which proves that, indeed, P_0 acts as an invertible operator from $[H_p \oplus H_0] \cap H_-$ onto $[H_p + H_-] \cap H_0$. This settles (5.11b).

The equalities (5.11c) are proved likewise. □

The next auxiliary proposition shows that H_0 is a Krein space with respect to a suitable indefinite inner product (see [6] for general information about Krein spaces).

<u>PROPOSITION 5.5</u>. *Let* (T,B) *be a semi-definite admissible pair on* H_0. *Then* H_0 *is a Krein space with respect to the indefinite inner product*

(5.12) $\quad [x,y] = \langle Tx,y\rangle \quad (x,y \in H_0).$

In fact,

(5.13) $\quad H_0 = \{[H_p \oplus H_-] \cap H_0\} \oplus \{[H_m \oplus H_-] \cap H_0\},$

where the decomposition (5.13) *is orthogonal with respect to the inner product* (5.12), *the space* $[H_p \oplus H_-] \cap H_0$ *is negative definite and the space* $[H_m \oplus H_+] \cap H_0$ *is positive definite.*

<u>PROOF</u>. Suppose $x_0 \in [H_p \oplus H_-] \cap H_0$. Then $x_0 = x_p + x_-$ for unique $x_p \in H_p$ and $x_- \in H_-$. So $\langle Tx_-,x_-\rangle = \langle Tx_0,x_0\rangle + \langle Tx_p,x_p\rangle - \langle Tx_0,x_p\rangle - \langle Tx_p,x_0\rangle$. By (3.11) we have $\langle Tx_0,x_0\rangle = \langle Tx_-,x_-\rangle - \langle Tx_p,x_p\rangle = \langle Tx_-,x_-\rangle - \langle Sx_p,x_p\rangle_A$. Since $\langle Tx_-,x_-\rangle \le 0$ and $\langle Sx_p,x_p\rangle_A \ge 0$, we have $[x_0,x_0] = \langle Tx_0,x_0\rangle \le 0$. Further, if $[x_0,x_0] = 0$, then $\langle Tx_-,x_-\rangle = \langle Sx_p,x_p\rangle_A = 0$, and therefore $x_- = x_p = 0$; but then x_0 would vanish. Hence, the subspace $[H_p \oplus H_-] \cap H_0$ is strictly negative in the indefinite inner product (5.12). In the same way one shows that $[H_m \oplus H_+] \cap H_0$ is strictly positive in the inner product (5.12).

Next, we show that the subspaces $[H_p \oplus H_-] \cap H_0$ and $[H_m \oplus H_+] \cap H_0$ are orthogonal in the indefinite inner product (5.12). Take $x_0 = x_p + x_- \in [H_p \oplus H_-] \cap H_0$ and $y_0 = x_m + x_+ \in [H_m \oplus H_+] \cap H_0$, where $x_p \in H_p$, $x_- \in H_-$, $x_m \in H_m$ and $x_+ \in H_+$. Using (3.9) we get $[x_0,y_0] = \langle Tx_0,x_m\rangle + \langle Tx_0,x_+\rangle = \langle Tx_0,x_+\rangle = \langle Tx_p,x_+\rangle + \langle Tx_-,x_+\rangle = \langle Tx_p,x_+\rangle = \langle Tx_p,y_0\rangle - \langle Tx_p,x_m\rangle = 0$. This settles our assertion.

As before, put $V = P_+P_p + P_-P_m$. According to Theorem 5.2 the operator is a Fredholm operator of index 0, $\operatorname{Ker} V = H_0$ and $\operatorname{Im} V = [H_p \oplus H_-] \cap [H_m \oplus H_-]$. Therefore, $\operatorname{codim}[H_p \oplus H_-] + \operatorname{codim}[H_m \oplus H_+] = \dim H_0$. By the identities (5.11b and (5.11c) one gets

$$\dim([H_p \oplus H_-] \cap H_0) + \dim([H_m \oplus H_+] \cap H_0) = \dim H_0.$$

Since the subspace $[H_p \oplus H_-] \cap H_0$ is strictly negative and $[H_m \oplus H_+] \cap H_0$ is strictly positive in the indefinite inner product (5.12), we see that these subspaces have a trivial intersection. Hence, (5.13) holds true.

Let ρ denote the projection of H_0 onto $[H_m \oplus H_+] \cap H_0$ along $[H_p \oplus H_-] \cap H_0$. Then H_0 is a Hilbert space with respect to the inner product

$$\{x,y\} = \langle T\rho x,\rho y\rangle - \langle T(I-\rho)x,(I-\rho)y\rangle \quad (x,y \in H_0).$$

With respect to this inner product the subspaces $[H_m \oplus H_+] \cap H_0$ and $[H_p \oplus H_-] \cap H_0$ are orthogonal and $[x,y] = \{\rho x,\rho y\} - \{(I-\rho)x,(I-\rho)y\}$ $(x,y \in H_0$. Hence, the space H_0 endowed with the indefinite inner product (5.12) is a Krein space indeed. □

<u>PROOF of the decompositions (5.1a), (5.1b) and (5.4)</u>. First observe that the subspaces H_p, H_- and $[H_m \oplus H_+] \cap H_0$ have a pairwise trivial intersection. Indeed, $H_p \cap H_- = \{0\}$ (cf. (5.10)), $H_p \cap ([H_m \oplus H_+] \cap H_0) \subset H_p \cap H_0 = \{0\}$ an $H_- \cap ([H_m \oplus H_+] \cap H_0) \subset (H_- \cap H_0) \cap ([H_m \oplus H_+] \cap H_0) \subset ([H_p \oplus H_-] \cap H_0) \cap ([H_m \oplus H_+] \cap$ $\cap H_0) = \{0\}$ (cf. (5.13)). Further, $H_p \oplus H_-$ is a finite-codimensional subspace of H and its codimension in H coincides with the dimension of the s space $[H_m \oplus H_+] \cap H_0$ (cf. (5.11c)). This settles (5.1a). The decomposition (5.1b) is derived analogously.

Next, observe that $\{[H_p \oplus H_-] \cap [H_m \oplus H_+]\} \cap H_0 = ([H_p \oplus H_-] \cap H_0) \cap$ $\cap ([H_m \oplus H_+] \cap H_0) = \{0\}$ (cf. (5.13)). Further, V is a Fredholm operator of index 0 with kernel H_0 and image $[H_p \oplus H_-] \cap [H_m \oplus H_+]$ (cf. Theorem 5.2). Therefore, $\mathrm{codim}\{[H_p \oplus H_-] \cap [H_m \oplus H_+]\} = \dim H_0$. But then the decomposition (5.4) is clear. □

PROOF of the second part of Theorem 5.1. Let P denote the projection of H onto the closed subspace $H_p \oplus \{[H_m \oplus H_+] \cap H_0\}$ along H_-, and let Q denote the projection of H onto the closed subspace $H_m \oplus \{[H_p \oplus H_-] \cap H_0\}$ along H_+. First we list a few orthogonality properties. Consider the indefinite inner product $[x,y] = \langle Tx,y\rangle$ on all of H (cf. (5.12)). By (the second paragraph of) the proof of Proposition 5.5 the subspaces $[H_m \oplus H_+] \cap H_0$ and $[H_p \oplus H_-] \cap H_0$ are orthogonal in $[\cdot,\cdot]$. By (3.9) the subspaces $[H_m \oplus H_+] \cap H_0$ ($\subset H_0$) and H_m, and also the subspaces H_p and $[H_p \oplus H_-] \cap H_0$ ($\subset H_0$) are orthogonal in $[\cdot,\cdot]$. Again by (3.9) the subspaces H_p and H_m are orthogonal in $[\cdot,\cdot]$. Hence,

$$(5.14) \qquad \langle Tx,y\rangle = 0; \quad x \in H_p \oplus \{[H_m \oplus H_+] \cap H_0\},\ y \in H_m \oplus \{[H_p \oplus H_-] \cap H_0\}.$$

For all $x,y \in H$ we have $\langle TPx,y\rangle = \langle TPx,y\rangle - \langle TPx,Qy\rangle = \langle TPx,(I-Q)y\rangle$ (cf. (5.14)). Since $H_+ = \mathrm{Ker}\, Q$ and $H_- = \mathrm{Ker}\, P$ are orthogonal, we have $\langle T(I-P)x,(I-Q)y\rangle = 0$. Hence, $\langle TPx,y\rangle = \langle TPx,(I-Q)y\rangle + \langle T(I-P)x,(I-Q)y\rangle = \langle Tx,(I-Q)y\rangle = \langle (I-Q^*)Tx,y\rangle$. But then $TP = (I-Q^*)T$. Taking adjoints we get $TQ = (I-P^*)T$. □

The proof of Theorems 5.1 and 5.2 is completed. For the isotropic case of the Transport Equation the operators P, Q and V have been introduced in [35,36] (see also [40]), where the identities (5.10) were proved, however, for (in our terminology) a positive definite pair. For positive definite pairs the intertwining properties (5.2) are due to HANGELBROEK. The decompositions (5.1) and the compactness of $I-V$ appear in literature for some special cases, which involve positive definite pairs (cf. [35,36,40,2]). For singular semi-definite pairs the decompositions (5.1) and (5.4) seem to be new.

6. Standard operations on hermitian admissible pairs

In this section we discuss certain elementary operations on hermitian admissible pairs and the canonical decompositions connected with them. We begin with the notion of similarity. For $i = 1,2$ let (T_i,B_i) be a hermitian admissible pair on a Hilbert space H_i. Then the pairs (T_1,B_1) and (T_2,B_2) are said to be *similar* if there exists an invertible operator $E: H_1 \to H_2$, such that

$$(6.1) \qquad ET_1 = T_2E, \quad EB_1 = B_2E.$$

In that case $\rho(T_1) = \rho(T_2)$, and for the corresponding symbols we have

$$EW_1(\lambda) = W_2(\lambda)E.$$

In formula (6.1) the operator E can be chosen to be unitary. To see this, we exploit the self-adjointness of the operators T_1, T_2, B_1 and B_2 and the polar decomposition $E = U(E^*E)^{\frac{1}{2}}$, where U is unitary (see Section 1.4.1 of [2], where in another situation a similar argument is used). We get $T_1E^* = (ET_1)^* = (T_2E)^* = E^*T_2$ and $B_1E^* = E^*B_2$. Then $T_1E^*E = E^*T_2E = E^*ET_2$ and $B_1E^*E = E^*EB_2$, and hence $T_1(E^*E)^{\frac{1}{2}} = (E^*E)^{\frac{1}{2}}T_1$ and $B_1(E^*E)^{\frac{1}{2}} = (E^*E)^{\frac{1}{2}}B_1$. Therefore, $UT_1(E^*E)^{\frac{1}{2}} = ET_1 = T_2E = T_2U(E^*E)^{\frac{1}{2}}$ and $UB_1(E^*E)^{\frac{1}{2}} = B_2U(E^*E)^{\frac{1}{2}}$. By the invertibility of the operator $(E^*E)^{\frac{1}{2}}$ we obtain

$$(6.2) \qquad UT_1 = T_2U, \quad UB_1 = B_2U,$$

which proves our assertion.

If the pairs (T_1,B_1) and (T_2,B_2) are regular and similar, then their associate operators $S_1 = (I-B_1)^{-1}T_1$ and $S_2 = (I-B_2)^{-1}T_2$ are similar too. In fact, from (6.2) one may conclude that S_1 and S_2 are unitarily equivale i.e.,

$$US_1 = S_2U.$$

In an obvious way the unitary equivalence U establishes a unitary equivalence between the spectral projections of T_1 and T_2 and of S_1 and S_2.

A hermitian admissible pair (T,B) on a Hilbert space H is called *inversion symmetric* if the hermitian admissible pairs (T,B) and (-T,B) are similar. Then there exists a unitary operator $U \in L(H)$, called an *inversio symmetry*, such that

$$(6.3) \qquad UT = -TU, \quad UB = BU.$$

Assume (6.3) holds. If $A = I - B$ and W denotes the symbol of the pair (T,B), then

$$UW(\lambda) = W(-\lambda)U, \quad \lambda \in \rho(T).$$

If, in addition, B is a trace class operator, then $W(\lambda) - I = -\lambda(\lambda - T)^{-1}B$ is a trace class operator too ($\lambda \in \rho(T)$), and therefore for the determinant one has

$$(6.4) \qquad \det W(\lambda) = \det W(-\lambda), \quad \lambda \in \rho(T).$$

PROPOSITION 6.1. *Let* (T,B) *be an inversion symmetric semi-definite admissible pair on a Hilbert space* H, *and let* U *be a unitary operator such that* (6.3) *holds. Then for the spectral projections of* T *and* $T^{-1}A$ *we have*

$$UP_+ = P_-U, \quad UP_p = P_mU, \quad UP_0 = P_0U.$$

Further, the spectra of the operators T *and* $T^{-1}A$ *are symmetric with respect to* $\lambda = 0$.

The proof of this proposition is straightforward. For the case of the one-speed Transport Equation, in which the pair involved is inversion symmetric (cf. Section VI.1), the second part of this proposition is due to MASLENNIKOV [50].

Next, we introduce the notion of Möbius transformation of hermitian admissible pairs. For characteristic operator functions and operator nodes related transformations have appeared in [7] and [2], respectively.

THEOREM 6.2. *Let* (T,B) *be a hermitian admissible pair on a Hilbert space* H, *and let* $k > 0$ *be a constant for which* $\sigma(T) \subset (-k,+\infty)$. *Then the pair* $(\tilde{T},\tilde{B})$, *defined by*

$$(6.5) \qquad \tilde{T} = kT(k+T)^{-1}, \quad \tilde{B} = k(k+T)^{-\frac{1}{2}}B(k+T)^{-\frac{1}{2}},$$

is a hermitian admissible pair on H *and its symbol is given by*

$$(6.6) \qquad \tilde{W}(\lambda) = (k+T)^{\frac{1}{2}}W\left(\frac{k\lambda}{k-\lambda}\right)(k+T)^{-\frac{1}{2}}.$$

Further, $(\tilde{T},\tilde{B})$ *is a regular hermitian admissible pair on* H *if and only if the operator* $T + kA$ *is invertible, and in that case the associate operator is equal to*

$$(6.7) \qquad (I-\tilde{B})^{-1}\tilde{T} = k(k+T)^{\frac{1}{2}}(T+kA)^{-1}T(k+T)^{-\frac{1}{2}}.$$

PROOF. Certainly, the operators $\tilde{T}$ and $\tilde{B}$ are self-adjoint, $\mathrm{Ker}\,\tilde{T} = \{0\}$ and $\tilde{B}$ is compact. Further, $I - \tilde{B} = (k+T)^{-\frac{1}{2}}(T+kA)(k+T)^{-\frac{1}{2}}$, and therefore $I - \tilde{B}$ is invertible if and only if $T + kA$ is invertible.

To prove Condition (C.3) we first note that the spectral subspaces of T and $\tilde{T}$ corresponding to the positive (negative) part of their spectrum coincide. Therefore, $|\tilde{T}|^{\alpha} = k^{\alpha}|T|^{\alpha}(k+T)^{-\alpha}$, $0 < \alpha < 1$. Hence, if $B = |T|^{\alpha}D$ for some $0 < \alpha < 1$ and $D \in L(H)$, then $\tilde{B} = |\tilde{T}|^{\alpha}\tilde{D}$, where

$$\tilde{D} = k^{1-\alpha}(k+T)^{\alpha-\frac{1}{2}}D(k+T)^{-\frac{1}{2}}.$$

But then $(\tilde{T},\tilde{B})$ is a hermitian admissible pair on H.

Finally, the identities (6.6) and (6.7) are deduced by straightforward computation. □

The transformations described in Theorem 6.2 will be referred to as Möbius transformations of hermitian admissible pairs. For $k > 0$ and $\sigma(T) \subset (-\infty,k)$ an analogue of Theorem 6.2 can be established.

Let $\tilde{T}$ and $\tilde{B}$ be as in Theorem 6.2. The possibly unbounded operator $\tilde{T}^{-1}\tilde{A}$ where $\tilde{A} = I - \tilde{B}$, is given by

$$\tilde{T}^{-1}\tilde{A} = (k+T)^{\frac{1}{2}}(k^{-1}+T^{-1}A)(k+T)^{-\frac{1}{2}}.$$

Now assume that the pair (T,B) is regular, and let $S = A^{-1}T$. Then the pair $(\tilde{T},\tilde{B})$ is regular if and only if $-k \notin \sigma(S)$, and in that case

$$\tilde{A}^{-1}\tilde{T} = (k+T)^{\frac{1}{2}}kS(k+S)^{-1}(k+T)^{-\frac{1}{2}}.$$

We continue this section with the following operation on semi-definite pairs.

THEOREM 6.3. *Let* (T,B) *be a semi-definite admissible pair on* H, *and let* ρ *be the projection of* H_0 *onto* $[H_m \oplus H_+] \cap H_0$ *along* $[H_p \oplus H_-] \cap H_0$. *For* $u > 0$ *put*

$$B_u x = B(I-P_0)x + (I+u^{-1}T)(I-\rho)P_0x + (I-u^{-1}T)\rho P_0 x, \quad x \in H.$$

Then (T,B_u) *is a positive definite admissible pair on* H.

PROOF. Recall that A acts as an invertible operator from H_1 onto $H_1^\dagger = H_0^\perp$ and that T acts as an invertible operator from H_0 onto $H_0^\dagger = H_1^\perp$ (cf. Section 3). For every invertible operator $\beta: H_0 \to H_0$ we define A_β by

$$(6.8) \qquad A_\beta = A(I-P_0) + T\beta^{-1}P_0.$$

Then A_β is invertible and $I - A_\beta$ is compact. Further, by Proposition 3.2 we have $A(T^{-1}A)x = 0$ ($x \in H_0$), and therefore $x = Bx + TBT^{-1}Ax$ for every $x \in H_0$. Since $B = |T|^\alpha D$ for some $0 < \alpha < 1$ and $D \in L(H)$, we conclude that $H_0 \subset \text{Im}\,|T|^\alpha$. By (6.8) there exists a $D_\beta \in L(H)$ such that

$$(6.9) \qquad I - A_\beta = |T|^\alpha D_\beta.$$

Let $[\cdot,\cdot]$ be the indefinite inner product on H_0 defined by (5.12). For $x,y \in H$ we have

$$\langle A_\beta x,y\rangle = \langle A(I-P_0)x,(I-P_0)y\rangle + [\beta^{-1}P_0x,P_0y];$$

$$\langle x,A_\beta y\rangle = \langle (I-P_0)x,A(I-P_0)y\rangle + [P_0x,\beta^{-1}P_0y],$$

where the orthogonality properties (3.8) and (3.9) have been used. Hence, A_β is self-adjoint (strictly positive) if and only if β is self-adjoint (strictly positive) with respect to the indefinite inner product (5.12).

So $(T,I-A_\beta)$ is a regular hermitian admissible pair on H if and only if β is self-adjoint in the inner product (5.12). Similarly, $(T,I-A_\beta)$ is a positive definite admissible pair on H if and only if β is strictly positive in the indefinite inner product (5.12).

To finish the proof we take a special β that is positive definite with respect to the inner product (5.12), namely the operator that with respect to the decomposition (5.13) of H_0 is given by

$$\beta = \begin{bmatrix} -uI_{[H_p \oplus H_-] \cap H_0} & 0 \\ 0 & +uI_{[H_m \oplus H_+] \cap H_0} \end{bmatrix}$$

Then $I - A_\beta$ coincides with the operator B_u in the statement of the theorem, and the proof is complete. □

REMARK 6.4. With respect to the decomposition $H_1 \oplus H_0 = H$ the operator $A_\beta^{-1}T$ is given by

$$(6.10) \qquad A_\beta^{-1}T = S \oplus \beta.$$

Hence, the pair $(T,I-A_\beta)$ can be viewed as a "regularized" version of the original semi-definite pair (T,B).

7. Inversion symmetry and the singular subspace

Throughout this section (T,B) will denote a semi-definite admissible pair on a Hilbert space H. We assume (T,B) to be inversion symmetric with a signature operator J as an inversion symmetry. So $J = J^* = J^{-1}$ and we ha

$$(7.1) \qquad JT = -TJ, \quad JB = BJ.$$

In this section we construct a canonical basis for the singular subspace H_0 of this pair and employ this basis to obtain some further decomposition of H_0.

From Proposition 6.1 it is clear that

$$J([H_p \oplus H_-] \cap H_0) = [H_m \oplus H_+] \cap H_0.$$

But then the decomposition (5.13) implies that H_0 has an even dimension. Put

$$H_0^+ = \{x \in H_0: Jx = x\}, \quad H_0^- = \{x \in H_0: Jx = -x\}.$$

Since for every $x \in H_0$ one has $\frac{1}{2}(I+J)x \in H_0^+$ and $\frac{1}{2}(I-J)x \in H_0^-$, one sees tha

$$(7.2) \qquad H_0^+ \oplus H_0^- = H_0.$$

THEOREM 7.1. *Let* (T,B) *be a semi-definite admissible pair on* H, *and let* $J \in L(H)$ *be a signature operator such that* (7.1) *holds. Then we have the decompositions*

$$(7.3a) \qquad H_p \oplus H_0^+ \oplus H_- = H_m \oplus H_0^+ \oplus H_+ = H;$$

$$(7.3b) \qquad H_p \oplus H_0^- \oplus H_- = H_m \oplus H_0^- \oplus H_+ = H.$$

PROOF. If $x \in H_0^+$, then $[x,x] = \langle Tx,x\rangle = \langle Tx,Jx\rangle = \langle JTx,x\rangle = -\langle TJx,x\rangle = -\langle Tx,x\rangle = -[x,x]$, and therefore $[x,x] = 0$. Similarly, if $x \in H_0^-$, then $[x,x] = 0$. Since $[H_p \oplus H_-] \cap H_0$ is negative definite and $[H_m \oplus H_+] \cap H_0$ is positive definite in the indefinite inner product (5.12), we get

$$(7.4) \qquad M \cap N = \{0\}; \quad M \in \{[H_p \oplus H_-] \cap H_0, [H_m \oplus H_+] \cap H_0\}, \quad N \in \{H_0^+, H_0^-\}.$$

Recall that the dimension of H_0 is even, $2n$ say. Then $\dim([H_p \oplus H_-] \cap H_0) = \dim([H_m \oplus H_+] \cap H_0) = n$. From (7.2) we have $\dim H_0^+ + \dim H_0^- = 2n$, and from (7.4) it follows that $\dim H_0^+ \le n$ and $\dim H_0^- \le n$. So $\dim H_0^+ = \dim H_0^- = n$. Hence,

$$\{[H_p \oplus H_-] \cap H_0\} \oplus N = \{[H_m \oplus H_+] \cap H_0\} \oplus N = H_0,$$

where N denotes either H_0^+ or H_0^-. □

THEOREM 7.2. *Let* (T,B) *be a semi-definite admissible pair on* H, *and let* $J \in L(H)$ *be a signature operator such that* (7.1) *holds. Then there exists a basis of* H_0, *with respect to which the restriction of* $T^{-1}A$ *to* H_0 *has the Jordan normal form and which consists of vectors from* H_0^+ *and* H_0^- *only.*

PROOF. Let S_0 denote the restriction of $T^{-1}A$ to H_0. Then, by Proposition III 3.2, $S_0^2 = 0$. By (7.1) we have $JS_0x = -S_0Jx$, $x \in H_0$. One easily checks the following identities:

$$\operatorname{Ker} S_0 = \operatorname{Ker} A, \quad S_0[H_0^+] \subset H_0^-, \quad S_0[H_0^-] \subset H_0^+.$$

Let $y_{11},\dots,y_{1p}$ be a basis of H_0^+ modulo $H_0^+ \cap \operatorname{Ker} A$, and let $y_{1(p+1)},\dots,y_{1(p+q)}$ be a basis of H_0^- modulo $H_0^- \cap \operatorname{Ker} A$. Put $y_{2j} = S_0y_{1j}$ $(j = 1,\dots,p+q)$. Suppose $x_1^-,\dots,x_\ell^-$ is a basis of $H_0^- \cap \operatorname{Ker} A$ modulo the linear span of the vectors $y_{21},\dots,y_{2p}$, and let $x_1^+,\dots,x_k^+$ be a basis of $H_0^+ \cap \operatorname{Ker} A$ modulo the span of the vectors $y_{2(p+1)},\dots,y_{2(p+q)}$; here we have used that S_0 maps H_0^+ into H_0^- and H_0^- into H_0^+. Now

$$\{x_1^-,\dots,x_\ell^-;\ y_{21},\dots,y_{2p};\ y_{1(p+1)},\dots,y_{1(p+q)}\} = b^-$$

is a basis of H_0^- and

$$\{x_1^+,\dots,x_k^+;\ y_{11},\dots,y_{1p};\ y_{2(p+1)},\dots,y_{2(p+q)}\} = b^+$$

is a basis of H_0^+. Since $\dim H_0^+ = \dim H_0^-$, one concludes that $k = \ell$. Finally S_0 has the Jordan normal form with respect to the basis $b^+ \cup b^-$ of H_0. □

The construction of a canonical basis of H_0 is related to the notion of "sign characteristics", which has been introduced in [27] (for the proofs we refer to [28]); to see this, observe that $iT^{-1}A|_{H_0}$ is self-adjoi with respect to the indefinite inner product $[x,y]_J = \langle Jx,y\rangle$ of H_0.

From the proof of Theorem 7.2 it is clear that for the operator $T^{-1}A|_{H_0}$ the number of Jordan blocks of order one is even. Later we shall see (cf. Section VI.4) that this is the only restriction on the Jordan normal form of $T^{-1}A|_{H_0}$.

Next we shall use the basis introduced in the proof of Theorem 7.2 to construct subspaces N of $\operatorname{Ker} A$ such that

$$(7.5) \qquad H = H_p \oplus N \oplus H_- = H_m \oplus N \oplus H_+.$$

An analogous result can be derived without assuming inversion symmetry. Pu

$$N^+ = \operatorname{span}\{x_1^+,\dots,x_k^+;\ y_{21},\dots,y_{2(p+q)}\},$$

$$N^- = \operatorname{span}\{x_1^-,\dots,x_\ell^-;\ y_{21},\dots,y_{2(p+q)}\}.$$

Note that N^+ and N^- are subspaces of $\operatorname{Ker} A$ and that their dimensions are equal to $\frac{1}{2}\dim H_0$. If $z \in N^+$, then $z = x + y$, where $x \in \operatorname{span}\{x_1^+,\dots,x_k^+\} \subset H_0^+$ and $y \in \operatorname{span}\{y_{21},\dots,y_{2(p+q)}\} \subset (T^{-1}A)[H_0]$. So $[x,x] = 0$ and $y = T^{-1}Aw$ for some $w \in H_0$. Then $[z,z] = [x,y] + [y,x] + [y,y] = \langle Tx,y\rangle + \langle Ty,x\rangle + \langle Ty,y\rangle = \langle x,Aw\rangle + \langle Aw,x\rangle + \langle Aw,y\rangle = \langle Ax,w\rangle + \langle w,Ax\rangle + \langle w,Ay\rangle = 0$, becaus $\{x,y\} \subset \operatorname{Ker} A$. Similarly, $[z,z] = 0$ for all $z \in N^-$. Since the subspaces $[H_p \oplus H_-] \cap H_0$ and $[H_m \oplus H_+] \cap H_0$ are negative definite and positive definite, respectively, and $\dim N^+ = \dim N^- = \dim([H_p \oplus H_-] \cap H_0) = \dim([H_m \oplus H_+] \cap H_0) = \frac{1}{2}\dim H_0$, one concludes that

$$M \oplus N = H_0; \quad M \in \{[H_p \oplus H_-] \cap H_0, [H_m \oplus H_+] \cap H_0\}, \quad N \in \{N^+, N^-\}.$$

But then we may conclude that (7.5) holds for $N = N^+$ or $N = N^-$. In (7.5) one may take $N = \operatorname{Ker} A$ if and only if all Jordan blocks of $T^{-1}A$ of $\lambda = 0$ have order 2.

In [47], for the conservative isotropic case of the Transport Equatio a decomposition appears that can be viewed as a special case of the decompositions (7.3) and (7.5).

CHAPTER IV

THE OPERATOR DIFFERENTIAL EQUATION
$(T\psi)' = -A\psi + f$

In this chapter we study the operator differential equation

$$(T\psi)'(t) = -A\psi(t) + f(t), \quad 0 < t < \tau.$$

The interval $(0,\tau)$ may be finite as well as infinite. Various boundary and growth conditions on the solutions will be considered.

1. Preliminaries

Let H be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$, and let (T,B) be a hermitian admissible pair on H. Put $A = I - B$. For $0 < \tau \le +\infty$ let $f\colon (0,\tau) \to H$ be some vector function. By a solution of the operator differential equation

$$(1.1) \qquad (T\psi)'(t) = -A\psi(t) + f(t), \quad 0 < t < \tau,$$

we mean a vector function $\psi\colon (0,\tau) \to H$ such that $T\psi$ is differentiable on the open interval $(0,\tau)$ and (1.1) holds.

If τ is finite, one says that a solution ψ of Eq. (1.1) has $\phi \in H$ as its *boundary value*, if the equalities

$$(1.2) \qquad \lim_{t\downarrow 0} P_+\psi(t) = P_+\phi, \quad \lim_{t\uparrow\tau} P_-\psi(t) = P_-\phi,$$

hold true. Here P_+ (P_-) is the spectral projection of the self-adjoint operator T corresponding to the positive (negative) part of its spectrum.

If τ is infinite, we say that a solution ψ of Eq. (1.1) has $\phi_+ \in H_+ = \mathrm{Im}\, P_+$ as its *boundary value* if the identity

$$(1.3) \qquad \lim_{t\downarrow 0} P_+\psi(t) = \phi_+$$

is valid.

The operator differential equation (1.1) with boundary conditions (1.2) is called the (abstract) *finite-slab problem* with boundary value ϕ. For $\tau = +\infty$ the operator differential equation (1.1) with boundary value ϕ_+ is called an (abstract) *half-space problem*. In Transport Theory concrete versions of these two problems have been studied intensively and in this monograph they will be dealt with in the final chapter.

2. The finite-slab problem (the homogeneous case)

Let (T,B) be a semi-definite admissible pair on a Hilbert space H, and let $A = I - B$. In this section we consider the homogeneous differential equation

$$(T\psi)'(t) = -A\psi(t), \quad 0 < t < \tau, \tag{2.1}$$

for finite τ. To solve this equation we need the following operator on H:

$$V_\tau = P_+[P_p + e^{+\tau T^{-1}A}P_m] + P_-[P_m + e^{-\tau T^{-1}A}P_p] + P_0 - \tau P_-T^{-1}AP_0. \tag{2.2}$$

Here P_+, P_- and P_p, P_m, P_0 are the spectral projections introduced in Section III.3. The next theorem may be viewed as a generalization of Theorem III 5.2.

THEOREM 2.1. *Let* (T,B) *be a semi-definite admissible pair on* H. *Then for* $0 < \tau < +\infty$ *the operator* V_τ *is invertible and* $I - V_\tau$ *is compact. Further, if* (T,B) *is a positive definite admissible pair, then*

$$\lim_{\tau\to+\infty} \|V_\tau - V\| = 0, \tag{2.3}$$

where $V = P_+P_p + P_-P_m$.

PROOF. Note that $I - V_\tau$ can be written as

$$I - V_\tau = [-P_0 + \tau P_-T^{-1}AP_0] + (I-V) +$$

$$-P_+(P_m - P_-)e^{+\tau T^{-1}A}P_m - P_-(P_p - P_+)e^{-\tau T^{-1}A}P_p.$$

Since the projection P_0 has a finite rank and the operators I-V, $P_m - P_-$ and $P_p - P_+$ are compact (see Theorem III 5.2 and Lemma III 5.3), the compactnes of the operator $I - V_\tau$ is clear.

So in order to prove that V_τ is invertible it suffices to show that $\mathrm{Ker}\, V_\tau = \{0\}$. Take $x \in \mathrm{Ker}\, V_\tau$, and put $x_p = P_px$, $x_m = P_mx$ and $x_0 = P_0x$. Further we define

$$y_- = x_p + e^{+\tau T^{-1}A}P_mx_m + x_0;$$

$$y_+ = x_m + e^{-\tau T^{-1}A}P_px_p + x_0 - \tau T^{-1}Ax_0.$$

Then $P_+y_- = P_+V_\tau x = 0$ and $P_-y_+ = P_-V_\tau x = 0$, and hence $\langle Ty_-,y_-\rangle \le 0$ and $\langle Ty_+,y_+\rangle \ge 0$. With the help of formulas (III 3.9) one rewrites these inequ ities and obtains

$$\langle Tx_p,x_p\rangle + \langle Te^{+\tau T^{-1}A}P_mx_m, e^{+\tau T^{-1}A}P_mx_m\rangle + \langle Tx_0,x_0\rangle \le 0;$$

$$\langle Tx_m,x_m\rangle + \langle Te^{-\tau T^{-1}A}P_px_p, e^{-\tau T^{-1}A}P_px_p\rangle +$$

$$+ \langle T[I-\tau T^{-1}A]x_0,[I-\tau T^{-1}A]x_0\rangle \ge 0.$$

By Proposition III 3.2 we first write

$$\langle T[I-\tau T^{-1}A]x_0,[I-\tau T^{-1}A]x_0\rangle = \langle Tx_0,x_0\rangle - 2\tau\langle Ax_0,x_0\rangle.$$

Put $S = A^{-1}T|_{H_1}$ (cf. formula (III 3.4)), and recall that S is self-adjoint on H_1 with respect to the inner product (III 3.5). So we can write our inequalities in the form

$$\langle Sx_p,x_p\rangle_A + \langle Se^{+2\tau T^{-1}A}P_mx_m,x_m\rangle_A + \langle Tx_0,x_0\rangle \le 0;$$

$$\langle Sx_m,x_m\rangle_A + \langle Se^{-2\tau T^{-1}A}P_px_p,x_p\rangle_A + \langle Tx_0,x_0\rangle - 2\tau\langle Ax_0,x_0\rangle \ge 0.$$

We now subtract these inequalities, rearrange terms and obtain

$$\begin{aligned}&\langle Sx_p,x_p\rangle_A - \langle Se^{-2\tau T^{-1}A}P_px_p,x_p\rangle_A + 2\tau\langle Ax_0,x_0\rangle \le \\ &\le \langle Sx_m,x_m\rangle_A - \langle Se^{+2\tau T^{-1}A}P_mx_m,x_m\rangle_A.\end{aligned} \tag{2.4}$$

Let F be the resolution of the identity of the self-adjoint operator S on H_1 (endowed with the inner product (III 3.5)). Then the inequality (2.4) can be written as

$$\int_0^{+\infty} t(1-e^{-2\tau/t})\langle F(dt)x_p,x_p\rangle_A + 2\tau\langle Ax_0,x_0\rangle \le$$

$$\le \int_{-\infty}^{0} t(1-e^{+2\tau/t})\langle F(dt)x_m,x_m\rangle_A .$$

The left- (right-)hand side of this inequality is nonnegative (nonpositive), because the operator A is (non-strictly) positive on H. Hence, all terms in the above expression vanish. Since Ker S = {0}, we get $x_p = x_m = 0$ and $Ax_0 = 0$. So the vector $x = x_p + x_m + x_0$, which we assumed to belong to Ker V_τ, belongs to Ker A. By (2.2) we have $V_\tau z = P_0 z$ for all $z \in$ Ker $A \subset H_0$. Hence, $x = x_0 = 0$. But then Ker $V_\tau = \{0\}$ and therefore the operator V_τ is invertible.

