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SPECTRAL ANALYSIS OF THE TRANSPORT EQUATION. I. NONDEGENERATE AND MULTIGROUP CASE.

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0. INTRODUCTION

In the theory of the one dimensional, linear Transport Equation there are two approaches. The first approach consists of the study of an operator differential equation of the form

$$(0.1) \qquad T\,\psi'(s) = -A\,\psi(s),\quad 0 < s < +\infty,$$

where T is the operator of multiplication by the independent variable, defined on the space $L_2[-1,+1]$, while A is the positive operator given by

$$(Af)(\mu) = f(\mu) - \int_{-1}^{+1} g(\mu,\mu')\, f(\mu')\, d\mu', \quad -1 \le \mu \le +1,$$

the function g being the scattering function. The solution of Eq. (0.1) is required to satisfy the following boundary condition:

$$(0.2) \qquad \lim_{s\downarrow 0} P_+\,\psi(s) = \varphi_+ .$$

Here P_+ is the orthogonal projection onto the maximal positive invariant subspace of T which corresponds here with the subspace of square integrable functions on [-1,+1] that vanish on the interval [-1,0].

The second approach focuses on the Wiener-Hopf operator integral equation

$$(0.3) \qquad \psi(s) - \int_0^{+\infty} H(s-t)\, B\,\psi(t)\, dt = \varphi(s),\quad 0 < s < +\infty,$$

where $\varphi(s)$ is a given vector function, while $H(t)$ is defined in terms of the operators T and P_+ introduced above and has the form

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$$H(t) = \begin{cases} +T^{-1}e^{-tT^{-1}} P_+ & t > 0. \\ -T^{-1}e^{-tT^{-1}}(I-P_+), & t < 0. \end{cases}$$

In this paper we forget about the special form the operators T and A have in the case of the Transport Equation and study Eqs (0.1) and (0.3), assuming only that T is an injective self-adjoint operator, B a compact operator for which A = I - B is strictly positive, while

$$B = |T|^{\alpha} D$$

for some $0 < \alpha < 1$ and some bounded operator D. This somewhat formal approach has the advantage that in applications to Transport Theory there is no difference between the treatment of the one-speed case and the symmetric multigroup case.

Under the conditions on T and B stated in the previous paragraph we prove that Eq. (0.1) with boundary condition (0.2) and the Wiener-Hopf operator integral equation (0.3) with right-hand side $\varphi(s) = e^{-sT^{-1}}\varphi_+$ are equivalent and we give explicit formulas for the solutions of both equations. Let m be a non-positive number such that the spectrum of T is contained in the interval $[m,+\infty)$. Then, for each $k > -m$, the solutions ψ of the differential equation (0.1) that satisfy

$$\int_0^{+\infty} e^{-t/k} \|\psi(t)\| \, dt < +\infty$$

have the form

$$\psi(s) = e^{-sT^{-1}A}\tilde{\varphi}, \; P_+ \tilde{\varphi} = \varphi_+.$$

In case one is searching for bounded solutions, there is a unique solution ψ to every boundary value φ_+, which has the form

$$(0.4) \qquad \psi(s) = e^{-sT^{-1}A}P \, \varphi_+, \; 0 < s < +\infty,$$

for some projection P. Because of the equivalence of the operator differential equation and the corresponding Wiener-Hopf integral equation, it is clear that the integrable solutions of Eq. (0.3) with right-hand side $\varphi(s) = e^{-sT^{-1}}\varphi_+$ are also given by (0.4). The solutions of the Wiener-Hopf equation (0.3) can also be obtained directly from a Wiener-Hopf factorization of the symbol of Eq. (0.3), using the method developed in [1], Chapter 1 and 6. By specifying our results for the operators appearing in the Transport Equation we obtain explicit formulas for the solutions of this equation, which are new both for a non-degenerate scattering function and for a multigroup case.

The two approaches hinge heavily upon the existence of the projection P appearing in (0.4). This projection has been introduced in [18]. In this paper we give a new proof of its existence which also applies to non-degenerate scattering functions.

Both the differential equation (0.1) and the Wiener-Hopf equation (0.3) have a long history in Transport Theory. An important step in solving (0.1) has been the use of the method of eigenfunction expansion by Case [3]. Although Case considered isotropic scattering only, his method stimulated a lot of research (reviewed in [22]). Hangelbroek first offered a mathematically rigorous approach [16]. An operator-theoretic approach is due to Hangelbroek and Lekkerkerker [18], where for the first time the basic projection P was introduced and the dispersion function (i.e., the perturbation determinant of the symbol of the Wiener-Hopf equation (0.3)) was factorized in order to obtain an explicit expression for P. The paper [18] only applies to isotropic scattering. Recently, Eq. (0.1) has been solved for the Transport Equation with a degenerate scattering function; also an explicit expression for P, the so-called half-range expansion formula, has been obtained (cf.[21]).

In comparison with the method of eigenfunction expansion, the Wiener-Hopf approach has not been used very often in mathematical physics (cf.[24]). However, judging from the generality of the results obtained, the Wiener-Hopf approach seems to be more promising. In [5] to [9], Feldman generalized the theory of systems of Wiener-Hopf integral equations, developed in [12] and [11], to the infinite-dimensional case and applied it to the Transport Equation with a non-degenerate scattering function. In this way he obtained the asymptotics of the solutions. Feldman, however, did not obtain a Wiener-Hopf factorization of the symbol of Eq. (0.3) nor solution formulas like (0.4). In this paper we shall supply these, making use of a geometric principle developed in [1], Chapter 1. In the case of the Transport Equation with a degenerate scattering function this factorization has been given in Chapter 6 of the same book.

After a preliminary section, we prove the existence of the projection P in Section 2. The third section is entirely devoted to the differential equation (0.1) which is solved with the help of semigroup theory. In Section 4 we prove the equivalence of the two approaches and obtain the asymptotics of the solutions of the two equations as a corollary.
In section 5 we construct a Wiener Hopf factorization of the symbol of Eq.(0.3) and obtain again the solution formula (0.4).
In Sections 6 and 7 we give the applications to the one-group and multigroup transport equation, respectively. In case of a degenerate scattering function we use the formulas for the factors of the Wiener-Hopf factorization obtained in Section 5 and get explicit half-range formulas. In contrast to similar formulas obtained earlier (cf.[18], for instance) there is no additional diagonalizing factor involved.

In Part II of the paper we shall present new results for the so-called finite-slab problem, i.e., the Transport Equation on a finite interval $(0,\tau)$. Again we shall consider both a differential equation (namely, Eq. (0.1) on the interval $(0,\tau)$) and a (finite section) Wiener-Hopf equation.

The approach offered in the present paper is essentially a Banach space approach. Basically, similar results can be obtained if we assume that T is an injective scalar operator acting on a Banach space with real spectrum, B a compact operator such that $A = I - B$ is invertible, while $B = |T|^{\alpha} D$ for some $0 < \alpha < 1$ and some bounded operator D. The main difference is the possible nonexistence of the projection P, due to the appearance of non-trivial bounded solutions of Eq.(0.1) with boundary value $\varphi_+ = 0$. A generalization to scalar operators T is desirable in order to treat the Transport Equation on the space $L_p[-1,+1]$, $p \neq 2$.

In a subsequent publication we shall deal with the case when A is a (non-strictly) positive operator and reduce this problem to the case considered here. Applications to the conservative case of the Transport Equation will be given.

1. PRELIMINARIES

In this paper $L(H)$ will denote the Banach algebra of bounded linear operators on a Hilbert space H, endowed with the usual operator norm $\|.\|$. The identity element of this algebra will be denoted by I. For each $T \in L(H)$ the spectrum of T will be denoted by $\sigma(T)$ and its resolvent set by $\rho(T)$. The null space (or kernel) and range of the operator T are denoted by Ker T and Im T, respectively.

Given a self-adjoint operator $T \in L(H)$ we denote by $m(T)$ and $M(T)$, respectively, the infimum and supremum of the numerical range of T, i.e.,

$$m(T) = \inf_{\|x\|\leq 1} \langle Tx,x\rangle,\quad M(T) = \sup_{\|x\|\leq 1} \langle Tx,x\rangle.$$

Here $\langle .,. \rangle$ denotes the inner product on the Hilbert space H. We have $\sigma(T) \subset [m(T),M(T)]$. For every continuous function $f\colon \sigma(T) \to \mathbb{C}$, with only a possible jump discontinuity at 0, we define

$$(1.1)\qquad f(T) = \int_{\sigma(T)} f(t)\, E(dt),$$

where E is the resolution of the identity of T. As it is known (cf.[4]),

$$(1.2)\qquad \|f(T)\| = \sup_{t\in\sigma(T)} |f(t)|.$$

From this equation we obtain for $0 < \alpha < 1$ and $\mathrm{Re}\,\lambda = 0$ $(\lambda \neq 0)$

$$(1.3)\qquad \| \; |T|^{\alpha}(T-\lambda)^{-1}\| \le \frac{c_\alpha}{|\mathrm{Im}\lambda|^{1-\alpha}},$$

where c_α is some constant depending on T and α only.

In Section 2 an important role is played by the following two propositions:

PROPOSITION 1.1. *For* $0 < \varphi < \frac{1}{2}\pi$, *let* $\Omega_\varphi = \{\lambda : |\frac{1}{2}\pi - \arg\lambda| \le \varphi\}$. *For every self-adjoint operator* T *with resolution of the identity* E *we have*

$$(1.4)\qquad \underset{\lambda\to 0,\lambda\in\Omega_\varphi}{\text{s-lim}} \; T(T-\lambda)^{-1} = I - E(\{0\}).$$

PROOF. By (1.1)

$$T(T-\lambda)^{-1} f = \int_{\sigma(T)} \frac{\mu}{\mu-\lambda} E(d\mu)f.$$

Since we have the estimate

$$\left|\frac{\mu}{\mu-\lambda}\right| \le 1 + \frac{1}{\cos\varphi} \quad (\lambda \in \Omega_\varphi,\ \mu \in \sigma(T) \subset \mathbb{R}),$$

we may apply the theorem of dominated convergence for vector-valued measures (cf. [4], Th. IV 10.10) and obtain

$$\lim_{\lambda\in\Omega_\varphi,\lambda\to 0} T(T-\lambda)^{-1} f = \int_{\sigma(T)} [1-\chi(\{0\})]\, E(d\mu)f.$$

Here χ denotes the characteristic function of the set $\{0\}$. From this, formula (1.4) is clear.

PROPOSITION 1.2. *For every nonnegative (self-adjoint) operator* T *with resolution of the identity* E *we have*

$$(1.5)\qquad \underset{\lambda\to 0,\mathrm{Re}\lambda\le 0}{\text{s-lim}} \; T(T-\lambda)^{-1} = I - E(\{0\}).$$

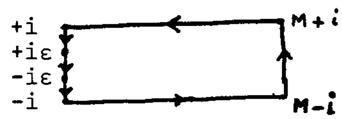
This proposition is proved in the same way as Proposition 1.1.

Next, we derive a formula for the values of the resolution of the identity of a self-adjoint operator. Although such a formula is well-known since the thirties (cf.[23,4]), we did not find in literature the form it has in the next proposition.

PROPOSITION 1.3. *Let* T *be a bounded self-adjoint operator with resolution of the identity* E. *If* M *is a finite constant exceeding* $\| T \|$, *we have*

$$\text{s-}\lim_{\varepsilon \downarrow 0} \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} (T-\lambda)^{-1} \, d\lambda = E(0,+\infty) + \tfrac{1}{2} E(\{0\}), \tag{1.6}$$

where Γ_ε *is the following curve*:



PROOF. Take a vector f. Using (1.1) we have

$$\frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} (T-\lambda)^{-1} f \, d\lambda = \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} \int_{\sigma(T)} (t-\lambda)^{-1} E(dt)f \quad d\lambda.$$

Applying a bounded linear functional to both sides of this identity, and observing that $(t,\lambda) \mapsto (t-\lambda)^{-1}$ is integrable on $\sigma(T) \times \Gamma_\varepsilon$, we may apply Fubini's theorem and obtain

$$\frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} (T-\lambda)^{-1} f d\lambda = \int_{\sigma(T)} \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} (t-\lambda)^{-1} \, d\lambda \quad E(dt)f.$$

We now compute the integral over Γ_ε at the right-hand side:

$$\frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} (t-\lambda)^{-1} \, d\lambda = \begin{cases} \frac{1}{2} + \frac{1}{\pi} \arctan \frac{t}{\varepsilon}, & 0 \le t < M, \\ 0, & t < 0 \text{ or } t > M, \end{cases}$$

which is bounded above by 1. Applying the theorem of dominated convergence for vector-valued measures (cf.[4], Th.IV 10.10) we get

$$\lim_{\varepsilon \downarrow 0} \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} (T-\lambda)^{-1} f d\lambda = \int_{\sigma(T)} [\chi(0,+\infty)(t) + \tfrac{1}{2}\chi(\{0\})(t)] \, E(dt)f.$$

Here $\chi(A)$ denotes the characteristic function of a set A. From this, formula (1.6) is clear.

If T is a nonnegative (self-adjoint) operator with Ker T = {0}, it follows from (1.2) that the set

$$\{\| [T(T-\lambda)^{-1}]^n \| : n = 0,1,2,\ldots; -\infty < \lambda < 0\}$$

is bounded. Therefore, the unbounded inverse of -T is the infinitesimal generator of a strongly continuous semigroup of order 0. This is clear from the Hille-Yosida-Phillips theorem (cf.[4], Th. VIII 1.13). In fact,

$$e^{-sT^{-1}} = \int_{\sigma(T)} e^{-s/t}\, E(dt),\ 0 < s < +\infty.$$

By (1.2) this implies that

$$\| e^{-sT^{-1}} \| \le e^{-s/\| T \|},\ 0 < s < +\infty.$$

Finally, note that all results of this section remain true for scalar operators on a Banach space whose spectrum is a compact subset of $\mathbb{R}$. In the norm estimates, however, we have to put a bounded constant.

2. A decomposition theorem.

Let H be a complex Hilbert space with inner product $\langle .,. \rangle$. A pair (T,B) of linear operators on H is called a <u>self-adjoint admissible pair</u> on H if

(C.1) $T : H \to H$ is a bounded self-adjoint operator with Ker T = {0};

(C.2) $B : H \to H$ is a bounded self-adjoint operator, B is compact and the spectrum of B is a subset of the open interval $(-\infty,1)$;

(C.3) there exists $0 < \alpha < 1$ and an operator $D \in L(H)$ such that

$$B = |T|^{\alpha} D.$$

The operator I - B will be denoted by A. Observe that A is invertible, while

$$C = A^{-1} - I = B(I-B)^{-1}$$

is compact.

With a self-adjoint admissible pair (T,B) on H we associate the operator function $W : \rho(T) \to L(H)$ given by

$$(2.1) \qquad W(\lambda) = I + T(T-\lambda)^{-1}C.$$

The function W is called the symbol of the pair (T,B). An important property of W is given by the following proposition.

PROPOSITION 2.1. *For* $0 < \varphi < \frac{1}{2}\pi$, *put* $\Omega_\varphi = \{\lambda\in\mathbb{C}: |\frac{1}{2}\pi-\arg\lambda|\le\varphi\}$. *In the norm of* $L(H)$ *we have*

$$(2.2) \qquad \lim_{\lambda\to 0,\lambda\in\Omega_\varphi} W(\lambda) = A^{-1}.$$

Further, $W(\lambda)$ *is invertible if and only if* $\lambda \in \rho(T) \cap \rho(A^{-1}T)$.

PROOF. According to Proposition 1.1 we have

$$\lim_{\lambda\to 0,\lambda\in\Omega_\varphi} T(T-\lambda)^{-1} f = f.$$

Note that the convergence is uniform in f on compact subsets of H. Since C is compact, formula (2.2) is clear.

The second statement is a direct consequence of the identity

$$(2.3) \qquad W(\lambda) = (T-\lambda)^{-1} A(A^{-1}T-\lambda)A^{-1}.$$

The operator $S = A^{-1}T$ will play an important role in what follows. In the terminology of [1] the operator S is the associate operator of the node (T,C,-T; H,H) (see also Section 5). The fact that A is strictly positive implies that

$$(2.4) \qquad \langle f,g\rangle_A = \langle Af,g\rangle, \quad f,g \in H,$$

defines an equivalent inner product $\langle .,.\rangle_A$ on H. With respect to this inner product the operator S is self-adjoint. For the special case of the Transport Equation the inner product (2.4) has been introduced earlier by Hangelbroek and Lekkerkerker [18].

As S is self-adjoint with respect to $\langle .,.\rangle_A$, we can use the Spectral Theorem to define the maximal spectral subspace $H_p(H_m)$ of S corresponding to the part of the spectrum of S on the positive (negative) real line. Since Ker S = {0}, we have

(2.5) $$H = H_m \oplus H_p.$$

The projection of H onto $H_p(H_m)$ along $H_m(H_p)$ is denoted by $P_p(P_m)$. Obviously, P_p and P_m are orthogonal projections in the Hilbert space $(H,\langle .,.\rangle_A)$.

Also

(2.6) $$H = H_- \oplus H_+,$$

where $H_+(H_-)$ is the maximal spectral subspace of T corresponding to the part of $\sigma(T)$ on the positive (negative) real line. We use the symbol P_+ (P_-) to denote the projection of H onto $H_+(H_-)$ along $H_-(H_+)$. The projections P_+ and P_- are orthogonal projections on H.

To get the decompositions (2.5) and (2.6) one does not need Condition (C.3) nor the compactness of B. However, both properties are heavily used in the proof of the following decomposition theorem.

THEOREM 2.2. *Let* (T,B) *be a self-adjoint admissible pair on* H. *Then*

(2.7) $$H = H_- \oplus H_p = H_+ \oplus H_m.$$

Further, if P *is the projection of* H *onto* H_p *along* H_- *and* Q *is the projection of* H *onto* H_m *along* H_+, *then*

(2.8) $$T\,P = (I-Q^*)T.$$

The theorem above is the main result of this section. For the case of the Transport Equation with a degenerate scattering function several authors have proved the first part of the theorem (cf.[18, 21]) ; the second part of this theorem is due to Hangelbroek (cf.[21]). Here the proof of Theorem 2.2 will be given in a more general way, not using the special form the operators T and B have in the case of the Transport Equation. Also, we shall give a proof of (2.8) different from the proof due to Hangelbroek. To obtain Theorem 2.2, we need two lemmas.

LEMMA 2.3. We have $(A[H_p])^{\perp} = H_m$, $(A[H_m])^{\perp} = H_p$.

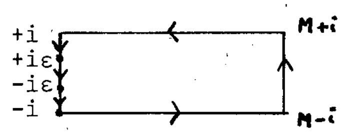
PROOF. For every subset M of H, we denote its orthogonal complement with respect to the inner product $\langle .,. \rangle_A$ by $M^{\perp\!\!\perp}$. Then an easy computation shows that $A[M^{\perp\!\!\perp}] = M^{\perp}$. As $H_p^{\perp\!\!\perp} = H_m$ and $H_m^{\perp\!\!\perp} = H_p$, the lemma is clear.

LEMMA 2.4. <u>The operator</u> $P_p - P_+$ <u>is a compact operator</u>.

PROOF. According to Proposition 1.3, we have

$$(2.9) \qquad P_p = \text{s-}\lim_{\varepsilon\downarrow 0} \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} (S-\lambda)^{-1} d\lambda, \quad P_m = \text{s-}\lim_{\varepsilon\downarrow 0} \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} (T-\lambda)^{-1} d\lambda.$$

Here Γ_ε is the following curve



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while M is a fixed constant exceeding $\max(\|T\|, \|A^{-1}T\|)$. From (2.9) we obtain

$$(2.10) \qquad (P_p - P_+) f = \lim_{\varepsilon\downarrow 0} \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} [(S-\lambda)^{-1} - (T-\lambda)^{-1}] f \, d\lambda.$$

A straightforward calculation, using (2.3), shows that

$$(2.11) \qquad (S-\lambda)^{-1} - (T-\lambda)^{-1} = -(T-\lambda)^{-1} C \, W(\lambda)^{-1} T (T-\lambda)^{-1}, \quad \lambda \in \rho(T) \cap \rho(S).$$

Now $C = BA^{-1}$ and $B = |T|^{\alpha} D$ for some $0 < \alpha < 1$ and $D \in L(H)$. We get

$$(S-\lambda)^{-1} - (T-\lambda)^{-1} = -|T|^{\alpha} (T-\lambda)^{-1} DA^{-1} W(\lambda)^{-1} T (T-\lambda)^{-1}, \quad \lambda \in \rho(T) \cap \rho(S).$$

Because of Proposition 2.1 and the boundedness of $T(T-\lambda)^{-1}$ on the set $\{\lambda \in \mathbb{C} : \text{Re}\,\lambda = 0, \lambda \neq 0\}$, it follows that there exists a constant N_1 such that

$$(2.12) \qquad \|(S-\lambda)^{-1} - (T-\lambda)^{-1}\| \le N_1 \| \, |T|^{\alpha} (T-\lambda)^{-1} \|, \quad \text{Re}\,\lambda = 0.$$

By a straightforward estimation, using (2.10), (2.12), (1.3) and a suitable upper bound for the lengths of the curves Γ_ε, one obtains

$$(2.13) \qquad \left\| \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon} [(S-\lambda)^{-1}-(T-\lambda)^{-1}]d\lambda \right\| \le N_2 \int_\varepsilon^1 u^{\alpha-1}\,du, \; 0 < \varepsilon < 1,$$

and

$$(2.14) \qquad \left\| \frac{-1}{2\pi i} \int_{\Gamma_\varepsilon \setminus \Gamma_\delta} [(S-\lambda)^{-1}-(T-\lambda)^{-1}]d\lambda \right\| \le N_2 \int_\varepsilon^\delta u^{\alpha-1}\,du, \quad 0 < \varepsilon < \delta < 1,$$

for some positive constant N_2. From (2.11) it is clear that $(S-\lambda)^{-1} - (T-\lambda)^{-1}$ is a compact operator for all $\lambda \in \Gamma_\varepsilon$, $0 < \varepsilon < 1$. Since the integral at the left hand side of (2.13) is absolutely convergent in the norm, it follows that this integral represents a compact operator. From (2.14) and the convergence of the improper integral $\int_0^1 u^{\alpha-1}\,du$, it appears that $P_p - P_+$ is the limit in the norm of a sequence of compact operators and therefore compact itself. This completes the proof.

In many cases it is possible to specify to which ideal of compact operators $P_p - P_+$ belongs. Recall that $B = |T|^\alpha D$ for some $0 < \alpha < 1$ and $D \in L(H)$. If we suppose, in addition to our previous assumptions, that D belongs to a symmetrically normed ideal J of compact operators (cf. [13] for the definition and main properties of such an ideal), then we can repeat the proof of Lemma 2.4, using a symmetric norm of the ideal J rather than the operator norm of $L(H)$, and deduce that $P_p - P_+$ belongs to the ideal J.

PROOF of Theorem 2.2. We first show that $H_p \cap H_- = H_m \cap H_+ = \{0\}$. Take $f \in H_p \cap H_-$. On the one hand, we have $f \in H_-$, and therefore $\langle Tf,f\rangle \le 0$. On the other hand, we have $f \in H_p$, and hence $\langle Tf,f\rangle = \langle Sf,f\rangle_A \ge 0$. Consequently, $\langle Tf,f\rangle = 0$, and thus $f = 0$. Hence, $H_p \cap H_- = \{0\}$. In the same way we prove that $H_m \cap H_+ = \{0\}$.

Consider the operator V, defined by

$$(2.15) \qquad V = P_+ P_p + P_- P_m .$$

Since $I - V = (P_- - P_+)(P_p - P_+)$, it follows from Lemma 2.4 that V is a Fredholm operator of index 0. Further, $\operatorname{Ker} V = [H_p \cap H_-]$

$\oplus$ $[H_m \cap H_+] = \{0\}$. So V is invertible. Since Im V can be expressed by Im V = $[H_p + H_-] \cap [H_m + H_+]$, we get $H_p + H_- = H_m + H_+ = H$. From this we obtain (2.7).

It remains to prove (2.8). Observe that $T\,P\,f = (I-Q^*)Tf = 0$, $f \in H_-$. The second part of this equation is clear from the identity $(I-Q^*)\,P_+ = [P_+(I-Q)]^* = I - Q^*$. Since $H_p + H_- = H$, it suffices to prove (2.8) for all $f \in H_p$. Take $f \in H_p$. On the one hand, $T\,P\,f = Tf$. On the other hand, $Tf = A(A^{-1}T\,f) \in A[H_p] = H_m^{\perp}$ (cf. Lemma 2.3). Since $\mathrm{Im}(I-Q^*) = \mathrm{Ker}(I-Q)^{\perp} = H_m^{\perp}$, it is clear that $(I-Q^*)Tf = Tf$. Hence, $T\,P\,f = (I-Q^*)Tf$. This completes the proof.