Next assume that the pair (T,B) is positive definite. Then $P_0 = 0$. Since the semigroups $e^{+\tau T^{-1}A}P_m$ and $e^{-\tau T^{-1}A}P_p$ are exponentially decreasing in the operator norm (see Section III.4), the identity (2.3) is immediate from the definition of V_τ. □

<u>THEOREM 2.2</u>. *Let* (T,B) *be a semi-definite admissible pair on* H, *and let* $0 < \tau < +\infty$. *For every boundary value* ϕ *the operator differential equation*

(2.5) $$(T\psi)'(t) = -A\psi(t), \quad 0 < t < \tau,$$

has a unique solution, namely

(2.6) $$\psi(t) = e^{-tT^{-1}A}P_pV_\tau^{-1}\phi + e^{(\tau-t)T^{-1}A}P_mV_\tau^{-1}\phi + (I-tT^{-1}A)P_0V_\tau^{-1}\phi \ (0<t<\tau),$$

where the invertible operator V_τ *is given by* (2.2).

<u>PROOF</u>. Let ψ: $(0,\tau) \to H$ be a solution of the operator differential equation (2.5). By definition the function $T\psi$ is differentiable on $(0,\tau)$ with derivative $-A\psi$. Using the intertwining properties (III 3.9) and (III 3.8) one sees that $t \mapsto TP_p\psi(t)$ and $t \mapsto TP_m\psi(t)$ are differentiable on $(0,\tau)$ with derivatives $-AP_p\psi(t)$ and $-AP_m\psi(t)$, respectively. So both vector functions are solutions of the operator differential equation $\dot\phi = -AT^{-1}\phi$.

As in Section III.4 we associate with AT^{-1} the strongly continuous semigroups $\{e^{-tAT^{-1}}P_p^*\}_{t\ge0}$ and $\{e^{+tAT^{-1}}P_m^*\}_{t\ge0}$ on the spaces Im $P_p^* = (H_m \oplus H_0)^\perp$ and Im $P_m^* = (H_p \oplus H_0)^\perp$, respectively. From the considerations of Section IX 1.3 of [45] it follows that on the subspaces $(H_m\oplus H_0)^\perp$ and $(H_p \oplus H_0)^\perp$ the initial value problems

$$\dot\phi(t) = -AT^{-1}\phi(t), \quad t \in [\tau_1,\tau_2] \subset (0,\tau),$$

with initial values $TP_p\psi(\tau_1) \in (H_m\oplus H_0)^\perp$ and $TP_m\psi(\tau_2) \in (H_p\oplus H_0)^\perp$ have unique solutions, which are given by the respective expressions

$$\phi_1(t) = e^{-(t-\tau_1)AT^{-1}}TP_p\psi(\tau_1) = Te^{-(t-\tau_1)T^{-1}A}P_p\psi(\tau_1), \quad t \ge \tau_1;$$

$$\phi_2(t) = e^{(\tau_2-t)AT^{-1}}TP_m\psi(\tau_2) = Te^{(\tau_2-t)T^{-1}A}P_m\psi(\tau_2), \quad t \le \tau_2.$$

Here we have employed (III 4.3). But the functions $t \mapsto TP_p\psi(t)$ and $t \mapsto TP_m\psi(t)$ are also solutions of these initial value problems. By the uniqueness of the solution of these problems and the injectivity of the operator T we obtain for $\tau_1 \le t \le \tau_2$:

(2.7) $$P_p\psi(t) = e^{-(t-\tau_1)T^{-1}A}P_p\psi(\tau_1), \quad P_m\psi(t) = e^{(\tau_2-t)T^{-1}A}P_m\psi(\tau_2).$$

By (III 3.9) and (III 3.8) the function $t \mapsto TP_0\psi(t)$ satisfies the differential equation $\dot\phi = -AT^{-1}\phi$ on the interval $(0,\tau)$. But T acts as an invertible operator from H_0 onto $H_0^\dagger$ (see Section III.3). So $P_0\psi$ is differential and satisfies (2.5). Since H_0 is a finite-dimensional space and $T^{-1}A$ acts as a bounded operator on H_0, by (III 4.4) we immediately have the solution in the form

(2.8) $$P_0\psi(t) = e^{-tT^{-1}A}\phi_0 = (I-tT^{-1}A)\phi_0, \quad 0 < t < \tau,$$

where $\phi_0 \in H_0$. We combine (2.7) and (2.8) and get for $\tau_1 \le t \le \tau_2$

(2.9) $$\psi(t) = e^{-(t-\tau_1)T^{-1}A}P_p\psi(\tau_1) + e^{(\tau_2-t)T^{-1}A}P_m\psi(\tau_2) + (I-tT^{-1}A)\phi_0.$$

Let us suppose that ϕ is the boundary value of the solution ψ, namely that

(2.10) $$\lim_{t\downarrow0} P_+\psi(t) = P_+\phi, \quad \lim_{t\uparrow\tau} P_-\psi(t) = P_-\phi.$$

Substituting $t = \tau_1$ ($t = \tau_2$) into (2.9), applying P_+ (P_-) to the left and taking the limit as $\tau_1 \downarrow 0$ ($\tau_2 \uparrow \tau$) it appears that the following two limits exist:

$$(2.11)\qquad \lim_{t\downarrow 0} P_+P_p\psi(t), \quad \lim_{t\uparrow\tau} P_-P_m\psi(t).$$

Now let P denote the bounded projection of H onto $H_p \oplus \{[H_m \oplus H_-] \cap H_0\}$ along H_-, and let Q denote the projection of H onto $H_m \oplus \{[H_p \oplus H_-] \cap H_0\}$ along H_+. Since Ker P = H_-, one has $P(P_+P_p) = PP_p - (PP_-)P_p = PP_p$. Because $H_p \subset \text{Im } P$, one gets $P(P_+P_p) = PP_p = P_p$. Similarly, one shows that $Q(P_-P_m) = P_m$. With the help of (2.11) and (2.8) one sees that the following vectors are well-defined:

$$\phi_p := \lim_{t\downarrow 0} P_p\psi(t), \quad \phi_m := \lim_{t\uparrow\tau} P_m\psi(t), \quad \phi_0 := \lim_{t\downarrow 0} P_0\psi(t).$$

With the help of (2.9) one obtains

$$(2.12)\qquad \psi(t) = e^{-tT^{-1}A}P_p\phi_p + e^{(\tau-t)T^{-1}A}P_m\phi_m + (I-tT^{-1}A)\phi_0, \quad 0 < t < \tau.$$

We now recall the definition (2.2) of the operator V_τ and replace (2.10) by the equivalent identity

$$(2.13)\qquad V_\tau(\phi_p + \phi_m + \phi_0) = \phi.$$

The solution formula (2.6) is now clear from (2.12), (2.13) and the invertibility of the operator V_τ.

Conversely, the vector function ψ given by (2.6) is, indeed, a solution of the operator differential equation (2.5) with boundary value ϕ. □

For a case when (in our terminology) the pair (T,B) is positive definite HANGELBROEK has announced the proof of formula (2.12) assuming the solution to be continuous on the closed interval $[0,\tau]$ (cf. [38,39]). The statement of the finite-slab problem in [38] stimulated the author to investigate this problem.

Later, independently of each other and for a positive definite pair only, both Hangelbroek and the present author showed the operator V_τ to be invertible. For semi-definite pairs Theorem 2.2 seems to be new in the abstract situation considered here.

3. The half-space problem (the homogeneous case)

Throughout this section (T,B) is a semi-definite admissible pair on H and A = I − B. In this section we describe the solutions of the operator differential equation

$$(3.1)\qquad (T\psi)'(t) = -A\psi(t), \quad 0 < t < +\infty,$$

under various boundary conditions. First we prove two results for positive definite pairs only. After this we continue with the general case of semi-definite pairs.

THEOREM 3.1. *Let* (T,B) *be a positive definite admissible pair on* H, *and let* P *be the projection of* H *onto* H_p *along* H_-. *Then for every boundary value* $\phi_+ \in H_+$ *there is a unique bounded solution of Eq.* (3.1), *namely*

$$(3.2)\qquad \psi(t) = e^{-tT^{-1}A}P\phi_+, \quad 0 < t < +\infty.$$

This solution is exponentially decreasing.

Theorem 3.1 will appear as a corollary of a more general theorem (Theorem 3.2 below). To state this theorem, let us recall some facts about the spectrum of $A^{-1}T$. Assume that (T,B) is a positive definite admissible pair on H. Then A = I − B is invertible and $A^{-1}T$ is self-adjoint in the equivalent inner product (III 3.5). So the spectrum $\sigma(A^{-1}T)$ of $A^{-1}T$ is real. Since $A^{-1}T$ is a compact perturbation of T, the part of $\sigma(A^{-1}T)$ outside $\sigma(T)$ consists of isolated eigenvalues of finite multiplicity. In particular, if $\sigma(T) \subset (-k,+\infty)$ for some $k > 0$, then $\sigma(A^{-1}T) \cap (-\infty,-k]$ is a finite set.

THEOREM 3.2. *Let* (T,B) *be a positive definite admissible pair on* H, *and let* $k > 0$ *be a constant such that* $\sigma(T) \subset (-k,+\infty)$. *Then for every boundary value* $\phi_+ \in H_+$ *the general solution* ψ *of Eq.* (3.1) *with the property that* $e^{-t/k}\psi(t)$ *is bounded on* $(0,+\infty)$, *is given by*

$$(3.3)\qquad \psi(t) = \sum_{i=1}^{r} e^{-t/\lambda_i}\phi_{0i} + e^{-tT^{-1}A}P\left(\phi_+ - \sum_{i=1}^{r}\phi_{0i}\right), \quad 0 < t < +\infty.$$

Here $\lambda_1,\dots,\lambda_r$ *are the eigenvalues of* $A^{-1}T$ *in* $(-\infty,-k]$ *and* $A^{-1}T\phi_{0i} = \lambda_i\phi_{0i}$ $(i = 1,\dots,r)$.

<u>PROOF</u>. Let K and L be the spectral subspaces of $A^{-1}T$ corresponding to the parts of its spectrum in $(-\infty,-k) \cup (0,+\infty)$ and $[-k,0)$. Note that $K \oplus L = H$. Since the spectral subspace corresponding to $\sigma(A^{-1}T) \cap (-\infty,-k)$ is finite-dimensional, the restriction of $-T^{-1}A$ to K generates a strongly continuous semigroup of order strictly less than $1/k$.

Suppose $\psi: (0,+\infty) \to H$ is a solution of Eq. (3.1) such that

$$\int_0^{+\infty} e^{-t/k} \|\psi(t)\| dt < +\infty.$$

First we prove the existence of a vector $x_0 \in K$ such that

$$A^{-1}T\,\psi(t) = e^{-tT^{-1}A}\,x_0, \quad 0 < t < +\infty. \tag{3.4}$$

Put $\phi(t) = e^{-t/k}\psi(t)$. This function satisfies the operator differential equation $k(A^{-1}T\phi)'(t) = -(k+A^{-1}T)\phi(t)$ $(0 < t < +\infty)$, while $\int_0^{+\infty}\|\phi(t)\|dt < +\infty$. For $0 < \gamma < \delta < +\infty$ we get $(k+A^{-1}T)\int_\gamma^\delta \phi(t)dt = kA^{-1}T(\phi(\gamma)-\phi(\delta))$, which implies the existence of the limits

$$x_0 = \lim_{t\downarrow 0} A^{-1}T\phi(t) = \lim_{t\downarrow 0} A^{-1}T\psi(t), \quad \hat{x} = \lim_{t\to+\infty} A^{-1}T\phi(t).$$

Similarly, for $0 < \gamma < \delta < +\infty$ we have

$$\lambda(k+A^{-1}T)\int_\gamma^\delta e^{t/\lambda}\,\phi(t)dt = -k\lambda[e^{t/\lambda}A^{-1}T\phi(t)]_{t=\gamma}^{\delta} +$$

$$+ kA^{-1}T\int_\gamma^\delta e^{t/\lambda}\,\phi(t)dt, \qquad \mathrm{Re}\,\lambda \le 0.$$

From this equation we see that $\lim_{t\to+\infty} e^{t/\lambda}A^{-1}T\,\phi(t)$ exists for all imaginary $\lambda \ne 0$, and hence $\hat{x} = 0$. (Indeed, if $\hat{x}$ would not vanish, choose $\hat{y} \in H$ such that $\langle\hat{x},\hat{y}\rangle \ne 0$; then $\lim_{t\to+\infty} e^{t/\lambda}\langle\hat{x},\hat{y}\rangle$ exists, which cannot be true.) So

$$\lambda(k+A^{-1}T)\int_0^{+\infty} e^{t/\lambda}\,\phi(t)dt = k\lambda x_0 + kA^{-1}T\int_0^{+\infty} e^{t/\lambda}\,\phi(t)dt, \quad \mathrm{Re}\,\lambda \le 0.$$

Recall that $\phi(t) = e^{-t/k}\,\psi(t)$. Note that the Möbius transformation $\zeta = k\lambda(k-\lambda)^{-1}$ maps the open left half-plane onto the interior domain Ω of the circle C_k with centre $-\frac{1}{2}k$ and radius $\frac{1}{2}k$. For $\zeta \in \bar{\Omega}\backslash\{0\}$ we get

$$\int_0^{+\infty} e^{t/\zeta}\,\psi(t)dt = \zeta(\zeta - A^{-1}T)^{-1}x_0. \tag{3.5}$$

Next, we show that $x_0 \in K$. Let P_k be the projection of H onto L along K. Applying $A^{-1}TP_k$ to both sides of (3.5) we obtain for $\zeta \in \bar{\Omega}\backslash\{0\}$

$$A^{-1}TP_k\int_0^{+\infty} e^{t/\zeta}\,\psi(t)dt = \zeta A^{-1}T(\zeta - A^{-1}T)^{-1}\,P_kx_0.$$

Now the right-hand side is analytic and continuous up to the boundary of the exterior domain of C_k (cf. Propositions III 1.1 and III 1.2). The left hand side is analytic and continuous up to the boundary of the interior domain Ω of C_k with boundary limit 0 at $\zeta = 0$. Hence, by Liouville's theor $P_kx_0 = 0$ and therefore $x_0 \in K$.

From (3.5) and Theorem VIII 1.11 of [13] we derive

$$A^{-1}T\,\psi(t) = e^{-tT^{-1}A}x_0, \quad 0 < t < +\infty,$$

which settles (3.4).

Next, we suppose that $\psi: (0,+\infty) \to H$ is a solution of Eq. (3.1) such that $e^{-t/k}\,\psi(t)$ is essentially bounded. For a sufficiently small $\varepsilon > 0$ the function ψ satisfies $\int_0^{+\infty} e^{-t(k-\varepsilon)}\|\psi(t)\|dt < +\infty$, while $\sigma(A^{-1}T) \cap (-k,-(k-\varepsilon))$ $= \emptyset$. So there exists a vector $x_0(\varepsilon)$ in the spectral subspace K_ε of $A^{-1}T$ corresponding to the part of its spectrum in $(-\infty,-(k-\varepsilon)) \cup (0,+\infty)$, which satisfies

$$A^{-1}T\,\psi(t) = e^{-tT^{-1}A}\,x_0(\varepsilon), \quad 0 < t < +\infty.$$

Taking the limit as $t \downarrow 0$ we see that $x_0(\varepsilon)$ does not depend on ε. Put $x_0 := x_0(\varepsilon)$. Then $x_0 \in \cap\{K_\varepsilon : \varepsilon > 0\}$, which is the spectral subspace of $A^{-1}T$ corresponding to its spectrum in $(-\infty,-k] \cup (0,+\infty)$, while

$$A^{-1}T\,\psi(t) = e^{-tT^{-1}A}\,x_0, \quad 0 < t < +\infty. \tag{3.6}$$

Finally, from Theorem III 5.1 we know that $H_p \oplus H_- = H$ and $TP = (I-Q^*)T$ where $P(Q)$ denotes the projection of H onto H_p (H_m) along H_- (H_+). Write

$$x_0 = x_p + A^{-1}T\,x_m; \quad x_p \in H_p,\ x_m \in H_m. \tag{3.7}$$

Here we used that the spectral subset $\sigma(A^{-1}T) \cap (-\infty,-k]$ contains finitely many eigenvalues of $A^{-1}T$ of finite geometric multiplicity only. Since $\operatorname{Ker} Q^* = H_m^{\perp}$, $P_+(I-Q) = I-Q$ and $Ax_p \in H_m^{\perp}$ (see (III 3.10)), we have $Ax_p = (I-Q^*)Ax_p = (I-Q^*)P_+Ax_p$. Inserting this into (3.7) we get $Ax_p = (I-Q^*)P_+Ax_0 - (I-Q^*)P_+Tx_m$.

Now assume the solution ψ has a boundary value $\phi_+ \in H_+$, i.e., that $P_+\psi(t) \to \phi_+$ (as $t \downarrow 0$). By (3.6) we have $P_+Ax_0 = T\phi_+$, and hence $Ax_p = (I-Q^*)TP_+(\phi_+ - x_m)$. By (III 5.2) we get $Ax_p = TP(\phi_+ - x_m)$. Hence,

$$x_0 = A^{-1}Tx_m + A^{-1}TP(\phi_+ - x_m). \tag{3.9}$$

Put $\sigma(A^{-1}T) \cap (-\infty,-k] = \{\lambda_1,\ldots,\lambda_m\}$ and $x_m = \sum_{i=1}^{r} \phi_{0i}$, where $(A^{-1}T - \lambda_i)\phi_{0i} = 0$ $(i = 1,\ldots,r)$. With the help of (3.6) - (3.8) and the identity $\operatorname{Ker} T = \{0\}$ we get

$$\psi(t) = \sum_{i=1}^{m} e^{-t/\lambda_i}\phi_{0i} + e^{-tT^{-1}A}P\Big(\phi_+ - \sum_{i=1}^{r}\phi_{0i}\Big), \quad 0 < t < +\infty,$$

which settles (3.3). Conversely, the function (3.3) is a solution of Eq. (3.1) with boundary value ϕ_+ such that $e^{-t/k}\psi(t)$ is bounded on $(0,+\infty)$. □

PROOF of Theorem 3.1. Let (T,B) be a positive definite admissible pair on H, and let ψ: $(0,+\infty) \to H$ be an essentially bounded solution of Eq. (3.1). For every $k > 0$ there exists a vector $x_0(k)$ in the spectral subspace of $A^{-1}T$ corresponding to its spectrum in $(-\infty,-k] \cup (0,+\infty)$ such that

$$A^{-1}T\,\psi(t) = e^{-tT^{-1}A}x_0(k), \quad 0 < t < +\infty.$$

Taking the limit as $t \downarrow 0$ we conclude that $x_0(k)$ does not depend on k and belongs to H_p. So

$$A^{-1}T\,\psi(t) = e^{-tT^{-1}A}P_p x_0, \quad 0 < t < +\infty, \tag{3.10}$$

for some $x_0 \in H_p$.

Now assume that the solution ψ has a boundary value $\phi_+ \in H_+$. Using (3.10) we see that $T\phi_+ = P_+Ax_0$. Apply Theorem III 5.1. Then $TP\phi_+ = (I-Q^*)T\phi_+ = (I-Q^*)P_+Ax_0 = (I-Q^*)Ax_0 = Ax_0 - Q^*Ax_0 = Ax_0$. The last equality is clear from formula (III 3.10). This establishes formula (3.2). The fact that (3.2) is a solution and is exponentially decreasing is trivial (cf. (III 4.2)). □

A vector function ψ: $(0,+\infty) \to H$ is called *polynomially bounded* if ψ i Bochner integrable on $(0,\tau)$ for every finite τ and there exists an integer $n \geq 0$ such that $\|\psi(t)\| = O(t^n)$ $(t \to +\infty)$.

THEOREM 3.3. *Let* (T,B) *be a semi-definite admissible pair on* H. *Then the polynomially bounded solutions of Eq.* (3.1) *with boundary value in* H_+ *are given by*

$$\psi(t) = e^{-tT^{-1}A}\phi_p + (I - tT^{-1}A)\phi_0, \quad 0 < t < +\infty, \tag{3.11}$$

where $\phi_p \in H_p$ *and* $\phi_0 \in H_0$. *Further, the function* ψ *is bounded if and only if* $\phi_0 \in \operatorname{Ker} A$, *and Bochner integrable if and only if* $\phi_0 = 0$.

PROOF. Consider the decomposition $H = H_0 \oplus H_1$. Let P_0 be the projection of H onto H_0 along H_1, and let $P_1 = I - P_0$. Put $\psi_i(t) = P_i\psi(t)$ $(i = 0,1)$. Then Eq. (3.1) is decomposed into the following two equations:

$$(T\psi_1)'(t) = -A\psi_1(t) \quad (\in H_0^{\perp}); \tag{3.12a}$$

$$(0 < t < +\infty)$$

$$\psi_0'(t) = -T^{-1}A\psi_0(t) \quad (\in H_0). \tag{3.12b}$$

Since H_0 is finite-dimensional and $(T^{-1}A)^2x = 0$ for $x \in H_0$, the general solution of Eq. (3.12b) is given by

$$\psi_0(t) = e^{-tT^{-1}A}\phi_0 = (I - tT^{-1}A)\phi_0, \quad 0 < t < +\infty, \tag{3.13}$$

where ϕ_0 is an arbitrary vector of H_0.

To find the solution of Eq. (3.12a) we apply Theorem III 6.3. Let $S \in L(H_1)$ be the associate operator of the pair (T,B) (see the second paragraph after the proof of Proposition III 3.2). Take a fixed $u > 0$ such that $\{-u,u\} \subset \rho(T) \cap \rho(S)$. Apply Theorem III 6.3 and construct a positive definite admissible pair (T,B_u) on H, whose associate operator is given by

$$S_u = S \oplus (-u)I_{[H_p \oplus H_-] \cap H_0} \oplus uI_{[H_m \oplus H_+] \cap H_0},$$

where $u > 0$ and $u \notin \sigma(S)$. Note that H_1 and H_0 are the spectral subspaces of S_u corresponding to the spectral subsets $\sigma(S)$ and $\{u,-u\}$, respectively. Put $A_u = I - B_u$. Then the operator differential equation

$$(3.14)\qquad (T\phi)'(t) = -A_u\phi(t) \qquad (0 < t < +\infty)$$

can be decomposed in a similar way as

$$(3.15a)\qquad (T\phi_1)'(t) = -A\phi_1(t) \qquad (\in H_0^{\perp});$$

$$(0 < t < +\infty)$$

$$(3.15b)\qquad \phi_0'(t) = -T^{-1}A_u\phi_0(t) \qquad (\in H_0).$$

Let $\psi: (0,+\infty) \to H$ be a polynomially bounded solution of Eq. (3.1) with boundary value in H_+. Then $\psi_1: (0,+\infty) \to H$, defined by $\psi_1(t) = P_1\psi(t)$ $(0 < t < +\infty)$, is also a polynomially bounded solution of Eq. (3.1) with boundary value in H_+. At the same time ψ_1 is a solution of Eq. (3.14), which has a boundary value in H_+ and satisfies $\int_0^{+\infty} e^{-t/k}\,\|\psi_1(t)\|\,dt < +\infty$ for every $k > 0$. Using Theorem 3.2 we easily prove the existence of a vector $\phi_p \in H_p$ such that

$$(3.16)\qquad \psi_1(t) = P_1\psi(t) = e^{-tT^{-1}A_p}\phi_p, \qquad 0 < t < +\infty.$$

The function $\psi_0: (0,+\infty) \to H$, defined by $\psi_0(t) = P_0\psi(t)$ $(0 < t < +\infty)$, satisfies Eq. (3.12b), and ψ_0 is given by (3.13). From formulas (3.13) and (3.16) the representation (3.11) is clear. The second part of the theorem is now easy to prove. □

Recall that a signature operator $J \in L(H)$ is said to be an inversion symmetry of the pair (T,B) (see Sections III.6 and III.7), if

$$TJ = -JT, \qquad BJ = JB.$$

<u>THEOREM 3.4</u>. *Let* (T,B) *be a semi-definite admissible pair on* H *with a signature operator as inversion symmetry. Then for every boundary value the operator differential equation* (3.1) *has at least one bounded solution. If all Jordan blocks of* $T^{-1}A$ *at* $\lambda = 0$ *have order 2, then this solution is uniquely determined by its boundary value. Further, there exists a non-trivial bounded solution of Eq.* (3.1) *with boundary value 0, if and only if* $T^{-1}A$ *has a Jordan block of order* 1 *at* $\lambda = 0$.

<u>PROOF</u>. By the remark following the proof of Theorem III 7.2 one has

$$(3.17)\qquad [H_p \oplus \text{Ker } A] + H_- = H,$$

and (3.17) is a direct sum (i.e., $H_p \oplus \text{Ker } A \oplus H_- = H$) if and only if all Jordan blocks of $T^{-1}A$ at $\lambda = 0$ have order 2.

According to Theorem 3.3 there exists a bounded solution ψ of Eq. (3 with boundary value ϕ_+ if and only if the equation

$$(3.18)\qquad P_+(\phi_p + \phi_0) = \phi_+$$

has a solution $\phi_p \in H_p$ and $\phi_0 \in \text{Ker } A$. In fact, all bounded solutions of Eq. (3.1) with boundary value ϕ_+ follow from a solution of Eq. (3.18). B Eq. (3.18) has at least one solution if and only if $\phi_+ \in [H_p \oplus \text{Ker } A] + H_-$ whereas for $\phi_+ = 0$ it has no non-trivial solutions if and only if $[H_p \oplus \text{Ker } A] \cap H_- = \{0\}$. Using (3.17) and the statement following this equality, the theorem is immediate. □

A solution ψ of Eq. (3.1) is called a *Milne solution* if

(M1) ψ is polynomially bounded;

(M2) ψ has a vanishing boundary value (i.e., $P_+\psi(t) \to 0$ as $t \downarrow 0$).

The problem of finding Milne solutions of the operator differential equa (3.1) is called an (abstract) *Milne problem*. Concrete versions of this p lem have been the object of study in astrophysics (cf. [8], for instance)

<u>THEOREM 3.5</u>. *Let* (T,B) *be a semi-definite admissible pair on* H. *Then for every Milne solution* ψ *of Eq.* (3.1) *there is a unique vector* $\phi_0 \in [H_p \oplus H_-]$ *such that*

$$(3.19)\qquad \psi(t) = e^{-tT^{-1}A}\phi_p + (I - tT^{-1}A)\phi_0, \qquad 0 < t < +\infty;$$

here ϕ_p *is the unique vector in* H_p *such that* $\phi_p + \phi_0 \in H_-$. *Conversely, if* $\phi_0 \in [H_p \oplus H_-] \cap H_0$ *and* $\phi_p \in H_p$ *satisfies* $\phi_p + \phi_0 \in H_-$, *then the function* (3. *is a Milne solution of the operator differential equation* (3.1).

<u>PROOF</u>. Let $\psi: (0,+\infty) \to H$ be a Milne solution of Eq. (3.1). Then ψ has 0 a a boundary value and is polynomially bounded. By Theorem 3.3 we have

$$\psi(t) = e^{-tT^{-1}A_p}\phi_p + (I - tT^{-1}A)\phi_0, \qquad 0 < t < +\infty,$$

where $P_+(\phi_p + \phi_0) = 0$ for some $\phi_p \in H_p$ and $\phi_0 \in H_0$. Therefore, $\phi_0 \in [H_p + H_-] \cap H_0$ and ϕ_p is the unique vector in H_p such that $\phi_p + \phi_0 \in H_-$ (the uniquen is clear from the equality $H_p \cap H_- = \{0\}$; see (III 5.10). Note that the vector ϕ_0 is also uniquely determined by ψ).

The converse part of this theorem is easily deduced. □

4. The propagator function

To derive solutions of the non-homogeneous versions of the finite-slab problem and the half-space problem we have to consider the function

$$(4.1)\qquad H(t) = \begin{cases} +T^{-1}e^{-tT^{-1}}P_+, & 0 < t < +\infty; \\ -T^{-1}e^{-tT^{-1}}P_-, & -\infty < t < 0. \end{cases}$$

Here T is a self-adjoint operator on a complex Hilbert space and Ker T = {0}. Further, $P_+ = E((0,+\infty))$ and $P_- = E((-\infty,0))$, where E denotes the resolution of the identity of T. The function $H(t)$ is called the *propagator function* of the operator T. If T is the main operator of a hermitian admissible pair (T,B) on H, we also say that $H(t)$ is the propagator function of the pair (T,B). In this section we derive the main properties of the propagator function.

If (T,B) is a positive definite admissible pair on H, then the associate operator $S = A^{-1}T$ is a self-adjoint operator on H endowed with the (equivalent) inner product (III 2.7) and Ker S = {0}. Analogously to (4.1) one defines the *associate propagator function* $H_S(t)$ by

$$(4.2)\qquad H_S(t) = \begin{cases} +T^{-1}A\, e^{-tT^{-1}A}P_p, & 0 < t < +\infty; \\ -T^{-1}A\, e^{-tT^{-1}A}P_m, & -\infty < t < 0. \end{cases}$$

Note that $H_S(t)$ is the propagator function of S on H endowed with the inner product (III 2.7).

PROPOSITION 4.1. *Let* $T \in L(H)$ *be a self-adjoint operator with* Ker T = {0}. *Then for all* $0 \le \alpha < 1$ *and* $0 \ne t \in \mathbb{R}$ *the operator* $|T|^\alpha H(t)$ *is bounded and*

$$(4.3a)\qquad \| |T|^\alpha H(t)\| = O(|t|^{\alpha-1}) \quad (t \to 0);$$

$$(4.3b)\qquad \| |T|^\alpha H(t)\| = O(e^{-|t|/\|T\|}) \quad (t \to \pm\infty).$$

PROOF. For $0 < t < +\infty$ the estimate (4.3a) with $t \downarrow 0$ is clear from the identity

$$\sup_{0<\mu<+\infty} |\mu^{\alpha-1} e^{-t/\mu}| = \left(\frac{1-\alpha}{e}\right)^{1-\alpha} \cdot t^{\alpha-1} \qquad (0 < t < +\infty)$$

and formula (III 1.2). The estimate (4.3a) with $t \uparrow 0$ follows analogously.

To obtain (4.3b) one uses a similar argument based on the identity

$$\sup_{0<\mu\le\|T\|} |\mu^{\alpha-1} e^{-t/\mu}| = \|T\|^{\alpha-1} e^{-t/\|T\|} \qquad (t \ge (1-\alpha)\|T\|)$$

and a related identity for $-\|T\| \le \mu < 0$. □

If (T,B) is a positive definite admissible pair on H, then its associate operator S is self-adjoint on H endowed with the equivalent inner product (III 2.7) and Ker S = {0}. Therefore, Proposition 4.1 applies and yields estimates for $\| |S|^\alpha H_S(t)\|$, where $0 \le \alpha < 1$.

From the definition (4.1) it is clear that the propagator function $H(\cdot)$ is strongly measurable, and hence the same is true for $H(\cdot)|T|^\alpha$, $0 < \alpha < 1$. By Proposition 4.1,

$$\int_{-\infty}^{+\infty} \|H(t)|T|^\alpha\| dt < +\infty.$$

So we may conclude that $H(\cdot)|T|^\alpha$ is Bochner integrable. By strong measurability we mean measurability with respect to Lebesgue measure as defined in [71], Section VI 3.1; all integrals of vector functions appearing in this section and later sections will be Bochner integrals with respect to Lebesgue measure.