3. THE OPERATOR DIFFERENTIAL EQUATION $T\psi' = -A\psi$

Let H be a complex Hilbert space with inner product $\langle .,. \rangle$. Suppose that (T,B) is a self-adjoint admissible pair on H. As in Section 2, put A = I - B. In this section we are interested in the operator differential equation

$$(3.1) \qquad T\psi' = -A\psi.$$

A function $\psi : (0.+\infty) \to H$ is called a <u>solution</u> of the equation (3.1) if

(I) the function $t \mapsto T\,\psi(t)$ is strongly differentiable on $(0,+\infty)$ with derivative $-A\psi(t)$, and

(II) $\lim_{t\downarrow 0} P_+\,\psi(t) = \varphi_+$.

Here φ_+ is a given function in the maximal spectral subspace H_+ of T corresponding to the part of $\sigma(T)$ on the positive real line. The function φ_+ is called the <u>boundary value</u> of the solution ψ. For later use we remark that any solution of (3.1) is strongly measurable.

To describe the solutions we shall use the notations introduced in the preceding section. In particular, P is the projection of H onto H_p along H_-, defined in Theorem 2.2.

In the three theorems below we describe the solutions of the operator differential equation $T\psi' = -A\psi$.

THEOREM 3.1. *For every $\varphi_+ \in H_+$, there is a unique bounded solution of the operator differential equation $T\psi = -A\psi$ with boundary value φ_+. This solution is integrable and given by*

$$(3.2)\qquad \psi(t) = e^{-tT^{-1}A} P\varphi_+, \quad 0 < t < +\infty.$$

THEOREM 3.2. *Let $-\kappa < \min(0,m(T))$. Then for every $\varphi_+ \in H_+$, there is at least one solution ψ of the operator differential equation $T\psi = -A\psi$ with boundary value φ_+ such that*

$$(3.3)\qquad \int_0^{+\infty} e^{-t/\kappa} \|\psi(t)\|\, dt < +\infty.$$

Suppose that $\lambda_1,\ldots,\lambda_m$ are the distinct eigenvalues of $A^{-1}T$ with $\lambda_i < -\kappa$ $(i=1,2,\ldots,m)$. Then all solutions with boundary value φ_+ that satisfy (3.3) *are given by*

$$\psi(t) = e^{-tT^{-1}A} P(\varphi_+ - \sum_{j=1}^{m}\varphi_{0j}) + \sum_{j=1}^{m} e^{-t/\lambda_j}\varphi_{0j},$$

where, for $i = 1,2,\ldots,m$, φ_{0i} ranges over the eigenspace of $A^{-1}T$ associated with the eigenvalue λ_i. Further, the number of linearly independent solutions with boundary value φ_+ coincides with the sum of the multiplicities of the eigenvalues of $A^{-1}T$ on $(-\infty,-\kappa)$.

A similar theorem holds if we are searching for solutions ψ such that

$$(3.4)\qquad \operatorname*{ess-sup}_{0<t<+\infty} e^{-t/\kappa}\|\psi(t)\| < +\infty.$$

If Condition (3.4) holds true, then for a sufficiently small $\varepsilon > 0$ we have Condition (3.3) with κ replaced by $\kappa - \varepsilon$. Using the previous theorem, it is clear that in this case we have to consider the eigenvalues of $A^{-1}T$ not exceeding $-\kappa$.

THEOREM 3.3. *Let $\kappa > \max(0,M(T))$. Suppose that H_κ is the maximal spectral subspace of $A^{-1}T$ corresponding to the part of $\sigma(A^{-1}T)$ on $[0,\kappa)$. Then, for a given $\varphi_+ \in H_+$, there is a solution of the operator differential equation $T\psi = -A\psi$ with boundary value φ_+ such that*

$$(3.5)\qquad \int_0^{+\infty} e^{+t/\kappa}\|\psi(t)\|dt < +\infty,$$

<u>if and only if</u> $\varphi_+ \in H_\kappa + H_-$. <u>In this case the solution</u> ψ <u>is unique and given by</u> (3.2). <u>Further, the codimension in</u> H_+ <u>of the subspace of all boundary values</u> φ_+ <u>for which a solution satisfying</u> (3.5) <u>exists coincides with the sum of the multiplicities of the eigenvalues</u> λ <u>of</u> $A^{-1}T$ <u>with</u> $\lambda \geq \kappa$.

PROOF of Theorem 3.2. Let E be the resolution of the identity of A^{-1} (as a self-adjoint operator in the inner product (2.4)). Take $\kappa > -\min(0,m(T))$. Let K be the image of $E((0,+\infty)\cup(-\infty,-\kappa))$ and $L = \operatorname{Im} E([-\kappa,0])$. As $A^{-1}T$ has a trivial kernel, we have $K \oplus L = H$. Put

$$(3.6)\qquad T(t) = e^{-tT^{-1}A}\big|_K \oplus I_L,\quad 0 \leq t < +\infty.$$

Then T is a strongly continuous semigroup on H which can be viewed of as the direct sum of a uniformly continuous semigroup on a finite-dimensional space and a strongly continuous semigroup with generator $(-T^{-1}A|H_p) \oplus 0_L$. As the infinitesimal generator of the former semigroup has the set $\{\lambda_1^{-1},\dots,\lambda_m^{-1}\}$ as its spectrum, where $\lambda_1,\dots,\lambda_m$ denote the eigenvalues of $A^{-1}T$ on $(-\infty,-\kappa)$, it is clear that its order is strictly less than $1/\kappa$. An easy application of the Hille-Yosida-Phillips theorem (cf.[4], Chapter VIII, for instance) shows that the latter semigroup is strongly continuous and bounded. Hence, T is a strongly continuous semigroup of H with infinitesimal generator $-T^{-1}A|_K \oplus 0_L$, whose order is strictly less than $1/\kappa$.

Suppose that ψ is a solution of $T\psi' = -A\psi$ satisfying (3.3). We shall first prove that there exists a vector $\tilde{\psi}$ in K such that

$$(3.7)\qquad A^{-1}T\,\psi(t) = T(t)\,\tilde{\psi},\quad 0 < t < +\infty.$$

We first note that, for all $\lambda \neq 0$ in the closed left half plane, the integral

$$\kappa\lambda \int_0^{+\infty} e^{t/\lambda}\, e^{-t/\kappa}\, \psi(t)\, dt$$

is absolutely convergent and represents a vector function, holomorphic on the open left half plane, continuous up to the boundary and with 0 as its continuous boundary value at $\lambda = 0$. Put $\zeta = \kappa\lambda(\kappa-\lambda)^{-1}$. Then this Möbius transformation maps the open left half plane onto the interior domain of the circle with centre $-\frac{1}{2}\kappa$ and radius $\frac{1}{2}\kappa$.

Note that $\varphi(t) = e^{-t/\kappa}\,\psi(t)$ satisfies the operator differential equation $\kappa A^{-1}T\dot{\varphi} = -(A^{-1}T+\kappa I)\varphi$, and is integrable on $(0,+\infty)$. As in the definition of a solution of $T\dot{\psi} = -A\psi$, we suppose that $T\varphi$ rather than φ is strongly differentiable. Integrating both sides of the former equation over the positive real axis, one sees that the following limits exist:

$$\tilde{\psi} = \lim_{t\downarrow 0} A^{-1}T\,\varphi(t), \quad \lim_{t\to+\infty} A^{-1}T\,\varphi(t).$$

Put $0 < \alpha < \beta < +\infty$. A straightforward computation yields

$$\lambda(A^{-1}T+\kappa I)\int_\alpha^\beta e^{t/\lambda}\varphi(t)\,dt = \kappa\lambda[e^{t/\lambda}\,A^{-1}T\varphi(t)]_\alpha^\beta + \kappa\,A^{-1}T\int_\alpha^\beta e^{t/\lambda}\,\varphi(t)dt, \quad \mathrm{Re}\,\lambda \le 0. \tag{3.8}$$

From this equation it is clear that $\lim_{t\to+\infty} e^{t/\lambda}\,A^{-1}T\,\varphi(t)$ exists for all imaginary $\lambda \neq 0$. Hence,

$$\lim_{t\to+\infty} A^{-1}T\,\varphi(t) = 0.$$

In (3.8), we take the limit as $\alpha\downarrow 0$ and $\beta \to +\infty$, and obtain

$$\lambda(A^{-1}T+\kappa I)\int_0^{+\infty} e^{t/\lambda}\,\varphi(t)dt = \kappa\lambda\,\tilde{\psi} + \kappa A^{-1}T\int_0^{+\infty} e^{t/\lambda}\varphi(t)dt, \quad \mathrm{Re}\,\lambda \le 0\ (\lambda\neq 0).$$

Recall that $\varphi(t) = e^{-t/\kappa}\,\psi(t)$. Inserting $\psi(t) = e^{t/\kappa}\,\varphi(t)$ and $\zeta = \kappa\lambda(\kappa-\lambda)^{-1}$, we obtain

$$\int_0^{+\infty} e^{t/\zeta}\,\psi(t)\,dt = \zeta(\zeta-A^{-1}T)^{-1}\,\tilde{\psi}, \tag{3.9}$$

for all $\zeta \neq 0$ for which $(\mathrm{Re}\,\zeta+\frac{1}{2}\kappa)^2 + (\mathrm{Im}\,\zeta)^2 \le (\frac{1}{2}\kappa)^2$. By the integrability of $\varphi(t) = e^{-t/\kappa}\,\psi(t)$, the vector function at the left hand side of (3.9) has 0 as a continuous boundary value at $\zeta = 0$.

Next, we prove that $\tilde{\psi} \in K$. Applying $A^{-1}T\, E([-\kappa,0])$ to (3.9), we get

$$A^{-1}T\, E([-\kappa,0]) \int_0^{+\infty} e^{t/\zeta}\, \psi(t)\, dt = \zeta A^{-1}T(\zeta - A^{-1}T)^{-1}\, E([-\kappa,0])\tilde{\psi},$$

for all $\zeta \neq 0$ for which $(\mathrm{Re}\,\zeta + \frac{1}{2}\kappa)^2 + (\mathrm{Im}\,\zeta)^2 \leq (\frac{1}{2}\kappa)^2$, the function at the left hand side having 0 as a continuous boundary value at $\zeta = 0$. The right hand side of this equation, however, is holomorphic on the exterior domain of the circle with centre $-\frac{1}{2}\kappa$ and radius $\frac{1}{2}\kappa$, and continuous on its closure. By virtue of the self-adjointness of $A^{-1}T$ in the equivalent inner product (2.4), the last part of this statement is a corollary of the propositions 1.1 and 1.2. The boundary value of the right hand side of $\zeta = 0$ vanishes too. So, by Liouville's theorem, it follows that $E([-\kappa,0])\, \tilde{\psi} = 0$. Therefore $\tilde{\psi} \in K$.

Recall that $-T^{-1}A|_K \oplus 0_L$ is the infinitesimal generator of a strongly continuous semigroup on H. From (3.9) and [4], Theorem VIII 1.11, we obtain (3.7). If φ_+ is the boundary value of ψ, we get, using (3.7),

$$(3.10) \qquad T\, \varphi_+ = P_+\, A\, \tilde{\psi},\ \tilde{\psi} \in K.$$

From the decomposition $K = H_p \oplus \mathrm{Im}\, E((-\infty,-\kappa))$ and the fact that $A^{-1}T$ maps $\mathrm{Im}\, E((-\infty,-\kappa))$ onto itself (to see this, note that $\mathrm{Im}\, E((-\infty,-\kappa))$ is finite-dimensional) we obtain

$$(3.11) \qquad \tilde{\psi} = \psi_p + A^{-1}T\, \varphi_0,$$

where $\psi_p \in H_p$ and $\varphi_0 \in \mathrm{Im}\, E((-\infty,-\kappa))$. Then $A\,\psi_p \in H_m^{\perp}$ (cf.Lemma 2.3). Since $\mathrm{Ker}\, Q^* = H_m^{\perp}$ and $P_+(I-Q) = I - Q$, we have $A\,\psi_p = (I-Q^*)A\,\psi_p = (I-Q^*)P_+\, A\,\psi_p$. By virtue of (3.11), we have $A\,\psi_p = (I-Q^*)P_+\, A\,\tilde{\psi} - (I-Q^*)P_+\, T\,\varphi_0$. Using (3.10) and $(I-Q^*)P_+ = I - Q^*$, we get $A\,\psi_p = (I-Q^*)\, T\,(\varphi_+ - \varphi_0)$. With the help of (2.8), we have $A\,\psi_p = T\, P(\varphi_+ - \varphi_0)$. Using (3.11), we eventually get

$$(3.12) \qquad \tilde{\psi} = A^{-1}T\, P(\varphi_+ - \varphi_0) + A^{-1}T\, \varphi_0.$$

From this equation and the injectivity of $A^{-1}T$, we obtain

$$(3.13)\qquad \psi(t) = T(t)[P(\varphi_+-\varphi_0) + \varphi_0],\ 0 < t < +\infty.$$

Note that $\sigma(A^{-1}T) \cap (-\infty,-\kappa)$ is a finite set of eigenvalues of rank one; the value of the rank is due to the self-adjointness of $A^{-1}T$ in the equivalent inner product (2.4). Denote the distinct eigenvalues of $A^{-1}T$ on $(-\infty,-\kappa)$ by $\lambda_1,\ldots,\lambda_m$. Then $\varphi_0 = \sum_{j=1}^{m}\varphi_{0j}$, where φ_{0j} is some vector in $\mathrm{Ker}(\lambda_j-A^{-1}T)$. Therefore,

$$(3.14)\qquad \psi(t) = e^{-tT^{-1}A}\,P(\varphi_+-\sum_{j=1}^{m}\varphi_{0j}) + \sum_{j=1}^{m} e^{-t/\lambda_j}\,\varphi_{0j},\qquad 0 < t < +\infty.$$

Conversely, every vector function of this form is a solution of $T\psi = -A\psi$ with boundary value φ_+ for which (3.3) holds. To see this, write $\varphi_0 = \sum_{j=1}^{m}\varphi_{0j} \in \mathrm{Im}\,E((-\infty,-\kappa))$. Then the vector function under consideration has the form (3.13), where $A^{-1}T[P(\varphi_+-\varphi_0) + \varphi_0]$ belongs to the domain of the infinitesimal generator $(-T^{-1}A|_K) \oplus 0_L$ of T. Hence,

$$\frac{d}{dt}\,A^{-1}T\,\psi(t) = -T^{-1}A\;T(t)\;A^{-1}T[P(\varphi_+-\varphi_0) + \varphi_0] = -\psi(t),\ 0 < t < +\infty.$$

Further, $\lim_{t\downarrow 0} P_+\,\psi(t) = P_+[P(\varphi_+-\varphi_0) + \varphi_0] = \varphi_+$. So every vector function of the form (3.14) is, indeed, a solution of $T\psi = -A\psi$ with boundary value φ_+. The estimate (3.3) for ψ is a consequence of the fact that T is a strongly continuous semigroup of order strictly less than $1/\kappa$.

Since $H_p \cap \mathrm{Im}\,E((-\infty,-\kappa)) = \{0\}$ and all solutions of the type under consideration have the form (3.14), it is clear that the number of linearly independent solutions of $T\psi = -A\psi$ with boundary value φ_+ coincides with the sum of the multiplicities of the eigenvalues of $A^{-1}T$ strictly less than $-\kappa$. This completes the proof of Theorem 3.2.

PROOF of Theorem 3.1. From the proof of Theorem 3.2 it is clear that the function (3.2) is bounded, integrable and a solution of Eq. (3.1) with boundary value φ_+.

Now, conversely, suppose that ψ is a bounded solution of $T\psi = -A\psi$ with boundary value φ_+. Then ψ satisfies (3.3) for all $-\kappa < \min(0,m(T))$. So for all such κ, there exists a vector $\varphi_\kappa \in \mathrm{Im}\, E((-\infty,-\kappa))$ such that

$$\psi(t) - T(t)\ P\ \varphi_+ = T(t)\ (I-P)\varphi_\kappa,\ 0 < t < +\infty.$$

Taking the limit as $t\downarrow 0$ (which is possible) and using the injectivity of I-P on $\mathrm{Im}\, E((-\infty,-\kappa))$, it appears that all vectors φ_κ coincide. Put $\varphi_0 = \varphi_\kappa$. Then $\varphi_0 \in \bigcap_\kappa \mathrm{Im}\, E((-\infty,-\kappa)) = \{0\}$. Hence,

$$\psi(t) = T(t)\ P\ \varphi_+ = e^{-tT^{-1}A}\ P\ \varphi_+,\ 0 \leq t < +\infty.$$

From this, Theorem 3.1 is clear.

PROOF of Theorem 3.3. If $\varphi_+ \in H_\kappa + H_-$, then $P\ \varphi_+ \in \mathrm{Im}\, E([0,\kappa)) = H_\kappa$. Since $(T(t)|H_\kappa)_\kappa$ is a strongly continuous semigroup on H_κ of order $\leq -1/(\kappa-\delta)$ for some $\delta > 0$, it follows that the vector function (3.2) satisfies Condition (3.5).

To prove the converse, take $\kappa > \max(0,M(T))$. Then $\sigma(A^{-1}T) \cap [\kappa,+\infty)$ consists of a finite number of eigenvalues of rank one. Let λ_0 be one of these eigenvalues. For every $\tilde{\psi} \in \mathrm{Im}\, E(\{\lambda_0\})$, we have

$$T(t)\ \tilde{\psi} = e^{-t/\lambda_0}\ \tilde{\psi},\ 0 \leq t < +\infty,$$

while $\tilde{\psi} \in \mathrm{Ker}(\lambda_0 - A^{-1}T)$. So $\|T(t)\tilde{\psi}\| = e^{-t/\lambda_0}\|\tilde{\psi}\|$, $t \geq 0$.

If ψ is a solution of $T\psi = -A\psi$ satisfying (3.5), then

$$\int_0^{+\infty} e^{+t/\kappa}\ \|E(\{\lambda_0\})\psi(t)\|\ dt < +\infty.$$

From Theorem 3.1 it is clear now that ψ has the form (3.2) with $E(\{\lambda_0\})\ P\ \varphi_+ = 0$. Since λ_0 is an arbitrarily chosen eigenvalue of $A^{-1}T$ on $[\kappa,+\infty)$, we have $E([\kappa,+\infty))\ P\ \varphi_+ = 0$, and therefore $P\ \varphi_+ \in \mathrm{Im}\, E([0,\kappa))$. Hence, $\varphi_+ \in H_\kappa + H_-$.

To finish the proof, note that the codimension in H_+ of the subspace of all boundary values φ_+ for which a solution satisfying (3.5) exists coincides with $\dim H_+/[(H_\kappa+H_-) \cap H_+]$, which equals the codimension of $H_\kappa + H_-$ in H. From the decomposition $H_p \oplus H_- = H$ (cf.Theorem 2.2) we also have $[H_\kappa + H_-] \oplus \text{Im } E([\kappa,+\infty)) = H$. This completes the proof.

4. A WIENER-HOPF OPERATOR INTEGRAL EQUATION

In this section we associate with a self-adjoint admissible pair (T,B) on a complex Hilbert space H a Wiener-Hopf operator integral equation, and we prove that this Wiener-Hopf integral equation is an equivalent form of the operator differential equation considered in the previous section.

Let H be a complex Hilbert space with inner product $\langle .,. \rangle$ and (T,B) a self-adjoint admissible pair on H. Consider the Wiener-Hopf operator integral equation

$$(4.1) \qquad \psi(s) - \int_0^{+\infty} \mathcal{H}(s-t)B\,\psi(t)\,dt = \varphi(s),\ 0 < s < +\infty,$$

where the kernel $\mathcal{H}$ is given by

$$(4.2) \qquad \mathcal{H}(t) = \begin{cases} +T^{-1}\,e^{-tT^{-1}}\,P_+ & ,\ t > 0. \\ -T^{-1}\,e^{-tT^{-1}}\,P_- & ,\ t < 0. \end{cases}$$

In (4.1) the given function $\varphi : (0,+\infty) \to H$ is assumed to be strongly measurable and

$$\int_0^{+\infty} \|\varphi(s)\|\,ds < +\infty.$$

By strong measurability we mean measurability with respect to the Lebesgue measure as defined in [25], Section VI. 31; all integrals of vector functions appearing in this section will be Bochner integrals with respect to the Lebesgue measure (cf.[25], Section VI. 31).

Let $\psi : (0,+\infty) \to H$ be strongly measurable. Then for each $0 < s < +\infty$ the function $\mathcal{H}(s-.)B\,\psi(.)$, defined on $(0,+\infty)$, is strongly measurable too. We call ψ a <u>solution</u> of Equation (4.1) if

$$(4.3) \qquad \int_0^{+\infty} \|\mathcal{H}(s-t)B\,\psi(t)\|\,dt < +\infty,\ 0 < s < +\infty,$$

and ψ satisfies (4.1).

LEMMA 4.1. *Let* $-\kappa < \min(0,m(T))$, *and let* $\psi : (0,+\infty) \to H$ *be strongly measurable. Then formula* (4.3) *is satisfied, whenever*

$$\operatorname*{ess-sup}_{0<t<+\infty} e^{-t/\kappa} \|\psi(t)\| < +\infty. \tag{4.4}$$

PROOF. Recall that there exists an $0 < \alpha < 1$ and a $D \in L(H)$ such that $B = |T|^{\alpha}D$. Put $\beta = 1 - \alpha$. Let E denote the resolution of the identity of T. Put $m = \min(0,m(T))$ and $M = \max(0,M(T))$. As

$$H(t)=\begin{cases} +\int_0^{+\infty} \mu^{-1} e^{-t/\mu} E(d\mu), & 0 < t < +\infty, \\ -\int_{-\infty}^{0} \mu^{-1} e^{-t/\mu} E(d\mu), & -\infty < t < 0, \end{cases} \tag{4.5}$$

a simple computation shows that

$$\| \, |T|^{\alpha} H(t)\| \le \begin{cases} (-m)^{-\beta} e^{-t/m}, & t \le \beta m, \\ \beta^{\beta} (-t)^{-\beta} e^{-\beta}, & \beta m \le t < 0, \\ \beta^{\beta} t^{-\beta} e^{-\beta}, & 0 < t \le \beta M, \\ M^{-\beta} e^{-t/M}, & t \ge \beta M. \end{cases} \tag{4.6}$$

If $m = 0$ $(-M=0)$, we read $(-m)^{-\beta} e^{-t/m}$ $(M^{-\beta} e^{-t/M})$ as 0.

To prove (4.3) one considers the integrand $H(s-.)B\,\psi(.)$ on the intervals $(0,s-\beta M)$, $[s-\beta M,s)$, $(s,s-\beta m]$ and $(s-\beta m,+\infty)$. Because of (4.4) the function ψ is essentially bounded on the first three bounded intervals and $e^{-t/\kappa}\,\psi(t)$ on the fourth unbounded interval. But then one can use the estimates (4.6) to obtain (4.3).

The next theorem provides the equivalence between the Wiener-Hopf operator integral equation (4.1) and the operator differential equation $T\psi' = -A\psi$ studied in the previous section.

THEOREM 4.2. *Let* $-\kappa < \min(0,m(T))$. *Suppose that* ψ *is a strongly measurable function such that* (4.4) *holds. Then* ψ *is a solution of the operator differential equation* $T\psi' = -A\psi$ *with boundary value* φ_+ *if and only if* ψ *is a solution of the Wiener-Hopf operator* *integral equation* (4.1) *with right-hand side* $\varphi(s) = e^{-sT^{-1}}\varphi_+$. *Further, every such solution* ψ *is continuous*.

PROOF. Let ψ be a solution of the Wiener-Hopf operator integral equation (4.1) with right-hand side $\varphi(s) = e^{-sT^{-1}}\varphi_+$, and assume that ψ satisfies (4.4). Consider the function

$$g(s) = \int_0^{+\infty} H(s-t)B\,\psi(t)dt.$$

We shall first prove that g is continuous.

Take h > 0. Using the definition of the kernel H one sees that

$$(4.7)\qquad g(s-h) - g(s) = [e^{-hT^{-1}}P_+ - P_+]\int_0^s H(s-t)B\,\psi(t)dt - [e^{hT^{-1}}P_- - P_-]\int_{s+h}^{+\infty} H(s+h-t)B\,\psi(t)dt + \int_s^{s+h}[H(s+h-t) - H(s-t)]B\,\psi(t)dt.$$

Note that

$$(4.8)\qquad \operatorname*{s-lim}_{h\downarrow 0} e^{-hT^{-1}}P_+ = P_+,\quad \operatorname*{s-lim}_{h\downarrow 0} e^{hT^{-1}}P_- = P_-.$$

Recall that $B = |T|^{\alpha}D$ for some $0 < \alpha < 1$ and some bounded operator D. Put $\tilde{B} = |T|^{\frac{1}{2}\alpha}D$ and $\tilde{\psi}(t) = e^{-t/\kappa}\psi(t)$. A straightforward estimation, using that $\tilde{\psi}$ is essentially bounded on $[s,+\infty)$ and $t \mapsto e^{t/\kappa}H(s-t)\tilde{B}$ is integrable on the interval $[s,+\infty)$ (cf.(4.6)), yields the boundedness of the set of vectors

$$(4.9)\qquad \int_{s+h}^{+\infty} H(s+h-t)\tilde{B}\,\psi(t)dt$$

for $0 \le h \le 1$.