PROPOSITION 4.2. *Let* h: $(-\infty,+\infty) \to H$ *be an essentially bounded (strongly measurable) vector function. For* $0 < \alpha < 1$ *put*

$$g(t) = \int_{-\infty}^{+\infty} H(t-s)|T|^\alpha h(s)ds, \quad t \in \mathbb{R}.$$

Then g *is a bounded and continuous vector function.*

PROOF. Since $H(\cdot)|T|^\alpha$ is Bochner integrable and the function h is essentially bounded, the function g is essentially bounded too. Hence, it suffices to show that g is continuous. But this may be proved in exactly the same way as one proves that the convolution product of a scalar L_1-functi and a scalar L_∞-function is continuous (cf. [71]; 30.17, 30.18, 31.7 and 31.9). □

PROPOSITION 4.3. *Let* h: $(-\infty,+\infty) \to$ H *be a bounded and piecewise continuous vector function. For* $0 < \alpha < 1$ *put*

$$g(t) = \int_{-\infty}^{+\infty} H(t-s)|T|^\alpha h(s)ds, \quad t \in \mathbb{R}.$$

Then with the possible exception of the finitely many discontinuities of the function h, *the vector function* Tg *is differentiable on the real line and its derivative is given by*

$$(4.4) \qquad (Tg)'(t) = -g(t) + |T|^\alpha h(t).$$

PROOF. Assume h is continuous in an open neighbourhood (a,b) of a fixed $t \in \mathbb{R}$. Using the definition (4.1) of the propagator function H one gets

$$(4.5) \qquad \varepsilon^{-1}[Tg(t+\varepsilon) - Tg(t)] = \varepsilon^{-1}[e^{-\varepsilon T^{-1}}P_+ - P_+]T\int_{-\infty}^{t} H(t-s)|T|^\alpha h(s)ds -$$

$$- \varepsilon^{-1}[e^{\varepsilon T^{-1}}P_- - P_-]T\int_{t}^{+\infty} H(t-s)|T|^\alpha h(s+\varepsilon)ds +$$

$$+ \varepsilon^{-1}\int_{t}^{t+\varepsilon} T[H(t+\varepsilon-s) - H(t-s)]|T|^\alpha h(s)ds.$$

Because of the equality

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}[e^{-\varepsilon T^{-1}}P_+ - P_+]Tx = -P_+x, \quad x \in H,$$

the first term at the right-hand side of (4.5) tends to the vector $-P_+g(t)$ as $\varepsilon \downarrow 0$. As the function h is bounded and continuous almost everywhere on $[t,+\infty)$ and since $H(t-s)|T|^\alpha$ is integrable for $s \in [t,+\infty)$, it follows from the theorem of dominated convergence for Bochner integrals (cf. [71]) that

$$\lim_{\varepsilon\downarrow 0}\int_{t}^{+\infty} H(t-s)|T|^\alpha h(s+\varepsilon)ds = \int_{t}^{+\infty} H(t-s)|T|^\alpha h(s)ds.$$

Using the identity

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}[e^{\varepsilon T^{-1}}P_- - P_-]Tx = P_-x, \quad x \in H,$$

and the principle of uniform boundedness it follows that the second term the right-hand side of (4.5) tends to $-P_-g(t)$ as $\varepsilon \downarrow 0$. Next we exploit t continuity of the integrand of the third term in (4.5) and infer that it converges to $P_+|T|^\alpha h(t) + P_-|T|^\alpha h(t) = |T|^\alpha h(t)$ as $\varepsilon \downarrow 0$. From (4.5) and the computations we made one gets

$$\lim_{\varepsilon\downarrow 0} \varepsilon^{-1}[Tg(t+\varepsilon) - Tg(t)] = -g(t) + |T|^\alpha h(t).$$

In a similar way, by considering $-T$ and $\tilde{h}(s) = h(-s)$ rather than T and h(one obtains

$$\lim_{\varepsilon\uparrow 0} \varepsilon^{-1}[Tg(t+\varepsilon) - Tg(t)] = -g(t) + |T|^\alpha h(t).$$

Hence, except for a finite set of exceptional points, Tg is differentiabl and its derivative is given by $-g + |T|^\alpha h$. □

We conclude this section with a remark related to Proposition 4.2. Assume that h: $(0,\tau) \to$ H is an essentially bounded (strongly measurable) vector function, and let $0 < \alpha < 1$. Consider the function

$$g(t) = \int_0^\tau H(t-s)|T|^\alpha h(s)ds, \quad 0 < t < \tau.$$

According to Proposition 4.2 the function g has a continuous extension to the full real line. In particular, g is continuous at the point zero. Not that

$$(4.6) \qquad \lim_{t\downarrow 0} P_+g(t) = 0.$$

If τ is finite, we also have

$$(4.7) \qquad \lim_{t\uparrow\tau} P_-g(t) = 0.$$

If $\tau = +\infty$, formula (4.7) takes the following form:

$$(4.8) \qquad e^{+tT^{-1}}P_-g(t) = -\int_t^{+\infty} e^{+sT^{-1}}P_-|T|^\alpha h(s)ds \to 0 \quad (t \to +\infty).$$

5. The finite-slab and half-space problems (the inhomogeneous case)

Throughout this section (T,B) is a positive definite admissible pair on the Hilbert space H and $A = I - B$. In this section we describe the solutions of the operator differential equation

$$(5.1)\qquad (T\psi)'(t) = -A\psi(t) + f(t), \quad 0 < t < \tau,$$

under various boundary conditions. The class of inhomogeneous terms $f: (0,\tau) \to H$, for which the above equation is solved, consists of vector functions f that can be written as

$$(5.2)\qquad f(t) = |T|^{\gamma} h(t), \quad 0 < t < \tau,$$

for some constant γ, $0 < \gamma < 1$, and some continuous vector function $h: (0,\tau) \to H$. To describe the solutions of the inhomogeneous equation (5.1) we shall employ the associate propagator function H_S defined by (4.2). Since $S = A^{-1}T$ is self-adjoint in the equivalent inner product (III 2.7) on H, the results of the previous section apply to H_S. Further, in the description of the solutions we shall use the operator V_τ introduced in Section 2.

THEOREM 5.1. *Let* (T,B) *be a positive definite admissible pair on* H, *and assume* $0 < \tau < +\infty$. *Put* $f(t) = |T|^{\gamma}h(t)$, $0 < t < \tau$, *where* $0 < \gamma < 1$ *and* $h: (0,\tau) \to H$ *is a bounded continuous vector function. Then for every boundary value* ϕ *there exists a unique solution of the operator differential equation* (5.1), *namely*

$$(5.3)\qquad \begin{aligned}\psi(t) &= e^{-tT^{-1}A}P_p V_\tau^{-1}(\phi-\chi) + e^{(\tau-t)T^{-1}A}P_m V_\tau^{-1}(\phi-\chi) + \\ &+ \int_0^\tau H_S(t-s)A^{-1}f(s)ds, \qquad 0 < t < \tau,\end{aligned}$$

where

$$(5.4)\qquad \chi = P_+ \int_0^\tau H_S(-s)A^{-1}f(s)ds + P_- \int_0^\tau H_S(\tau-s)A^{-1}f(s)ds.$$

PROOF. By Proposition III 2.2 there exists $0 < \alpha \le \gamma$ and a compact operator D such that $B = |T|^{\alpha}D$. From the proof of Theorem III 2.3 we know that $|T|^{\alpha} = |S|^{\alpha}E$ for some invertible operator E. It follows that

$$f(t) = |S|^{\alpha}\tilde{h}(t), \quad 0 < t < \tau,$$

where $\tilde{h}: (0,\tau) \to H$ is the bounded continuous vector function given by $\tilde{h}(t) = E|T|^{\gamma-\alpha}h(t)$, $0 < t < \tau$. Put

$$g(t) = \int_0^\tau H_S(t-s)A^{-1}f(s)ds, \quad 0 \le t \le \tau.$$

Notice that $A^{-1}f(s) = |S|^{\alpha}[\tilde{h}(s) + EDA^{-1}f(s)]$, $0 < s < \tau$, where the functi[on] $s \mapsto \tilde{h}(s) + EDA^{-1}f(s)$ is bounded and continuous on $(0,\tau)$. But, with respe[ct] to the equivalent inner product (III 2.7) on H, the operator S is self-adjoint and Ker S = {0}. By Proposition 4.3 the function Sg is differenti[able] on $(0,\tau)$ and

$$(Sg)'(t) + g(t) = A^{-1}f(t), \quad 0 < t < \tau.$$

Premultiplying this by the operator A we see that g is a solution of Eq. (5.1). By virtue of Proposition 4.2 the solution g can be extended to a function that is continuous on the closed interval $[0,\tau]$ and satisfies the equalities

$$\lim_{t\downarrow 0} P_+g(t) = P_+\chi, \qquad \lim_{t\uparrow\tau} P_-g(t) = P_-\chi,$$

where χ is defined by (5.4). This means that χ is the boundary value of t[he] solution g.

Finally, to obtain the unique (bounded) solution ψ of Eq. (5.1) with a given boundary value ϕ we have to add to the solution g the unique (boun[d]ed) solution of the homogeneous differential equation (2.1) with boundary value $\phi - \chi$. The latter solution can be obtained using Theorem 2.2, and formula (5.3) is clear. □

THEOREM 5.2. *Let* (T,B) *be a positive definite admissible pair on* H, *and assume* $\tau = +\infty$. *Put* $f(t) = |T|^{\gamma}h(t)$, $0 < t < +\infty$, *where* $0 < \gamma < 1$ *and* $h: (0,+\infty) \to H$ *is a bounded continuous vector function. Then for every boundary value* $\phi_+ \in H_+$ *there exists a unique bounded solution of the opera[*]*tor differential equation* (5.1), *namely*

$$(5.5)\qquad \psi(t) = e^{-tT^{-1}A}P\Big(\phi_+ - \int_0^{+\infty} H_S(-s)A^{-1}f(s)ds\Big) + \int_0^{+\infty} H_S(t-s)A^{-1}f(s)ds$$

for $0 < t < +\infty$.

PROOF. As in the proof of the previous theorem one shows that for $\tau = +\infty$ the vector function $g\colon [0,+\infty) \to H$, defined by

$$g(t) = \int_0^{+\infty} H_S(t-s)A^{-1}f(s)ds, \quad 0 \le t < +\infty,$$

is a bounded solution of Eq. (5.1) with boundary value

$$\lim_{t \downarrow 0} P_+ g(t) = P_+ \int_0^{+\infty} H_S(-s)A^{-1}f(s)ds.$$

The unique bounded solution ψ of Eq. (5.1) with a given boundary value $\phi_+ \in H_+$ is obtained by adding to the function g the unique bounded solution of the homogeneous differential equation (3.1) with boundary value $\phi_+ - P_+ \int_0^{+\infty} H_S(-s)A^{-1}f(s)ds$. The latter solution is derived using Theorem 3.1, and formula (5.5) is clear. □

For semi-definite admissible pairs analogous results can be derived. The operation on semi-definite pairs that has been discussed in Theorem III 6.3 can be employed to obtain these analogous results as corollaries of Theorems 5.1 and 5.2 (see the proof of Theorem 3.3, where an example of the use of this operation can be found).

CHAPTER V

THE CONVOLUTION EQUATION

$$\psi(t) - \int_0^{\tau} H(t-s)B\psi(s)ds = \omega(t)$$

In this chapter we investigate the operator convolution equation

$$\psi(t) - \int_0^{\tau} H(t-s)B\psi(s)ds = \omega(t), \quad 0 < t < \tau.$$

Here $H(\cdot)$ is the propagator function of the self-adjoint operator T that appears in a given hermitian admissible pair (T,B). The interval $(0,\tau)$ ma be finite as well as infinite. The equivalence to the operator differenti equation of the previous chapter is proved and, if (T,B) is a positive definite pair, formulas for the resolvent kernel are given. Finally, for this case the canonical factorizations of the symbol are constructed.

1. Preliminaries

Let (T,B) be a hermitian admissible pair on H. In this chapter we stu the convolution equation

$$(1.1) \qquad \psi(t) - \int_0^{\tau} H(t-s)B\psi(s)ds = \omega(t), \quad 0 < t < \tau.$$

Here $0 < \tau \le +\infty$. The vector function ω is given and the problem is to find the solution ψ. We shall assume that ω is an element of $L_p((0,\tau);H)$ and we look for solutions ψ in the same space.

From Section IV.4 we know that $H(\cdot)B$ is Bochner integrable on the rea line. Hence, the operator K defined by

$$(1.2) \qquad (K\psi)(t) = \int_0^{\tau} H(t-s)B\psi(s)ds \quad (0 < t < \tau),$$

acts as a bounded linear operator on $L_p((0,\tau);H)$ $(0 < \tau \le +\infty;\ 1 \le p \le +\infty)$. Mimicking the proof of Lemma 1.1 of [22] one sees that for finite τ the operator K is a compact operator on $L_p((0,\tau);H)$ $(1 \le p \le +\infty)$.

As usual the symbol of Eq. (1.1) is given by

$$S(\lambda) = I - \int_{-\infty}^{+\infty} e^{\lambda t} H(t)B\,dt, \qquad \text{Re}\,\lambda = 0.$$

Note that

$$(1.3) \qquad S(\lambda^{-1}) = W(\lambda) = (T-\lambda)^{-1}(T-\lambda A), \qquad \text{Re}\,\lambda = 0.$$

In other words, up to a trivial change of variable, the symbol of Eq. (1.1) coincides with the symbol of the pair (T,B) (cf. (III 2.1)).

2. Equivalence to the finite-slab problem

In this section we consider the convolution equation (1.1) on a finite interval $(0,\tau)$. We prove that for a large class of right-hand sides ω the convolution equation (1.1) is equivalent to an inhomogeneous finite-slab problem, which has been dealt with in the previous chapter.

THEOREM 2.1. *Fix* $0 < \tau < +\infty$. *Let* (T,B) *be a hermitian admissible pair on* H, *and put* $A = I - B$. *Suppose that* $\omega: [0,\tau] \to H$ *is a continuous vector function, and let* $T\omega$ *be differentiable on* $(0,\tau)$. *A vector function* $\psi \in L_\infty((0,\tau);H)$ *is a solution of the operator differential equation*

$$(2.1) \qquad (T\psi)'(t) = -A\psi(t) + (T\omega)'(t) + \omega(t), \qquad 0 < t < \tau,$$

with boundary value $\phi = P_+\omega(0) + P_-\omega(\tau)$ *if and only if* ψ *is a solution of the convolution equation* (1.1) *with right-hand side* ω.

From this theorem one easily derives the following: let $0 < \gamma < 1$, and let $h: (0,\tau) \to H$ be a bounded and continuous vector function. Then a vector function $\psi \in L_\infty((0,\tau);H)$ is a solution of the operator differential equation

$$(T\psi)'(t) = -A\psi(t) + |T|^\gamma h(t), \qquad 0 < t < \tau,$$

with boundary value ϕ if and only if ψ is a solution of the convolution equation (1.1) with right-hand side

$$(2.2) \qquad \omega(t) = e^{-tT^{-1}} P_+\phi + e^{(\tau-t)T^{-1}} P_-\phi + \int_0^\tau H(t-s)|T|^\gamma h(s)\,ds.$$

One easily checks that ω is continuous (cf. Proposition IV 4.2), $T\omega$ is differentiable on $(0,\tau)$ and

$$(2.3) \qquad (T\omega)'(t) = -\omega(t) + |T|^\gamma h(t), \qquad 0 < t < \tau$$

(see Proposition IV 4.3). Finally, from (IV 4.6) and (IV 4.7) one gets the boundary conditions

$$\lim_{t\downarrow 0} P_+\omega(t) = P_+\phi, \qquad \lim_{t\uparrow\tau} P_-\omega(t) = P_-\phi.$$

PROOF of Theorem 2.1. Let $\psi \in L_\infty((0,\tau);H)$ be a solution of the convolution equation (1.1) with right-hand side ω. Here $\omega: [0,\tau] \to H$ is assumed to be continuous and $T\omega$ is supposed to be differentiable on $(0,\tau)$. Recall that $B = |T|^\alpha D$ for some $0 < \alpha < 1$ and $D \in L(H)$. According to this identity, the essential boundedness of $D\psi$ and Proposition IV 4.2, the vector

$$(2.4) \qquad g(t) = \int_0^\tau H(t-s)B\psi(s)\,ds, \qquad t \in \mathbb{R},$$

depends continuously on $t \in \mathbb{R}$. Since $\psi = \omega + g$ (cf. (1.1)), it follows that ψ is continuous. In fact, it follows that ψ has a continuous extension to the closed interval $[0,\tau]$.

Define $h: (-\infty,+\infty) \to H$ by $h(t) = D\psi(t)$ for $0 < t < \tau$ and by $h(t) = 0$ for $t \le 0$ or $t \ge \tau$. Then $g(t) = \int_0^\tau |T|^\alpha H(t-s)h(s)\,ds$ $(t \in \mathbb{R})$ and h is a bounded function that is continuous on $\mathbb{R}\setminus\{0,\tau\}$. By virtue of Proposition IV 4.3 it is clear that Tg is differentiable on all of $\mathbb{R}$, except possibly at $t = 0$ and $t = \tau$. Its derivative is given by

$$(Tg)'(t) = \begin{cases} -g(t) + B\psi(t), & 0 < t < \tau; \\ -g(t), & t \notin (0,\tau). \end{cases}$$

Since $\psi = \omega + g$ and $T\omega$ is differentiable on $(0,\tau)$, it follows that $T\psi$ is differentiable on $(0,\tau)$ and

$$(T\psi)'(t) = \{-g(t) + B\psi(t)\} + (T\omega)'(t), \qquad 0 < t < \tau,$$

and therefore ψ is a solution of the operator differential equation (2.1).

From Proposition IV 4.2 it follows that

$$g(0) = \int_0^\tau H(-s)B\psi(s)ds \in H_-, \quad g(\tau) = \int_0^\tau H(\tau-s)B\psi(s)ds \in H_+.$$

So $P_+g(0) = P_-g(\tau) = 0$. Because of the continuity of ω on the closed interval $[0,\tau]$ and the equality $\psi = \omega+g$, the vector $\phi = P_+\omega(0) + P_-\omega(\tau)$ is the boundary value of the solution ψ.

Conversely, let ψ be a solution of Eq. (2.1) with boundary value $\phi = P_+\omega(0) + P_-\omega(\tau)$, and let $\psi \in L_\infty((0,\tau);H)$. Since $B = |T|^\alpha D$ for some $0 < \alpha < 1$ and $D \in L(H)$ and $D\psi \in L_\infty((0,\tau);H)$, it is clear from Proposition IV 4.2 that the function g defined by (2.4) is continuous on the closed interval $[0,\tau]$. Put $\chi = \psi-\omega$. Then $T\chi$ is differentiable on $(0,\tau)$, $\chi \in L_\infty((0,\tau);H)$ and

$$(2.5) \qquad (T\chi)'(t) + \chi(t) = B\psi(t), \quad 0 < t < \tau.$$

So for $0 \le t \le \tau$ the Bochner integral $\int_0^\tau H(t-s)[(T\chi)'(s) + \chi(s)]ds$ is well-defined, absolutely convergent and depends continuously on $t \in [0,\tau]$.

By the analyticity of the semigroups $(e^{-tT^{-1}}P_+)_{t\ge0}$ and $(e^{+tT^{-1}}P_-)_{t\ge0}$ the propagator function $H(z)$ is differentiable for all $0 \ne z \in \mathbb{R}$. Fix $0 < t < \tau$. Take $0 < \tau_1 < t < \tau_2 < \tau$. By the differentiability of the propagator function partial integration is allowed and yields

$$\int_0^{\tau_1} H(t-s)[(T\chi)'(s) + \chi(s)]ds = [e^{-(t-s)T^{-1}}P_+\chi(s)]_{s=0}^{\tau_1};$$

$$\int_{\tau_2}^{\tau} H(t-s)[(T\chi)'(s) + \chi(s)]ds = [-e^{-(t-s)T^{-1}}P_-\chi(s)]_{s=\tau_2}^{\tau}.$$

Recall that ψ has the vector $\phi = P_+\omega(0) + P_-\omega(\tau)$ as its boundary value. Therefore,

$$\lim_{t\downarrow0} P_+\chi(t) = \lim_{t\uparrow\tau} P_-\chi(\tau) = 0.$$

Now one takes the limits as $\tau_1 \uparrow t$ and $\tau_2 \downarrow t$ and gets

$$\int_0^\tau H(t-s)[(T\chi)'(s) + \chi(s)]ds = \chi(t), \quad 0 < t < \tau.$$

With the help of (2.5) and the equality $\chi = \psi-\omega$ it appears that ψ is a solution of the convolution equation (1.1) with right-hand side ω. □

3. Equivalence to the half-space problem

In this section we take $\tau = +\infty$ and prove that on the half-line $(0,+\infty)$ the convolution equation (1.1) is equivalent to a half-space problem.

THEOREM 3.1. *Take $\tau = +\infty$. Let (T,B) be a hermitian admissible pair on the Hilbert space H, and put $A = I - B$. Let $0 < k \le +\infty$ be a constant such that $\sigma(T) \subset (-k,+\infty)$. Suppose that ω: $[0,+\infty) \to H$ is a continuous vector function such that $e^{-t/k}\omega(t)$ is bounded and $T\omega$ is differentiable on $(0,+\infty)$. Then a vector function ψ such that $e^{-t/k}\psi(t) \in L_\infty((0,+\infty);H)$ is a solution of the operator differential equation*

$$(3.1) \qquad (T\psi)'(t) = -A\psi(t) + (T\omega)'(t) + \omega(t), \quad 0 < t < +\infty,$$

with boundary value $\phi_+ = P_+\omega(0)$ if and only if ψ is a solution of the convolution equation (1.1) with right-hand side ω.

PROOF. First we consider the case when $k = +\infty$. Note that in this case $e^{-t/k} \equiv 1$; so for $k = +\infty$ the theorem applies to functions ψ in $L_\infty((0,+\infty);H)$. So take $\psi \in L_\infty((0,+\infty);H)$, and let h: $(0,+\infty) \to H$ be a bounded and continuous vector function. If ψ is a solution of the convolution equation (1.1) (for $\tau = +\infty$) with right-hand side ω, then one can use the same arguments as in the proof of Theorem 2.1 to show that ψ is a solution of the operator differential equation (3.1) with boundary value $\phi_+ = P_+\omega(0)$.

Conversely, let ψ be a solution of the operator differential equation (3.1) with boundary value $\phi_+ = P_+\omega(0)$, and let $\psi \in L_\infty((0,+\infty);H)$. As in the proof of Theorem 2.1, it appears that for $\chi = \psi-\omega$ the expression

$$\int_0^{+\infty} H(t-s)[(T\chi)'(s) + \chi(s)]ds$$

represents a bounded continuous function on $[0,+\infty)$. Fix $0 < t < +\infty$, and take $0 < \tau_1 < t < \tau_2 < \tau_3 < +\infty$. Partial integration yields

$$\int_0^{\tau_1} H(t-s)[(T\chi)'(s) + \chi(s)]ds = [e^{-(t-s)T^{-1}}P_+\psi(s)]_{s=0}^{\tau_1};$$

$$\int_{\tau_2}^{\tau_3} H(t-s)[(T\chi)'(s) + \chi(s)]ds = [-e^{-(t-s)T^{-1}}P_-\chi(s)]_{s=\tau_2}^{\tau_3}.$$

Since T is a bounded self-adjoint operator and $\chi \in L_\infty((0,+\infty);H)$, one has

$$\left\| e^{-(t-\tau_3)T^{-1}} P_-\chi(\tau_3) \right\| \le M\, e^{-(\tau_3 - t)/\|T\|},$$

where M is the norm of χ in the Banach space $L_\infty((0,+\infty);H)$. Taking the limits as $\tau_1 \uparrow t$, $\tau_2 \downarrow t$ and $\tau_3 \to +\infty$ we get

$$\int_0^{+\infty} H(t-s)[(T\chi)'(s) + \chi(s)]ds = \chi(t), \quad 0 < t < +\infty.$$

Recall that $(T\chi)'(t) + \chi(t) = B\psi(t)$ and $\chi(t) = \psi(t) - \omega(t)$ $(0 < t < +\infty)$. Then it is clear that ψ is a solution of the convolution equation (1.1) (with $\tau = +\infty$) with right-hand side ω.

Next, let $0 < k < +\infty$ be a constant such that $\sigma(T) \subset (-k,+\infty)$, and suppose that $e^{-t/k}\psi(t) \in L_\infty((0,+\infty);H)$. Consider the Möbius transformation

$$\tilde{T} = kT(k+T)^{-1}, \quad \tilde{B} = k(k+T)^{-\frac{1}{2}}B(k+T)^{-\frac{1}{2}}. \tag{3.2}$$

By Theorem III 6.2 the pair $(\tilde{T},\tilde{B})$ is a hermitian admissible pair on H. Note that the spectral projection of $\tilde{T}$ corresponding to the positive (negative) part of its spectrum coincides with P_+ (P_-). Therefore, if $\tilde{H}(t)$ denotes the propagator function of the self-adjoint operator $\tilde{T}$ (cf. Section IV.4), then

$$\tilde{H}(t)\tilde{B} = (k+T)^{\frac{1}{2}}\cdot e^{-t/k}H(t)B\cdot(k+T)^{-\frac{1}{2}}, \quad 0 < t < +\infty. \tag{3.3}$$

Consider the convolution equations

$$\psi(t) - \int_0^{+\infty} H(t-s)B\psi(s)ds = \omega(t), \quad 0 < t < +\infty; \tag{3.4a}$$

$$\tilde{\psi}(t) - \int_0^{+\infty} \tilde{H}(t-s)\tilde{B}\tilde{\psi}(s)ds = (k+T)^{\frac{1}{2}}\cdot e^{-t/k}\omega(t), \quad 0 < t < +\infty. \tag{3.4b}$$

Then a function ψ such that $e^{-t/k}\psi(t) \in L_\infty((0,+\infty);H)$ is a solution of Eq. (3.4a) if and only if $\tilde{\psi}(t) = (k+T)^{\frac{1}{2}}\cdot e^{-t/k}\psi(t)$ is a solution of Eq. (3.4b) in $L_\infty((0,+\infty);H)$.

Observe that $\tilde{A} := I - \tilde{B} = (k+T)^{-\frac{1}{2}}(kA+T)(k+T)^{-\frac{1}{2}}$. Then a function ψ such that $e^{-t/k}\psi(t) \in L_\infty((0,+\infty);H)$ is a solution of the operator differential equation (3.1) with boundary value $\phi_+ = P_+\omega(0)$ if and only if the function $\tilde{\psi}(t) = (k+T)^{\frac{1}{2}}\cdot e^{-t/k}\psi(t)$ is a solution in the space $L_\infty((0,+\infty);H)$ of the operator differential equation

$$(\tilde{T}\tilde{\psi})'(t) = -\tilde{A}\tilde{\psi}(t) + (\tilde{T}\tilde{\omega})'(t) + \tilde{\omega}(t), \quad 0 < t < +\infty, \tag{3.5}$$

with boundary value $P_+\tilde{\omega}(0)$. Here $\tilde{\omega}$ denotes the right-hand side of Eq. (3.4

Finally, by the first part of the proof a vector function $\tilde{\psi} \in L_\infty((0,+\infty$ is a solution of the convolution equation (3.4b) if and only if $\tilde{\psi}$ is a solu tion of the operator differential equation (3.5) with boundary value $P_+\tilde{\omega}(0$ From this fact one easily completes the proof. □

As to Theorem 3.1 a remark analogous to the one following the stateme of Theorem 2.1 can be made. Let $0 < k \le +\infty$ be a constant such that $\sigma(T) \subset$ $\subset (-k,+\infty)$. Let $0 < \gamma < 1$, and let $h: (0,+\infty) \to H$ be a continuous vector func tion such that $e^{-t/k}h(t) \in L_\infty((0,+\infty);H)$. Then a vector function ψ such that $e^{-t/k}\psi(t) \in L_\infty(0,+\infty);H)$ is a solution of the operator differential equatio

$$(T\psi)'(t) = -A\psi(t) + |T|^\gamma h(t), \quad 0 < t < +\infty,$$

with boundary value ϕ_+ if and only if ψ is a solution of the convolution equation (1.1) (with $\tau = +\infty$) with right-hand side

$$\omega(t) = e^{-tT^{-1}}P_+\phi_+ + \int_0^{+\infty} H(t-s)|T|^\gamma h(s)ds, \quad 0 < t < +\infty.$$

For $k = +\infty$ we reduce this statement to Theorem 3.1 with the help of the sam method of reduction as the one described in Section V.2. For $0 < k < +\infty$ one employs, as in the proof of Theorem 3.1, the Möbius transformation (3.2) to reduce this statement to the case when $k = +\infty$.

4. The convolution equation on the finite interval

In this section we exploit the equivalence theorem of Section 2 and the solution of the inhomogeneous finite-slab problem (cf. Theorem IV 5.1) to show that the convolution equation

$$\psi(t) - \int_0^{\tau} H(t-s)B\psi(s)ds = \omega(t) \quad (0 < t < \tau < +\infty) \tag{4.1}$$

is uniquely solvable in $L_p((0,\tau);H)$ $(1 \le p \le +\infty)$ and to derive a formula for its resolvent kernel.

THEOREM 4.1. *Let* (T,B) *be a positive definite admissible pair on* H, *and let* $1 \le p \le +\infty$. *For every* $\omega \in L_p((0,\tau);H)$ *there is a unique solution* ψ *of the convolution equation* (4.1) *in the space* $L_p((0,\tau);H)$, *namely*

$$\psi(t) = \omega(t) + \int_0^\tau \gamma(t,s)\omega(s)ds \quad (0 < t < \tau), \tag{4.2}$$

where the resolvent kernel $\gamma(t,s)$ *has the form*

$$\begin{aligned}\gamma(t,s) &= H_S(t-s)C - \\ &- [e^{-tT^{-1}A}P_p + e^{(\tau-t)T^{-1}A}P_m]V_\tau^{-1}[P_+H_S(-s)C + P_-H_S(\tau-s)C].\end{aligned} \tag{4.3}$$

Here $A = I - B$, $S = A^{-1}T$ *and* $C = BA^{-1}$.

PROOF. First we assume that there exist $0 < \gamma < 1$ and an H-valued polynomial g such that $\omega(t) = |T|^\gamma g(t)$, $0 < t < \tau$. Then ω is continuous on the closed interval $[0,\tau]$, $T\omega$ is differentiable on $(0,\tau)$ and

$$(T\omega)'(t) + \omega(t) = |T|^\gamma h(t) \quad (0 < t < \tau), \tag{4.4}$$

where h is a bounded and continuous function on $(0,\tau)$ (which is a polynomial, but this is not relevant to the present proof).

By Theorem 2.1 every solution $\psi \in L_\infty((0,\tau);H)$ of Eq. (4.1) with right-hand side ω is also a solution of the operator differential equation

$$(T\psi)'(t) = -A\psi(t) + |T|^\gamma h(t) \quad (0 < t < \tau) \tag{4.5}$$

with boundary value $\phi := P_+\omega(0) + P_-\omega(\tau)$, and conversely. According to Theorem IV 5.1 the differential equation (4.5) has a unique solution, namely

$$\begin{aligned}\psi(t) &= e^{-tT^{-1}A}P_pV_\tau^{-1}(\phi-\chi) + e^{(\tau-t)T^{-1}A}P_mV_\tau^{-1}(\phi-\chi) + \\ &+ \int_0^\tau H_S(t-s)A^{-1}|T|^\gamma h(s)ds, \quad 0 < t < \tau,\end{aligned} \tag{4.6}$$

where

$$\chi = P_+\int_0^\tau H_S(-s)A^{-1}|T|^\gamma h(s)ds + P_-\int_0^\tau H_S(\tau-s)A^{-1}|T|^\gamma h(s)ds. \tag{4.7}$$

Put

$$g(t) = \int_0^\tau H_S(t-s)A^{-1}|T|^\gamma h(s)ds, \quad 0 \le t \le \tau. \tag{4.8}$$

By Theorem III 2.3 the pair (S,−C), where $C = BA^{-1} = A^{-1}-I$, is a positive definite admissible pair on H endowed with the inner product (III 2.7). Since $A^{-1}|T|^\gamma h(s) = |S|^\beta\tilde{h}(s)$ for some $0 < \beta \le \gamma < 1$ and bounded continuous function $\tilde{h}: (0,\tau) \to H$, the function g is continuous on $[0,\tau]$ (by Propositic IV 4.2). By substituting (4.4) into (4.8) one obtains

$$\begin{aligned}g(t) &= \int_0^\tau H_S(t-s)[(S\omega)'(s) + \omega(s) + C\omega(s)]ds = \\ &= [e^{-(t-s)T^{-1}A}P_p\omega(s)]_{s=0}^t + [-e^{-(t-s)T^{-1}A}P_m\omega(s)]_{s=t}^\tau + \\ &+ \int_0^\tau H_S(t-s)C\omega(s)ds = \omega(t) - e^{-tT^{-1}A}P_p\omega(0) \\ &- e^{(\tau-t)T^{-1}A}P_m\omega(\tau) + \int_0^\tau H_S(t-s)C\omega(s)ds.\end{aligned} \tag{4.9}$$

By the continuity of g and the terms at the last side of the previous equality the above identity also holds for t = 0 and t = τ. Hence, by (4.7) and the way the vector ϕ has been defined, we have

$$\begin{aligned}\phi - \chi &= [P_+\omega(0) + P_-\omega(\tau)] - [P_+g(0) + P_-g(\tau)] = \\ &= P_+[P_p\omega(0) + e^{+\tau T^{-1}A}P_m\omega(\tau)] + P_-[P_m\omega(\tau) + e^{-\tau T^{-1}A}P_p\omega(0)] - \\ &- P_+\int_0^\tau H_S(-s)C\omega(s)ds - P_-\int_0^\tau H_S(\tau-s)C\omega(s)ds = \\ &= V_\tau P_p\omega(0) + V_\tau P_m\omega(\tau) - P_+\int_0^\tau H_S(-s)C\omega(s)ds - P_-\int_0^\tau H_S(\tau-s)C\omega(s)\end{aligned}$$

We substitute this identity and the equality (4.9) into (4.6) and conclude that

$$\psi(t) = \omega(t) + \int_0^\tau \gamma(t,s)ds \quad (0 < t < \tau),$$

where $\gamma(t,s)$ is given by (4.3).

Let K be the operator defined by (1.2) and put

$$(4.10)\qquad (L\omega)(t) = \int_0^\tau \gamma(t,s)\omega(s)ds, \quad 0 < t < \tau.$$

Since $H(\cdot)B$ is Bochner integrable, the operator K acts as a bounded linear operator on $L_p((0,\tau);H)$ $(1 \le p \le +\infty)$. Recall that $(S,-C)$ is a positive definite admissible pair on H endowed with some equivalent inner product. In particular, $-C = |S|^{\alpha}\tilde{D}$ for some $0 < \alpha < 1$ and some compact operator $\tilde{D}$. We shall prove that for $\alpha^{-1} < p \le +\infty$ the operator L acts as a bounded linear operator on $L_p((0,\tau);H)$.

Since $(S,-C)$ is a positive definite admissible pair on H endowed with some equivalent inner product, the function $H_S(\cdot)C$ is Bochner integrable. Further, by Proposition IV 4.1 we have

$$\|H_S(-s)C\| = O(s^{\alpha-1})\ (s\downarrow 0), \quad \|H_S(\tau-s)C\| = O((\tau-s)^{\alpha-1})(s\uparrow\tau).$$

Therefore,

$$(4.11)\qquad \int_0^\tau \|P_+H_S(-s)C + P_-H_S(\tau-s)C\|^q ds < +\infty, \quad 1 \le q < (1-\alpha)^{-1}.$$

Further, there exists a constant $M \ge 0$ such that

$$(4.12)\qquad \|e^{-tT^{-1}A_p}P_p + e^{(\tau-t)T^{-1}A_p}P_m\| \le M, \quad 0 < t < \tau.$$

From the estimates (4.11) and (4.12) and the fact that $H_S(\cdot)C$ is Bochner integrable it follows that for $\alpha^{-1} < p \le +\infty$ the operator L acts as a bounded operator on the Banach space $L_p((0,\tau);H)$.

In the first part of this proof we have shown that

$$(I-K)(I+L)\omega = \omega = (I+L)(I-K)\omega, \qquad \omega \in \mathbb{D},$$

where $\mathbb{D} := \{\omega: (0,\tau) \to H \mid \omega(t) = |T|^{\gamma} q(t)$ for some $0 < \gamma < 1$ and some polynomial $q(t)\}$. Observe that for $1 \le p < +\infty$ the set $\mathbb{D}$ is a dense linear subspace of $L_p((0,\tau);H)$. Hence, on each of the spaces $L_p((0,\tau);H)$ $(\alpha^{-1} < p < +\infty)$ the operator $I-K$ is invertible and its inverse coincides with the operator $I+L$. This settles the present theorem for $\alpha^{-1} < p < +\infty$.