We now show that the right-hand side of Eq.(4.7) tends to zero as $h\downarrow 0$. From (4.8), (4.9) and the identity

$$\lim_{h\downarrow 0} \| \, |T|^{\frac{1}{2}\alpha}[e^{hT^{-1}}P_- - P_-] \| = 0$$

it is clear that the first and second term at the right-hand side of Eq.(4.7) vanish as $h\downarrow 0$. The vanishing of the third term of the right-hand side of this equation is clear from the estimates

$$H(s-t)B = O(|s-t|^{\alpha-1}),\ H(s+h-t)B = O(|s+h-t|^{\alpha-1}),$$

which hold true for $s < t < s + h$ and a sufficiently small h (cf. (4.6)), and the essential boundedness of the function ψ on the interval $[s,s+1]$. It follows that

$$\lim_{h\downarrow 0} \{g(s+h)-g(s)\} = 0.$$

In a similar way we prove that

$$\lim_{h\uparrow 0} \{g(s+h)-g(s)\} = 0.$$

Hence, g is continuous.

Also φ is continuous. From (4.1) we get that the function ψ is continuous.

Next, we shall prove that the function Tg is strongly differentiable and we shall compute its derivative.

Take $h > 0$. Using the definition of the kernel H, we get

$$\begin{aligned} (4.10) \qquad & h^{-1}[T\, g(s+h) - T\, g(s)] = \\ & = h^{-1}[e^{-hT^{-1}} P_+ - P_+]T \int_0^s H(s-t)B\, \psi(t)dt \\ & - h^{-1}[e^{hT^{-1}} P_- - P_-]T \int_s^{+\infty} H(s-t)B\, \psi(t+h)dt \\ & + h^{-1} \int_s^{s+h} T[H(s+h-t) - H(s-t)]B\, \psi(t)dt. \end{aligned}$$

Because of the equation

$$\operatorname{s-lim}_{h\downarrow 0} h^{-1}[e^{-hT^{-1}} P_+ - P_+]T = -P_+,$$

the first term at the right-hand side of (4.10) tends to the vector $-P_+ g(s)$ as $h\downarrow 0$. As the function ψ is continuous and the function $t \mapsto e^{t/\kappa} H(s-t)B$ is integrable on $[s,+\infty)$, it follows from the theorem of dominated convergence for Bochner integrals [25] that

$$\lim_{h\downarrow 0} \int_s^{+\infty} H(s-t)B\, \psi(t+h)dt = \int_s^{+\infty} H(s-t)B\, \psi(t)dt.$$

Using that

$$\text{s-}\lim_{h\downarrow 0} h^{-1}[e^{hT^{-1}} P_- - P_-]T = P_-,$$

it follows that the second term at the right-hand side of Eq. (4.10) tends to $-P_-\ g(s)$ as $h\downarrow 0$. By the continuity of the integrand the third term at the right-hand side of Eq.(4.10) tends to $P_+\ B\ \psi(s) + P_-\ B\ \psi(s) = B\ \psi(s)$ as $h\downarrow 0$. Using Eq.(4.10) and the results of the computations we made we get

$$\lim_{h\downarrow 0} h^{-1}[T\ g(s+h) - T\ g(s)] = -\ g(s) + B\ \psi(s).$$

In a similar way we prove that

$$\lim_{h\uparrow 0} h^{-1}\{T\ g(s+h) - T\ g(s)\} = -\ g(s) + B\ \psi(s).$$

Hence, Tg is strongly differentiable and its derivative is equal to $-g + B\psi$.

Also $T\varphi$ is strongly differentiable and $T\dot{\varphi} = -\varphi$. Using this in (4.1) one obtains that $T\psi$ is strongly differentiable, while

$$T\dot{\psi} = T\dot{g} + T\dot{\varphi} = -g + B\psi - \varphi = -A\psi.$$

Next, we prove that ψ is, indeed, a solution of $T\dot{\psi} = -A\psi$ with boundary value φ_+. Applying P_+ to both sides of (4.1) we get

$$P_+\ \psi(s) - \int_0^s H(s-t)B\ \psi(t)dt = e^{-sT^{-1}}\ \varphi_+,\ 0 < s < +\infty.$$

Recall that we have (4.3) (cf.Lemma 4.1). Therefore,

$$\lim_{s\downarrow 0} \int_0^s H(s-t)B\ \psi(t)dt = 0.$$

Consequently, ψ is a solution of the differential equation $T\dot{\psi} = -A\psi$ with boundary value φ_+.

Conversely, let ψ be a solution of $T\dot{\psi} = -A\psi$ with boundary value φ_+ that satisfies (4.4). As we noted in Section 3, the function ψ is strongly measurable. By virtue of Lemma 4.1 we have (4.3).

Since $T\psi = -A\psi = -\psi + B\psi$, we perform a partial integration and obtain for $\varphi(s) = e^{-sT^{-1}}\varphi_+$:

$$\int_0^s e^{-(s-t)T^{-1}} P_+ T\psi(t)dt =$$

$$= [e^{-(s-t)T^{-1}} P_+ T\psi(t)]_{t=0}^s - \int_0^s T^{-1} e^{-(s-t)T^{-1}} P_+ T\psi(t)dt =$$

$$TP_+\psi(s) - T\varphi(s) - \int_0^s T\, H(s-t)\, \psi(t)dt, \quad 0 < s < +\infty.$$

So we have

$$\int_0^s T\, H(s-t)[T\psi(t) + \psi(t)]dt = T[P_+ \psi(s) - \varphi(s)].$$

Using (4.3), the continuity of $T\psi$ and the norm estimate

$$\|T^{-1} e^{+tT^{-1}} P_-\| \le |m|^{-1} e^{+t/m}, \quad t \ge -m,$$

where $m = \min(0, m(T))$, we have

$$\lim_{t\to+\infty} e^{+tT^{-1}} P_- \psi(t) = 0.$$

We now repeat the argument above and obtain

$$\int_s^{+\infty} T\, H(s-t)[T\psi(t) + \psi(t)]dt = T[P_-\psi(s)].$$

Therefore,

$$T \int_0^{+\infty} H(s-t)B\, \psi(t)dt = T[\psi(s) - \varphi(s)], \quad 0 < s < +\infty.$$

Using that Ker $T = \{0\}$, we get Equation (4.1). This completes the proof.

From Theorem 4.2 and the results of Section 3, we obtain the solutions of the Wiener-Hopf equation (4.1) with right-hand side $\varphi(s) = e^{-sT^{-1}}\varphi_+$. The next three theorems describe the solutions of (4.1) (cf. the remark after Theorem 3.2).

THEOREM 4.3. For every $\varphi_+ \in H_+$, there is a unique bounded solution of the Wiener-Hopf integral equation (4.1) with right-hand side $\varphi(s) = e^{-sT^{-1}}\varphi_+$. This solution is integrable and given by

$$(4.11) \qquad \psi(s) = e^{-sT^{-1}A} P \varphi_+, \; 0 < s < +\infty .$$

THEOREM 4.4. Let $-\kappa < \min(0,m(T))$. Then for every $\varphi_+ \in H_+$, there exists at least one solution of the Wiener-Hopf integral equation (4.5) with right-hand side $\varphi(s) = e^{-sT^{-1}} \varphi_+$, for which

$$(4.12) \qquad \operatorname*{ess-sup}_{0<t<+\infty} e^{-t/\kappa}\|\psi(t)\| < +\infty.$$

Suppose that $\lambda_1,\dots,\lambda_m$ are the distinct eigenvalues of $A^{-1}T$ with $\lambda_i \leq -\kappa (i=1,2,\dots,m)$. Then the solutions with right-hand side $\varphi(s) = e^{-sT^{-1}} \varphi_+$ that satisfy (4.12) are given by

$$(4.13) \qquad \psi(s) = e^{-sT^{-1}A} P(\varphi_+ - \sum_{j=1}^{m}\varphi_{0j}) + \sum_{j=1}^{m} e^{-s/\lambda_j} \varphi_{0j},$$

where, for $i = 1,2,\dots,m$, φ_{0i} ranges over the eigenspace of $A^{-1}T$ associated with the eigenvalue λ_i. Further, the number of linearly independent solutions with right-hand side $\varphi(s) = e^{-sT^{-1}} \varphi_+$ coincides with the sum of the multiplicities of the eigenvalues of $A^{-1}T$ on $(-\infty,-\kappa]$.

THEOREM 4.5. Let $\kappa > \max(0,M(T))$. Suppose that H_κ is the maximal spectral subspace of $A^{-1}T$ corresponding to the part of $\sigma(A^{-1}T)$ on $[0,\kappa)$. Then, for a given $\varphi_+ \in H_+$, there is a solution of the Wiener-Hopf integral equation (4.1) with right-hand side $\varphi(s) = e^{-sT^{-1}} \varphi_+$ such that

$$(4.14) \qquad \int_0^{+\infty} e^{+t/\kappa} \|\psi(t)\|dt < +\infty ,$$

if and only if $\varphi_\kappa \in H_\kappa + H_-$. In this case the solution ψ is unique and given by (4.11). Further, the codimension in H_+ of the subspace of all vectors φ_+ inducing a right-hand side $\varphi(s) = e^{-sT^{-1}}\varphi_+$ for which a solution satisfying (4.14) exists coincides with the sum of the multiplicities of the eigenvalues λ of $A^{-1}T$ with $\lambda \geq \kappa$.

Theorem 4.4 is related to Theorem 2 in [6]. In [6] Feldman studied the Transport Equation (see Section 6 of the present paper) by transforming the equation into a Wiener-Hopf integral equation. Among other things he proved that every solution satisfying (4.12) has the form

$$(4.15)\qquad \psi(s) = \psi_p(s) + \sum_{j=1}^{m} e^{-s/\lambda_j}\varphi_{0j},\ 0 < s < +\infty,$$

where $\lambda_1,\ldots,\lambda_m$ are the distinct eigenvalues of $A^{-1}T$ not exceeding $-\kappa$, while, for $i = 1,2,\ldots,m$, φ_{0i} belongs to the eigenspace of $A^{-1}T$ corresponding to the eigenvalue λ_i and ψ_p is integrable. Note that Equation (4.15) is a direct consequence of (4.13) and represents the asymptotics of the solutions of the Wiener-Hopf integral equation (4.1) with right-hand side $\varphi(s) = e^{-sT^{-1}}\varphi_+$.

5. FACTORIZATION OF THE SYMBOL.

In the previous section we have reformulated the operator differential equation $T\psi' = A\psi$ as a Wiener-Hopf operator integral equation of the form

$$(5.1)\qquad \psi(s) - \int_0^{+\infty} H(s-t)B\,\psi(t)\,dt = \varphi(s),\ 0 < s < +\infty,$$

and we have used the equivalence of the two equations to obtain the solution of Eq. (5.1) for the case when $\varphi(s) = e^{-sT^{-1}}\varphi_+$. In this section we solve Eq.(5.1) directly by establishing a Wiener-Hopf factorization of its symbol.

In (5.1) the kernel $H(t)$ is defined by (4.2). It follows that the symbol of Eq. (5.1), i.e., the Fourier transform of the operator function $I - H(t)B$, is, up to a trivial change of variable, given by

$$I - \int_{-\infty}^{+\infty} e^{t/\lambda}\,H(t)B\,dt = (T-\lambda)^{-1}\,(T-\lambda A),\ \mathrm{Re}\,\lambda = 0.$$

Both sides of this equation tend to I in the norm as $\lambda \to 0$ ($\mathrm{Re}\,\lambda=0$). Clearly the symbol of Eq. (5.1) coincides with the operator function

$$W(\lambda)A = [I+T(T-\lambda)^{-1}C]A,\ \mathrm{Re}\,\lambda = 0,$$

which, up to a constant invertible factor at the right, is just the symbol of the self-adjoint admissible pair (T,B) on the Hilbert space H.