In order to prove Theorem 4.1 in general, we apply the theory of GOHBERG & HEINIG of convolution equations of the type

$$\psi(t) - \int_0^\tau k(t-s)\psi(s)ds = \omega(t) \quad (0 < t < \tau < +\infty)$$

(cf. [22]). This theory has only been stated for the finite-dimensional case, but a close inspection shows that none of the proofs changes essentially, if one assumes that $k(\cdot)$ is a Bochner integrable operator function whose values are compact operators on a Hilbert space. Since the kern $H(\cdot)B$ of Eq. (4.1) is of this type, we may apply the Gohberg-Heinig theory in this case.

By Lemma 1.1 of [22] the operator K defined by (1.2) is a compact operator on all of the spaces $L_p((0,\tau);H)$ and Ker(I-K) consists of absolut continuous functions only $(1 \le p \le +\infty)$. But for $\alpha^{-1} < p < +\infty$ the operator $I-K$ has been shown to be invertible on $L_p((0,\tau);H)$. Hence, $I-K$ is an invertible operator on all of the spaces $L_p((0,\tau);H)$ $(1 \le p \le +\infty)$. By Theore 2.1 of [22] (and its proof) the solution of Eq. (4.1) on $L_p((0,\tau);H)$ $(1 \le p \le +\infty)$ has the form

$$(4.13)\qquad \psi(t) = \omega(t) + \int_0^\tau \tilde{\gamma}(t,s)\omega(s)ds \quad (0 < t < \tau),$$

where the resolvent kernel $\tilde{\gamma}(t,\cdot) \in L_1((0,\tau);L(H))$ $(0 < t < \tau)$. Further, by (4.3), (4.11), (4.12) and the fact that $H_S(\cdot)C$ is Bochner integrable, the operator function $\gamma(t,\cdot) \in L_1((0,\tau);L(H))$ $(0 < t < \tau)$. For $\omega \in L_p((0,\tau);H)$ $(\alpha^{-1} < p < +\infty)$ we have

$$\int_0^\tau [\gamma(t,s) - \tilde{\gamma}(t,s)]\omega(s)ds = 0, \quad 0 < t < \tau.$$

Choosing ω to be an H-valued polynomial and exploiting the fact that the polynomials are dense in $L_p((0,\tau);H)$ $(\alpha^{-1} < p < +\infty)$ we obtain

$$\gamma(t,s) = \tilde{\gamma}(t,s), \quad 0 < t \ne s < \tau.$$

Hence, on all of the spaces $L_p((0,\tau);H)$ $(1 \le p \le +\infty)$ the unique solution of Eq. (4.1) is given by (4.2). □

From the Gohberg-Heinig theory [22] one may also deduce that for an arbitrary semi-definite pair (T,B) on H the convolution equation (4.1) has a unique solution on all of the spaces $L_p((0,\tau);H)$ $(1 \le p \le +\infty)$. To prove this, it suffices to show that the operator $I-K$, where K is defined by (1.2), is injective on the space $L_\infty((0,\tau);H)$. But this is clear from the equivalence theorem of Section 2 and the solution of the homogeneous finite-slab problem, which is given by Theorem IV 2.2.

5. The convolution equation on the half-line

In this section we turn our attention to the Wiener-Hopf operator integral equation

$$\psi(t) - \int_0^{+\infty} H(t-s)B\psi(s)ds = \omega(t), \quad 0 < t < +\infty. \tag{5.1}$$

Here, as before, (T,B) is a positive definite admissible pair on a Hilbert space H and $H(t)$ is the propagator function of T. To deal with Eq. (5.1) three different methods are available. In the present section we shall use the equivalence theorem of Section 3 and the solution of the inhomogeneous half-space problem (cf. Theorem IV 5.2) to solve Eq. (5.1). In the next section we shall apply the "projection method" (see [17,21]), by which the solution of Eq. (5.1) is obtained as the limit for $\tau \to +\infty$ of the solution of the corresponding convolution equation on the finite interval $(0,\tau)$, which we considered in the previous section. In Section V.7 we shall use the method of factorization (see Theorem II 3.2) to solve Eq. (5.1).

THEOREM 5.1. *Let* (T,B) *be a positive definite admissible pair on* H, *and let* $1 \le p \le +\infty$. *For any* $\omega \in L_p((0,+\infty);H)$ *there is a unique solution* ψ *of the Wiener-Hopf operator integral equation* (5.1) *in the space* $L_p((0,+\infty);H)$, *namely*

$$\psi(t) = \omega(t) + \int_0^{+\infty} \gamma(t,s)\omega(s)ds, \quad 0 < t < +\infty, \tag{5.2}$$

where the resolvent kernel $\gamma(t,s)$ *has the form*

$$\gamma(t,s) = H_S(t-s)C - e^{-tT^{-1}A}PH_S(-s)C. \tag{5.3}$$

Here $A = I - B$, $S = A^{-1}T$ *and* $C = BA^{-1}$.

PROOF. First we assume that there exist $0 < \gamma < 1$, a constant $r > 0$ and an H-valued polynomial q such that $\omega(t) = e^{-rt}|T|^{\gamma}q(t)$ $(0 < t < +\infty)$. Then ω is a bounded and continuous function on $[0,+\infty)$, $T\omega$ is differentiable on $(0,+\infty)$ and

$$(T\omega)'(t) + \omega(t) = |T|^{\gamma} e^{-rt}h(t) \quad (0 < t < +\infty), \tag{5.4}$$

where h is a bounded and continuous function on $(0,+\infty)$.

By Theorem 3.1 every function $\psi \in L_\infty((0,+\infty);H)$ that is a solution of Eq. (5.1), is also a solution of the operator differential equation

$$(T\psi)'(t) = -A\psi(t) + |T|^{\gamma} e^{-rt}h(t) \quad (0 < t < +\infty) \tag{5.5}$$

with boundary value $\phi_+ := P_+\omega(0) \in H_+$, and conversely. According to Theorem IV 5.2 the differential equation (5.5) has a unique solution, namely

$$\psi(t) = e^{-tT^{-1}A}P\left(\phi_+ - P_+\int_0^{+\infty} H_S(-s)A^{-1}|T|^{\gamma} e^{-rs}h(s)ds\right) + \int_0^{+\infty} H_S(t-s)A^{-1}|T|^{\gamma} e^{-rs}h(s)ds. \tag{5.6}$$

As in the proof of Theorem 4.1 one shows that for $0 \le t < +\infty$

$$g(t) := \int_0^{+\infty} H_S(t-s)A^{-1}|T|^{\gamma} e^{-rs}h(s)ds = \int_0^{+\infty} H_S(t-s)[(S\omega)'(s) + \omega(s) + C\omega(s)]ds =$$

$$= \omega(t) - e^{-tT^{-1}A}P_p\omega(0) + \int_0^{+\infty} H_S(t-s)C\omega(s)ds,$$

and therefore

$$\phi_+ - \int_0^{+\infty} H_S(-s)A^{-1}|T|^{\gamma} e^{-rs}h(s)ds = P_+\omega(0) - P_+g(0) =$$

$$= P_+P_p\omega(0) - P_+\int_0^{+\infty} H_S(-s)C\omega(s)ds.$$

Substituting the above two equalities into (5.6) and using that $P(P_+P_p) = 1$ one shows that the solution ψ is given by (5.2).

Put

$$(K\psi)(t) = \int_0^{+\infty} H(t-s)B\psi(s)ds, \quad (L\omega)(t) = \int_0^{+\infty} \gamma(t,s)\omega(s)ds; \; 0 < t < +\infty \tag{5.7}$$

We know that the operator K acts as a bounded linear operator on $L_p((0,+\infty);H)$. As in the proof of Theorem 4.1 one shows that for some $0 < \alpha < 1$ and every $\alpha^{-1} < p \le +\infty$ the operator L acts as a bounded linear operator on $L_p((0,+\infty);H)$. Further, if $\mathbb{D} = \{\omega\colon (0,+\infty) \to H \mid \omega(t) = |T|^{\gamma} e^{-rt} q(t)$ for some $0 < \gamma < 1$, some $r > 0$ and some H-valued polynomial $q(t)\}$, then we have shown that

$$(I-K)(I+L)\omega = \omega = (I+L)(I-K)\omega, \quad \omega \in \mathbb{D}.$$

Since for $1 \le p < +\infty$ the set $\mathbb{D}$ is a dense linear subspace of $L_p((0,+\infty);H)$, it follows that on each of the spaces $L_p((0,+\infty);H)$ $(\alpha^{-1} < p < +\infty)$ the operator $I-K$ is invertible with inverse $I+L$. Hence, for $\alpha^{-1} < p < +\infty$ we have established Theorem 5.1.

To get the general result we apply Theorem II 3.1 to the Wiener-Hopf equation (5.1) and conclude that $I-K$ is invertible on all of the spaces $L_p((0,+\infty);H)$ $(1 \le p \le +\infty)$ and the symbol of Eq. (5.1) has a left canonical factorization with respect to the imaginary line. By Theorem II 3.2 we have

$$((I-K)^{-1}\omega)(t) = \omega(t) + \int_0^{+\infty} \tilde{\gamma}(t,s)\omega(s)ds, \quad 0 < t < +\infty,$$

where $\tilde{\gamma}(t,s)$ is some resolvent kernel; this formula is correct on $L_p((0,+\infty);H)$ for every $1 \le p \le +\infty$. In the same way as in the proof of Theorem 4.1 one shows that $\tilde{\gamma}(t,s) = \gamma(t,s)$ $(0 < t \ne s < +\infty)$, and Theorem 5.1 is clear in general. □

6. The projection method

In this section we give an alternative proof of Theorem 5.1 based on the projection method in [17,21].

THEOREM 6.1. *Let* (T,B) *be a positive definite admissible pair on* H, *and let* $1 \le p \le +\infty$ *and* $\omega \in L_p((0,+\infty);H)$. *Put* $\omega_\tau(t) = \omega(t)$ *for* $0 < t < \tau < +\infty$, *and let* $\psi_\tau \in L_p((0,\tau);H)$ *be the unique solution of*

$$\psi_\tau(t) - \int_0^{\tau} H(t-s)B\psi_\tau(s)ds = \omega_\tau(t), \quad 0 < t < \tau.$$

Then in $L_p((0,+\infty);H)$ *the limit*

$$\lim_{\tau\to+\infty} \psi_\tau = \psi$$

exists and ψ *is the unique solution in* $L_p((0,+\infty);H)$ *of the Wiener-Hopf equation*

$$(6.1)\qquad \psi(t) - \int_0^{+\infty} H(t-s)B\psi(s)ds = \omega(t), \quad 0 < t < +\infty.$$

PROOF. First we consider the case when $\omega \in L_\infty((0,+\infty);H)$. From Theorem 4.1 one has

$$(6.2)\qquad \psi_\tau(t) = \omega_\tau(t) + \int_0^{\tau} \gamma_\tau(t,s)\omega_\tau(s)ds \quad (0 < t < \tau),$$

where the resolvent kernel $\gamma_\tau(t,s)$ has the form

$$(6.3)\qquad \begin{aligned}\gamma_\tau(t,s) &= H_S(t-s)C - \\ &- [e^{-tT^{-1}}A_{P_p} + e^{(\tau-t)T^{-1}}A_{P_m}]V_\tau^{-1}[P_+H_S(-s)C + P_-H_S(\tau-s)C].\end{aligned}$$

Put $\tilde{\psi}_\tau(t) = \psi_\tau(t)$ $(0 < t < \tau)$ and $\tilde{\psi}_\tau(t) = 0$ $(t \ge \tau)$, and define L_τ by

$$(L_\tau\omega_\tau)(t) = \int_0^{\tau} \gamma_\tau(t,s)\omega_\tau(s)ds \quad (0 < t < \tau;\ \omega_\tau \in L_\infty((0,\tau);H)).$$

Observe that

$$(6.4a)\qquad \sup_{0<t<\tau} \int_0^{\tau} \|H_S(t-s)C\|ds \le \int_{-\infty}^{-\infty} \|H_S(z)C\|dz < +\infty;$$

$$(6.4b)\qquad \sup_{0<t<\tau} \left\|e^{-tT^{-1}}A_{P_p} + e^{(\tau-t)T^{-1}}A_{P_m}\right\| \le$$

$$\le \sup_{0\le t<+\infty} \left\|e^{-tT^{-1}}A_{P_p}\right\| + \sup_{0\le t<+\infty} \left\|e^{+tT^{-1}}A_{P_m}\right\| < +\infty;$$

$$(6.4c)\qquad \sup_{0<\tau<+\infty} \|V_\tau^{-1}\| < +\infty;$$

$$(6.4d)\qquad \int_0^{\tau} \|P_+H_S(-s)C + P_-H_S(\tau-s)C\|ds \le \int_{-\infty}^{+\infty} \|H_S(z)C\|dz < +\infty.$$

The inequality (6.4c) follows from the fact that $\lim_{\tau\to+\infty} \|V_\tau^{-1} - V^{-1}\| = 0$ (see Theorem IV 2.1). Note that none of the upper bounds appearing in the inequalities (6.4) depends on τ. Using the inequalities (6.4) one shows that there is a constant M, not depending on τ, such that

(6.5) $\qquad \| I + L_\tau \|_{L_\infty((0,\tau);H)} \le M < +\infty, \quad 0 < \tau < +\infty.$

We now apply the "projection method" (namely, Theorem I of [17]) and conclude that for a given $\omega \in L_\infty((0,+\infty);H)$ the Wiener-Hopf operator integral equation (5.1) has a unique solution $\psi \in L_\infty((0,+\infty);H)$, which satisfies

$$\lim_{\tau\to+\infty} \| \psi - \tilde{\psi}_\tau \|_{L_\infty((0,+\infty);H)} = 0.$$

Taking this limit in the expressions (6.2) and (6.3) one obtains

$$\psi(t) = \omega(t) + \int_0^{+\infty} \gamma(t,s)\omega(s)ds, \quad 0 < t < +\infty,$$

where $\gamma(t,s)$ is computed by taking the limit of $\gamma_\tau(t,s)$ as $\tau \to +\infty$. Since

$$\lim_{\tau\to+\infty} \| V_\tau^{-1} - V^{-1} \| = 0, \quad \lim_{\tau\to+\infty} \| e^{(\tau-t)T^{-1}} A_{P_m} \| = 0, \quad \lim_{\tau\to+\infty} \| H_S(\tau-s) \| = 0,$$

one gets (5.3).

Finally, one applies Theorem II 3.1 and II 3.2 and extends formula (5.3) to right-hand sides $\omega \in L_p((0,+\infty);H)$, where $1 \le p \le +\infty$. □

7. The factorization method

In this section we solve the Wiener-Hopf equation (5.1) by constructing a canonical factorization of its symbol and using the factors to compute the resolvent kernel (5.3).

THEOREM 7.1. *Consider the operator integral equation*

(7.1) $\qquad \psi(t) - \int_0^{+\infty} H(t-s)B\psi(s)ds = \omega(t), \quad 0 < t < +\infty,$

where (T,B) *is a positive definite admissible pair on* H *and* $H(t)$ *is the propagator function of the pair* (T,B). *Let*

$$S(\lambda) = I - \int_{-\infty}^{-\infty} e^{\lambda t} H(t)B\,dt, \quad \text{Re}\,\lambda = 0,$$

be the symbol of the equation (7.1). *Put*

(7.2a) $\qquad y(t) = (I-P)H_S(t)C, \quad -\infty < t < 0;$

(7.2b) $\qquad x(t) = H_S(t)PB(I-PB)^{-1}, \quad 0 < t < +\infty,$

where $C = B(I-B)^{-1}$, H_S *is the propagator function of the operator* $S = A^{-1}$, *and* P *the projection of* H *onto* H_p *along* H_-. *Then*

$$S(\lambda)^{-1} = \left[I + \int_0^{+\infty} e^{\lambda t}x(t)dt\right]\left[I + \int_{-\infty}^{0} e^{\lambda t}y(t)dt\right], \quad \text{Re}\,\lambda = 0,$$

and this factorization is a left canonical factorization of $S(\lambda)^{-1}$ *with respect to the imaginary axis.*

To prove Theorem 7.1 we first consider the symbol of the pair (T,B), which up to a trivial change of the variable is equal to $S(\lambda)$. The followi two theorems describe a left and a right canonical factorization of the symbol of the pair (T,B).

THEOREM 7.2. *Let* (T,B) *be a positive definite admissible pair on* H. *Then its symbol* W *has a left canonical factorization with respect to the imaginary axis, namely*

(7.3a) $\qquad W(\lambda) = W_+(\lambda)W_-(\lambda)A, \quad \text{Re}\,\lambda = 0,$

where the factors and their inverses have the form

(7.3b) $\qquad W_+(\lambda) = I + T(T-\lambda)^{-1}(I-P)C;$

(7.3c) $\qquad W_-(\lambda) = I + TP(T-\lambda)^{-1}C;$

(7.3d) $\qquad W_+(\lambda)^{-1} = I - T(I-P)(S-\lambda)^{-1}C;$

(7.3d) $\qquad W_-(\lambda)^{-1} = I - T(S-\lambda)^{-1}PC.$

Here $A = I - B$, $C = A^{-1} - I$, $S = A^{-1}T$ *and* P *is the projection of* H *onto* H_p *along* H_-.

THEOREM 7.3. *Let* (T,B) *be a positive definite admissible pair on* H. *Then its symbol* W *has a right canonical factorization with respect to the imaginary axis, namely*

(7.4a) $$W(\lambda) = W_-(\lambda)W_+(\lambda)A, \quad \operatorname{Re}\lambda = 0,$$

where the factors and their inverses have the form

(7.4b) $$W_-(\lambda) = I + T(T-\lambda)^{-1}(I-Q)C;$$

(7.4c) $$W_+(\lambda) = I + TQ(T-\lambda)^{-1}C;$$

(7.4d) $$W_-(\lambda)^{-1} = I - T(I-Q)(S-\lambda)^{-1}C;$$

(7.4e) $$W_+(\lambda)^{-1} = I - T(S-\lambda)^{-1}QC.$$

Here $A = I - B$, $C = A^{-1} - I$, $S = A^{-1}T$ *and* Q *is the projection of* H *onto* H_m *along* H_+.

Theorem 7.3 is easily deduced from Theorem 7.2 if one considers the pair $(-T,B)$ rather than (T,B). Therefore, we only give the proof of Theorem 7.2.

PROOF of Theorem 7.2. For $C = A^{-1} - I$ we have

$$W(\lambda)A^{-1} = I + T(T-\lambda)^{-1}C, \quad \operatorname{Re}\lambda = 0.$$

Hence, $W(\lambda)A^{-1}$ is the transfer function of the node $\theta = (T,C,-T,I;H,H)$; its associate node is given by $\theta^\times = (S,C,T,I;H,H)$. The projection P, whose range H_p is invariant under the associate operator S of the node θ and whose kernel H_- is invariant under the main operator T of θ, is a supporting projection of θ. By Theorem II 1.1 formula (7.3a) represents, indeed, a factorization of the symbol W of the pair (T,B).

To show that (7.3a) is a left canonical factorization of W with respect to the imaginary axis, one rewrites W_+ and W_-^{-1} as

$$W_+(\lambda) = I + [T(T-\lambda)^{-1}P_-](I-P)C; \quad W_-(\lambda)^{-1} = I - A[S(S-\lambda)^{-1}P_p]PC.$$

Therefore, W_+ and W_-^{-1} have analytic continuations to the open right and open left half-plane, respectively. By Proposition III 1.2 we have $\|T(T-\lambda)^{-1}P_-x - P_-x\| \to 0$ $(\lambda\to 0, \operatorname{Re}\lambda \ge 0)$ and $\|S(S-\lambda)^{-1}P_px - P_px\| \to 0$ $(\lambda \to 0, \operatorname{Re}\lambda \le 0)$, uniformly in x on compact subsets of H. Since C is a compact operator, it follows that

(7.5) $$\lim_{\lambda\to 0, \operatorname{Re}\lambda\ge 0} \|W_+(\lambda) - [I+(I-P)C]\| = 0, \quad \lim_{\lambda\to 0, \operatorname{Re}\lambda\le 0} \|W_-(\lambda)^{-1} - [I-APC]\| = 0;$$

and hence W_+ and W_-^{-1} are continuous up to the imaginary axis.

Next, we use the intertwining property (III 5.2) and rewrite W_- and W_+^{-1} as

$$W_-(\lambda) = I + (I-Q^*)[T(T-\lambda)^{-1}P_+]C; \; W_+(\lambda)^{-1} = I - Q^*A[S(S-\lambda)^{-1}P_m]C.$$

Therefore, W_- and W_+^{-1} have analytic continuations to the open left and open right half-plane, respectively. By Proposition III 1.2 we have $\|T(T-\lambda)^{-1}P_+x - P_+x\| \to 0$ $(\lambda\to 0, \operatorname{Re}\lambda \le 0)$ and $\|S(S-\lambda)^{-1}P_mx - P_mx\| \to 0$ $(\lambda \to 0, \operatorname{Re}\lambda \ge 0)$, uniformly in x on compact subsets of H. Because the operator C is compact, we get

(7.6) $$\lim_{\lambda\to 0, \operatorname{Re}\lambda\le 0} \|W_-(\lambda) - [I+(I-Q^*)C]\| = 0, \quad \lim_{\lambda\to 0, \operatorname{Re}\lambda\ge 0} \|W_+(\lambda)^{-1} - [I-Q^*B]\| = 0,$$

and hence W_- and W_+^{-1} are continuous up to the imaginary axis.

Finally, one concludes that (7.3a) is, indeed, a left canonical factorization of W with respect to the imaginary axis. □

The factorization formulas (7.3) (or equivalently, (7.4)) appeared earlier (cf. [2,53]).

PROOF of Theorem 7.1. Let W be the symbol of the pair (T,B). Then $S(\lambda) = W(1/\lambda)$ is the symbol of the equation (5.1) (see Section 1). According to Theorem II 3.2 there exist functions $x \in L_1((0,+\infty);L(H))$ and $y \in L_1((-\infty,0);L(H))$ such that

(7.7) $$W(\lambda)^{-1} = \left[I + \int_0^{+\infty} e^{t/\lambda}x(t)\,dt\right]\left[I + \int_{-\infty}^0 e^{t/\lambda}y(t)\,dt\right], \quad \operatorname{Re}\lambda = 0,$$

is a left canonical factorization of W^{-1} with respect to the imaginary line. As known (see [21], for instance), left canonical factorizations are unique up to a displacement of a constant invertible operator as a factor. Using Theorem 7.2 one gets

$$I + \int_{-\infty}^{0} e^{t/\lambda} y(t)\,dt = [I-Q^*B]^{-1}[I - T(I-P)(S-\lambda)^{-1}C] =$$
$$= I + \lambda[I-Q^*B]^{-1}Q^*A(\lambda-S)^{-1}C;$$
$$I + \int_{0}^{+\infty} e^{t/\lambda} x(t)\,dt = A^{-1}[I-T(S-\lambda)^{-1}PC][I-APC]^{-1}A =$$
$$= I + \lambda(\lambda-S)^{-1}PC[I-APC]^{-1}A.$$

Here we have used that the factors of (7.7) tend to I as $\lambda \to 0$ ($\mathrm{Re}\,\lambda = 0$), together with (7.5) and (7.6). Formulas (7.5) and (7.6) also yield $A = [I-APC][I-Q^*B]$. From the semigroup properties of S (see Section III.4) and Theorem VIII 1.11 of [13] one obtains

$$\text{(7.8a)} \qquad y(t) = [I-Q^*B]^{-1}Q^*A\,H_S(t)C, \qquad -\infty < t < 0;$$

$$\text{(7.8b)} \qquad x(t) = H_S(t)PC[I-APC]^{-1}A, \qquad 0 < t < +\infty.$$

Now $[I-Q^*B]^{-1}Q^*A = A^{-1}[I-APC]Q^*A$. But $\mathrm{Ker}\,P = \mathrm{Im}\,Q^* = H_-$ and $\mathrm{Im}\,A^{-1}(I-Q^*) = \mathrm{Im}\,P = H_p$. So $PCQ^*A = PA^{-1}Q^*A - (PQ^*)A = PA^{-1}Q^*A = P - PA^{-1}(I-Q^*)A = P - A^{-1}(I-Q^*)A$. Therefore,

$$\text{(7.9)} \qquad [I-Q^*B]^{-1}Q^*A = I - P,$$

which settles (7.2a). Finally, to transform (7.8b) into (7.2b) one employs the intertwining property $[I-APC]A = A[I-PB]$. □

To conclude this section we employ (7.2a) - (7.2b) to compute the resolvent kernel $\gamma(t,s)$ by the method of Theorem II 3.2. In the above proof we showed that $PCQ^*A = P - A^{-1}(I-Q^*)A$. For $0 < r < \min(t,s)$ we have

$$\text{(7.10)} \qquad x(t-r)y(r-s) = H_S(t-r)PH_S(r-s)C - H_S(t-r)A^{-1}(I-Q^*)AH_S(r-s) =$$
$$= \frac{d}{dr}\{H_S(t-r)A^{-1}(I-Q^*)TH_S(r-s)C\}.$$

To see this, denote by F the resolution of the identity of S and note that in the operator norm

$$\frac{d}{dr} H_S(t-r) = \frac{d}{dr}\int_0^{+\infty} \mu^{-1}e^{-(t-r)/\mu}F(d\mu) = \int_0^{+\infty} \mu^{-2}e^{-(t-r)/\mu}F(d\mu);$$
$$(0 < r < \min(t,s))$$
$$\frac{d}{dr} H_S(r-s) = -\frac{d}{dr}\int_{-\infty}^{0} \mu^{-1}e^{-(r-s)/\mu}F(d\mu) = \int_{-\infty}^{0} \mu^{-2}e^{-(r-s)/\mu}F(d\mu).$$

Then (7.10) is clear from these identities and the intertwining property (III 5.2). Next we apply Theorem II 3.2. For $0 < t < s < +\infty$ we employ (III 5.2) and write

$$x(t-r)y(r-s) = \frac{d}{dr}\left\{e^{-(t-r)T^{-1}A_{P_p}}\cdot PH_S(r-s)C\right\}, \qquad 0 < r < t = \min(t,$$

Therefore,

$$\gamma(t,s) = y(t-s) + \int_0^t x(t-r)y(r-s)\,dr =$$
$$= (I-P)H_S(t-s)C + PH_S(t-s)C - e^{-tT^{-1}A_{P_p}}\cdot PH_S(-s)C =$$
$$= H_S(t-s)C - e^{-tT^{-1}A}PH_S(-s)C.$$

For $0 < s < t < +\infty$ one writes

$$x(t-r)y(r-s) = \frac{-d}{dr}\left\{H_S(t-r)A^{-1}(I-Q^*)A\,e^{-(r-s)T^{-1}A_{P_m}}C\right\}.$$

Hence,

$$\gamma(t,s) = x(t-s) + \int_0^s x(t-r)y(r-s)\,dr =$$
$$= H_S(t-s)PB(I-PB)^{-1} - H_S(t-s)A^{-1}(I-Q^*)AP_mC +$$
$$+ H_S(t)A^{-1}(I-Q^*)A\,e^{+sT^{-1}A_{P_m}}C.$$

The third term at the right-hand side is rewritten as

$$-H_S(t)A^{-1}(I-Q^*)TH_S(-s)C = -e^{-tT^{-1}A}PH_S(-s)C,$$

where we employed (III 5.2). Therefore,

$$\gamma(t,s) = H_S(t-s)[PB(I-PB)^{-1} + A^{-1}Q^*A(I-P_p)C] - e^{-tT^{-1}}A_{PH_S}(-s)C =$$

$$= H_S(t-s)[PB(I-PB)^{-1} + A^{-1}Q^*B] - e^{-tT^{-1}}A_{PH_S}(-s)C =$$

$$= H_S(t-s)C - e^{-tT^{-1}}A_{PH_S}(-s)C.$$

This completes the proof. □

8. Factorization and asymptotics

Let (T,B) be a positive definite admissible pair on a Hilbert space H with symbol

$$W(\lambda) = I - \lambda(\lambda - T)^{-1}B, \quad \lambda \in \rho(T).$$

In this section we relate Wiener-Hopf factorization properties of W to the asymptotics of the solutions of Eq. (5.1).

THEOREM 8.1. *Let* (T,B) *be a positive definite admissible pair on* H *with symbol* W, *and let* $k > 0$ *be a constant such that* $\sigma(T) \subset (-k,+\infty)$ *and* $-k$ *is not an eigenvalue of the associate operator* $S = A^{-1}T$. *Denote by* C_k *the positively oriented circle with centre* $-\frac{1}{2}k$ *and radius* $\frac{1}{2}k$. *Then* W *admits a left Wiener-Hopf factorization with respect to* C_k, *all the non-zero left factorization indices are equal to* -1 *and their number is precisely the number of eigenvalues of* S *on* $(-\infty,-k)$, *counted according to multiplicity.*

PROOF. To the pair (T,B) we apply the Möbius transformation

$$\tilde{T} = kT(k+T)^{-1}, \quad \tilde{B} = k(k+T)^{-\frac{1}{2}}B(k+T)^{-\frac{1}{2}}.$$

By Theorem III 6.2 the pair $(\tilde{T},\tilde{B})$ is a regular hermitian admissible pair on H, its symbol $\tilde{W}$ and its associate operator $\tilde{S} = (I-\tilde{B})^{-1}\tilde{T}$ are given by

$$\tilde{W}(\lambda) = (k+T)^{\frac{1}{2}}\cdot W\left(\frac{k\lambda}{k-\lambda}\right)\cdot(k+T)^{-\frac{1}{2}}, \quad \tilde{S} = (k+T)^{\frac{1}{2}}\cdot kS(k+S)^{-1}\cdot(k+T)^{-\frac{1}{2}}.$$

Observe that the Möbius transform $\lambda(\zeta) = k\zeta(k+\zeta)^{-1}$ maps the inner circular region of C_k onto the open left half-plane and the inverse map is $\zeta(\lambda) = k\lambda(k-\lambda)^{-1}$. According to Proposition III 2.1 the operator $\tilde{W}$ is Hölder continuous on the extended imaginary line, and therefore W is Hölder continuous on C_k. Since the spectrum of W equals the set of eigenvalues of $S = A^{-1}T$ on $\rho(T)$ and $-k \notin \sigma(S)$, it appears that $W(\zeta)$ is invertible for all $\zeta \in C_k$, and therefore $\tilde{W}(\lambda)$ is invertible for all λ on the extended imagina line. Further, for $\zeta \in C_k$ the operator $I - W(\zeta)$ is compact, and thus $I - \tilde{W}(\lambda)$ is compact for all λ on the extended imaginary axis. By these properties W has a left Wiener-Hopf factorization with respect to C_k and $\tilde{W}$ has a left Wiener-Hopf factorization with respect to the imaginary line (cf. [31], Theorems 6.1 and 6.2; see also [46], Theorem II of the introduction)

Consider the Wiener-Hopf operator integral equations

$$(8.1) \qquad \psi(t) - \int_0^{+\infty} H(t-s)B\psi(s)ds = \omega(t), \quad 0 < t < +\infty;$$

$$(8.2) \qquad \tilde{\psi}(t) - \int_0^{+\infty} \tilde{H}(t-s)\tilde{B}\tilde{\psi}(s)ds = \tilde{\omega}(t), \quad 0 < t < +\infty,$$

where $\tilde{H}(t)$ is given by (3.3). Let ω be a given function such that $e^{-t/k}\omega(t) \in L_\infty((0,+\infty);H)$, and let $\tilde{\omega}(t) = (k+T)^{\frac{1}{2}}\cdot e^{-t/k}\omega(t) \in L_\infty((0,+\infty);H)$. Then a vector function ψ such that $e^{-t/k}\psi(t) \in L_\infty((0,+\infty);H)$ is a solution of Eq. (8.1) if and only if $\tilde{\psi}(t) = (k+T)^{\frac{1}{2}}\cdot e^{-t/k}\psi(t)$ is a solution of Eq. (8.2) in $L_\infty((0,+\infty);H)$. Since, up to a trivial change of variable, the symbol of Eq. (8.2) coincides with $\tilde{W}$ (cf. Section 1) and this operator function has a left canonical factorization with respect to the imaginary axis, it is clear that the left-hand side of Eq. (8.2) defines a Fredholm operator on the space $L_\infty((0,+\infty);H)$, $I-\tilde{K}$ say (cf. Theorem II 3.1). Further, if $\kappa_1,\dots,\kappa_n$ are the left indices of $\tilde{W}$, then

$$(8.3) \qquad \dim \operatorname{Ker}(I-\tilde{K}) = -\sum_{\kappa_i<0}\kappa_i, \quad \operatorname{codim} \operatorname{Im}(I-\tilde{K}) = +\sum_{\kappa_i>0}\kappa_i$$

(cf. Theorem II 3.1). If $\omega \in L_\infty((0,+\infty);H)$, then, according to Theorem 5.1, Eq. (8.1) has at least one solution ψ such that $e^{-t/k}\psi(t) \in L_\infty((0,+\infty);H)$. Thus Eq. (8.2) has at least one solution $\tilde{\psi} \in L_\infty((0,+\infty);H)$ for every right-hand side $\tilde{\omega}$ such that $e^{+t/k}\tilde{\omega}(t) \in L_\infty((0,+\infty);H)$. Thus the set $\mathbb{D} = \{\omega \in L_\infty((0,+\infty);H) : e^{+t/k}\omega(t) \in L_\infty((0,\infty);H)\}$ is a dense linear subspace of $L_\infty((0,+\infty);H)$ contained in $\operatorname{Im}(I-\tilde{K})$. But then $\operatorname{Im}(I-\tilde{K}) = L_\infty((0,+\infty);H)$ and all left indices of $\tilde{W}$ are non-positive. Hence, all left indices of W (with respect to the circle C_k) are non-positive.

By Theorem 3.1 and IV 3.2 the solutions ψ of the homogeneous Wiener-Hopf equation (8.1) (with $\omega = 0$) for which $e^{-t/k}\psi(t) \in L_\infty((0,+\infty);H)$ are given by

$$(8.4)\qquad \psi(t) = \sum_{i=1}^{r} e^{-t/\lambda_i}\phi_{0i} - \sum_{i=1}^{r} e^{-tT^{-1}A}P\phi_{0i} \qquad (0 \le t < +\infty).$$

Here $\lambda_1 < \dots < \lambda_r < -k$ are the different eigenvalues of $A^{-1}T$ on $(-\infty,-k)$ and $(A^{-1}T-\lambda_i)\phi_{0i} = 0$ $(i = 1,\dots,r)$. Since $\phi_{0i} \in H_m$ $(i = 1,\dots,r)$ and $\mathrm{Ker}(I-P) \cap H_m = \{0\}$, there is a one-to-one correspondence between solutions of the form (8.4) and arrays of vectors $(\phi_{01},\dots,\phi_{0r})$. Hence, the dimension of the space of such solutions is the sum of the multiplicities of the eigenvalues $\lambda_1,\dots,\lambda_r$. By (8.3) the sum of the negative indices of W (with respect to the circle C_k) equals the opposite of the sum of the multiplicities of the eigenvalues of $A^{-1}T$ on $(-\infty,-k)$.