To obtain the solutions of Eq. (5.1) directly we shall use the method of Wiener-Hopf factorization. By this we shall mean the following. Let Γ be an oriented simple closed rectifiable Jordan contour on the Riemann sphere. The inner domain of Γ will be denoted by F_+ and the outer domain by F_-. Suppose we have a continuous operator function

$$W : \Gamma \to GL(H).$$

Here $GL(H)$ denotes the group of invertible operators on H. By definition, a left Wiener-Hopf factorization of W with respect to the contour Γ is a representation of W in the form

$$W(\lambda) = W_+(\lambda)\, D(\lambda)\, W_-(\lambda),\ \lambda \in \Gamma, \tag{5.2}$$

where $W_\pm$ is holomorphic on and continuous up to the boundary of $F_\pm$, the values of $W_\pm$ on $F_\pm \cup \Gamma$ are invertible and the diagonal factor D has the form

$$D(\lambda) = P_0 + \sum_{i=1}^{r} \left(\frac{\lambda-\lambda_-}{\lambda-\lambda_+}\right)^{\kappa_i} P_i,\ \lambda \in \Gamma.$$

Here $\lambda_\pm \in F_\pm \setminus\{\infty\}$, while $P_1,\ldots,P_r$ are disjoint one-dimensional projections with sum $I - P_0$. Finally, $\kappa_1,\ldots,\kappa_r$ are non-vanishing integers satisfying $\kappa_1 \geq \ldots \geq \kappa_r$.

It is easy to prove that the integers $\kappa_1 \geq \ldots \geq \kappa_r$ are uniquel determined by W and Γ ([10], Proof of Theorem 1.1 . They are called the left (partial) indices of W with respect to the contour Γ. The points λ_+ and λ_- can be arbitrarily chosen within F_+ and F_-, respectively, without affecting the values of the left partial indices of W with respect to Γ.

If the left partial indices all vanish then (5.2) is reduced to

$$W(\lambda) = W_+(\lambda)\, W_-(\lambda),\ \lambda \in \Gamma;$$

such a factorization of W is called a left canonical (Wiener-Hopf) factorization of W with respect to Γ. The factors W_+ and W_- are unique up to a constant invertible factor at the right and left, respectively.

Analogously, by interchanging the order of the factors in (5.2) and keeping unchanged the form of the diagonal factor D, we define a _right_ (_canonical_) _Wiener-Hopf factorization_ and _right_ (_partial_) _indices_ of W with respect to Γ.

In this section we only consider the case when one region contains the open right half-plane, whereas the other one is contained in the open left half-plane. We therefore use the convention that F_+ contains the right half-plane and F_- is contained in the left half-plane.

THEOREM 5.1. _Let_ (T,B) _be a self-adjoint admissible pair on the Hilbert space_ H _and_ P _the projection of_ H _onto_ H_p _along_ H_-. _Then a left canonical Wiener-Hopf factorization of the symbol_ W _with respect to the imaginary axis exists and is given by_

$$(5.3\ a)\qquad W(\lambda) = [I+T(T-\lambda)^{-1}(I-P)C][I+TP(T-\lambda)^{-1}C],$$

where the inverses of the factors have the following form:

$$(5.3\ b)\qquad [I+T(T-\lambda)^{-1}(I-P)C]^{-1} = I - T(I-P)(A^{-1}T-\lambda)^{-1}C,$$

$$(5.3\ c)\qquad [I+TP(T-\lambda)^{-1}C]^{-1} = I - T(A^{-1}T-\lambda)^{-1}PC.$$

THEOREM 5.2. _Let_ (T,B) _be a self-adjoint admissible pair on the Hilbert space_ H _and_ Q _the projection of_ H _onto_ H_m _along_ H_+. _Then a right canonical Wiener-Hopf factorization of the symbol_ W _with respect to the imaginary axis exists and is given by_

$$(5.4\ a)\qquad W(\lambda) = [I+T(T-\lambda)^{-1}(I-Q)C][I+TQ(T-\lambda)^{-1}C],$$

where the inverses of the factors have the following form:

$$(5.4\ b)\qquad [I+T(T-\lambda)^{-1}(I-Q)C]^{-1} = I - T(I-Q)(A^{-1}T-\lambda)^{-1}C,$$

$$(5.4\ c)\qquad [I+TQ(T-\lambda)^{-1}C]^{-1} = I - T(A^{-1}T-\lambda)^{-1}QC.$$

PROOF of the Theorem 5.1. Consider the operator node (cf. [1]) $\theta = (T,C,-T,I;H,H)$ whose transfer function is the symbol W of the pair (T,B). The associate node θ^x of θ is then given by $\theta^x = (A^{-1}T,C,T,I;H,H)$. Since, as a consequence of Theorem 2.2, the state space H can be written as the direct sum $H = H_- \oplus H_p$, where H_- is invariant under the main operator T of the node θ and H_p is invariant under the associate operator $A^{-1}T$ of θ, it follows that the projection P with kernel H_- and image H_p is a supporting projection of θ. Then (5.3 a) is just a corollary of Theorem 1.1 of [1]. Noting that $I - P$ is a supporting projection of the associate node θ^x, the formulas (5.3 b) and (5.3 c) are clear from the same theorem.

We now prove that (5.3 a) is, indeed, a left canonical Wiener Hopf factorization of W with respect to the imaginary axis Γ. Since

$$I + T(T-\lambda)^{-1}(I-P)C = I + [T(T-\lambda)^{-1}P_-](I-P)C,$$

this operator function has an analytic continuation to the open right half-plane F_+. By Proposition 1.2 and the compactness of $(I-P)C$, it tends to $I + (I-P)C$ in the norm, as $\lambda \to 0$ from the closed right half-plane $F_+ \cup \Gamma$.

Since

$$I - T(A^{-1}T-\lambda)^{-1}PC = I - A[A^{-1}T(A^{-1}T-\lambda)^{-1}P_p]PC,$$

this operator function has an analytic continuation to the open left half-plane F_-. By the self-adjointness of $A^{-1}T$ in the equivalent inner product (2.4), Proposition 2.1 and the compactness of PC, the value of this operator function tends to the operator $I - APC$ in the norm, as $\lambda \to 0$ from the closed left half-plane $F_- \cup \Gamma$.

In a similar way, we prove that $I + TP(T-\lambda)^{-1}C[I-T(I-P)(A^{-1}T-\lambda)^{-1}C]$ has an analytic continuation to the open left [right] half-plane $F_-[F_+]$, while its value tends to the operator $I + (I-Q^*)C$ $[I-Q^*AC]$ in the norm, as $\lambda \to 0$ from the closed left [right] half-plane $F_- \cup \Gamma$ $[F_+ \cup \Gamma]$. To be able to apply Proposition 1.2 again, we have to use the identities $TP = (I-Q^*)T$ and $T(I-P) = Q^*C$ (cf.Theorem 2.2). This completes the proof.

Theorem 5.2 is proved likewise. We remark that the existence of the Wiener-Hopf factorizations obtained in Theorem 5.1 and 5.2 is also clear from [15], Theorem 4.3 (or 4.4). If either $\sigma(B) \subset (-1,+1)$ or $\sigma(B) \subset (-\infty,\frac{1}{2})$, the existence of a canonical factorization of W with respect to the imaginary axis is clear from [14], Theorem 4.1. First we take B in such a way that $\sigma(B) \subset (-1,+1)$. Then the operator $A = I - B$ is invertible, while

$$\max_{\text{Re}\,\lambda=0} \|W(\lambda)\ A-I\| \leq \|B\| < 1,$$

proving the existence of both a left and a right canonical factorization of W with respect to the imaginary axis. If we choose B in such a way that $\sigma(B) \subset (-\infty,\frac{1}{2})$, then $\sigma(C) \subset (-1,+1)$, and therefore

$$\max_{\text{Re}\,\lambda=0} \|W(\lambda) - I\| \leq \|C\| < 1.$$

From this we have the existence of both a left and a right canonical factorization with respect to the imaginary axis in case $\sigma(B) \subset (-\infty,\frac{1}{2})$. Hence, if B satisfies the extra condition that either $\sigma(B) \subset (-1,+1)$ or $\sigma(B) \subset (-\infty,\frac{1}{2})$, there exists both a left and a right canonical factorization of W with respect to the imaginary axis. In the isotropic case of the Transport Equation (cf.Section 6) the operator B has only one eigenvalue, which is strictly less than +1. Hence, in this case W has both a left and a right canonical factorization. Note, however, that the Gohberg-Leiterer theory [14,15] does not yield the factors explicitly.

Now that the right-hand side of Eq.(5.1) has the special form $\varphi(s) = e^{-sT^{-1}}\varphi_+$ and a left canonical Wiener-Hopf factorization of the symbol of this equation is explicitly known, we are able to derive an explicit formula for the (unique) integrable solution ψ of Eq.(5.1). Put

$$\varphi(s) = 0,\ \psi(s) = \int_0^{+\infty} H(s-t)B\ \psi(t)dt,\ -\infty < s < 0.$$

Then Eq.(5.1) holds true for all $s \in \mathbb{R}\setminus\{0\}$. For integrable ψ, put

$$\hat{\Psi}_-(\lambda) = \int_0^{+\infty} e^{t/\lambda}\psi(t)dt,\ \hat{\Psi}_+(\lambda) = \int_{-\infty}^{0} e^{t/\lambda}\ \psi(t)dt,$$

$$\hat{H}(\lambda) = \int_{-\infty}^{+\infty} e^{t/\lambda} H(t)dt, \ \hat{\varphi}_-(\lambda) = \int_0^{+\infty} e^{t/\lambda} \varphi(t)dt$$

for Re $\lambda = 0$. Then we have

$$[I-\hat{H}(\lambda)B] \hat{\psi}_-(\lambda) + \hat{\psi}_+(\lambda) = \hat{\varphi}_+(\lambda), \ \mathrm{Re}\,\lambda = 0.$$

Since $I - \hat{H}(\lambda)B = W(\lambda) A$, we have for Re $\lambda = 0$

$$(5.5) \qquad W_-(\lambda)A \hat{\psi}_-(\lambda) + W_+(\lambda)^{-1} \hat{\psi}_+(\lambda) = W_+(\lambda)^{-1} \hat{\varphi}_-(\lambda),$$

where W_- and W_+ are the factors of a left canonical factorization of W with respect to the imaginary axis, obtained in Theorem 5.1, i.e.,

$$(5.6) \qquad W_-(\lambda) = I + TP(T-\lambda)^{-1}C, \ W_+(\lambda)^{-1} = I - T(I-P)(A^{-1}T-\lambda)^{-1}C.$$

Further, by [4], Theorem VIII 1.11, we have

$$(5.7) \qquad \hat{\varphi}_-(\lambda) = \lambda T(\lambda-T)^{-1} \varphi_+.$$

A straightforward computation, using Liouville's theorem and Eqs. (5.5)-(5.7), yields

$$\hat{\psi}_-(\lambda) = \lambda A^{-1}T(A^{-1}T-\lambda)^{-1} P\varphi_+, \ \hat{\psi}_+(\lambda) = -\lambda T(T-\lambda)^{-1}(I-P)\varphi_+.$$

By [4], Theorem VIII 1.11, we have

$$\psi(s) = e^{-sT^{-1}A} P\varphi_+, \ 0 < s < +\infty,$$

which corresponds to the solution obtained in Theorem 4.3.

Next, we consider those solutions of Eq.(5.1) with right-hand side $\varphi(s) = e^{-sT^{-1}} \varphi_+$ that satisfy the condition

$$(5.8) \qquad \operatorname*{ess\text{-}sup}_{0<t<+\infty} e^{-t/\kappa} \|\psi(t)\| < +\infty,$$

for some $\kappa > -\min(0,m(T))$. These solutions coincide with the essentially bounded solutions of a Wiener-Hopf operator integral equation with right-hand side $\varphi_0(s) = e^{+s/\kappa} \varphi(s)$ and symbol

$$(5.9) \qquad W((\lambda^{-1}-\kappa^{-1})^{-1})A, \ \mathrm{Re}\ \lambda = 0.$$

This Wiener-Hopf equation can be solved by making a left Wiener-Hopf factorization of the operator function (5.9) with respect to the imaginary axis or, what turns out to be equivalent, by making a left Wiener-Hopf factorization of the symbol W of the pair (T,B) with respect to the circle Γ_κ with centre $-\frac{1}{2}\kappa$ and radius $\frac{1}{2}\kappa$. In this way, making use of the solution formula (4.9), we obtain information about the left partial indices.