Finally, to finish the proof it suffices to show that all left indices of W exceed -2. Note that every solution ψ of the form (8.4) has the property that $\psi(t) \equiv 0$ whenever $\psi(0) = 0$. By the operator-valued version of Theorem 9.2 of [24] this implies that one cannot have indices equal to or less than -2. □

COROLLARY 8.2. *Let* (T,B) *be a positive definite admissible pair on* H, *and let* $k > 0$ *be a constant such that* $\sigma(T) \subset (-k,+\infty)$ *and* $-k$ *is not an eigenvalue of the associate operator* $S = A^{-1}T$. *Then for every right-hand side* $\omega \in L_\infty((0,+\infty);H)$ *the solutions* ψ *of the Wiener-Hopf operator integral equation* (8.1) *such that* $e^{-t/k}\psi(t) \in L_\infty((0,+\infty);H)$ *have the form*

$$\psi(t) = \sum_{i=1}^{r} e^{-t/\lambda_i}\phi_{0i} + \hat{\psi}(t), \qquad 0 < t < +\infty.$$

Here $\{\lambda_1,\dots,\lambda_r\} = \sigma(A^{-1}T) \cap (-\infty,-k)$, $(A^{-1}T-\lambda_i)\phi_{0i} = 0$ $(i = 1,\dots,r)$ *and* $\hat{\psi} \in L_\infty((0,+\infty);H)$.

This corollary is immediate from the proof of the previous theorem and describes the asymptotics of the solutions of Eq. (5.1). For the one-speed Transport Equation this corollary has been obtained before by FELDMAN [16], who derived it within the context of a general theory of asymptotics of solutions of Wiener-Hopf operator integral equations.

CHAPTER VI

APPLICATIONS TO TRANSPORT THEORY

In this chapter the theory developed in Chapters III to V is applied to Transport Theory. Both the one-speed and the symmetric multigroup Transport Equation are shown to fit into the present language of hermitian admissible pairs. As to one-speed theory more explicit formulas are derived The dispersion function is proved to have a canonical factorization. For the degenerate case formulas for the basic projections P and Q are improved and linked up with astrophysical theory. The structure of the singular subspace is described and the isotropic Milne problem is completely solved. Analytic solutions for the isotropic finite-slab problem are obtained.

1. Hermitian admissible pairs

Throughout this chapter, except in Section 7, we consider an integro-differential equation of the form

$$(1.1)\qquad \mu\frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) = \int_{-1}^{+1} g(\mu,\mu')\psi(x,\mu')d\mu' + f(x,\mu), \qquad 0 < x < \tau,$$

where

$$(1.2)\qquad g(\mu,\mu') = (2\pi)^{-1}\int_{0}^{2\pi} \hat{g}(\mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\cos\alpha)d\alpha, \qquad -1 \le \mu,\mu' \le +1.$$

Here $\hat{g}$ is the so-called *scattering indicatrix* or *phase function*, which is assumed to be given as is the inhomogeneous term $f(x,\mu)$. The problem is to compute the unknown function ψ under certain boundary conditions.

In the *finite-slab problem*, where τ is finite, one considers the boundary conditions

$$(1.3a)\qquad \psi(0,\mu) = \phi_+(\mu) \quad (0 \le \mu \le +1); \qquad \psi(\tau,\mu) = \phi_-(\mu) \quad (-1 \le \mu \le 0).$$

Here ϕ_+ and ϕ_- are given functions on [0,1] and [-1,0], respectively. In the *half-space problem*, where τ is infinite, a possible pair of boundary conditions is

$$(1.3b)\qquad \psi(0,\mu) = \phi_+(\mu) \quad (0 \le \mu \le +1); \quad \lim_{x\to+\infty} \psi(x,\mu) = 0 \quad (-1 \le \mu \le 0),$$

but at infinity the boundary condition is often replaced by a growth condition on the solution ψ.

For physical reasons the indicatrix $\hat{g}$ has to be nonnegative and $\int_{-1}^{+1} \hat{g}(t)dt < +\infty$. Here we assume that $\hat{g}$ is a real-valued function that belongs to $L_r[-1,+1]$ for some $r > 1$. For a nonnegative indicatrix $\hat{g}$ one distinguishes between the *non-conservative case* when $c = \int_{-1}^{+1} \hat{g}(t)dt < 1$, the *conservative case* when $c = \int_{-1}^{+1} \hat{g}(t)dt = 1$, and the *multiplying medium case* when $c = \int_{-1}^{+1} \hat{g}(t)dt > 1$. The latter case is relevant to neutron physics only and will not be considered here in detail.

As in the introduction we write Eq. (1.1) with boundary values as a vector equation. Thus one introduces the Hilbert space $H := L_2[-1,+1]$ of square integrable functions on the closed interval [-1,+1], the vectors $\psi(x) \in H$ and $f(x) \in H$, the operators $T: H \to H$ and $B: H \to H$, and the projections $P_+: H \to H$ and $P_-: H \to H$ by setting

$$(1.4a)\qquad \psi(x)(\mu) = \psi(x,\mu); \quad f(x)(\mu) = f(x,\mu); \qquad (0 < x < \tau;\ -1 \le \mu \le +1)$$

$$(1.4b)\qquad (Th)(\mu) = \mu h(\mu)\ ; \quad (Bh)(\mu) = \int_{-1}^{+1} g(\mu,\mu')h(\mu')d\mu';$$

$$(1.4c)\qquad (P_+h)(\mu) = \begin{cases} h(\mu), & 0 \le \mu \le +1; \\ 0, & -1 \le \mu < 0; \end{cases} \qquad (P_-h)(\mu) = \begin{cases} 0, & 0 \le \mu \le +1; \\ h(\mu), & -1 \le \mu < 0. \end{cases}$$

With the help of the above vectors and operators and the vectors $\phi_+ \in H_+ := L_2[0,1]$ and $\phi_- \in H_- := L_2[-1,0]$ one rewrites the integro-differential equation (1.1) with boundary conditions (1.3) as

$$(1.5)\qquad (T\psi)'(x) = -(I-B)\psi(x) + f(x), \quad 0 < x < \tau;$$

$$(1.6a)\qquad \lim_{x\downarrow 0} P_+\psi(x) = \phi_+, \quad \lim_{x\uparrow\tau} P_-\psi(x) = \phi_-;$$

$$(1.6b)\qquad \lim_{x\downarrow 0} P_+\psi(x) = \phi_+, \quad \lim_{x\to+\infty} P_-\psi(x) = 0.$$

Here (1.6a) and (1.6b) correspond to the boundary conditions (1.3a) and (1.3b) of the finite-slab and half-space problem, respectively. The second one of the boundary conditions (1.6b) of the half-space problem can be replaced by a growth condition on the norm $\|\psi(x)\|$ of $\psi(x)$ in the space $H = L_2[-1,+1]$ as $x \to +\infty$.

VLADIMIROV has proved that the operator B is a compact operator on th Hilbert space $H = L_2[-1,+1]$, even under the general condition that $\hat{g} \in L_1[-1,+1]$ (cf. [69], Appendix XII.8). If one denotes by p_n the L_2-normalized Legendre polynomial of degree n with positive leading coefficie i.e., the polynomial

$$(1.7)\qquad p_n(\mu) = \sqrt{n+\tfrac{1}{2}}\,(2^n\cdot n!)^{-1}\left(\frac{d}{d\mu}\right)^n(\mu^2-1)^n, \quad n \ge 0,$$

then, as Vladimirov shows, $Bp_n = a_n p_n$ $(n = 0,1,2,\dots)$, where

$$(1.8)\qquad a_n = (n+\tfrac{1}{2})^{-\frac{1}{2}} \int_{-1}^{+1} \hat{g}(t)p_n(t)dt, \quad n = 0,1,2,\dots\ .$$

In fact, the polynomials $(p_n)_{n=0}^{+\infty}$ form a complete set of eigenfunctions of B and

$$(1.9)\qquad (Bh)(\mu) = \sum_{n=0}^{+\infty} a_n p_n(\mu) \int_{-1}^{+1} h(\mu')p_n(\mu')d\mu', \quad -1 \le \mu \le +1.$$

For $n \ge 0$ the polynomial is related to the usual Legendre polynomial P_n (which is used, for instance, in [49,68]) by $p_n = \sqrt{n+\tfrac{1}{2}}\,P_n$. By Theorem 7.3.3 of [68] we have the basic estimate $(\sin\theta)^{\frac{1}{2}}|P_n(\cos\theta)| \le (2(n\pi)^{-1})^{\frac{1}{2}}$ $(0 < \theta < \pi)$, and therefore

$$(1.10)\qquad |p_n(\mu)| \le \left(\frac{3}{\pi}\right)^{\frac{1}{2}}(1-\mu^2)^{-\frac{1}{4}}, \quad -1 < \mu < +1.$$

Finally, the Legendre polynomials $(p_n)_{n=0}^{+\infty}$ form an orthonormal basis of $L_2[-1,+1]$.

The operator T defined by (1.4b) is a self-adjoint operator on $H = L_2[-1,+1]$ with spectrum [-1,+1] and empty eigenvalue spectrum. A cyclic vector of T is given by $e(\mu) \equiv 1$ (note that $e = \sqrt{2}\,p_0$) and the polynomials $(p_n)_{n=0}^{+\infty}$ can be retrieved by applying the Gramm-Schmidt orthonormalization algorithm to the set $(T^n e)_{n=0}^{+\infty}$ (cf. [68]).

The next two theorems will allow us to apply the approach of Chapters III to V to the present one-speed Transport Equation.

<u>THEOREM 1.1</u>. *Suppose that for some* $r > 1$ *the indicatrix* $\hat{g}$ *is real-valued and belongs to* $L_r[-1,+1]$. *Then the pair* (T,B) *of operators* T *and* B, *defined by*

$$(Th)(\mu) = \mu h(\mu),$$

$$(Bh)(\mu) = \int_{-1}^{+1} \left[(2\pi)^{-1} \int_0^{2\pi} \hat{g}(\mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\,\cos\alpha)\,d\alpha \right] h(\mu')\,d\mu',$$

is a hermitian admissible pair on $L_2[-1,+1]$. *This pair is semi-definite (positive definite) if and only if for* $n = 0,1,2,\ldots$ *the numbers* $a_n = (n+\frac{1}{2})^{-\frac{1}{2}} \cdot \int_{-1}^{+1} \hat{g}(t)p_n(t)\,dt$ *do not exceed* $+1$ *(resp. are strictly less than* $+1$*). Further, the signature operator* J, *defined by*

$$(1.11)\qquad (Jh)(\mu) = h(-\mu), \quad -1 \le \mu \le +1,$$

is an inversion symmetry of the pair (T,B).

<u>THEOREM 1.2</u>. *For some* $p > 2$ *let the vector function* $f\colon (0,\tau) \to L_p[-1,+1]$ *be continuous. Then for every* $0 < \alpha < (p-2)(2p)^{-1}$ *there exists a vector function* $h\colon (0,\tau) \to L_2[-1,+1]$ *such that*

$$f(t) = |T|^{\alpha}\, h(t), \quad 0 < t < \tau.$$

If f *is a bounded vector function, then the function* h *is bounded too.*

<u>PROOF of Theorem 1.1</u>. We shall prove that the pair (T,B) satisfies Conditions (C.1) - (C.3) stated in Section III.2. By the properties of T and B we have derived so far Conditions (C.1) and (C.2) are trivially fulfilled. To settle (C.3) we shall prove that for every $0 < \alpha < (r-1)(2r)^{-1}$ there exists a compact operator $D_\alpha\colon L_2[-1,+1] \to L_2[-1,+1]$ such that $B = |T|^{\alpha}D_\alpha$. Here $|T|^{\alpha}$ is, of course, given by $(|T|^{\alpha}h)(\mu) = |\mu|^{\alpha}h(\mu)$ $(-1 \le \mu \le +1)$.

By Theorem 1(2.X) of [44] the operator B acts as a bounded linear operator from $L_2[-1,+1]$ into $L_{2r}[-1,+1]$. Denoting this operator by $B_r\colon L_2[-1,+1] \to L_{2r}[-1,+1]$, we have the norm estimate

$$(1.12)\qquad \|B_r\| \le c_r \|\hat{g}\|_r,$$

where c_r depends on r only. By Theorem 3(2.X) of [44] the operator B_r is compact.

By Hölder's inequality the operator S_α, defined by

$$(1.13)\qquad (S_\alpha k)(\mu) = |\mu|^{-\alpha}\, k(\mu) \quad (-1 \le \mu \le +1,\ \text{almost everywhere}),$$

acts as a bounded linear operator from $L_{2r}[-1,+1]$ into $L_2[-1,+1]$, provide[d] $0 < \alpha < (r-1)(2r)^{-1}$. Put $D_\alpha = S_\alpha B_r\colon L_2[-1,+1] \to L_2[-1,+1]$, $0 < \alpha < (r-1)(2r)^{-1}$. Then D_α is a compact operator, $B = |T|^{\alpha}D_\alpha$ and Condition (C[.3]) is established.

Finally, put $A = I - B$. Then the operator A is positive (strictly positive) if and only if $\sigma(B) = \overline{\{a_n : n \ge 0\}}$ is contained in $(-\infty,+1]$ $((-\infty,+1))$. From this one easily deduces the second part of this theorem. The third part about inversion symmetry is immediate from (1.2). □

<u>PROOF of Theorem 1.2</u>. For $0 < \alpha < (p-2)(2p)^{-1}$ the operator S_α defined by (1.13) acts as a bounded linear operator from $L_p[-1,+1]$ into $L_2[-1,+1]$. F[or] a fixed $0 < \alpha < (p-2)(2p)^{-1}$ we put $h(t) = S_\alpha f(t) \in L_2[-1,+1]$ $(0 < t < \tau)$. Now this theorem is easily proved. □

The proof of Theorem 1.1 does not give the least upper bound of all $0 < \alpha < 1$ for which $B = |T|^{\alpha}D$ for some bounded operator D. For instance, [if] $\hat{g} \in L_2[-1,+1]$, then the theorem yields $B = |T|^{\alpha}D_\alpha$ for all $0 < \alpha < \frac{1}{4}$. In fact, a much sharper result holds.

<u>PROPOSITION 1.3</u>. *Let the pair* (T,B) *be as in Theorem 1.1. If the indicatr[ix]* $\hat{g} \in L_2[-1,+1]$, *then for every* $0 < \alpha < \frac{1}{2}$ *one has* $B = |T|^{\alpha}D_\alpha$, *where* D_α *is a Hilbert-Schmidt operator. If, in addition,* B *is a trace class operator, th[en] for* $0 < \alpha < \frac{1}{2}$ *the operator* D_α *is of the same type.*

<u>PROOF</u>. For $0 < \alpha < \frac{1}{2}$ put

$$q_n^{(\alpha)}(\mu) = |\mu|^{-\alpha}\, p_n(\mu) \quad (-1 \le \mu \le +1,\ n = 0,1,2,\ldots).$$

Using (1.10) one sees that

$$\|q_n^{(\alpha)}\|_2^2 = \int_{-1}^{+1} |\mu|^{-2\alpha}|p_n(\mu)|^2\,d\mu \le \frac{6}{\pi}\int_0^1 |\mu|^{-2\alpha}(1-\mu^2)^{-\frac{1}{2}}\,d\mu = \frac{3}{\pi}B(\tfrac{1}{2}-\alpha,\tfrac{1}{2}) < +\infty,$$

where B denotes Binet's beta-function (cf. [49], p.7), and therefore the sequence $(q_n^{(\alpha)})_{n=0}^{+\infty}$ is bounded in $L_2[-1,+1]$. Hence, formally we may write

$$(D_\alpha h)(\mu) = \sum_{n=0}^{+\infty} a_n q_n^{(\alpha)}(\mu) \int_{-1}^{+1} h(\mu')p_n(\mu')d\mu', \quad 0 < \alpha < \tfrac{1}{2}.$$

If B is a trace class operator, then its nuclear norm equals $\|B\|_1 = \sum_{n=0}^{+\infty} |a_n| < +\infty$. Then $\sum_{n=0}^{+\infty} |a_n| \|q_n^{(\alpha)}\| \|p_n\| \le [3\pi^{-1}B(\tfrac{1}{2}-\alpha,\tfrac{1}{2})]^{\frac{1}{2}} \cdot \|B\|_1 < +\infty$, and therefore D_α is a trace class operator $(0 < \alpha < \tfrac{1}{2})$.

If B is a Hilbert-Schmidt operator, then its Hilbert-Schmidt norm equals $\|B\|_2 = (\sum_{n=0}^{+\infty} |a_n|^2)^{\frac{1}{2}} = \|\hat{g}\|_2 < +\infty$. Now the operator D_α is, formally speaking, an integral operator on $L_2[-1,+1]$ with kernel $|\mu|^{-\alpha} g(\mu,\mu') = \sum_{n=0}^{+\infty} a_n q_n^{(\alpha)}(\mu)p_n(\mu')$, where $g(\mu,\mu') = \sum_{n=0}^{+\infty} a_n p_n(\mu)p_n(\mu')$. Since $(p_n)_{n=0}^{+\infty}$ is an orthonormal basis of $L_2[-1,+1]$, one easily estimates that

$$\int_{-1}^{+1}\int_{-1}^{+1} |\mu|^{-2\alpha}|g(\mu,\mu')|^2 d\mu d\mu' = \sum_{n=0}^{+\infty} |a_n|^2 \|q_n^{(\alpha)}\|^2 \le$$

$$\le \frac{3}{\pi} B(\tfrac{1}{2}-\alpha,\tfrac{1}{2})\|B\|_2^2 < +\infty,$$

and therefore D_α is a Hilbert-Schmidt operator, indeed. □

Because of Theorems 1.1 and 1.2 the theory of Chapters III to V is applicable to one-speed Transport Theory. The only, probably quite essential, restriction is that the indicatrix $\hat{g} \in L_r[-1,+1]$ for some $r > 1$. In the remaining sections we shall specify our results further. In particular, we shall provide explicit expressions for the projection P that appears in the solution formula for the half-space problem.

Proposition 1.3 is of some independent interest. It answers a question raised implicitly in [39]. In [2] the operator $I - V$, which is studied here in Section III.5, has been shown to be a trace class operator, if the indicatrix $\hat{g}$ is degenerate (i.e., $a_n = 0$ for $n \ge N+1$). Subsequently, assuming that $\sum_{n=0}^{+\infty} |a_n| < +\infty$, HANGELBROEK stated $I - V$ to be a Hilbert-Schmidt operator and asked whether $I - V$ would be trace class (cf. [39]). This question is answered in the affirmative by Proposition 1.3. To see this we refer to the paragraph following the proof of Lemma III 5.3.

We conclude this section with a remark about the integro-differential equation (1.1). This equation is a simplified (in astrophysical terms, the azimuth-averaged) version of the plane-symmetric equation

$$(1.14) \qquad (\cos\theta)\frac{\partial\psi}{\partial x}(x,\omega) + \psi(x,\omega) = (2\pi)^{-1}\int_\Omega \hat{g}(\omega\cdot\omega')\psi(x,\omega')d\omega' + f(x,\omega),$$

$$0 < x < \tau,$$

where Ω is the unit sphere in $\mathbb{R}^3$ and $\omega = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \in$ Instead of in $L_2[-1,+1]$ this equation is usually studied in the Hilbert space $L_2(\Omega)$, endowed with the inner product $\langle h_1,h_2\rangle = (2\pi)^{-1}\int_\Omega h_1(\omega')\overline{h_2(\omega')}$ Consider on $L_2(\Omega)$ the operators $\hat{T}$ and $\hat{B}$ defined by

$$(1.15) \qquad (\hat{T}h)(\omega) = (\cos\theta)h(\omega), \quad (\hat{B}h)(\omega) = (2\pi)^{-1}\int_\Omega \hat{g}(\omega\cdot\omega')h(\omega')d\omega'.$$

It turns out that again this pair $(\hat{T},\hat{B})$ is a hermitian admissible pair on $L_2(\Omega)$, whenever the indicatrix $\hat{g}$ belongs to $L_r[-1,+1]$ for some $r > 1$.

It is interesting to observe that the pair (T,B), defined by (1.4b), may be viewed as a "restriction" of the pair $(\hat{T},\hat{B})$ given by (1.15). Indee let $j: L_2[-1,+1] \to L_2(\Omega)$ be defined by $(jh)(\omega) = h(\cos\theta)$. Then j is an isometric embedding and

$$\hat{T}j = jT, \quad \hat{B}j = jB.$$

Note that the image of j consists of all $h \in L_2(\Omega)$ that depend on θ only. Eq. (1.1) may be viewed as the one-dimensional version of Eq. (1.14).

In this chapter we only deal with Eq. (1.1). However, in the work of MASLENNIKOV & FELDMAN (cf. [50],[15] to [19]) its counterpart (1.14) is tl object of study, but as far as their (abstract) results are concerned, th differences do not seem to be essential.

2. The dispersion function

Throughout this section we assume that the expansion coefficients $a_n = \int_{-1}^{+1} \hat{g}(t)P_n(t)dt$ of the indicatrix $\hat{g}$ satisfy the relations

$$(2.1) \qquad a_n < +1 \ (n = 0,1,2,\ldots); \quad \sum_{n=0}^{+\infty} |a_n| < +\infty.$$

Then for this indicatrix the pair (T,B) in (1.4b) is a positive definite admissible pair on $L_2[-1,+1]$, the operator B is trace class and its nuclear norm $\|B\|_1$ is given by the infinite sum appearing in (2.1). In this section for such an indicatrix the dispersion function is defined and its main properties are investigated.

Let W denote the symbol of the pair (T,B). According to Proposition 1.3, for every $0 < \alpha < \frac{1}{2}$ one has $B = |T|^{\alpha} D_{\alpha}$ for some trace-class operator D_{α}. Using (the proof of) Proposition III 2.1 one sees that on the extended imaginary line the symbol W is a Hölder continuous function of exponent $0 < \alpha < \frac{1}{2}$. In fact, one has

$$(2.2) \qquad |\lambda|^{-\alpha} \| W(\lambda) - I \|_1 \to 0 \quad (\lambda \to 0,\ \mathrm{Re}\,\lambda = 0;\ 0 < \alpha < \tfrac{1}{2}),$$

where the limit is taken in the nuclear norm.

Since B is a trace-class operator, the expression

$$(2.3) \qquad \Lambda(\lambda) = \det W(\lambda) = \det[I - \lambda(\lambda - T)^{-1} B]$$

is well-defined for all $\lambda \in \mathbb{C}_{\infty}\backslash[-1,+1]$. The function Λ is called the *dispersion function*. By Section IV 1.8 of [25] Λ is analytic on the Riemann sphere cut along $[-1,+1]$. Putting $\Lambda(0) = 1$ one sees from (2.2) that Λ is Hölder continuous of exponent α on the extended imaginary line ($0 < \alpha < \frac{1}{2}$) (see [25], Corollary IV 1.1). Further, for $\lambda \in \mathbb{C}_{\infty}\backslash[-1,+1]$ we have

$$(2.4) \qquad \overline{\Lambda(\bar{\lambda})} = \det W(\bar{\lambda})^{*} = \det[I - \lambda B(\lambda - T)^{-1}] =$$

$$= \det[I - \lambda(\lambda - T)^{-1} B] = \Lambda(\lambda),$$

and therefore $\Lambda(\lambda) \in \mathbb{R}$ for $\lambda \in \mathbb{R}\backslash[-1,+1]$. By the inversion symmetry of the pair (T,B) (see Theorem 1.1) a straightforward application of formula (III 6.4) yields

$$(2.5) \qquad \Lambda(\lambda) = \Lambda(-\lambda), \qquad \lambda \in \mathbb{C}_{\infty}\backslash[-1,+1].$$

So Λ is real-valued on the imaginary axis as well.

Since for a trace class operator K the operator $I - K$ is invertible if and only if $\det(I - K) \neq 0$ (cf. [62], Theorem 3.3.13), it is clear that

$$\sigma(A^{-1}T)\backslash[-1,+1] = \{\lambda \in \mathbb{C}_{\infty}\backslash[-1,+1] : W(\lambda) \text{ is not invertible}\} =$$

$$= \{\lambda \in \mathbb{C}_{\infty}\backslash[-1,+1] : \Lambda(\lambda) = 0\}.$$

From (2.4) and (2.5) one sees that $\Lambda(\lambda) = \overline{\Lambda(-\bar{\lambda})}$. Since Λ is analytic on $\mathbb{C}_{\infty}\backslash[-1,+1]$, the Cauchy-Schwarz reflection principle yields that $\Lambda(\lambda)$ is real for imaginary λ. Since $\Lambda(0) = 1$, $\Lambda(\infty) = \det A > 0$ and Λ is continuou on the extended imaginary line, a simple argument involving connectedness shows that $\Lambda(\lambda)$ is strictly positive on the extended imaginary line.

<u>THEOREM 2.1</u>. *The dispersion function* Λ *has a canonical factorization of the form*

$$(2.6) \qquad \Lambda(\lambda) = H(\lambda)^{-1} H(-\lambda)^{-1}, \qquad \mathrm{Re}\,\lambda = 0,$$

where the function H *is given by*

$$(2.7) \qquad H(\lambda) = \exp[-(2\pi i)^{-1} \int_{-i\infty}^{+i\infty} (t - \lambda)^{-1} \log \Lambda(t)\,dt], \qquad \mathrm{Re}\,\lambda > 0,$$

and has continuous and non-zero boundary values up to the imaginary line. In particular, the function H^{-1} *has an analytic continuation to* $\mathbb{C}_{\infty}\backslash[-1,0]$.

<u>PROOF</u>. Take the following properties into account: for $0 < \alpha < \frac{1}{2}$ the dispersion function Λ is Hölder continuous of exponent α on the extended imaginary line; on the same line Λ is positive definite. The theorem is no clear from a well-known result of MUSKHELISHVILI [59] (see also [61], Theorem III 4.1). □

Another proof of the factorization (2.6) appeared in [40], but this proof can be simplified with the help of Proposition 1.3, which shows $I - V$ to be a trace class rather than a Hilbert-Schmidt operator. Still another proof can be based on Theorem V 7.2. However, none of these proofs produces Plemelj's formula (2.7).

For a degenerate indicatrix (i.e., $a_n = 0$ for $n \geq N+1$) the next proposition provides explicit formulas for the dispersion function Λ.

<u>PROPOSITION 2.2</u>. *Let* $H_0, H_1, H_2, \ldots$ *be the sequence of polynomials defined by the following recurrence relation:*

$$(2.8a) \qquad (2\ell+1)(1-a_{\ell})\mu H_{\ell}(\mu) = (\ell+1)H_{\ell+1}(\mu) + \ell H_{\ell-1}(\mu);$$

$$(2.8b) \qquad H_{-1}(\mu) \equiv 0, \quad H_0(\mu) \equiv 1, \quad H_1(\mu) = (1-a_0)\mu.$$

Then for a degenerate indicatrix (i.e., $a_n = 0$ *for* $n \geq N+1$*) the dispersion function is given by either one of the following expressions:*

$$\Lambda(\lambda) = 1 - a_0 + \sum_{\ell=0}^{N} a_\ell(\ell+\tfrac{1}{2})H_\ell(\lambda)\int_{-1}^{+1}\frac{\mu P_\ell(\mu)}{\mu-\lambda}\,d\mu, \quad \lambda \in \mathbb{C}_\infty\backslash[-1,+1]; \tag{2.9}$$

$$\Lambda(\lambda) = 1 + \lambda\int_{-1}^{+1}(\mu-\lambda)^{-1}\Psi(\mu)\,d\mu, \quad \lambda \in \mathbb{C}_\infty\backslash[-1,+1], \tag{2.10}$$

where Ψ *is the even real continuous function on* $[-1,+1]$ *given by*

$$\Psi(\mu) = \sum_{\ell=0}^{N} a_\ell(\ell+\tfrac{1}{2})P_\ell(\mu)H_\ell(\mu). \tag{2.11}$$

PROOF. Formula (2.9) has been derived by HANGELBROEK (cf. formula (13) of [37]; formula (23) of [38]). Actually the result has been stated for $0 < a_0 < 1$ and $|a_n| \le a_0$ $(n = 1,2,\dots,N)$, but the proof can be generalized to the case when $a_n < 1$ $(n = 0,1,\dots,N)$. Another proof has appeared in [48] (namely, Proposition 1 and Eq. (3.11)).

Next, one writes

$$\Lambda(\lambda) = 1 - a_0 + \sum_{\ell=0}^{N} a_\ell(\ell+\tfrac{1}{2})H_\ell(\lambda)\left[\int_{-1}^{+1}P_\ell(\mu)\,d\mu + \lambda\int_{-1}^{+1}\frac{P_\ell(\mu)}{\mu-\lambda}\,d\mu\right] =$$

$$= 1 - a_0 + a_0 + \lambda\sum_{\ell=0}^{N} a_\ell(\ell+\tfrac{1}{2})\left[\int_{-1}^{+1}\frac{H_\ell(\mu)P_\ell(\mu)}{\mu-\lambda}\,d\mu - \int_{-1}^{+1}\frac{H_\ell(\mu)-H_\ell(\lambda)}{\mu-\lambda}P_\ell(\mu)\,d\mu\right].$$

From this expression formula (2.10) is clear, because for a polynomial h of degree $\le \ell-1$ one has $\int_{-1}^{+1} h(\mu)P_\ell(\mu)\,d\mu = 0$. □

The functions Λ, H and Ψ and the polynomials $(H_\ell)_{\ell=0}^{+\infty}$ appeared probably for the first time in the work of AMBARTSUMIAN [1]. In astrophysics the equation $\Lambda(\lambda) = 0$ is called the *characteristic equation*, its zeros *characteristic roots*, the function H the *H-function* and the function Ψ the *characteristic function* (cf. [10,65]). Originally Ambartsumian has introduced the H-functions as solutions of the so-called H-equation

$$\frac{1}{H(\mu)} = 1 - \mu\int_0^1\frac{\Psi(\nu)H(\nu)}{\nu+\mu}\,d\nu, \quad 0 < \mu \le +1 \tag{2.12}$$

(cf. [1]). To see that for a degenerate indicatrix the function H in (2.7) satisfies Eq. (2.12), indeed, one considers the function

$$\phi(\lambda) = H(\lambda)\Lambda(\lambda) - \lambda\int_0^1\frac{\Psi(\nu)H(\nu)}{\nu-\lambda}\,d\nu. \tag{2.13}$$

By (2.6) the function ϕ is analytic on $\mathbb{C}_\infty\backslash[0,1]$. Next one inserts (2.10) into (2.13) and gets

$$\phi(\lambda) = H(\lambda) + \lambda H(\lambda)\int_{-1}^{0}\frac{\Psi(\nu)\,d\nu}{\nu-\lambda} - \lambda\int_0^1\Psi(\nu)\,\frac{H(\lambda)-H(\nu)}{\lambda-\nu}\,d\nu, \ \lambda \notin [-1, \tag{2.14}$$

But the latter expression yields an analytic continuation of ϕ to $\mathbb{C}_\infty\backslash[-1,$ This is clear, because Ψ is continuous on [0,1] and H is analytic and uni formly Hölder continuous up to the boundary of the open right half-plane. Thus ϕ has an analytic continuation to $\mathbb{C}_\infty\backslash\{0\}$. Respectively from (2.13) a (2.14) one gets

$$\lim_{\lambda\to 0,\,\mathrm{Re}\,\lambda\le 0}\phi(\lambda) = H(0)\Lambda(0) = 1, \quad \lim_{\lambda\to 0,\,\mathrm{Re}\,\lambda\ge 0}\phi(\lambda) = H(0) = 1.$$

But then Liouville's theorem applies and $\phi(\lambda) \equiv 1$, $\lambda \in \mathbb{C}_\infty$. Taking $\lambda = -\mu$ in (2.13) the H-equation (2.12) is immediate. Now the connection between the H-functions appearing in Theorem 2.1 and the H-functions studied in astrophysics is made explicit.

After its appearance in Ambartsumian's work [1], CHANDRASEKHAR [10] has studied the H-equation in detail and has employed it frequently. Late the functions Λ, Ψ, H and the polynomials $(H_\ell)_{\ell=0}^{+\infty}$ turned up in neutron Transport Theory, often under different names and notations (cf. [54,9], f instance). The term "dispersion function" and its notation by Λ are commo ly used by neutron physicists.

The connection between solutions of the H-equation (2.12) and the factorization of the dispersion function in the form (2.6) has been obser ed probably for the first time by CRUM [11] (see also Section 40 of [10]). For degenerate (and to some degree also for non-degenerate) indicatrices such that the characteristic function Ψ is nonnegative, the complete solution of the H-equation (2.12) has been found by BUSBRIDGE [8]. The method used by her is mathematically rigorous. Later it appeared that the conditi that Ψ is nonnegative can be dropped (cf. [58], Theorem 2).

For the isotropic case and for the degenerate anisotropic case HANGELBROEK identified the dispersion function as a determinant (see [35,36; 37,38]; also [48] and Proposition 1 of [40]). For the non-degenera case the Hölder continuity of the dispersion function seems to be new. Her for the first time this property is exploited to get the factorization (2.

3. The projections P and Q

Throughout this section the indicatrix $\hat{g}$ will be degenerate and non-conservative, i.e.,

$$\hat{g}(t) = \sum_{n=0}^{N} a_n(n+\tfrac{1}{2})P_n(t) \quad (-1 \le t \le +1);\ 0 \le a_0 < 1,\ |a_n| \le a_0 \quad (n = 1,2,\dots,N). \tag{3.1}$$

Then the corresponding pair of operators (T,B) (see (1.4b)) is a positive definite admissible pair on $L_2[-1,+1]$. According to Theorem III 5.1 we have the decompositions

$$H_p \oplus H_- = H_m \oplus H_+ = L_2[-1,+1],$$

where $H_+ = L_2[0,1]$ and $H_- = L_2[-1,0]$. In this section explicit formulas are deduced for the projection P of $L_2[-1,+1]$ onto H_p along H_- and these formulas are related to astrophysical theory (for instance, [10,8]). Analogous formulas may be derived for the projection Q of $L_2[-1,+1]$ onto H_m along H_+. In fact, since Q = JPJ, where J is the inversion symmetry (1.11), any formula for the projection P can be transformed into one for Q and conversely.

In this section H will stand for the H-function and $H_0,H_1,H_2,\dots$ for the polynomials defined by (2.10). By definition

$$\Psi(\nu,\mu) = \sum_{k=0}^{N} a_k(k+\tfrac{1}{2})H_k(\nu)P_k(\mu).$$

This function is called the *characteristic binomial.*

<u>THEOREM 3.1</u>. *For every* $h \in L_2[-1,+1]$ *one has*

$$(Ph)(\mu) = \begin{cases} \sum_{k=0}^{N} a_k(k+\tfrac{1}{2})(-1)^k q_k(-\mu)H(-\mu) \int_0^1 \nu(\nu-\mu)^{-1} q_k(\nu)H(\nu)h(\nu)d\nu; \\ \qquad -1 \le \mu < 0; \\ h(\mu), \quad 0 \le \mu \le +1. \end{cases} \tag{3.2}$$

Here $q_0,q_1,\dots,q_N$ *are certain polynomials of degree* $\le N$*, which are the unique polynomial solutions of the equation*

$$q_m(\lambda) = H_m(\lambda) + \lambda \int_0^1 \frac{\Psi(\lambda,\mu)q_m(\mu) - \Psi(\mu,\mu)q_m(\lambda)}{\mu-\lambda} H(\mu)d\mu; \quad \lambda \notin [0,1] \tag{3.3}$$

<u>THEOREM 3.2</u>. *For every* $h \in L_2[-1,+1]$ *one has*

$$(Ph)(\mu) = \begin{cases} \sum_{k=0}^{N} a_k(1-a_k)^{-1}(k+\tfrac{1}{2})(-1)^k r_k(-\mu)H(-\mu)\ \cdot \\ \cdot \int_0^1 \nu(\nu-\mu)^{-1} r_k(\nu)H(\nu)h(\nu)d\nu; \quad (-1 \le \mu < 0), \\ h(\mu), \quad 0 \le \mu \le +1. \end{cases} \tag{3.4}$$

Here $r_0,r_1,\dots,r_N$ *are certain polynomials of degree* $\le N$*, which are the unique polynomial solutions of the equation*

$$\left\{\prod_{k=0}^{N}(1-a_k)\right\}^{\frac{1}{2}} r_m(\lambda) = (1-a_m)H_m(\lambda) + \int_0^1 \mu \frac{\Psi(\lambda,\mu)r_m(\mu) - \Psi(\mu,\mu)r_m(\lambda)}{\mu-\lambda} H(\mu)d\mu; \quad \lambda \notin [0,1] \tag{3.5}$$

To prove the above theorems we shall employ a diagonalization of A^{-1}. The following proposition is essentially known (see [38], Eqs (30) and (3 [48], Theorems 2 and 5; as to formulas (3.8) - (3.9), see [54], Eq. (2.10)

<u>PROPOSITION 3.3</u>. *Put* $I = [-1,+1]$ *and* $N = \sigma(A^{-1}T)$. *Then there exists a finite Borel measure* σ *on* N *and an invertible operator* F *from* $L_2(I)$ *onto the Hilbert space* $L_2(N)_\sigma$ *of* σ*-square-integrable functions on* N *with the following properties:*

$$(FA^{-1}Th)(\nu) = \nu(Fh)(\nu); \quad \nu \in N,\ h \in L_2(I); \tag{3.7a}$$

$$\int_N (Fh_1)(\nu)\overline{(Fh_2)(\nu)}d\sigma(\nu) = \langle Ah_1,h_2\rangle; \quad h_1,h_2 \in L_2(I); \tag{3.7b}$$

$$(FP_n)(\nu) = (1-a_n)H_n(\nu); \quad \nu \in N,\ n = 0,1,2,\dots . \tag{3.7c}$$

In terms of a Cauchy principal value one has

$$(Fh)(\nu) = \lambda(\nu)h(\nu) - \fint_{-1}^{1} \mu(\mu-\nu)^{-1}\Psi(\nu,\mu)h(\mu)d\mu; \quad \nu \in N,\ h \in L_2(I) \tag{3.8}$$

where $\lambda(\nu)$ *can be expressed in terms of the dispersion function* Λ *by*

$$\text{(3.9)} \qquad \lambda(\nu) = \lim_{\varepsilon\downarrow 0} \tfrac{1}{2}\{\Lambda(\nu+i\varepsilon)+\Lambda(\nu-i\varepsilon)\} = $$

$$= 1 - a_0 + \fint_{-1}^{+1} \mu(\mu-\nu)^{-1}\Psi(\nu,\mu)\,d\mu.$$

The operator F *maps the functions on* I *that are Hölder continuous of exponent* $0 < \alpha < 1$ *except for a possible jump discontinuity at the point* 0, *onto functions on* N *of the same type.*

PROOF. Except for formula (3.8) and some notational differences the present proposition is well-known and the methods to prove it can be found in [38] and [48]. The existence of the measure σ and the operator F such that (3.7a) and (3.7b) hold true is clear, because $h(\mu) \equiv 1-a_0$ is a cyclic vector of $A^{-1}T$. By the recurrence relations for the Legendre polynomials and the polynomials $(H_n)_{n=0}^{+\infty}$ (cf. (2.10)) and the identity $AP_n = (1-a_n)P_n$ $(n = 0,1,2,\dots)$ one gets (3.7c). As in [38] one computes $[F(T-\lambda)^{-1}P_0](\nu)$ and from the result of this calculation one derives the identity

$$\text{(3.10)} \qquad (Fh)(\nu) = (1-a_0)h(\nu) - \sum_{k=0}^{N} a_k(k+\tfrac{1}{2})H_k(\nu) \int_{-1}^{+1} \mu P_k(\mu)\frac{h(\mu)-h(\nu)}{\mu-\nu}\,d\mu, \quad \nu\in N$$

(cf. Eq. (30) of [38]; note the differences caused by the use of another cyclic vector). For a polynomial h the following expression is bounded for $\mu \neq \nu \in N$ and $\varepsilon > 0$:

$$\frac{h(\mu)-h(\nu)}{\mu-\nu} - \tfrac{1}{2}\left\{\frac{h(\mu)-h(\nu)}{\mu-\nu-i\varepsilon} + \frac{h(\mu)-h(\nu)}{\mu-\nu+i\varepsilon}\right\} = \frac{\varepsilon^2}{(\mu-\nu)^2+\varepsilon^2}\,\frac{h(\mu)-h(\nu)}{\mu-\nu},$$

and thus the application below of the theorem of dominated convergence is justified and yields

$$\int_{-1}^{+1} \mu P_k(\mu)\frac{h(\mu)-h(\nu)}{\mu-\nu}d\mu = \lim_{\varepsilon\downarrow 0}\int_{-1}^{+1} \mu P_k(\mu)\tfrac{1}{2}\left\{\frac{h(\mu)-h(\nu)}{\mu-\nu-i\varepsilon} + \frac{h(\mu)-h(\nu)}{\mu-\nu+i\varepsilon}\right\}d\mu, \ \nu\in N.$$

Inserting Cauchy's principal value one gets

$$(Fh)(\nu) = (1-a_0)h(\nu) - \sum_{k=0}^{N} a_k(k+\tfrac{1}{2})H_k(\nu) \fint_{-1}^{+1} \mu P_k(\mu)\frac{h(\mu)-h(\nu)}{\mu-\nu}d\mu, \quad \nu\in N.$$

On $L_2[-1,+1]$ one considers the singular integral operators S_+ and S_-, which are given by

$$\text{(3.11)} \qquad (S_\pm h)(\nu) = \lim_{\varepsilon\downarrow 0} (\pi i)^{-1}\int_{-1}^{+1} (\mu-\nu\mp i\varepsilon)^{-1}h(\mu)\,d\mu, \quad \nu\in[-1,+1].$$

These operators are well-defined and bounded (cf. [61], Section 3.4.4). Using (3.9) one gets

$$\text{(3.12)} \qquad (Fh)(\nu) = \lambda(\nu)h(\nu) - \sum_{k=0}^{N} a_k(k+\tfrac{1}{2})H_k(\nu) \fint_{-1}^{+1} \mu(\mu-\nu)^{-1}P_k(\mu)h(\mu)\,d\mu, \ \nu$$

Next we employ that the set $N\backslash I$ is finite and the measure σ is abso lutely continuous on I with bounded Radon-Nikodym derivative

$$\text{(3.13)} \qquad (d\sigma/d\nu) = [\lambda^2(\nu) + \pi^2\nu^2\Psi(\nu)^2]^{-1} \qquad (-1 < \nu < +1)$$

(see [37], Theorem 3 and Lemma 3(iii); [38], Theorem 4 and Lemma 3.3(iii) [48], Theorem 3 and Eq. (3.12)). So the Hilbert space $L_2(I)$ can be embed in a natural and continuous way into $L_2(N)_\sigma$. Since the operators S_+ and S in (3.11) are bounded on $L_2(I)$, the sum at the right-hand side of (3.12) represents a bounded linear operator on $L_2(I)$. Because $\lambda(\nu)^2(d\sigma/d\nu) \le 1$, the multiplication by $\lambda(\nu)$ acts as a bounded linear operator from $L_2(I)$ into $L_2(N)_\sigma$. Hence, both sides of (3.12) represent bounded linear operat from $L_2(I)$ into $L_2(N)_\sigma$. Since the polynomials are dense in $L_2(I)$, formula (3.8) is immediate. The last statement of the theorem follows from Theore 5 of [48]. □

In a similar way one constructs an invertible operator $F^\dagger : L_2(I) \to L_2(N)_\sigma$ that satisfies $F^\dagger TA^{-1} = T_N F^\dagger$ rather than (3.7a). Here T_N is the multiplication operator on $L_2(N)_\sigma$. Clearly, $F^\dagger = FA^{-1}$ and thus $F^\dagger T = FA^{-1}T = T_N F$ (cf. (3.7a)). So the operator $F^\dagger$ is given by

$$\text{(3.14)} \qquad (F^\dagger h)(\nu) = \lambda(\nu)h(\nu) - \fint_{-1}^{1} \nu(\mu-\nu)^{-1}\Psi(\nu,\mu)h(\mu)\,d\mu; \ \nu\in N,\ h\in L_2(I),$$

where $\lambda(\nu)$ is given by (3.9). Since A maps the functions on I that are Hölder continuous of exponent $0 < \alpha < 1$ except for a possible jump discontinuity at the point 0, onto functions on I of the same type, the operato $F^\dagger$ maps these functions onto functions on N of the same type. In particul one has $F^\dagger P_\ell = FA^{-1}P_\ell = (1-a_\ell)^{-1}FP_\ell = H_\ell$ $(\ell = 0,1,2,\dots)$.

Next we derive an auxiliary formula for the polynomial H_ℓ in which th projection P will appear (see (3.15)). Later (see Proposition 3.4 below) this formula will allow us to describe the action of P^* and $I-Q$ on the Legendre polynomials.

Let $P_{+,N}$ be the operator on $L_2(N)_\sigma$, defined by

$$\begin{cases}(P_{+,N}k)(\nu) = k(\nu) & (\nu \in N \cap [0,+\infty));\\ (P_{+,N}k)(\nu) = 0 & (\nu \in N \cap (-\infty,0)).\end{cases}$$

Then the similarity $F^\dagger(TA^{-1}) = T_N F^\dagger$ yields that $F^\dagger P_p^* = P_{+,N}F^\dagger$, where P_p^* is the spectral projection of TA^{-1} corresponding to its spectrum on $[0,+\infty)$. But $P_p^* = P_p^* P^*$. So $P_{+,N}F^\dagger = F^\dagger P_p^* P^* = P_{+,N}F^\dagger P^*$. By (3.14) one has for all $h \in L_2[-1,+1]$ and $\nu \in N \cap [0,+\infty)$:

$$(3.15) \qquad H_\ell(\nu) = \lambda(\nu)(P^*P_\ell)(\nu) - \fint_0^1 \nu(\mu-\nu)^{-1}\Psi(\nu,\mu)(P^*P_\ell)(\mu)d\mu; \quad \ell = 0,1,2,\dots$$

In this expression the functions P^*P_ℓ are Hölder continuous of exponent $0 < \alpha < 1$ on $[0,1]$. (Recall that $F^\dagger$ maps functions on I that are Hölder continuous except for a possible jump discontinuity at $\mu = 0$, onto functions on N of the same type. Further, $P_{+,N}F^\dagger P^* P_\ell = P_{+,N}F^\dagger P_\ell$.) By the proof of the auxiliary formula (3.15) we have completed the first part of the proof of Theorems 3.1 and 3.2.

The proof of the next proposition is inspired by an argument of PAHOR ([60], (2.14) - (2.24)), who derived Eq. (3.3) from (an analogue of) formula (3.15) but did not give complete proofs.

<u>PROPOSITION</u> 3.4. *For $\ell = 0,1,2,\dots$ one has*

$$(3.16) \qquad (P^*P_\ell)(\nu) = q_\ell(\nu)H(\nu), \quad ((I-Q)P_\ell)(\nu) = r_\ell(\nu)H(\nu); \; \nu \in [0,1].$$

Here the polynomials $(q_\ell)_{\ell=0}^{+\infty}$ are the unique polynomial solutions of Eq. (3.3), whereas the polynomials $(r_\ell)_{\ell=0}^{+\infty}$ are the unique polynomial solutions of Eq. (3.5). For $\ell \geq 0$ the degrees of q_ℓ and r_ℓ do not exceed $\max(\ell,N)$.

<u>PROOF</u>. Write $q_\ell(\mu) = H(\mu)^{-1}(P^*P_\ell)(\mu)$ $(0 \leq \mu \leq +1;\ \ell = 0,1,2,\dots)$. Since H is Hölder continuous and non-zero on $[0,1]$, the functions q_ℓ are Hölder continuous of exponent $0 < \alpha < 1$ on $[0,1]$ too. For $\lambda \in \mathbb{C}\backslash[-1,+1]$ define $f_\ell(\lambda)$ by setting

$$(3.17) \qquad H_\ell(\lambda) = \Lambda(\lambda)H(\lambda)f_\ell(\lambda) - \sum_{k=0}^N a_k(k+\tfrac{1}{2})H_k(\lambda)\int_0^1 \frac{\lambda P_k(\mu)q_\ell(\mu)H(\mu)}{\mu-\lambda}d\mu.$$

Note that this defines f_ℓ as a meromorphic function on $\mathbb{C}\backslash[0,1]$ (and not only on $\mathbb{C}\backslash[-1,+1]$; see (2.6)), and that the poles of f_ℓ are contained in the s of zeros of $\Lambda(\lambda)$. But all these zeros are simple (see [19], second paragr of page 238; for a later proof, see [37], Lemma 3(i); [48], Theorem 3) an comprise the set $N \cap (1,+\infty)$. From (3.15) (applied for $\nu \in N \cap (1,+\infty)$, whe $\lambda(\nu) = \Lambda(\nu) = 0$) one sees that $\lim_{\lambda\to\nu} \Lambda(\lambda)H(\lambda)f_\ell(\lambda) = 0$. But then f_ℓ is analytic on $\mathbb{C}\backslash[0,1]$.

Next, observe that P^*P_ℓ is a real function. So q_ℓ is a real function too and $\overline{f_\ell(\bar\lambda)} = f_\ell(\lambda)$, $\lambda \in \mathbb{C}\backslash[0,1]$. In (3.17) one substitutes $\lambda = \nu\pm i\varepsilon$ $(0 < \nu < 1,\ \varepsilon > 0)$. Because q_ℓ is Hölder continuous on $[0,1]$, a well-know result of MUSKHELISHVILI [59] yields that for $\varepsilon \downarrow 0$ all terms in (3.17) (with $\lambda = \nu\pm i\varepsilon$) have a limit which is Hölder continuous on the open inter $(0,1)$. So if λ approaches a point $\nu \in (0,1)$ from either the upper or the lower half-plane, the function f_ℓ has a limit. Write

$$\lim_{\varepsilon\downarrow 0} f_\ell(\nu\pm i\varepsilon) = \alpha_\ell(\nu)\pm i\beta_\ell(\nu), \quad 0 < \nu < 1.$$

In (3.17) we take the limits as $\varepsilon \downarrow 0$ for both $\lambda = \nu+i\varepsilon$ and $\lambda = \nu-i\varepsilon$, compute their sum and difference, incorporate (3.15), divide the two resultin expressions by the non-zero number $2H(\nu)$ and get the following linear system of equations:

$$\begin{bmatrix}\lambda(\nu) & -\pi\nu\Psi(\nu)\\ \pi\nu\Psi(\nu) & \lambda(\nu)\end{bmatrix}\begin{bmatrix}\alpha_\ell(\nu)\\ \beta_\ell(\nu)\end{bmatrix} = \begin{bmatrix}\lambda(\nu)q_\ell(\nu)\\ \pi\nu\Psi(\nu)q_\ell(\nu)\end{bmatrix}, \quad 0 < \nu < 1.$$

The determinant $\lambda^2(\nu) + \pi^2\nu^2\Psi(\nu)^2$ of this system does not vanish (see [37 Lemmas 3(v) and 4(ii); [38], Lemmas 3.3(v) and 4.1(ii); [48], Theorem 3 and Eq. (3.12)) and therefore the above system has a unique solution, name $\alpha_m(\nu) = q_m(\nu)$, $\beta_m(\nu) = 0$. Hence, for $0 < \nu < 1$ the number $f_\ell(\nu\pm i\varepsilon)$ tends t $q_\ell(\nu)$ as $\varepsilon \downarrow 0$. We may conclude that q_ℓ has an analytic continuation to $\mathbb{C}\backslash\{0,1\}$, where the possible singularities 0 and 1 are isolated. We shall denote the continuation by q_ℓ too.

Since $P_k(\mu)q_\ell(\mu)H(\mu)$ is bounded on $[0,1]$, it follows from (3.17) that

$$\Lambda(0)H(0)\lim_{\lambda\to 0,\mathrm{Re}\,\lambda\geq 0} q_\ell(\lambda) = H_\ell(0), \quad H(0)^{-1}\lim_{\lambda\to 0,\mathrm{Re}\,\lambda\leq 0} q_\ell(\lambda) = H_\ell$$

(for the first limit, employ Eq. (29.4) of [59]; to obtain the second limi use the principle of dominated convergence). But $\Lambda(0) = H(0) = 1$, and thus q_ℓ is analytic at $\lambda = 0$.

To prove that q_ℓ is analytic at $\lambda = 1$, one has to consider two cases. In case $\Psi(1,1) = \sum_{k=0}^{N} a_k(k+\frac{1}{2})H_k(1)P_k(1) = 0$, the dispersion function $\Lambda(\lambda)$ has a finite limit for $\lambda \to 1$ and $\lambda \notin [0,1]$, which does not vanish (see [37], Lemma 4; [38], Lemma 4.1; [48], page 313). But the sum at the right-hand side of (3.17) tends to $\Psi(1,1) = 0$ for $\lambda \to 1$ and $\lambda \notin [0,1]$ (cf. [59], Eq. (29.4)). Since $H(1) \neq 0$, Eq. (3.17) yields that q_ℓ is bounded in a neighbourhood of $\lambda = 1$. But then q_ℓ is analytic at $\lambda = 1$. Next, assume $\Psi(1,1) \neq 0$. If q_ℓ is not analytic at $\lambda = 1$, then q_ℓ has an essential singularity at $\lambda = 1$. According to the Casorati-Weierstrass theorem, for every $c \in \mathbb{C}$ there is a path Γ_c in $\mathbb{C}\setminus[0,1]$ such that $|q_\ell(\lambda)-c| \to 0$ for $\lambda \to 1$ along the path Γ_c. Now one substitutes (2.9) into (3.17). With the help of Eq. (29.4) of [59] one sees that up to terms that are bounded on the path Γ_c, the right-hand side of (3.17) has the form

$$\sum_{k=0}^{N} a_k(k+\tfrac{1}{2})H_k(\lambda)P_k(1)[cH(\lambda)-\lambda q_\ell(1)H(1)]\log(\lambda-1); \quad \lambda \in \Gamma_c, \; c \in \mathbb{C}.$$

Here the branch cut of $\log(\lambda-1)$ is chosen to be the half-line $(-\infty,1]$. Since the left-hand side of (3.17) is bounded on Γ_c, the above expression has to be bounded too. But $\Psi(1,1) = \sum_{k=0}^{N} a_k(k+\frac{1}{2})H_k(1)\,P_k(1) \neq 0$. So for $c \neq q_\ell(1)$ a contradiction arises. Hence, also in the case $\Psi(1,1) \neq 0$ the function q_ℓ is analytic at $\lambda = 1$.

We may conclude that q_ℓ is an entire function and satisfies

$$H_\ell(\lambda) = H(-\lambda)^{-1}q_\ell(\lambda) - \lambda \sum_{k=0}^{N} a_k(k+\tfrac{1}{2})H_k(\lambda)\int_0^1 \frac{P_k(\mu)q_\ell(\mu)H(\mu)}{\mu-\lambda}\,d\mu, \qquad \lambda \notin [0,1] \tag{3.18}$$

(cf. (2.6)). Since $H(-\lambda)^{-1} \to (\det A)^{\frac{1}{2}}$ as $\lambda \to \infty$ and for $r \geq 0$ the polynomial H_r has degree r, it follows that q_ℓ is a polynomial of degree $\leq \max(\ell,N)$. In fact, for $\ell \geq N+1$ the degree of q_ℓ equals ℓ. The equation (3.3) for q_ℓ is immediate from (3.18) and the H-equation (2.12) (more precisely, the continuation of (2.12) for $\mu \notin [-1,0]$).

From the linear independence of the polynomials $(H_\ell)_{\ell=0}^{+\infty}$ and (3.3) one easily sees that the polynomials $(q_\ell)_{\ell=0}^{+\infty}$ are linearly independent. So for every $n \geq N$ the polynomials $q_0,q_1,\ldots,q_n$ form a basis of the linear space of polynomials of degree $\leq n$. But then the uniqueness of the polynomial solutions of Eq. (3.3) is clear.

Apply the intertwining property $TP^* = (I-Q)T$ (cf. (III 5.2)) and the recurrence relation $(2\ell+1)TP_\ell = (\ell+1)P_{\ell+1} + \ell P_{\ell-1}$ $(\ell = 0,1,2,\ldots)$ of the Legendre polynomials. Then

$$\nu[(I-Q)P_\ell](\nu) = \left\{\frac{\ell+1}{2\ell+1}q_{\ell+1}(\nu) + \frac{\ell}{2\ell+1}q_{\ell-1}(\nu)\right\}H(\nu), \qquad 0 \leq \nu \leq +1$$

Note that H is continuous and non-zero on $[0,1]$ and $(I-Q)P_\ell$ is an L_2-function. So there exists a unique polynomial r_ℓ such that

$$(2\ell+1)\nu r_\ell(\nu) = (\ell+1)q_{\ell+1}(\nu) + \ell q_{\ell-1}(\nu); \quad 0 \leq r \leq +1, \; \ell = 0,1,2 \tag{3.19}$$

In terms of the polynomials $(r_m)_{m=0}^{+\infty}$ one has the second one of the identi (3.16). From (3.19) one concludes that for $n \geq N$ the polynomials $r_0,r_1,\ldots$ r_n form a basis of the linear space of polynomials of degree $\leq n$. Finall to derive (3.5) from (3.3) one employs (3.19) and the identity

$$1 - \int_0^1 \Psi(\mu,\mu)H(\mu)\,d\mu = (\det A)^{\frac{1}{2}} = \left\{\prod_{k=0}^{N}(1-a_k)\right\}^{\frac{1}{2}}.$$

This identity is immediate from the H-equation (2.12) and the factorizat (2.6). □

<u>PROOF of Theorem 3.1.</u> Put $C = A^{-1} - I$. Because the range of P is invariant under $A^{-1}T$ and its kernel under T, one has

$$(I-P)CTP = (I-P)A^{-1}TP - (I-P)TP = PT - TP. \tag{3.20}$$

Since $T[\mathrm{Im}\,P] \subset A[\mathrm{Im}\,P] = \mathrm{Ker}\,Q^*$ and $\mathrm{Im}(I-P) = \mathrm{Im}\,Q^* = H_-$, one gets

$$Q^*BPT = Q^*PT = Q^*TP + Q^*(I-P)CTP = (I-P)CTP = PT - TP. \tag{3.21}$$

Repeated application of this identity yields

$$PT^m - T^mP = \sum_{j=0}^{m-1} T^{m-1-j}Q^*BPT^{j+1}; \quad m = 1,2,3,\ldots . \tag{3.22}$$

So for $m = 1,2,3,\ldots$ and $h \in L_2[-1,+1]$ it follows that

$$(PT^m h) = \mu^m(Ph)(\mu) - \int_0^{+1} \nu\left(\sum_{j=0}^{m-1}\mu^{m-1-j}\nu^j\right)h(\nu)\,\cdot$$

$$\cdot \sum_{k=0}^{N} a_k(Q^*p_k)(\mu)(P^*p_k)(\nu)\,d\nu.$$

First one replaces the sum $\sum_{j=0}^{m-1} \mu^{m-1-j}\nu^j$ by $(\nu^m-\mu^m)/(\nu-\mu)$. Next one puts $\phi(\mu) = \mu^m$, and for $-1 \le \mu < 0$ one gets the expression

$$(3.23)\qquad (P\phi h)(\mu) = \phi(\mu)(Ph)(\mu) - \int_0^1 \nu \frac{\phi(\nu)-\phi(\mu)}{\nu-\mu} h(\nu) \cdot \sum_{k=0}^{N} a_k (Q^*p_k)(\mu)(P^*p_k)(\nu)d\nu.$$

Observe that this identity is correct for $\phi(\mu) \equiv 1$. By linearity it is correct if ϕ is a polynomial. Since $\nu(\nu-\mu)^{-1} \sum_{k=0}^{N} a_k(Q^*p_k)(\mu)(P^*p_k)(\nu)$ is square integrable for $(\mu,\nu) \in [-1,0]\times[0,1]$, it follows that for each $h \in L_\infty[-1,+1]$ the right-hand side of (3.23) is a bounded operator on $L_2[-1,+1]$. Note that P is bounded and the polynomials are dense in $L_2[-1,+1]$. Hence, formula (3.23) holds true for all pairs of functions $\phi \in L_2[-1,+1]$ and $h \in L_\infty[-1,+1]$.

Put

$$(Kf)(\mu) = \begin{cases} (Pf)(\mu) + \int_0^1 \nu(\nu-\mu)^{-1} f(\nu) \sum_{k=0}^{N} a_k(Q^*p_k)(\mu)(P^*p_k)(\nu)d\nu, & -1 \le \mu < 0; \\ (Pf)(\mu), & 0 \le \mu \le +1. \end{cases}$$

Then K acts as a bounded operator on $L_2[-1,+1]$. Taking $\phi(\mu) = \mu$ in (3.23) one gets

$$(3.24)\qquad TKh = K(Th), \quad h \in L_\infty[-1,+1].$$

Since K and T are bounded on $L_2[-1,+1]$ and $L_\infty[-1,+1]$ is a dense linear subspace of $L_2[-1,+1]$, the equality (3.24) is true for every $h \in L_2[-1,+1]$ and thus TK = KT. But T is a self-adjoint operator with a simple spectrum. By [66], Theorem 8.1, there exists a bounded measurable function χ: $\sigma(T) = [-1,+1] \to \mathbb{C}$ such that $K = \chi(T)$. But $P - K$ is a compact operator and the same is true for $P - P_+$ (use Theorem III 5.2, Lemma III 5.3 and the identity $P - P_+ = (P_p - P_+)V^{-1} + P_+V^{-1}(I-V)$). So $K - P_+ = \chi(T) - P_+$ is a compact operator, which is given by

$$[(K-P_+)f](\mu) = \begin{cases} 0 &, \quad 0 \le \mu \le +1; \\ \chi(\mu)f(\mu), & -1 \le \mu < 0. \end{cases}$$

Therefore, $K - P_+$ is a normal operator. To prove that $K - P_+ = 0$, it suffices to show that $\sigma(K-P_+) = \{0\}$. If $0 \ne \lambda \in \sigma(K-P_+)$, then, by the compactness of $K - P_+$, there exists $0 \ne f \in L_2[-1,+1]$ such that $f(\mu) = 0$ for $\mu \ge 0$ and $\chi(\mu)f(\mu) = \lambda f(\mu)$ for $\mu < 0$. Since $f \ne 0$, there is a subinterval (a,b) of (-1,0) such that $\chi(\mu) = \lambda$ for $\mu \in (a,b)$. But then all functions in $L_2[-1,+1]$ which have their support on [a,b], belong to the kernel of $K - P_+ - \lambda$, and thus $\mathrm{Ker}(K - P_+ - \lambda)$ is infinite dimensional. But this contradicts the compactness of $K - P_+$. Hence, one has $\sigma(K-P_+) = \{0\}$, indeed. Thus $K = P_+$. The theorem is now immediate from (3.23), the identities $K = P_+$ and $(Q^*P_k)(\mu) = (-1)^k(P^*P_k)(-\mu)$ and Proposition 3.4. □

<u>PROOF of Theorem 3.2</u>. The proof of this theorem closely resembles the proc of the previous one. Instead of (3.21) one applies (3.20) repeatedly. With the help of the intertwining property (III 5.2) one gets for $\phi(\mu) \equiv \mu^m$ and $h \in L_2[-1,+1]$:

$$(3.25)\qquad [(I-P)\phi h](\mu) = \phi(\mu)[(I-P)h](\mu) - \int_0^1 \nu\frac{\phi(\nu)-\phi(\mu)}{\nu-\mu}h(\nu) \cdot \sum_{k=0}^{N} a_k(1-a_k)^{-1}[(I-P)p_k](\mu)[(I-Q)p_k](\nu)d\nu.$$

As in the previous proof one extends this expression to all $(\phi,h) \in L_2[-1,+1]\times L_\infty[-1,+1]$. The rest of the present proof follows the method of the previous one. □

Using the method sketched in the proof of Theorem 3.2, formula (3.25) (with $h(\mu) \equiv 1$) has been found before (see [53,39]).

The projection P is related to the so-called scattering function $S(\nu,\mu)$ $(0 \le \nu,\mu \le +1)$, which has been introduced by CHANDRASEKHAR [10]. The relationship is given by

$$(3.25)\qquad (Ph)(-\mu) = \tfrac{1}{2}\int_0^1 \mu^{-1}S(\nu,\mu)h(\nu)d\nu, \quad 0 \le \mu \le +1$$

(see [60], (2.1) - (2.3); [51], (7) - (8)). Explicit formulas for $S(\nu,\mu)$ are now immediate from Theorems 3.1 and 3.2. For example,

$$(3.26)\qquad S(\nu,\mu) = 2\mu\nu(\mu+\nu)^{-1} \sum_{k=0}^{N} a_k(k+\tfrac{1}{2})(-1)^k q_k(\mu)q_k(\nu)H(\mu)H(\nu).$$

The first explicit expressions for $S(\nu,\mu)$ appear in [10] for the case when $N \le 2$. The general formula (3.26) is due to BUSBRIDGE ([8], (48.51)), who supplied a rather intricate method to compute what would later be calle the Busbridge polynomials $q_0,\dots,q_N$. Her calculations were greatly simplified by PAHOR [60] and later by SOBOLEV [64], who both obtained Eq. (3.3). However, sometimes their derivations suffer from mathematical incompleteness.

For the isotropic case of the neutron Transport Equation and later for anisotropic cases too the projection P and an analytic expression for it are due to HANGELBROEK (cf. [35,36]). For the degenerate anisotropic case the existence of the projection P has been proved by HANGELBROEK (however, in literature a full, but different proof first appeared in Section 6.4 of [2]). For the degenerate anisotropic case an explicit expression for the projection P has been published by LEKKERKERKER (cf. [48], (5.7)). Because in [48] eigendistributions are used, actually a formula for FP was supplied, where F is the diagonalizing map described in Proposition 3.3. In recent years existence proofs for the projection P have been given for non-degenerate indicatrices too (see [53]; a preliminary version of the proof contained in [53] has appeared in Section 6.4 of [2]), but for the non-degenerate case analytic expressions for P are unknown.

The expressions for P that appear in this section are as expedient as the ones found in astrophysical literature. From (3.3) the polynomials $q_0,q_1,\dots,q_N$ can be found by expanding q_m, H_m and $\Psi(\lambda,\mu)$ in powers of λ and solving a system of N+1 linear equations in N+1 unknown coefficients of q_m. Since the Busbridge polynomials $q_0,q_1,\dots,q_N$ form a basis of the linear vector space of polynomials of degree at most N, the determinant of this system does not vanish. For $N \leq 3$ astrophysicists have computed these polynomials explicitly and observed the invertibility of the linear system connected with Eq. (3.3). The polynomials $r_0,r_1,\dots,r_N$ can be found in a similar way with the help of Eq. (3.5).

4. The conservative case

In this section we analyze the conservative case of the one-speed Transport Equation. We describe the structure of the singular subspace and for the isotropic case all bounded solutions are obtained.

Let $a_n = \int_{-1}^{+1} \hat{g}(t)P_n(t)dt$ be the n-th expansion coefficient of the indicatrix $\hat{g}$, and assume that $a_n \leq +1$ $(n = 0,1,2,\dots)$. By M we denote the set of integers n for which $a_n = +1$. Since $\lim_{n\to\infty} a_n = 0$, the set M is finite. In particular, if the operators T and B are defined by (1.4b) and $A = I - B$, then the kernel of A is given by

$$\text{Ker } A = \text{span}\{p_n : n \in M\}.$$

By a *cycle* of M one denotes the convex components of M with respect to th linear order inherited by M from the set of integers, and the *length* of a cycle is its cardinality. In other words, a subset $\{k+1,k+2,\dots,k+n\}$ of M is called a cycle of M of length n if and only if k and k+n+1 do not belong to M.

In the proof of the next theorem we need the three-terms recurrence relation of the Legendre polynomials $(p_n)_{n=0}^{+\infty}$. Using the operator T in (1.4 one has

$$(4.1)\qquad Tp_n = \alpha_n p_{n+1} + \alpha_{n-1}p_{n-1} \quad (n = 0,1,2,\dots);\ p_{-1} = 0,$$

where $\alpha_n = \langle Tp_n,p_{n+1}\rangle = \int_{-1}^{+1} \mu p_n(\mu)p_{n+1}(\mu)d\mu$ is a non-zero real constant. Its precise value is $\alpha_n = \frac{1}{2}(n+1)[(n+\frac{1}{2})(n+\frac{3}{2})]^{-\frac{1}{2}}$ (cf. [68]), but for our pur pose this value is of no concern. The next theorem describes the Jordan structure of the singular subspace H_0 of the pair (T,B).

<u>THEOREM 4.1</u>. *Let* m(M) *be the number of elements of* M, *and let* c(M) *be the number of its cycles of odd length. Then the dimension of the singular subspace* H_0 *equals* m(M) + c(M), *the number of Jordan blocks of* $T^{-1}A$ *of order 2 at* $\lambda = 0$ *equals* c(M) *and the number of Jordan blocks of* $T^{-1}A$ *of order 1 at* $\lambda = 0$ *equals* m(M) − c(M).