THEOREM 5.3. *Let (T,B) be a self-adjoint admissible pair on a Hilbert space, and let $-\kappa < \min(0,m(T))$. Then the symbol W of the pair (T,B) has a left (right) Wiener-Hopf factorization with respect to the circle Γ_κ with centre $-\frac{1}{2}\kappa$ and radius $\frac{1}{2}\kappa$ if and only if $-\kappa$ is not an eigenvalue of the operator $A^{-1}T$. If this is the case, all left partial indices of the symbol W with respect to the circle Γ_κ have the value -1, while the number of these indices coincides with the number of eigenvalues of $A^{-1}T$, strictly less than $-\kappa$, counted according to multiplicity.*

PROOF. Let ψ be a solution of Eq.(5.1) with right-hand side $\varphi(s) = e^{-sT^{-1}}\varphi_+$ satisfying Condition (5.8). Put

$$\psi_0(s) = e^{-s/\kappa}\psi(s)\ (0<s<+\infty);\ H_0(t) = e^{-t/\kappa}H(t)\ (t\in\mathbb{R}\setminus\{0\}).$$

Then ψ_0 is an integrable solution of the Wiener-Hopf operator integral equation

$$\psi_0(s) - \int_0^{+\infty} H_0(s-t)B\,\psi_0(t)\,dt = e^{+s/\kappa}\varphi(s),\ 0 < s < +\infty. \tag{5.10}$$

The symbol of this equation is, up to a trivial change of variable, given by

$$I - \int_{-\infty}^{+\infty}\exp[t(\lambda^{-1}-\kappa^{-1})]\,H(t)B\,dt = W(\zeta)A,$$

where $\zeta^{-1} = \lambda^{-1} - \kappa^{-1}$, Re $\zeta = 0$, and W is the symbol of the pair (T,B). Then ζ ranges over the circle Γ_κ with centre $-\frac{1}{2}\kappa$ and radius $\frac{1}{2}\kappa$. Clearly the problem of solving Eq.(5.10) amounts to constructing a left Wiener-Hopf factorization of the operator function W with respect to Γ_κ. According to [15], Theorem 4.3 (or 4.4), such a factorization exists if and only if all values of the operator function W on the circle Γ_κ are invertible. With the help of Proposition 2.1 it is easy to see that this is the case if and only if $-\kappa$ is not an eigenvalue of $A^{-1}T$.

According to a well-known result (cf., for instance, [11], Theorem VIII 6.1, and [8], Theorem I), the number of linearly independent solutions of a homogeneous Wiener-Hopf operator integral equation is, up to a sign, equal to the sum of the negative left indices. By this same result, the operator defined on the Banach space of H-valued L_∞-functions on the interval $[0,+\infty)$ by the left-hand side of Eq.(5.10) has a closed range whose codimension equals the sum of the positive left indices. However, from the theorem describing the solutions of Eq.(5.10) (i.e., Theorem 4.5) it is clear that the sum of the negative left indices coincides with the opposite of the number of eigenvalues of $A^{-1}T$ on the interval $(-\infty,-\kappa]$, while the sum of the positive left indices vanishes. Therefore, all left indices are negative.

It remains to prove that none of the left indices is less than -1. To see this we argue as follows. Since every integrable solution of Eq.(5.10) has the form

$$\psi(s) = e^{-sT^{-1}A}\varphi_0, \quad 0 < s < +\infty,$$

for some vector φ_0, it is clear that such a solution vanishes identically if and only if it vanishes at $s = 0$. The assertion is clear now from [12], Theorem 9.2. This completes the proof.

We conclude with a remark. Take $-\kappa_1 < -\kappa_2 < \min(0,m(T))$ and assume that neither $-\kappa_1$ nor $-\kappa_2$ is an eigenvalue of $A^{-1}T$. From the previous theorem it follows that the sums of the left indices of the operator function W with respect to the circles Γ_{κ_1} and Γ_{κ_2}, respectively, differ by the number of eigenvalues of the operator $A^{-1}T$ on the interval $(-\kappa_1,-\kappa_2)$, counted according to multiplicity. This statement has been proved earlier by Feldman (cf.[7], Lemma 1; [9], Theorem 5).

6. THE TRANSPORT EQUATION: ONE-SPEED CASE.

In the present and the next section we apply our results to the Transport Equation. This equation describes the time-independent transport of particles through a homogeneous, semi-infinite medium. Let x be a position coordinate ranging over $(0,+\infty)$, μ the sine of the angle between the velocity and the surface of the medium, g the scattering function and ψ the angular density. Assuming, in addition, that the particles do not interact and have a (nearly) constant speed, one obtains

$$(6.1) \qquad \mu \frac{\partial\psi}{\partial x}(x,\mu) + \psi(x,\mu) = \int_{-1}^{+1} g(\mu,\mu')\, \psi(x,\mu')d\mu'.$$

The solution ψ has to satisfy a boundary condition of the form

$$(6.2) \qquad \lim_{x\downarrow 0} \psi(x,\mu) = \varphi_+(\mu), \ 0 \le \mu \le 1,$$

and the following growth condition

$$\operatorname*{ess-sup}_{0<x<+\infty} e^{-x/\kappa} \left[\int_{-1}^{+1} |\psi(x,\mu)|^2 d\mu\right]^{1/2} < +\infty,$$

for some fixed $\kappa > 1$.

The scattering function g in Eq.(6.1) will be assumed to be a real symmetric function satisfying the following two conditions:

$$\operatorname*{ess-sup}_{-1\le\mu\le+1} \int_{-1}^{+1} |\, g(\mu,\mu') \,|^r d\mu' < +\infty,$$

$$\operatorname*{ess-sup}_{-1\le\mu'\le+1} \int_{-1}^{+1} |\, g(\mu,\mu') \,|^r d\mu < +\infty,$$

for some fixed $1 < r < 2$. According to [19], Theorem 3(2.X), this implies that the operator $B : L_2[-1,+1] \to L_2[-1,+1]$, defined by

$$(6.3) \qquad (Bf)(\mu) = \int_{-1}^{+1} g(\mu,\mu')\, f(\mu')d\mu', \ -1 \le \mu \le +1,$$

is a well-defined compact self-adjoint operator. We shall assume that $\sigma(B)$ belongs to the open interval $(-\infty,+1)$.

By writing $\psi(x)(\mu) = \psi(x,\mu)$, we may consider the unknown function ψ as a vector function with values in the Hilbert space $L_2[-1,+1]$ of square-integrable functions on $[-1,+1]$. In this way Equation (5.1) can be written as an operator differential equation of the form

(6.4) $T\psi = -A\psi$,

where $A = I - B$ and T is the operator of multiplication on $L_2[-1.+1]$, i.e.,

$$(Tf)(\mu) = \mu f(\mu), \quad -1 \le \mu \le +1.$$

Note that T is a bounded self-adjoint operator with Ker $T = \{0\}$. The next lemma allows us to apply the theory developed in Section 3 to the operator differential equation (6.4).

LEMMA 6.1. _The pair_ (T,B) _of operators introduced above is a self-adjoint admissible pair on_ $L_2[-1,+1]$.

PROOF. It remains to show that the pair (T,B) satisfies Condition (C.3) of Section 2. In order to do this take $0 < \alpha < (2r)^{-1}(r-1)$, where r follows from the two conditions imposed on the scattering function g. For such an α the operator S_α, defined by

$$(S_\alpha f)(\mu) = |\mu|^{-\alpha} f(\mu), \quad -1 \le \mu \le 1,$$

is a well-defined and bounded linear operator from $L_{2r}[-1,+1]$ into $L_2[-1,+1]$. This is a direct consequence of Hölders inequality.

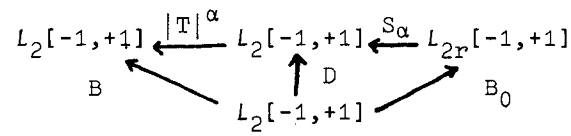
The operator given by the formula (6.3) is a well-defined compact operator from $L_2[-1,+1]$ into $L_{2r}[-1,+1]$ (cf.[19], Theorem 3(2.X)); we shall denote this operator by B_0. Put

$$(Df)(\mu) = \int_{-1}^{+1} |\mu|^{-\alpha} g(\mu,\mu') f(\mu')d\mu', \quad -1 \le \mu \le +1.$$

By virtue of the identity $D = S_\alpha B_0$, the operator D is a compact linear operator on $L_2[-1,+1]$. As the operator $|T|^\alpha \in L(L_2[-1,+1])$ acts by the formula

$$(|T|^\alpha f)(\mu) = |\mu|^\alpha f(\mu), \quad -1 \le \mu \le +1,$$

we have the following commutative diagram:



From this diagram Lemma 6.1. is clear.

By the previous lemma, the theory developed in Sections 2 to 5 can be applied. In particular, the main results of these sections, i.e., the decomposition $H_p \oplus H_- = H_m \oplus H_+ = H$ (Theorem 2.2), the theorems describing the solutions of the differential equation $T\psi = -A\psi$ (Theorems 3.1, 3.2 and 3.3), the equivalence result for the two approaches (Theorem 4.2) and the factorization results (Theorems 5.1 and 5.2), are applicable to the Transport Equation. For H, H_+ and H_- we have to read $L_2[-1,+1]$, $L_2[0,1]$ and $L_2[-1,0]$, respectively. As an example, we mention the fact that for every $\varphi_+ \in L_2[0,1]$ the Transport Equation (6.1) has a unique bounded solution with φ_+ as its boundary value. This solution is given by the formula

$$(6.5) \qquad \psi(x) = e^{-xT^{-1}A} P \varphi_+, \; 0 < x < +\infty.$$

Here, as before, P is the projection of H along H_- onto H_p.

A formula like (6.5) does not have any practical meaning as long as we are not able to find an explicit expression for the projection P. Such explicit expressions are provided by the next two theorems. In these theorems we restrict ourselves to the case when the scattering function is degenerate. More precisely, when the scattering function g is given by the expression

$$(6.6) \qquad g(\mu,\mu') = \sum_{j=0}^{n} a_j \, p_j(\mu) \, p_j(\mu'), \; -1 \le \mu,\mu' \le +1.$$

For $j = 0,1,\ldots,n$, the function p_j is the normalized (real) Legendre polynomial of degree j, while $a_j < 1$. For $j = 0,1,\ldots,n$, put $b_j = a_j(1-a_j)^{-1}$. Then we have

$$B = \sum_{j=0}^{n} a_j \langle \cdot, p_j \rangle p_j, \quad C = \sum_{j=0}^{n} b_j \langle \cdot, p_j \rangle p_j.$$

To get an explicit formula for the projection P, we introduce the auxiliary functions Λ, Λ_+ and Λ_-. Let $\Lambda(\lambda)$ be the determinant of the operator

$$(A^{-1}T-\lambda)\,(T-\lambda)^{-1}, \; \lambda \notin [-1,+1].$$

Then Λ has a left canonical factorization $\Lambda = \Lambda_+ \Lambda_-$ with respect to the imaginary axis (cf.Theorem 5.1). Because $\Lambda(\infty) = 1$, we normalize Λ_+ and Λ_- by the condition $\Lambda_+(\infty) = \Lambda_-(\infty) = 1$.

For $j = 0,1,\ldots,n$, we put

$$(6.7)\qquad \hat{p}_j(\mu) = \begin{cases} \Lambda_+(\mu)((I-P)p_j)(\mu), & -1 \le \mu < 0, \\ \Lambda_-(\mu)((I-Q)p_j)(\mu), & 0 < \mu \le +1, \end{cases}$$

where $P(Q)$ is the projection of H onto $H_p(H_m)$ along $H_+(H_-)$, introduced earlier. We have the following result.