<u>PROOF</u>. In this proof we give an explicit description of a basis of H_0 with respect to which $T^{-1}A|_{H_0}$ has the Jordan normal form. From Proposition III 3.: we know that $T^{-1}A|_{H_0}$ does not have Jordan chains of length > 2, and knowin this greatly simplifies the proof. It is important to observe that H_0 consists of polynomials only.

We cut the proof into the proof of four separate statements.

1. *If* $n+1 \in M$, *then there does not exist a polynomial of degree* n *in the range of* $T^{-1}A$.

Indeed, assume that $f = T^{-1}Ag$. Then on the one hand $\langle Tf,p_{n+1}\rangle = \langle Ag,p_{n+1}\rangle = \langle g,Ap_{n+1}\rangle = (1-a_{n+1})\langle g,p_{n+1}\rangle = 0$. On the other hand, if f would be a polynomial of degree n, then Tf is a polynomial of degree n+1, and thus $\langle Tf,p_{n+1}\rangle \neq 0$. Contradiction.

2. *If* {k+1} *is a cycle of* M *of length* 1, *then for* $\gamma_{k+2} = (1-a_{k+2})^{-1}\alpha_{k+}$ *and* $\gamma_k = (1-a_k)^{-1}\alpha_k$ *one has* $T^{-1}A(\gamma_{k+2}p_{k+2}+\gamma_k p_k) = p_{k+1}$.

Certainly, $A(\gamma_{k+2}p_{k+2}+\gamma_k p_k) = \alpha_{k+1}p_{k+2}+\alpha_k p_k$, and the statement is clear from (4.1).

3. *If* $\{k+1,k+2,\dots,k+2n+1\}$ *is a cycle of* M *of length* 2n+1, *then for* $\gamma_{k+2n+2} = (1-a_{k+2n+2})^{-1}\alpha_{k+2n+1}$ *and* $\gamma_k = (1-a_k)^{-1}\alpha_k$ *one has*

$$T^{-1}A\left(\gamma_{k+2n+2}p_{k+2n+2} + (-1)^n \frac{\alpha_{k+2}\alpha_{k+4}\cdots\alpha_{k+2n}}{\alpha_{k+1}\alpha_{k+3}\cdots\alpha_{k+2n-1}}\gamma_k p_k\right) = \tag{4.2}$$

$$= \sum_{r=0}^{n} \xi_r(k,n)p_{k+2r+1},$$

where $\xi_0(k,n),\xi_1(k,n),\dots,\xi_n(k,n)$ *are complex numbers.*

Put $\xi_n(k,n) = 1$ and

$$\xi_m(k,n) = (-1)^{n-m}\frac{\alpha_{k+2m+2}\alpha_{k+2m+4}\cdots\alpha_{k+2n}}{\alpha_{k+2m+1}\alpha_{k+2m+3}\cdots\alpha_{k+2n-1}} \qquad (m = 0,1,\dots,n-1).$$

Then, using the recurrence relation (4.1), one has

$$T\sum_{r=0}^{n}\xi_r(k,n)p_{k+2r+1} = \alpha_{k+2n+1}p_{k+2n+2}+(-1)^n\frac{\alpha_{k+2}\alpha_{k+4}\cdots\alpha_{k+2n}}{\alpha_{k+1}\alpha_{k+3}\cdots\alpha_{k+2n-1}}\alpha_k p_k,$$

which settles (4.2).

4. *If* $\{k+1,k+2,\dots,k+2n\}$ *is a cycle of* M *of length* 2n, *then there does not exist a polynomial* f *of degree* k+2n *in the range of* $T^{-1}A$ *such that* $\langle f,p_k\rangle = 0$.

Suppose f is a polynomial of degree k+2n such that $\langle f,p_k\rangle = 0$, and assume that there exists a polynomial g such that Tf = Ag. Then for certain coefficients one has

$$f = \sum_{s=0}^{k+2n}\zeta_s p_s, \quad \zeta_{k+2n} \neq 0,\ \zeta_k = 0; \quad g = \sum_{r=0}^{k+2n+1}\eta_r p_r.$$

Inserting (4.1) into the equality Tf − Ag = 0 one gets that

$$\zeta_{k+2n}\alpha_{k+2n}p_{k+2n+1} + \sum_{s=k+1}^{k+2n}(\zeta_{s-1}\alpha_{s-1}+\zeta_{s+1}\alpha_s)p_s - \eta_{k+2n+1}(1-a_{k+2n+1})p_{k+2n+1}$$

is a polynomial of degree ≤ k, where ζ_{k+2n+1} is assumed to be zero. Using that $\zeta_k = 0$ and $\alpha_m \neq 0$ (m = 0,1,2,...) one successively concludes that $\zeta_{k+2} = 0$, $\zeta_{k+4} = 0,\dots,\zeta_{k+2n-2} = 0$, $\zeta_{k+2n} = 0$. Contradiction.

From the four statements we have proved that a basis of the singular subspace H_0, with respect to which $T^{-1}A|_{H_0}$ has the Jordan normal form, is given by the following set:

$$\{p_r:\ r \in M,\ \text{but } r \text{ is not the maximum of a cycle of } M \text{ of odd length}\}$$
$$\cup \bigcup_{(k,n)}\left\{\sum_{r=0}^{n}\xi_r(k,n)p_{k+2r+1},\ \gamma_{k+2n+2}p_{k+2n+2}+(-1)^n\frac{\alpha_{k+2}\alpha_{k+4}\cdots\alpha_{k+2n}}{\alpha_{k+1}\alpha_{k+3}\cdots\alpha_{k+2n-1}}\gamma_k \tag{4.3}$$

Here we take the union over all pairs of integers (k,n) such that {k+1,k+2 ...,k+2n+1} is a cycle of M. □

We emphasize two quite unexpected peculiarities. First the number of Jordan blocks of $T^{-1}A$ of order 2 and the number of Jordan blocks of order at λ = 0 only depend on the set $M = \{n\geq 0:\ a_n = 1\}$ and not on the expansion coefficients a_n that are strictly less than 1. Secondly, the singular subspace H_0 itself and the Jordan basis of $T^{-1}A|_{H_0}$ only depend on the set $M = \{n\geq 0:\ a_n = 1\}$ and the coefficients a_k and a_{k+2n+2} for which {k+1,k+2, ...,k+2n+1} is a cycle of M of odd length 2n+1.

Observe that the basis (4.3) of H_0 consists of even and odd functions only. Recall that an inversion symmetry of the pair (T,B) is given by the signature operator J, defined by

$$(Jh)(\mu) = h(-\mu), \quad -1 \leq \mu \leq +1.$$

Then the subspaces $H_0^+ = \{h \in H_0:\ Jh = h\}$ and $H_0^- = \{h \in H_0:\ Jh = -h\}$ consis of the even functions of H_0 and the odd functions of H_0, respectively. Therefore, the basis (4.3) has the special form described in the proof of Theorem III 7.2.

The physically most relevant conservative indicatrices $\hat{g}$ have the property that $a_0 = 1$ and $a_n < 1$ (n = 1,2,3,...). Then M = {0}, $H_0 = \operatorname{span}\{p_0,p_1\}$ and a basis of H_0 with respect to which $T^{-1}A|_{H_0}$ has the Jordan normal form, is $\{p_0,\frac{1}{3}(1-a_1)^{-1}\sqrt{3}\,p_1\}$, and this basis consists of only one Jordan block. Also for anisotropic cases this fact has been notice by MASLENNIKOV (cf. [50]).

Now we turn to the isotropic case when the indicatrix equals $\hat{g}(t) \equiv \frac{1}{2}c$

for some $0 \le c \le 1$ and the pair (T,B) in (1.4b) has the form

$$(Th)(\mu) = \mu h(\mu), \quad (Bh)(\mu) = \tfrac{1}{2}c \int_{-1}^{+1} h(\nu)d\nu; \quad -1 \le \mu \le +1.$$

For $0 \le c < 1$ the pair (T,B) is positive definite; for $c = 1$ it is singular and semi-definite. We now specify Theorem 3.1 and the H-equation (2.12) for the non-conservative isotropic case.

PROPOSITION 4.2. *For* $0 \le c < 1$ *the projection* P *of* $H = L_2[-1,+1]$ *onto* H_p *along* $H_- = L_2[-1,0]$ *is given by*

$$(4.4) \qquad (P\phi)(\mu) = \begin{cases} \phi(\mu) & , \quad 0 \le \mu \le +1; \\ \tfrac{1}{2}c \int_0^1 \nu(\nu-\mu)^{-1}H(-\mu)H(\nu)\phi(\nu)d\nu, & -1 \le \mu < 0. \end{cases}$$

The H-*function appearing in* (4.4) *is a solution of the* H-*equation*

$$(4.5) \qquad H(\mu)^{-1} = 1 - \tfrac{1}{2}c\mu \int_0^1 \frac{H(\nu)d\nu}{\nu+\mu}, \quad 0 \le \mu \le +1.$$

PROOF. For the isotropic case one has $a_0 = c$ and $a_n = 0$ $(n = 1,2,\ldots)$. So for the polynomials defined by (2.8) one has $H_0(\mu) \equiv 1$ and $H_1 = (1-c)\mu$. The characteristic binomial is given by $\Psi(\nu,\mu) = \tfrac{1}{2}a_0H_0(\nu)P_0(\mu) \equiv \tfrac{1}{2}c$. Now (4.5) is immediate from (2.12). From (3.3) one gets the Busbridge polynomial $q_0(\mu) \equiv 1$. Formula (4.4) is a straightforward application of Theorem 3.1. □

Next we deal with the conservative case $c = 1$. Then we have

THEOREM 4.3. *For* $c = 1$ *the projection* P *of* $H = L_2[-1,+1]$ *onto* $H_p \oplus \text{span}\{p_0\}$ *along* $H_- = L_2[-1,0]$ *exists and is given by*

$$(4.6) \qquad (P\phi)(\mu) = \begin{cases} \phi(\mu) & , \quad 0 \le \mu \le +1; \\ \tfrac{1}{2}\int_0^1 \nu(\nu-\mu)^{-1}H(-\mu)H(\nu)\phi(\nu)d\nu, & -1 \le \mu < 0. \end{cases}$$

Here the H-*function* H *is analytic on* $\mathbb{C}\setminus[-1,0]$, *continuous and non-zero on* $\mathbb{C}\setminus[-1,0)$, *has a simple pole at infinity and satisfies the identity*

$$(4.7) \qquad \Lambda(\lambda) = H(\lambda)^{-1}H(-\lambda)^{-1}, \quad \operatorname{Re}\lambda = 0,$$

where Λ *is the dispersion function. The function* H *also satisfies the* H-*equation*

$$(4.7) \qquad H(\mu)^{-1} = 1 - \tfrac{1}{2}\mu \int_0^1 \frac{H(\nu)d\nu}{\nu+\mu}, \quad 0 \le \mu \le +1.$$

PROOF. From Theorem 2.2 one computes that for $0 \le c \le 1$ the dispersion fu tion is given by

$$\Lambda_c(\lambda) = 1 + \tfrac{1}{2}c\lambda \int_{-1}^{+1} (\mu-\lambda)^{-1}d\mu, \quad \lambda \in \mathbb{C}_\infty\setminus[-1,+1].$$

Then for $0 < c < 1$ the function Λ_c has two simple zeros $\nu_c > +1$ and $\nu_c <$ for $c \uparrow 1$ one has $\nu_c \to +\infty$. If $c = 1$, then Λ_c has a double zero at infinit only. (These properties are easily derived from the monotony of Λ_c on $(1,+\infty)$. They are well-known.) Further, $\Lambda_c(\lambda) \to \Lambda(\lambda)$ as $c \uparrow 1$, uniformly i λ on the extended imaginary line.

From Theorem 2.1 it follows that for $0 \le c < 1$ the function Λ_c is Hölder continuous of exponent $0 < \alpha < \tfrac{1}{2}$ on the imaginary line (in fact, direct computation shows the Hölder exponent to be $\alpha = 1$). Hence, the fun tions $\nu_c^2(\nu_c^2-\lambda^2)^{-1}(1-\lambda^2)\Lambda_c(\lambda)$ $(0 < c < 1)$ and $(1-\lambda^2)\Lambda(\lambda)$ are Hölder contin ous and strictly positive on the extended imaginary line, while

$$\lim_{c\uparrow 1} \sup_{\operatorname{Re}\lambda=0} \left| \frac{\nu_c^2(1-\lambda^2)}{\nu_c^2-\lambda^2}\Lambda_c(\lambda) - (1-\lambda^2)\Lambda(\lambda) \right| = 0.$$

By Theorem III 4.1 of [61] the function $(1-\lambda^2)\Lambda(\lambda)$ has a canonical factor zation with respect to the imaginary line, namely

$$(1-\lambda^2)\Lambda(\lambda) = (1+\lambda)H(\lambda)^{-1}\cdot(1-\lambda)H(-\lambda)^{-1}, \quad \operatorname{Re}\lambda = 0,$$

where H is described in the statement of this theorem. Using the stability of a canonical factorization (cf. [61]; Section I.5) one sees that

$$\lim_{c\uparrow 1} \left| \frac{\nu_c(1+\lambda)}{\nu_c+\lambda} H_c(\lambda)^{-1} - (1+\lambda)H(\lambda)^{-1} \right| = 0,$$

uniformly in λ on the closed right half-plane. Here H_c denotes the H-function for $0 < c < 1$. But then $|H_c(\mu) - H(\mu)| \to 0$ as $c \uparrow 1$, uniformly in μ on $[0,1]$. Employing (4.5) the H-equation (4.7) is clear.

It remains to prove (4.6). This proof involves a stability argument. Because we shall use (spectral) projections connected to the pair (T,B) fo several values of the parameter c, we shall often use the subscript c to avoid possible confusion. Consider the operator P given by (4.6). Clearly, this is a bounded projection with kernel H_-. Substituting $\phi(\mu) \equiv 1$ into (4.6) and employing (4.8) one easily shows that $P\phi = \phi$, and thus $\mathrm{span}\{p_0\} \subset \subset \mathrm{Im}\, P$.

Let Γ be the positively oriented rectangle with vertices $-i$, $2-i$, $2+i$ and $+i$, and let Γ_ε be the line obtained from Γ by omitting the segment from $+i\varepsilon$ to $-i\varepsilon$, with its orientation inherited from Γ $(0 < \varepsilon < 1)$. Fix $0 < c_0 < 1$ such that for $c_0 < c < 1$ the zero ν_c of Λ_c does not belong to the inner region F_+ of Γ. Putting $A_c = I - c\langle\cdot,p_0\rangle p_0$ one defines the bounded projection $P_{\Gamma,c}$ by

$$P_{\Gamma,c}\, h = \lim_{\varepsilon\downarrow 0} (-2\pi i)^{-1} \int_\Gamma (T-\lambda A_c)^{-1} A_c h\, d\lambda; \quad h \in L_2[-1,+1],\ c_0 < c \le 1.$$

Then $P_{\Gamma,1} = P_p$ (cf. Proposition III 3.3) and for $c_0 < c < 1$ the operator $P_{\Gamma,c}$ is the spectral projection $A_c^{-1}T$ corresponding to the part of its spectrum on the interval $[0,1]$. One has

$$\text{(4.9)} \qquad \lim_{c\uparrow 1} \| P_{\Gamma,c} - P_p \| = 0.$$

To see this, one first computes that for $-1 \le \mu \le +1$, $\lambda \notin [-1,+1] \cup \{\nu_c, -\nu_c\}$

$$[(T-\lambda A_c)^{-1}h](\mu) = (\mu-\lambda)^{-1}h(\mu) - \tfrac{1}{2}c\lambda\Lambda_c(\lambda)^{-1}(\mu-\lambda)^{-1}\int_{-1}^{+1}(\nu-\lambda)^{-1}h(\nu)\,d\nu.$$

Since $|\Lambda_c(\lambda)^{-1} - \Lambda(\lambda)^{-1}| \to 0$ as $c \uparrow 1$, uniformly in λ on $\Gamma\backslash\{0\}$, formula (4.9) is clear. From the identities (4.9) and $\|P_c - P\| \to 0$ as $c \uparrow 1$, where P_c is defined by (4.4) and P by (4.6), one gets

$$P P_p = P_p,$$

and thus $H_p \subset \mathrm{Im}\, P$. Therefore, $H_p \oplus \mathrm{span}\{p_0\} \subset \mathrm{Im}\, P$. By the considerations of Section III.7 one has $H_p \oplus \mathrm{span}\{p_0\} \oplus H_- = H = L_2[-1,+1]$. Thus $\mathrm{Im}\, P = H_p \oplus \mathrm{span}\{p_0\}$, and formula (4.6) is established. □

Using other methods rigorous proofs of the H-equations (4.5) and (4.8) have been given by BUSBRIDGE [8]. Expressions for the projection P have been provided by HANGELBROEK for the case $0 \le c < 1$ (see [35,36]) and by LEKKERKERKER for the case $c = 1$ (cf. [47]), but their expressions involve a diagonalization operator as an additional factor. Formulas (4.4) and (4.7) seem to be new.

5. The Milne problem (the isotropic case)

In this section we study the Milne problem in an isotropic medium. I[n] Section IV.3 we constructed the solution of the Milne problem in an abst[ract] way (cf. Theorem IV 3.5). Specified for the conservative isotropic case t[he] Milne problem amounts to obtaining all (locally integrable) polynomially bounded solutions in $L_2[-1,+1]$ of the integro-differential equation

$$\text{(5.1a)} \qquad \mu\frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) = \tfrac{1}{2}\int_{-1}^{+1}\psi(x,\mu')\,d\mu', \quad 0 \le x < +\infty,$$

with the boundary condition $\psi(0,\mu) = 0$ $(0 \le \mu \le +1)$. To normalize a solut[ion] we impose the extra condition

$$\text{(5.1b)} \qquad \lim_{x\to+\infty}\int_{-1}^{+1}\mu\psi(x,\mu)\,d\mu = -\tfrac{1}{2}F,$$

where F denotes the radiative flux coming from the interior of the star (cf. [10], Eq. (86) of Chapter I). By Theorem IV 3.5 all solutions have the form

$$\text{(5.2)} \qquad \psi(x) = e^{-xT^{-1}A}x_p + (I - xT^{-1}A)x_0, \quad 0 \le x < +\infty,$$

where $x_0 \in [H_p \oplus H_-] \cap H_0$, $x_p \in H_p$ and $x_p + x_0 \in H_- = L_2[-1,0]$. Using the proj[ec]tion P of $L_2[-1,+1]$ onto $H_p \oplus \mathrm{span}\{p_0\}$ along $H_- = L_2[-1,0]$ (see Theorem 4.[…]) one gets

$$\text{(5.3)} \qquad \psi(0) = x_p + x_0 = (I-P)x_0, \quad x_p = -Px_0.$$

To insert the flux condition (5.1b) one computes that for $e(\mu) \equiv 1$ the identity

$$\text{(5.4)} \qquad \int_{-1}^{+1}\mu\psi(x,\mu)\,d\mu = \langle T\psi(x),e\rangle = \langle Te^{-xT^{-1}A}x_p,e\rangle + \langle Tx_0,e\rangle - x\langle Ax_0,e\rangle = \langle Tx_0,e\rangle$$

holds true. Note that we used that $T[H_p] \subset \mathrm{Im}\, A = (\mathrm{Ker}\, A)^\perp$ and $Ae = 0$. So the radiative flux $2\int_{-1}^{+1}\mu\psi(x,\mu)\,d\mu$ at optical depth x within the stellar atmosphere is a constant $-F$ that does not depend on $x \in (0,+\infty)$. (The physical explanation of this phenomenon is the absence of absorption within the medium; see [10], Chapter I.) Hence, from (5.1b) and (5.4) one gets

$$(5.5) \qquad F = -2\langle Tx_0, e\rangle = -2\int_{-1}^{+1} \mu x_0(\mu)\,d\mu.$$

The next theorems provide the solution of the Milne problem in the isotropic case.

<u>THEOREM 5.1</u>. *For* $x = 0$ *the solution of the Milne problem is given by*

$$(5.6) \qquad \psi(0,\mu) = \tfrac{1}{4}F\sqrt{3}\ H(-\mu), \quad -1 \le \mu < 0.$$

Moreover, $\psi(0) = x_p + x_0$, *where* x_p *belongs to* H_p *and* $x_0 \in [H_p \oplus H_-] \cap H_0$ *is given by*

$$(5.7) \qquad x_0(\mu) = \tfrac{3}{4}F\left\{\tfrac{1}{2}\sqrt{3}\int_0^1 \nu^2 H(\nu)\,d\nu - \mu\right\}, \quad -1 \le \mu \le +1.$$

<u>PROOF</u>. Let $e(\mu) \equiv 1$, and recall that $H_0 = \mathrm{span}\{e, Te\}$. So the vector x_0 appearing in (5.2) to (5.5) has the form $x_0 = F(\xi e - \tfrac{3}{4}Te)$, where (5.5) has been taken into account and ξ is some constant to be determined later. From Theorem 4.3 and the identity $(I-P)e = 0$ one gets

$$\psi(0,\mu) = \tfrac{3}{4}F[(I-P)Te](\mu) = \tfrac{3}{8}FH(-\mu)\int_0^1 \nu(\nu-\mu)^{-1}H(\nu)\nu\,d\nu =$$

$$= \tfrac{3}{8}FH(-\mu)\left\{\int_0^1 \nu H(\nu)\,d\nu + \mu\int_0^1 \nu(\nu-\mu)^{-1}H(\nu)\,d\nu\right\}, \quad -1 \le \mu < 0.$$

But $Pe = 0$, and therefore

$$(5.8) \qquad \psi(0,\mu) = \tfrac{3}{8}FH(-\mu)\int_0^1 \nu H(\nu)\,d\nu, \quad -1 \le \mu < 0.$$

Next, recall that $H_0 = \mathrm{span}\{e, Te\}$ and $H_1 = T[H_0]^{\perp} = \{Te, T^2e\}^{\perp}$. Since $\langle T^2e, e\rangle = \langle Te, Te\rangle = \tfrac{2}{3}$, one easily checks that

$$P_0 h = \tfrac{3}{2}\{\langle h, T^2e\rangle e + \langle h, Te\rangle Te\}, \quad h \in L_2[-1,+1].$$

Now $x_0 = P_0\psi(0)$. Therefore,

$$x_0(\mu) = \tfrac{9}{16}F\int_0^1 \nu H(\nu)\,d\nu\left\{\int_0^1 \nu^2 H(\nu)\,d\nu - \mu\int_0^1 \nu H(\nu)\,d\nu\right\}, \quad -1 \le \mu \le +1.$$

With the help of the identity $x_0(\mu) = F(\xi - \tfrac{3}{4}\mu)$ one gets $\tfrac{1}{2}F = \tfrac{9}{16}F[\int_0^1 H(\nu)\,d\nu]^2 \cdot \tfrac{2}{3}$, and thus

$$\int_0^1 \nu H(\nu)\,d\nu = \tfrac{2}{3}\sqrt{3}$$

(cf. [8], (12.15)). But then (5.7) is clear. Finally one simplifies (5.: to get (5.6). □

<u>THEOREM 5.2</u>. *For* $0 \le x < +\infty$ *the solution of the Milne problem is given b*

$$(5.9) \qquad \psi(x,\mu) = \begin{cases} +\ \tfrac{3}{4}F\mu^{-1}\int_0^x e^{-(x-y)/\mu} J(y)\,dy, & 0 < \mu \le 1; \\ -\ \tfrac{3}{4}F\mu^{-1}\int_x^{+\infty} e^{-(x-y)/\mu} J(y)\,dy, & -1 \le \mu < 0. \end{cases}$$

Here the connection between the function J *and the* H*-function is given b*

$$(5.10) \qquad \sqrt{3}\int_0^{+\infty} e^{-s/\mu} J(s)\,ds = \mu H(\mu), \quad 0 < \mu < +\infty.$$

<u>PROOF</u>. Using the equivalence theorem of Section V.3 one transforms Eq. (5 (with boundary condition $\psi(0,\mu) = 0$, $0 \le \mu \le +1$) into the convolution equation

$$(5.11) \qquad \psi(x) - \tfrac{1}{2}\int_0^{+\infty} \langle\psi(y), e\rangle\, H(x-y)e\,dy = 0, \quad 0 \le x < +\infty,$$

where $e(\mu) \equiv 1$ and $H(t)$ denotes the propagator function. Putting $J(x) = (\tfrac{3}{2}F)^{-1}\langle\psi(x), e\rangle$ $(0 \le x < +\infty)$, one obtains (5.9). Comparing the solutions (5.9) with (5.6) one gets (5.10). □

The first results on the solution of the isotropic Milne problem are due to MILNE [55], who reduced the equation (5.1a) to the Schwarzschild-Milne integral equation

$$(5.12) \qquad J(t) - \tfrac{1}{2}\int_0^{+\infty} Ei(t-s)J(s)\,ds = 0, \quad 0 < t < +\infty,$$

where $Ei(t) = +\int_1^{+\infty} z^{-1}e^{-z|t|}\,dz$ is the exponential integral function. In the present terminology Eq. (5.12) is immediate from Eq. (5.11), because $Ei(t) = \langle H(t)e, e\rangle$ $(0 \ne t \in \mathbb{R})$. A study of the unbounded function J has been made by HOPF (cf. [42], Chapter II; see also [8], Chapter 4). From (5.2) and the identity $J(x) = (\tfrac{3}{2}F)^{-1}\langle\psi(x), e\rangle$ we see that

$$J(x) = (\tfrac{3}{2}F)^{-1}\left\{\langle x_0,e\rangle - x\langle T^{-1}Ax_0,e\rangle + \langle e^{-xT^{-1}A}x_p,e\rangle\right\} = \qquad (0 < x < +\infty)$$

$$= (\tfrac{3}{2}F)^{-1}\left\{F(\xi + \tfrac{3}{2}x) + \langle e^{-xT^{-1}A}x_p,e\rangle\right\} = x + q_\infty + [q(x) - q_\infty],$$

where $q_\infty = \frac{2}{3}\xi = \frac{1}{4}\sqrt{3}\int_0^1 \nu^2 H(\nu)d\nu$ (cf. (5.7)) and $q(\cdot) - q_\infty$ is an exponentially decreasing function. The function q is the so-called Hopf function and has been studied in detail in [42] and [8]. Formula (5.6) has been found by CHANDRASEKHAR with the help of principles of invariance (cf. [10], Eq. (52) of Chapter IV). Later LEKKERKERKER applied the method of singular eigendistributions to the Milne problem (cf. [47]).

6. The finite-slab problem (the isotropic case)

In this section we solve in an analytic way the finite-slab problem in an isotropic medium. In fact, we assume that $\hat{g}(\mu) \equiv \frac{1}{2}c$, where $0 < c \le 1$. The analytic solution will require two auxiliary scalar functions X_τ and Y_τ that are defined on [0,1]. To define these functions let Ei and Ei_2 be the exponential integral functions

$$Ei(t) = \int_1^{+\infty} z^{-1}e^{-|t|z}dt, \quad Ei_2(t) = \int_1^{+\infty} z^{-2}e^{-|t|z}dt; \quad t \in \mathbb{R}\backslash\{0\},$$

and consider the convolution equation

$$\text{(6.1)} \qquad x_\tau(t) - \tfrac{1}{2}c\int_0^\tau Ei(t-s)x_\tau(s)ds = \tfrac{1}{2}cEi(t), \quad 0 < t < \tau.$$

Since $\frac{1}{2}c\int_{-\tau}^{+\tau}|Ei(t)|dt = c\int_0^{+\tau} Ei(t)dt = c[1 - Ei_2(\tau)] < 1$ ($0 < c \le 1$ and $0 < \tau < +\infty$; $0 < c < 1$ and $\tau = +\infty$), it is clear that Eq. (6.1) has a unique solution $x_\tau \in L_1(0,\tau)$, which is nonnegative. Hence, for $1 \le p \le +\infty$ the convolution equation

$$f(t) - \tfrac{1}{2}c\int_0^\tau Ei(t-s)f(s)ds = \omega(t), \quad 0 < t < \tau,$$

has a unique solution $f \in L_p(0,\tau)$ for every $\omega \in L_p(0,\tau)$, which is given by

$$\text{(6.2)} \qquad f(t) = \omega(t) + \int_0^\tau \delta_\tau(t,s)\omega(s)ds, \quad 0 < t < \tau.$$

Because $\frac{1}{2}cEi$ is an even function, the resolvent kernel $\delta_\tau(t,s)$ has the form

$$\text{(6.3)} \qquad \delta_\tau(t,s) = \begin{cases} x_\tau(|t-s|) + \int_0^{\min(t,s)} [x_\tau(t-r)x_\tau(s-r) - \\ \qquad - x_\tau(\tau+r-t)x_\tau(r-s+\tau)]dr; \\ \qquad\qquad (0 < t \ne s < \tau) \\ x_\tau(|t-s|) + \int_{\max(t,s)}^{\tau} [x_\tau(r-t)x_\tau(r-s) - \\ \qquad - x_\tau(\tau+t-r)x_\tau(\tau+s-r)]dr \end{cases}$$

(cf. [4,32]; [21], p.99). In particular, $\delta_\tau(0,s) = x_\tau(t)$ and $\delta_\tau(\tau,s) = x_\tau(\tau-s)$.

For $0 < \mu \le +1$ one introduces the unique L_∞-solution $y_{\tau,\mu}$ of the convolution equation

$$\text{(6.4a)} \qquad y_{\tau,\mu}(t) - \tfrac{1}{2}c\int_0^\tau Ei(t-s)y_{\tau,\mu}(s)ds = e^{-t/\mu}, \quad 0 < t < \tau.$$

Put

$$\text{(6.4b)} \qquad X_\tau(\mu) = y_{\tau,\mu}(0), \quad Y_\tau(\mu) = y_{\tau,\mu}(\tau); \quad 0 < \mu \le +1.$$

From (6.4a) and (6.2) one gets

$$\text{(6.4c)} \qquad y_{\tau,\mu}(t) = e^{-t/\mu} + \int_0^\tau \delta_\tau(t,s)e^{-s/\mu}; \quad 0 < \mu \le +1,\ 0 < t < \tau.$$

But $\delta_\tau(0,s) = x_\tau(s)$ and $\delta_\tau(\tau,s) = x_\tau(\tau-s)$, and therefore

$$\text{(6.5a)} \qquad X_\tau(\mu) = 1 + \int_0^\tau e^{-s/\mu}x_\tau(s)ds;$$

$$\qquad\qquad (0 < \mu \le 1)$$

$$\text{(6.5b)} \qquad Y_\tau(\mu) = e^{-\tau/\mu} + \int_0^\tau e^{-(\tau-s)/\mu}x_\tau(s)ds.$$

Hence, X_τ and Y_τ are continuous on the closed interval [0,1], and $X_\tau(0) = 1$ and $Y_\tau(0) = 0$.

The auxiliary functions X_τ and Y_τ appear for the first time in the work of CHANDRASEKHAR [10], who denoted them by X and Y, respectively. A thorough study of these functions has been made by BUSBRIDGE [8] and MULLIKIN [57]. Chandrasekhar has introduced X_τ and Y_τ as solutions of the nonlinear integral equations

$$(6.6a)\qquad X_\tau(\mu) = 1 + \tfrac{1}{2}c\mu \int_0^1 (\nu+\mu)^{-1}[X_\tau(\mu)X_\tau(\nu) - Y_\tau(\mu)Y_\tau(\nu)]d\nu;$$

$$(0 \le \mu \le 1,\ 0 < \tau < +\infty)$$

$$(6.6b)\qquad Y_\tau(\mu) = e^{-\tau/\mu} + \tfrac{1}{2}c\mu \int_0^1 (\nu-\mu)^{-1}[X_\tau(\mu)Y_\tau(\nu) - Y_\tau(\mu)X_\tau(\nu)]d\nu.$$

Busbridge showed that the functions X_τ and Y_τ defined by (6.4b) satisfy the system (6.6). Mullikin obtained all other solutions of Eqs (6.6) and derived linear constraints, which force the functions (6.4b) to become the unique solution of Eqs (6.6).

If $\tau \to +\infty$, then, according to the "projection method" (cf. [21], for instance),

$$\lim_{\tau\to+\infty} |X_\tau(\mu) - X_\infty(\mu)| = 0, \qquad \lim_{\tau\to+\infty} |Y_\tau(\mu)| = 0 \qquad (0 \le \mu \le 1).$$

Here we restrict ourselves to the case when $0 < c < 1$, and put

$$X_\infty(\mu) = 1 + \int_0^{+\infty} e^{-s/\mu} x_\infty(s)ds, \qquad 0 \le \mu \le 1,$$

where x_∞ is the unique L_1-solution of the Wiener-Hopf equation

$$x_\infty(t) - \tfrac{1}{2}c \int_0^{+\infty} Ei(t-s)x_\infty(s)ds = \tfrac{1}{2}cEi(t), \qquad 0 < t < +\infty.$$

Note that the symbol of this equation is given by

$$1 - \tfrac{1}{2}c \int_{-\infty}^{+\infty} e^{t/\lambda}Ei(t)dt = (1-c) + \tfrac{1}{2}c \int_{-1}^{+1} \mu(\mu-\lambda)^{-1}d\mu, \quad \lambda \notin [-1,+1].$$

On the other hand, with the non-conservative indicatrix $\hat{g}(t) \equiv \tfrac{1}{2}c$, one associates the dispersion function Λ (cf. Section VI.2) of the form

$$\Lambda(\lambda) = \det[I - \lambda(\lambda-T)^{-1}B] = (1-c) + \tfrac{1}{2}c \int_{-1}^{+1} \mu(\mu-\lambda)^{-1}d\mu, \ \lambda \notin [-1,+1].$$

One sees that

$$\Lambda(\lambda) = X_\infty(\lambda)^{-1}\, X_\infty(-\lambda)^{-1}, \qquad \text{Re } \lambda = 0$$

(cf. Theorem II 3.2). Hence, X_∞ coincides with the H-function (cf. (2.5) - (2.6); note that $H(0) = 1$ and $X_\infty(0) = 1$).

<u>THEOREM 6.1</u>. *At the boundary points the solution ψ of the homogeneous finite-slab problem with boundary value $\phi \in L_2[-1,+1]$ is given by*

$$(6.7a)\qquad \psi(0,\mu) = e^{\tau/\mu}\phi(\mu) + \tfrac{1}{2}c \int_0^1 \nu(\nu-\mu)^{-1}[X_\tau(-\mu)X_\tau(\nu) - Y_\tau(\nu)Y_\tau(-\mu)]\phi$$

$$(-1 \le \mu < 0)$$

$$+ \tfrac{1}{2}c \int_{-1}^0 \nu(\nu-\mu)^{-1}[X_\tau(-\mu)Y_\tau(-\nu) - X_\tau(-\nu)Y_\tau(-\mu)]\cdot \phi(\nu)d\nu;$$

$$(6.7b)\qquad \psi(\tau,\mu) = e^{-\tau/\mu}\phi(\mu) + \tfrac{1}{2}c \int_{-1}^0 \nu(\nu-\mu)^{-1}[X_\tau(\mu)X_\tau(-\nu) - Y_\tau(-\nu)Y_\tau(\mu)]\phi$$

$$(0 < \mu \le +1)$$

$$+ \tfrac{1}{2}c \int_0^1 \nu(\nu-\mu)^{-1}[X_\tau(\mu)Y_\tau(\nu) - X_\tau(\nu)Y_\tau(\mu)]\phi(\nu)$$

In order to prove this theorem two propositions are needed. We now state the first one of them.

<u>PROPOSITION 6.2</u>. *Let $1 \le p \le +\infty$. For every function $\omega \in L_p((0,\tau);L_2[-1,+1$ there is a unique L_p-solution ψ of the convolution equation*

$$(6.8)\qquad \psi(t) - \int_0^\tau H(t-s)B\psi(s)ds = \omega(t) \qquad (0 < t < \tau),$$

namely

$$(6.9)\qquad \psi(t,\mu) = \begin{cases} \omega(t,\mu) + \frac{c}{2\mu} \int_{-1}^{+1} \int_0^t \omega(s,\nu)[e^{-(t-s)/\mu} + \int_0^t e^{-(t-r)/\mu}\delta_\tau(t,r)dr]drd\nu; & (0 < \mu \le +1) \\ \omega(t,\mu) - \frac{c}{2\mu} \int_{-1}^{+1} \int_t^{+\infty} \omega(s,\nu)[e^{-(t-s)/\mu} + \int_t^\tau e^{-(t-r)/\mu}\delta_\tau(t,r)dr]drd\nu, & (-1 \le \mu < 0), \end{cases}$$

where $\delta_\tau(t,s)$ can be expressed in terms of the function x_τ by the formula (6.3).

<u>PROOF</u>. For $0 < c \le 1$ the pair (T,B), with T and B defined by $(Th)(\mu) = \mu h(\mu)$ and $(Bh)(\mu) = \tfrac{1}{2}c \int_{-1}^{+1} h(\mu')d\mu'$ $(-1 \le \mu \le +1)$, is a semi-definite admissible pair on $L_2[-1,+1]$ (cf. Theorem 1.1). Then for $1 \le p \le +\infty$ and $\omega \in L_p((0,\tau);L_2[-1,+1])$ the convolution equation (6.8) has a unique solut

$L_p((0,\tau);L_2[-1,+1])$ (for $0 < c < 1$, when the pair (T,B) is positive definite, this follows from Theorem V 4.1; for $c = 1$, when this pair is singularly semi-definite, this is clear by the remark at the end of Section V.4). This unique solution ψ is given by the formula

$$(6.10) \qquad \psi(t) = \omega(t) + \int_0^\tau \gamma(t,s)\omega(s)ds \quad (0 < t < \tau),$$

where the resolvent kernel satisfies the convolution equation

$$\gamma(t,s) - \int_0^\tau \gamma(t,r)H(r-s)B dr = H(t-s)B \quad (0 < t \neq s < \tau)$$

(cf. [22], formula (2.5)). From this equation one sees that $\gamma(t,s)h \equiv 0$ for all $h \in \operatorname{Ker} B$, and therefore $\gamma(t,s)h = \frac{1}{2}\langle h,e\rangle\gamma(t,s)e$ $(0 < t \neq s < \tau)$. Here $e(\mu) \equiv 1$, and $\langle\cdot,\cdot\rangle$ denotes the usual inner product on $L_2[-1,+1]$. From formula (2.4) of [22] it follows that the resolvent kernel $\gamma(t,s)$ also satisfies the convolution equation

$$(6.11) \qquad \gamma(t,s) - \int_0^\tau H(t-r)B\gamma(r,s)dr = H(t-s)B \quad (0 < t \neq s < \tau).$$

Substituting $B = \frac{1}{2}c\langle\cdot,e\rangle e$ into this equation and putting $\delta_\tau(t,s) = \frac{1}{2}c\langle\gamma(t,s)e,e\rangle$ we obtain

$$(6.12) \qquad \gamma(t,s)e = c\left[H(t-s)e + \int_0^\tau \delta_\tau(r,s)H(t-r)e\,dr\right], \quad 0 < t \neq s < \tau.$$

From Eq. (6.11) one directly computes that $\delta_\tau(t,s)$ satisfies the convolution equation

$$(6.13) \qquad \delta_\tau(t,s) - \tfrac{1}{2}c\int_0^\tau \mathrm{Ei}(t-r)\delta_\tau(r,s)dr = \tfrac{1}{2}c\mathrm{Ei}(t-s), \quad 0 < t \neq s < \tau.$$

With the help of the uniqueness part of Theorem 2.1 in [22] it follows that $\delta_\tau(t,s)$ is given by (6.3). From (6.10) - (6.12) we obtain the identity

$$\psi(t) = \omega(t) + \tfrac{1}{2}c\int_0^\tau \langle\omega(s),e\rangle\left[H(t-s)e + \int_0^\tau \delta_\tau(r,s)H(t-r)e\,dr\right]ds.$$

Note that $\langle\omega(s),e\rangle = \int_{-1}^{+1}\omega(s,\nu)d\nu$, $H(t)(\mu) = 0$ for $t\mu < 0$ and $H(t)(\mu) = |\mu|^{-1}e^{-t/\mu}$ for $t\mu > 0$. But then (6.9) is clear. □

The second proposition is immediate from the first one, if one inser $\omega(t,\mu) = e^{-t/\mu}\phi(\mu)$ $(0 < \mu \le 1,\ 0 < t < \tau)$ and $\omega(t,\mu) = e^{(\tau-t)/\mu}\phi(\mu)$ $(-1 \le \mu < 0,\ 0 < t < \tau)$.

PROPOSITION 6.3. *To every boundary value $\phi \in L_2[-1,+1]$ there is a unique solution ψ of the homogeneous finite-slab problem, namely*

$$(6.14a) \qquad \psi(t,\mu) = \begin{cases} e^{-t/\mu}\phi(\mu) + \frac{1}{2}c\int_{-1}^{+1}K_t(\mu,\nu)\phi(\nu)d\nu, & 0 < \mu \le +1; \\ e^{(\tau-t)/\mu}\phi(\mu) + \frac{1}{2}c\int_{-1}^{+1}K_t(\mu,\nu)\phi(\nu)d\nu, & -1 \le \mu < 0. \end{cases}$$

Here the kernel $K_t(\mu,\nu)$ has the form

$$(6.14b) \qquad K_t(\mu,\nu) = \begin{cases} +\mu^{-1}\int_0^t p(s,\nu)[e^{-(t-s)/\mu} + \int_0^t \delta_\tau(r,s)e^{-(t-r)/\mu}dr]ds, \\ \qquad 0 < \mu \le +1; \\ -\mu^{-1}\int_t^\tau p(s,\nu)[e^{-(t-s)/\mu} + \int_t^\tau \delta_\tau(r,s)e^{-(t-r)/\mu}dr]ds, \\ \qquad -1 \le \mu < 0, \end{cases}$$

where $p(s,\mu) = e^{-s/\mu}$ $(0 < \mu \le +1)$ *and* $p(s,\mu) = e^{(\tau-s)/\mu}$ $(-1 \le \mu < 0)$.

PROOF of Theorem 6.1. For $-1 \le \mu < 0$ we substitute $t = 0$ into Eqs (6.14) and obtain

$$(6.15a) \qquad \begin{aligned} \psi(0,\mu) = e^{\tau/\mu}\phi(\mu) - \tfrac{1}{2}c\mu^{-1}\int_0^1\int_0^\tau e^{-s/\nu}\left[e^{+s/\mu} + \int_0^\tau \delta_\tau(r,s)e^{+r/\mu}dr\right]ds\,\phi(\cdot \\ (-1 \le \mu < 0) \\ - \tfrac{1}{2}c\mu^{-1}\int_{-1}^0\int_0^\tau e^{(\tau-s)/\nu}\left[e^{+s/\mu} + \int_0^\tau \delta_\tau(r,s)e^{+r/\mu}dr\right]ds\,\phi(\nu)d\nu. \end{aligned}$$

For $0 < \mu \le +1$ we substitute $t = \tau$ into (6.14) and obtain

$$(6.15b) \qquad \begin{aligned} \psi(\tau,\mu) = e^{-\tau/\mu}\phi(\mu) + \tfrac{1}{2}c\mu^{-1}\int_{-1}^0\int_0^\tau e^{(\tau-s)/\nu}\,\cdot \\ \cdot\left[e^{-(\tau-s)/\mu} + \int_0^\tau \delta_\tau(r,s)e^{-(\tau-r)/\mu}dr\right]ds\,\phi(\nu)d\nu \\ (0 < \mu \le +1) \\ + \tfrac{1}{2}c\mu{-1}\int_0^1\int_0^\tau e^{-s/\nu}\left[e^{-(\tau-s)/\mu} + \int_0^\tau \delta_\tau(r,s)e^{-(\tau-r)/\mu}dr\right]ds\,\phi(\nu)d\nu \end{aligned}$$

We consider the double integrals over the variables r and s appearing in the above expressions. They can be considered as double Laplace transforms of $\delta(r-s) + \delta_\tau(r,s)$, where $\delta(r-s)$ denotes Diracs delta function, and they can be computed with the help of a result of DYM & GOHBERG (cf. [14], Theorems 7.2 and 7.3). In fact we have

$$\int_0^\tau e^{-s/\nu}\left[e^{+s/\mu} + \int_0^\tau \delta_\tau(r,s)e^{+r/\mu}\,dr\right]ds = \frac{\mu\nu}{\mu-\nu}e^{-\tau/\nu}\{X_\tau(-\mu)Y_\tau(-\nu) - X_\tau(-\nu)Y_\tau(-\mu)\};$$
$$(-1 \le \mu < 0,\ 0 < \nu \le +1)$$

$$\int_0^\tau e^{(\tau-s)/\nu}\left[e^{+s/\mu} + \int_0^\tau \delta_\tau(r,s)e^{+r/\mu}\,dr\right]ds = \frac{\mu\nu}{\mu-\nu}\{X_\tau(-\mu)Y_\tau(-\nu) - X_\tau(-\nu)Y_\tau(-\mu)\};$$
$$(-1 \le \mu < 0,\ -1 \le \nu < 0)$$

$$\int_0^\tau e^{(\tau-s)/\nu}\left[e^{-(\tau-s)/\mu} + \int_0^\tau \delta_\tau(r,s)e^{-(\tau-r)/\mu}\,dr\right]ds = \frac{\mu\nu}{\nu-\mu}e^{+\tau/\nu}\{X_\tau(\mu)Y_\tau(\nu) - X_\tau(\nu)Y_\tau(\mu)\};$$
$$(0 < \mu \le +1,\ -1 \le \nu < 0)$$

$$\int_0^\tau e^{-s/\nu}\left[e^{-(\tau-s)/\mu} + \int_0^\tau \delta_\tau(r,s)e^{-(\tau-r)/\mu}\,dr\right]ds = \frac{\mu\nu}{\nu-\mu}\{X_\tau(\mu)Y_\tau(\nu) - X_\tau(\nu)Y_\tau(\mu)\}.$$
$$(0 < \mu \le +1,\ 0 < \nu \le +1)$$

To deduce the latter pair of identities from the former pair of identities, we use that $\delta_\tau(t,s) = \delta_\tau(\tau-t,\tau-s)$ $(0 < t \neq s < \tau)$ (cf. (6.3)). In the first and third identity we have defined X_τ and Y_τ for negative values of their arguments by (6.5a) and (6.5b), respectively. By (6.5a) and (6.5b) we get $e^{-\tau/\nu}\,Y_\tau(-\nu) = X_\tau(\nu)$ and $e^{-\tau/\nu}\,X_\tau(-\nu) = Y_\tau(\nu)$ $(0 < \nu \le +1)$ and $e^{+\tau/\nu}\,Y_\tau(\nu) = X_\tau(-\nu)$ and $e^{+\tau/\nu}\,X_\tau(\nu) = Y_\tau(-\nu)$ $(-1 \le \nu < 0)$. Then the first and third identity become as follows:

$$\int_0^\tau e^{-s/\nu}\left[e^{+s/\mu} + \int_0^\tau \delta_\tau(r,s)e^{+r/\mu}\,dr\right]ds = \frac{\mu\nu}{\mu-\nu}\{X_\tau(-\mu)X_\tau(\nu) - Y_\tau(\nu)Y_\tau(-\mu)\};$$
$$(-1 \le \mu < 0,\ 0 < \nu \le +1)$$

$$\int_0^\tau e^{(\tau-s)/\nu}\left[e^{-(\tau-s)/\mu} + \int_0^\tau \delta_\tau(r,s)e^{-(\tau-r)/\mu}\,dr\right]ds = \frac{\mu\nu}{\nu-\mu}\{X_\tau(\mu)X_\tau(-\nu) - Y_\tau(-\nu)Y_\tau(\mu)\}.$$
$$(0 < \mu \le +1,\ -1 \le \nu < 0)$$

The theorem is clear if we substitute these four (modified) identities into (6.15). □

We have derived an analytic solution of the finite-slab problem in a homogeneous medium without deriving an explicit formula for the inverti operator V_τ^{-1}. To obtain a formula for $V_\tau^{-1}\phi$ one first calculates the solui ψ of the homogeneous finite-slab problem with boundary value ϕ. If P_p (P_r denotes, as usual, the spectral projection of $S = A^{-1}T$ corresponding to t positive (negative) part of its spectrum (note that we now restrict ourselves to the non-conservative case $0 < c < 1$), then

$$P_p V_\tau^{-1}\phi = P_p\psi(0), \quad P_m V_\tau^{-1}\phi = P_m\psi(\tau)$$

(cf. Section IV.2). For the non-conservative isotropic case the projectic P_p and P_m are known explicitly (cf. [36], for instance).

On the basis of physical arguments the finite-slab problem has been analyzed by CHANDRASEKHAR [10] and SOBOLEV [63]. With the help of invariance principles Chandrasekhar reduced the problem to the computation of t so-called reflection and transmission functions. For several cases, inclu ing isotropic scattering, he produced explicit solutions, which were expr sed in terms of auxiliary functions X and Y that satisfy a coupled system of non-linear integral equations (such as (6.6a) - (6.6b)). These results were improved and generalized by SOBOLEV (see [63,65]). In [63] Sobolev produced the convolution equation (6.13), but employs iteration as a tool To some extent the present approach can be viewed as a mathematical justi fication of the physical methods in [10] and [63].

7. The symmetric multigroup Transport Equation

In this section the symmetric multigroup Transport Equation is the object of investigation. This equation describes the time-independent transport of particles through a homogeneous semi-infinite medium. We assı that the particles are divided into N groups with a (nearly) constant spe Let x be a position coordinate from $(0,+\infty)$, the number $\mu \in [-1,+1]$ the cosine of the scattering angle and $\psi_1,\dots,\psi_N$ the angular densities within the respective groups. Further, let $\sigma_1,\dots,\sigma_N$ be numbers proportionate to the mean speeds within the groups, ordered in such a way that $\sigma_1 \ge \dots \ge \sigma_N$ By Σ we denote the N×N-diagonal matrix with diagonal entries $\sigma_1,\dots,\sigma_N$.

The scattering is described by the N×N-scattering matrix $\hat{G}(\mu)$, which depends on $\mu \in [-1,+1]$ and all of whose entries $\hat{g}_{ij}(\mu)$ are real-valued L_r-functions on $[-1,+1]$ for some $r > 1$. For $i,j = 1,2,\dots,N$ one writes

$$g_{ij}(\mu,\mu') = (2\pi)^{-1} \int_{-1}^{+1} \hat{g}_{ij}(\mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\,\cos\alpha)\,d\alpha,$$

and $G(\mu,\mu')$ denotes the N×N-matrix with entries $g_{ij}(\mu,\mu')$ $(i = 1,\dots,N;\ j = 1,\dots,N)$. Let $\psi(x,\mu)$ be the column vector with entries $\psi_1(x,\mu),\dots,\psi_N(x,\mu)$, and let $f(x,\mu)$ be the column vector with entries $f_1(x,\mu),\dots,f_N(x,\mu)$. Here $f_i(x,\mu)$ is an inhomogeneous term accounting for sources of particles within the i-th group $(i = 1,\dots,N)$. Then the multigroup transport process is described by the vector-valued integro-differential equation

$$(7.1)\qquad \mu\frac{\partial\psi}{\partial x}(x,\mu) + \Sigma\psi(x,\mu) = \int_{-1}^{+1} G(\mu,\mu')\psi(x,\mu')\,d\mu' + f(x,\mu), \qquad 0 < x < +\infty,$$

with boundary conditions

$$(7.2)\qquad \lim_{x\downarrow 0} \psi_i(x,\mu) = \phi_{+,i}(\mu) \qquad (0 \le \mu \le +1;\ i = 1,2,\dots,N);$$
$$\sum_{i=1}^{N} \sigma_i \int_{-1}^{+1} |\psi_i(x,\mu)|^2\,d\mu = O(1) \qquad (x \to +\infty).$$

The column with entries $\phi_{+,1}(\mu),\dots,\phi_{+,N}(\mu)$ will be denoted by $\phi_+(\mu)$. Additionally we impose the following symmetry condition:

$$(7.3)\qquad \hat{g}_{ij}(t) = \hat{g}_{ji}(t); \quad -1 \le t \le +1,\ 1 \le i,j \le N.$$

We shall consider Eq. (7.1) with boundary conditions (7.2) on the space $H = L_2([-1,+1];\mathbb{C}^N)$ of $\mathbb{C}^N$-valued square-integrable functions on $[-1,+1]$, endowed with the (weighted) inner product

$$(7.4)\qquad \langle f,g\rangle = \sum_{i=1}^{N} \sigma_i \int_{-1}^{+1} f_i(\mu)\overline{g_i(\mu)}\,d\mu,$$

where f (g) is the column of functions $f_1,\dots,f_N$ $(g_1,\dots,g_N)$. Define the operators $B_{ij}\colon L_2[-1,+1] \to L_2[-1,+1]$ by

$$(B_{ij}h)(\mu) = \int_{-1}^{+1} \left[(2\pi)^{-1} \int_0^{2\pi} \hat{g}_{ij}(\mu\mu' + \sqrt{1-\mu^2}\,\sqrt{1-\mu'^2}\,\cos\alpha)\,d\alpha \right] h(\mu')\,d\mu', \quad -1 \le \mu \le$$

where $1 \le i,j \le N$. According to [69], Appendix XII.8 (see also Section VI all operators B_{ij} are compact and self-adjoint. We define the operator $B\colon H \to H$ by

$$(7.5)\qquad (Bf)_i(\mu) = \sigma_i^{-1} \sum_{j=1}^{N} (B_{ij}f_j)(\mu), \quad -1 \le \mu \le +1.$$

By the symmetry condition (7.3) one has $B_{ij}^* = B_{ji}$ $(1 \le i,j \le N)$, and ther fore the operator $B\colon H \to H$ is self-adjoint and compact. Put $A = I - B$ and define the operator $T\colon H \to H$ by

$$(7.6)\qquad (Tf)_i(\mu) = \sigma_i^{-1}\,\mu f(\mu); \quad -1 \le \mu \le +1,\ 1 \le i \le N.$$

Then T is a self-adjoint operator with spectrum $\bigcup_{i=1}^{N}[-\sigma_i^{-1},\sigma_i^{-1}] = [-1,+1]$. Its kernel is trivial.

<u>THEOREM 7.1</u>. *Let* H *be the Hilbert space* $L_2([-1,+1];\mathbb{C}^N)$ *endowed with the inner product* (7.4), *and let the operators* T *and* B *on* H *be defined by*

$$(Tf)_i(\mu) = \sigma_i^{-1}\,\mu f(\mu),$$
$$(Bf)_i(\mu) = \sigma_i^{-1} \sum_{j=1}^{N} \int_{-1}^{+1}\left[(2\pi)^{-1} \int_0^{2\pi} \hat{g}_{ij}(\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}\,\cos\alpha)\,d\alpha \right] f_j(\mu')\,d$$
$$(-1 \le \mu \le +1,\ 1 \le i \le N$$

Then the pair (T,B) *is a hermitian admissible pair on* H. *This pair is inversion symmetric with signature operator* $J\colon H \to H$, *defined by*

$$(Jf)_i(\mu) = f_i(-\mu); \quad -1 \le \mu \le +1,\ 1 \le i \le N,$$

as an inversion symmetry.

<u>PROOF</u>. Clearly, T is self-adjoint with Ker T = {0} and B is compact and self-adjoint. Further, there exists $r > 1$ such that $\hat{g}_{ij} \in L_r[-1,+1]$ $(1 \le i,j \le N)$. By Theorem 1.1 we have $B_{ij} = |T_0|^{\alpha} D_{ij}$ for some $0 < \alpha < (r-1)(2r)^{-1}$ and some compact operator D_{ij} $(1 \le i,j \le N)$. Here T_0 i

the self-adjoint operator on $L_2[-1,+1]$, defined by $(T_0 h)(\mu) = \mu h(\mu)$ $(-1 \le \mu \le +1)$. It follows that

$$B = |T|^{\alpha} D,$$

where D is the compact operator defined by $(Df)_i(\mu) = \sigma_i^{\alpha-1} \sum_{j=1}^{N} (D_{ij} f_j)(\mu)$ $(-1 \le \mu \le +1;\ 1 \le i \le N)$. Hence, the pair (T,B) fulfills the conditions (C.1) - (C.3) of Section III.2, and therefore (T,B) is a hermitian admissible pair on H. □

The theory developed in Chapters III to V is now applicable. If $A = I - B$ is a strictly positive operator, then to every boundary value ϕ_+ there is a unique solution of the integro-differential equation (7.1) such that

$$\sum_{i=1}^{N} \sigma_i \int_0^{+\infty} \left[\int_{-1}^{+1} |\psi_i(x,\mu)|^2 \, d\mu \right]^{\frac{1}{2}} dx < +\infty \qquad (0 < x < +\infty).$$

A related result has been obtained before by GREENBERG [34]. A preliminary version of the present treatment of the symmetric multigroup case can be found in [53].

LITERATURE

[1] AMBARTSUMIAN, V.A., *Diffusion of light by planetary atmospheres*, Astron. Zhurnal 19, 30-46, 1942 (Russian).

[2] BART, H., GOHBERG, I. & M.A. KAASHOEK, *Minimal factorization of matr and operator functions. Operator Theory: Advances and Applications 1*, Birkhäuser Verlag, 1979.

[3] BART, H., GOHBERG, I., KAASHOEK, M.A. & P. VAN DOOREN, *Factorization of transfer functions*, Report 107, Vrije Universiteit, Amster 1979.

[4] BAXTER, G. & I.I. HIRSCHMANN, *An explicit inversion formula for fin section Wiener-Hopf operators*, Bull. A.M.S. 70, 820-823, 1964.

[5] BOER, B. DEN, *Linearization of operator functions on arbitrary open sets*, Integral equations and operator theory 1, 19-27, 1978.

[6] BOGNÁR, J., *Indefinite inner product spaces*, Berlin, Springer Verlag 1974.

[7] BRODSKII, V.M., GOHBERG, I.C. & M.G. KREIN, *On characteristic functi of an invertible operator*, Acta Sci. Math. (Szeged) 32, 141-16 1975 (Russian).

[8] BUSBRIDGE, I.W., *The mathematics of radiative transfer*, Cambridge Tracts in Mathematics and Mathematical Physics, vol. 50, Cambridge, University Press, 1960.

[9] CASE, K.M. & P.F. ZWEIFEL, *Linear transport theory*, Addison-Wesley, 1967.

[10] CHANDRASEKHAR, S., *Radiative transfer*, second revised edition, New Y Dover Publ. Inc., 1960.

[11] CRUM, M.M., *On an integral equation of Chandrasekhar*, The quarterly Journal of Mathematics 18, 244-252, 1947.

[12] DUDERSTADT, J.J. & W.R. MARTIN, *Transport Theory*, A Wiley-Interscien Publication, John Wiley & Sons, 1979.

[13] DUNFORD, N. & J.T. SCHWARZ, *Linear operators I, II, III*, New York/ London, Interscience, 1958.

[14] DYM, H. & I. GOHBERG, *On an extension problem, generalized Fourier analysis, and an entropy formula*, Integral Equations and Operator Theory 3, 143-215, 1980.

[15] FELDMAN, I.A., *The radiative transport equation and operator Wiener-Hopf equations*, Funct. Anal. Appl. 5, 262-264, 1971 = Funkcional. Anal. i Priložen 5 (3), 106-108, 1971 (Russian).

[16] FELDMAN, I.A., *Wiener-Hopf operator equations and their application to the transport equation*, Integral Equations and Operator Theory 3, 43-61, 1980 = Matem. Issled. 6 (3), 115-132, 1971 (Russian).

[17] FELDMAN, I.A., *On some projection methods for the solution of the equation of radiative energy transfer*, Matem. Issled. 7 (4), 228-236, 1972 (Russian).

[18] FELDMAN, I.A., *On Wiener-Hopf equations with weakly integrable operator-valued kernels*, Matem. Issled. 8 (4), 101-110, 1973 (Russian).

[19] FELDMAN, I.A., *On the discrete spectrum of the characteristic equation in the theory of radiative transfer*, Matem. Issled. 10 (1), 236-243, 1975 (Russian).

[20] GOHBERG, I.C., *The factorization problem for operator functions*, Izvestija Akad. Nauk SSSR, Serija Mat. 28, 1055-1082, 1964 (Russian).

[21] GOHBERG, I.C. & I.A. FELDMAN, *Convolution equations and projection methods for their solution*, Transl. Math. Monographs, vol. 41, A.M.S., Providence, R.I., 1974 = Moscow, "Nauka", 1971 (Russian).

[22] GOHBERG, I.C. & G. HEINIG, *On matrix integral operators on a finite interval with kernels depending on the difference of their arguments*, Rev. Roum. Math. Pures et Appl. 20, 55-73, 1975 (Russian).

[23] GOHBERG, I., KAASHOEK, M.A. & D.C. LAY, *Equivalence, linearization and decomposition of holomorphic operator functions*, J. Funct. Anal. 28, 102-144, 1978.

[24] GOHBERG, I.C. & M.G. KREIN, *Systems of integral equations on a half-line with kernels depending on the difference of arguments*, A.M.S. Transl. 14, 217-287, 1960 = Uspehi Mat. Nauk 13 (2), 3- (Russian).

[25] GOHBERG, I.C. & M.G. KREIN, *Introduction to the theory of linear non selfadjoint operators in Hilbert space*, Transl. Math. Monographs, vol. 18, A.M.S., Providence, R.I., 1969 = Moscow, "Nauka", 1965 (Russian).

[26] GOHBERG, I., LANCASTER, P. & L. RODMAN, *Spectral analysis of matrix polynomials, I, Canonical forms and divisors*, Linear Algebra Appl. 20, 1-44, 1978.

[27] GOHBERG, I., LANCASTER, P. & L. RODMAN, *On self-adjoint matrix polynomials*, Integral Equations and Operator Theory 1, 434-439 1978.

[28] GOHBERG, I., LANCASTER, P. & L. RODMAN, *Spectral analysis of self-adjoint matrix polynomials*, Research paper 419, University of Calgary, Canada, 1979.

[29] GOHBERG, I.C. & J. LEITERER, *Factorization of operator functions wit respect to a contour, I. Finite-meromorphic operator functions* Math. Nachrichten 52 259-282, 1972 (Russian).

[30] GOHBERG, I.C. & J. LEITERER, *Factorization of operator functions wit respect to a contour, II. Canonical factorization of operator functions, close to the identity*, Math. Nachrichten 54, 41-74, 1972 (Russian).

[31] GOHBERG, I.C. & J. LEITERER, *Factorization of operator functions wit respect to a contour, III. Factorization in algebras*, Math. Nachrichten 55, 33-61, 1973 (Russian).

[32] GOHBERG, I.C. & A.A. SEMENÇUL, *On the invertibility of finite Toeplitz matrices and their continuous analogues*, Matem. Issled. 7 (2), 201-223, 1972 (Russian).

[33] GOHBERG, I.C. & E.I. SIGAL, *An operator generalization of the logarithmic residue theorem and the theorem of Rouché*, Math. USSR, Sbornik 13, 603-625, 1971 = Mat. Sbornik 84 (126), 607-629, 1971 (Russian).

[34] GREENBERG, W., *Functional calculus for the symmetric multigroup neutron transport operator*, J. Math. Phys. 17, 159-162, 1976.

[35] HANGELBROEK, R.J., *A functional-analytic approach to the linear transport equation*, Thesis, University of Groningen, 1973.

[36] HANGELBROEK, R.J., *Linear analysis and solution of neutron transport problems*, Transport Theory and Statistical Physics 5, 1-85, 1976.

[37] HANGELBROEK, R.J., *On the dispersion function of Transport Theory*, Transport Theory and Statistical Physics 8, 133-146, 1979.

[38] HANGELBROEK, R.J., *On the derivation of some formulas in linear transport theory for media with anisotropic scattering*, Report 7720, University of Nijmegen, 1978.

[39] HANGELBROEK, R.J., *Time-independent neutron transport equation with anisotropic scattering in absorbing media*, Report ANL 80-60, Argonne National Laboratory, Illinois, 1980.

[40] HANGELBROEK, R.J. & C.G. LEKKERKERKER, *Decompositions of a Hilbert space and factorization of a W-A determinant*, SIAM J. Math. Anal. 8, 458-472, 1977.

[41] HEINIG, G., *On a continuous analogue of the companion matrix of a polynomial and the linearization of some classes of holomorphic operator functions*, Beiträge zur Analysis 13, 111-126, 1979 (German).

[42] HOPF, E., *Mathematical problems of radiative equilibrium*, Cambridge University Press, 1934.

[43] KALMAN, R.E., FALB, P.F. & M.A. ARBIB, *Topics in mathematical systems theory*, New York, McGraw-Hill, 1969.

[44] KANTOROVIČ, L.V. & G.P. AKILOV, *Functional analysis in normed spaces*, Series of Monographs in Pure and Appl. Math., vol. 46, New York, McMillan, 1964 = Moscow, Fizmatgiz, 1959 (Russian).

[45] KATO, T., *Perturbation theory for linear operators*, Berlin/Heidelberg, Springer Verlag, 1966.

[46] LEITERER, J., *Factorization of operator functions with respect to a contour*, Habilitationsschrift, Part I, Akademie der Wissenschaften DDR, 1974 (German).

[47] LEKKERKERKER, C.G., *The linear transport equation. The degenerate ca c = 1, I. Full-range theory; II. Half-range theory*, Proc. Roya Soc. Edinburgh 75A, 259-282 & 283-295, 1975.

[48] LEKKERKERKER, C.G., *On eigendistributions in linear transport theory* Proc. Royal Soc. Ediburgh 83A, 303-326, 1979.

[49] MAGNUS, W., OBERHETTINGER, F. & R.P. SONI, *Formulas and theorems for the special functions of mathematical physics*, Third English edition, Heidelberg, Springer Verlag, 1966.

[50] MASLENNIKOV, M.V., *The Milne problem with anisotropic scattering*, Proc. of the Steklov institute of mathematics, vol. 97, 1968 = Trudy matem. instituta im. V.A. Steklova, AN SSSR, 97, 1968 (Russian).

[51] McCORMICK, N.J. & I. KUŠČER, *Surface densities and currents in half-space transport problems*, Transport Theory and Statistical Physics 1, 1-4, 1971.

[52] MEE, C.V.M. VAN DER, *Realization and linearization*, Report 109, Vrije Universiteit, Amsterdam, 1979.

[53] MEE, C.V.M. VAN DER, *Spectral analysis of the Transport Equation, I. Nondegenerate and multigroup case*, Integral Equations and Operator Theory 3, 529-573, 1980.

[54] MIKA, J.R., *Neutron transport with anisotropic scattering*, Nuclear science and engineering 11, 415-427, 1961.

[55] MILNE, E.A., *Radiative equilibrium in the outer layers of a star: the temperature distribution and the law of darkening*, Monthly notices R.A.S. 81, 361-375, 1921.

[56] MITIAGIN, B., *Linearization of holomorphic operator functions, I, II*, Integral Equations and Operator Theory 1, 114-131 & 226-249, 1978.

[57] MULLIKIN, T.W., *Chandrasekhar's X- and Y-equations*, Trans. A.M.S. 112, 316-332, 1964.

[58] MULLIKIN, T.W., *Wiener-Hopf factorizations*, Transport Theory and Statistical Physics 3, 215-227, 1973.

[59] MUSKHELISHVILI, N.I., *Singular integral equations*, Groningen, Noordhc 1953 = Moscow, "Nauka", 1967 (Russian).

[60] PAHOR, S., *A new approach to half-space transport problems*, Nuclear Science and Engineering 26, 192-197, 1966.

[61] PRÖSSDORF, S., *Some classes of singular equations*, North-Holland Publ. Cie., 1978.

[62] RINGROSE, J.R., *Compact non-selfadjoint operators*, London, Van Nostrand Reinhold Cie., 1971.

[63] SOBOLEV, V.V., *A treatise on radiative transfer*, Princeton (N.J.)., Van Nostrand Cie., Inc., 1963 = Radiative energy transfer in atmospheres of stars and planets, Moscow, Gosud. Izdat. Techn.-teoret. Lit., 1956 (Russian).

[64] SOBOLEV, V.V., *Anisotropic scattering of light in a semi-infinite atmosphere, II*, Soviet astronomy 12, 420-426, 1968 = Astrom. Zhurnal 45, 528-537, 1968 (Russian).

[65] SOBOLEV, V.V., *Light scattering in planetary atmospheres*, Oxford, Pergamon Press, 1975.

[66] STONE, M.H., *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Colloquium Publ., vol. 15, 1932.

[67] STUMMEL, F., *Discrete convergence of linear operators, II*, Math. Zeitschrift 120, 231-264, 1971 (German).

[68] SZEGÖ, G., *Orthogonal polynomials*, A.M.S. Colloquium Publ., vol. 23, Providence, R.I., United States, 1939.

[69] VLADIMIROV, V.S., *Mathematical problems of the one-speed particle transport theory*, Trudy matem. instituta im. V.A. Steklova, AN SSSR, 61, 1961 (Russian).

[70] WIENER, N. & E. HOPF, *On a class of singular integral equations*, S.B. Preuss. Akad. Wiss., 696-706, 1931 (German).

[71] ZAANEN, A.C., *Integration*, Amsterdam, North-Holland Publ. Cie., 1967.

AUTHOR INDEX

-o-o-o-

SUBJECT INDEX

-o-o-o-

SYMBOL INDEX

Subspaces and projections

Remaining symbols

-o-o-o-