THEOREM 6.2. <u>If the scattering function</u> g <u>has the form</u> (6.6) <u>and</u> $\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_n$ <u>are defined by</u> (6.7), <u>then the projection</u> P <u>is given by the equation</u>

$$((I-P)\varphi)(\mu) = 2^{1/2}\,\frac{\varphi(\mu)\hat{p}_0(\mu)}{\Lambda_+(\mu)} - \int_0^1 v.\ \frac{\varphi(v)-\varphi(\mu)}{v-\mu}\ \frac{K(\mu,v)}{\Lambda_+(\mu)\Lambda_-(v)}\,dv,$$

for $-1 \le \mu \le 0$. For $0 < \mu \le +1$, <u>we have</u> $((I-P)\varphi)(\mu) = 0$. <u>Here the kernel</u> K <u>is given by</u>

$$K(\mu,v) = \sum_{j=0}^{n} b_j\ \hat{p}_j(\mu)\ \hat{p}_j(v).$$

PROOF. Clearly, $(I-P)T = T(I-P) - (I-P)CT\,P$. By induction we easily prove that

$$(I-P)T^\kappa = T^\kappa(I-P) - \sum_{i=0}^{\kappa-1} T^i(I-P)CT\ P\ T^{\kappa-1-i},\quad \kappa \in \mathbb{N}.$$

Inserting the explicit form for the operator C, we get

$$(I-P)T^\kappa f = T^\kappa(I-P)f - \sum_{i=0}^{\kappa-1}\sum_{j=0}^{n} b_j \langle TP\ T^{\kappa-1-i} f,\ p_j\rangle\ T^i(I-P)p_j,\quad \kappa \in \mathbb{N}.$$

Using Theorem 2.2, we have for every $g \in H$:

$$\begin{aligned}\langle TPg, p_j\rangle &= \langle Tg, (I-Q)p_j\rangle \\ &= \int_0^1 \mu g(\mu)\ \frac{\hat{p}_j(\mu)}{\Lambda_-(\mu)}\,d\mu\quad (j=0,1,\ldots,n).\end{aligned}$$

From this equation and the fact that $\operatorname{Ker} P = H_- = L_2[-1,0]$, it is clear that we have for $-1 \le \mu < 0$:

$$((I-P)T^{\kappa}f)(\mu) = \mu^{\kappa}((I-P)f)(\mu) - \sum_{i=0}^{\kappa-1} \sum_{j=0}^{n} b_j \mu^i \int_0^1 \nu^{\kappa-i} f(\nu) \frac{\hat{p}_j(\mu)\hat{p}_j(\nu)}{\Lambda_+(\mu)\Lambda_-(\nu)} d\nu.$$

Because of the identity

$$\sum_{i=0}^{\kappa-1} \mu^i \nu^{\kappa-1-i} = \frac{\nu^{\kappa}-\mu^{\kappa}}{\nu-\mu} ,$$

and the way in which the kernel K is defined, we obtain the theorem for all functions φ of the form $\varphi(t) = t^{\kappa}$. By linearity, the theorem is true if φ is a polynomial. But the right-hand side of Eq.(6.8) is an integral operator on the space $L_2[0,1]$ with values in $L_2[-1,0]$, whose kernel is essentially bounded. Such an integral operator, however, is bounded. Because the polynomials are dense in $L_2[-1,+1]$, the theorem is clear.

In case of isotropic scattering we have a constant scattering function $g(\mu,\mu') \equiv \frac{1}{2} c$ where $c < 1$. Then $n = 0$, $a_0 = c$, $b_0 = c(1-c)^{-1}$. Put $e(\mu) \equiv 1$. Then $p_0 = \frac{1}{2}\sqrt{2}\, e$. In view of (6.8), it suffices to compute $(I-P)e$ in order to obtain an explicit formula for the projection P in which $\hat{p}_0$ is known. This computation will use the factorization formulas obtained in Theorem 5.1 and 5.2.

THEOREM 6.3. _If the scattering function_ g _has the form_ $g(\mu,\mu') \equiv \frac{1}{2}c$ _for some constant_ $c < 1$, _then for_ $-1 \le \mu \le 0$ _we have_

$$((I-P)\varphi)(\mu) = \frac{\varphi(\mu)}{\Lambda_+(\mu)} - \tfrac{1}{2}c(1-c)^{-1} \int_0^1 \nu . \frac{\varphi(\nu)-\varphi(\mu)}{\nu-\mu} \frac{d\nu}{\Lambda_+(\mu)\Lambda_-(\nu)}.$$

PROOF. We have $C = \frac{1}{2}c(1-c)^{-1} \langle .,e\rangle\, e$, while the determinant Λ has the form

$$\Lambda(\lambda) = 1 + \tfrac{1}{2}c(1-c)^{-1} \int_{-1}^{+1} \mu(\mu-\lambda)^{-1}\, d\mu.$$

By direct computation (cf.[4], Exercise VIII 9.11) we have

$$\lim_{\varepsilon\downarrow 0} \frac{\Lambda(t+i\varepsilon)-\Lambda(t-i\varepsilon)}{2i} = \frac{\pi c(1-c)^{-1}t}{2}, \quad -1 < t < +1.$$

From Theorem 5.1 we have

$$(6.9) \qquad \Lambda_-(\lambda) = 1 + \tfrac{1}{2}c(1-c)^{-1} < T(T-\lambda)^{-1}(I-P)e,\ e > .$$

Let Γ_- be an oriented contour in the closed left half plane that cuts the real line at $\lambda = 0$ perpendicularly and encloses the parts of the spectra of T and $A^{-1}T$ in the open left half plane once in the positive direction. From (6.9) we have for $\kappa = 0,1,2,\ldots$

$$(6.10) \qquad (-2\pi i)^{-1} \int_{\Gamma_-} \lambda^\kappa \Lambda_-(\lambda)\, d\lambda = \tfrac{1}{2}c(1-c)^{-1} <T^{\kappa+1}(I-P)e,\ e>.$$

On the other hand, it follows from the analyticity of Λ_- outside $[-1,0]$ and the continuity and non-vanishing of Λ_+ in the closed left half plane that

$$(6.11) \qquad (-2\pi i)^{-1} \int_{\Gamma_-} \lambda^\kappa \Lambda_-(\lambda)\, d\lambda = \frac{1}{\pi}\int_{-1}^{0} \frac{t^\kappa}{\Lambda_+(t)} \lim_{\varepsilon\downarrow 0} \frac{\Lambda(t+i\varepsilon)-\Lambda(t-i\varepsilon)}{2i}\, dt.$$

From (6.10) and (6.11) we have

$$<T(I-P)e,\ T^\kappa e> = \int_{-1}^{0} \frac{t}{\Lambda_+(t)} t^\kappa\, dt, \quad \kappa = 0,1,2,\ldots.$$

Since $\{T^\kappa e\}_{\kappa=0}^{+\infty}$ spans a dense linear subspace of $H = L_2[-1,+1]$, we obtain

$$((I-P)e)(\mu) = \begin{cases} \Lambda_+(\mu)^{-1}, & -1 \le \mu < 0. \\ 0 & 0 < \mu \le +1. \end{cases}$$

In a similar way, we get

$$((I-Q)e)(\mu) = \begin{cases} \Lambda_-(\mu)^{-1}, & 0 < \mu \le +1. \\ 0, & -1 \le \mu < 0. \end{cases}$$

Hence, $\hat{p}_0(\mu) \equiv 1/\sqrt{2}$ and $K(\mu,v) \equiv \tfrac{1}{2}c(1-c)^{-1}$. Applying the previous theorem we obtain the present one.

As a concluding remark we note that results similar to Theorem 6.3 have been obtained by several authors (for instance, [3,16, 18,20,22]), while a result similar to Theorem 6.2 has been obtained in [21]. All these authors, however, need an additional diagonalizing factor in the formulas for the projection P.

7. THE MULTIGROUP TRANSPORT EQUATION.

In the present section we apply our results to the multigroup transport equation. This equation describes the time-independent transport of particles through a homogeneous semi-infinite medium. Now it is assumed that the particles do not interact and are divided into N groups of particles with (nearly) constant speed. Let x be a position coordinate ranging over $(0,+\infty)$ and μ the sine of the angle between the velocity and the surface of the medium. For $i = 1,\dots,N$, let ψ_i be the angular density within the i-th group and σ_i a scalar number proportionate to the mean speed of the i-th group. Here $\sigma_1,\dots,\sigma_N$ are ordered in such a way that $\sigma_1 \geq \dots \geq \sigma_N = 1$.

Let Σ denote the $N \times N$ -diagonal matrix with diagonal elements $\sigma_1,\dots,\sigma_N$. For $i,j = 1,\dots,N$, the function g_{ij} describes the scattering from group j to group i. Put

$$G(\mu,\mu') = (g_{ij}(\mu,\mu'))_{i,j=1}^{N}.$$

Then G is called the scattering function. Let $\psi(x,\mu)$ be the column vector with elements $\psi_1(x,\mu),\dots, \psi_N(x,\mu)$. According to [2], the particle transport is described by the vectorvalued integro-differential equation

$$(7.1) \qquad \mu \frac{\partial\psi}{\partial x}(x,\mu) + \Sigma\, \psi(x,\mu) = \int_{-1}^{+1} G(\mu,\mu')\, \psi(x,\mu')d\mu', \quad 0 < x < +\infty,$$

with boundary conditions

$$(7.2) \qquad \lim_{x\downarrow 0} \psi_i(x,\mu) = \varphi_{+,i}(\mu), \quad 0 \leq \mu \leq +1, \quad i = 1,2,\dots,N.$$

We shall denote the column vector with elements $\varphi_{+,i}(\mu)$ by $\varphi_+(\mu)$. Besides the boundary conditions (7.2) the solution ψ has to satisfy a growth condition of the form

$$\underset{0<x<+\infty}{\text{ess-sup}}\; e^{-x/\kappa} \left[\int_{-1}^{+1} |\psi_i(x,\mu)|^2 d\mu\right]^{1/2} < +\infty,$$

for some fixed $\kappa > 1$ and $1 \le i \le N$.

The scattering function G in Eq.(7.1) will be assumed to be an $N \times N$-matrix of real symmetric functions satisfying the following conditions for $i = 1,\dots,N$:

$$\underset{-1\le\mu\le+1}{\text{ess-sup}} \int_{-1}^{+1} |g_{ij}(\mu,\mu')|^r \, d\mu' < +\infty, \quad \underset{-1\le\mu'\le+1}{\text{ess-sup}} \int_{-1}^{+1} |g_{ij}(\mu,\mu')|^r \, d\mu < +\infty,$$

for some fixed $1 < r < 2$. According to [19], Theorem 3 (2.X), this implies that all the operators $B_{ij} : L_2[-1,+1] \to L_2[-1,+1]$, defined by

$$(B_{ij}f)(\mu) = \int_{-1}^{+1} g_{ij}(\mu,\mu')\, f(\mu')d\mu', \quad -1 \le \mu \le +1,$$

are well-defined compact operators $(1\le i,j\le N)$. Assuming, in addition, that $g_{ij}(\mu,\mu') = g_{ji}(\mu',\mu)$, we have $B_{ij}^* = B_{ji}$.

We shall consider Eq.(6.1) in the space $H = L_2([-1,+1], \mathbb{C}^N)$ of all $\mathbb{C}^N$-valued L_2-functions on $[-1,+1]$, endowed with the inner product

$$\langle f,g\rangle = \sum_{i=1}^{N} \int_{-1}^{+1} f_i(\mu)\, \overline{g_i(\mu)}\, d\mu,$$

where $f(g)$ is the column of functions $f_1,\dots,f_N$ $(g_1,\dots,g_N)$. Define the operator $B : H \to H$ by

$$(Bf)_i = \sum_{j=1}^{N} B_{ij}\, f_j, \quad 1 \le i \le N.$$

Then B is a self-adjoint compact operator on the space H. Our final assumption will be that the operator $A = I - \Sigma B$ is strictly positive.

We define the operator $T : H \to H$ by

$$(Tf)_i\,(\mu) = \mu f_i(\mu), \quad 1 \le i \le N, \; -1 \le \mu \le +1.$$

Then T is a self-adjoint operator with spectrum $[-1,+1]$, whose null space is trivial. The same holds true for the operator $\Sigma^{-1}\, T$.

LEMMA 7.1. The pair $(\Sigma^{-1}T, \Sigma^{-1}B)$ of operators introduced above is a self-adjoint admissible pair on $L_2([-1,+1], \mathbb{C}^N)$.

PROOF. It remains to show that the pair $(\Sigma^{-1}T, \Sigma^{-1}B)$ satisfies Condition (C.3) of Section 2. In order to do this take $0 < \alpha < (2r)^{-1}(r-1)$, where r follows from the conditions imposed on the scattering function G. Note that $|\Sigma^{-1}T|^{\alpha} = \Sigma^{-\alpha}|T|^{\alpha}$. The lemma will be clear now from the argument used in the proof of Lemma 6.1 and the specific form of the operators T and B.

By the lemma, the theory developed in Section 2 to 5 can be applied. Put $C = \Sigma^{-1}[(I-\Sigma^{-1}B)^{-1}-I]$. Then the roles of $B,A,C,T,A^{-1}T$ are played by $\Sigma^{-1}B$, A, ΣC, $\Sigma^{-1}T$, $A^{-1}\Sigma^{-1}T$, respectively. From Theorem 2.2 it follows that $H_p \oplus H_- = H$, where $H_p(H_-)$ is the spectral projection of $A^{-1}\Sigma^{-1}T$ $(\Sigma^{-1}T)$ corresponding to the positive (negative) part of its spectrum. From Theorem 3.1 it is clear that for every $\varphi_+ \in L_2([0,1], \mathbb{C}^N)$ the multigroup transport equation (7.1) has a unique bounded solution with φ_+ as its column vector of boundary values; this solution is given by the formula

$$\psi(x) = e^{-xT^{-1}\Sigma A} P \varphi_+, \quad 0 < x < +\infty,$$

where P is the projection of H onto H_p along H_-.

As a concluding remark we note that the existence and uniqueness of a bounded solution of Eq.(7.1) has been proved earlier in [2] under somewhat more restricted conditions for degenerate scattering functions.

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