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REALIZATION AND LINEARIZATION

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0. INTRODUCTION.

In this report the connection between realization and linearization is studied from the point of view developed in [3], Chapter 2. Most results on linearization, known in literature, are derived using this connection. Further, the analogue of linearization for meromorphic operator functions is introduced and results similar to those for holomorphic operator functions are deduced.

The operator functions considered here appear as realizations. In the case considered mostly, namely that of a monic realization, they have the form

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \tag{0.1}$$

where λ is a complex parameter and A, B, C and D are bounded linear operators acting between appropriate Banach spaces. When D is an invertible operator, by the connection between realization and linearization referred to above, the function $W(\lambda)$ admits a linearization on the resolvent set $\rho(A)$ of A of the form

$$\lambda I - (A - BD^{-1}C), \tag{0.2}$$

provided either B is left invertible or C is right invertible. In fact, there exists an extension space Z such that the Z-extension of W is analytically equivalent on $\rho(A)$ to the linear pencil (0.2). If the invertibility condition on B or C is dropped, then still some extension of W is equivalent on $\rho(A)$ to some extension of the linear pencil (0.2).

If W is a holomorphic operator function on a bounded open subset Ω of $\mathbb{C}$, then W can be written in the form (0.1) where $\Omega \subset \rho(A)$, D is invertible and either B is left invertible or C is right invertible. For several classes of holomorphic operator functions, such as the classes of (comonic) operator polynomials, holomorphic functions with continuous or non-continuous boundary values, we derive in this way a concrete linearization of the form (0.2). Linearization results of Gohberg and Rodman [12], Gohberg, Kaashoek and Lay [10], Mitiagin [18] and Den Boer [8] appear as special cases. Also a linearization for arbitrary entire operator functions is obtained in this way.

If W is a meromorphic operator function on a bounded open subset Ω of $\mathbb{C}$, then W can be written as a realization of the form (0.1), where

D is invertible, B is left invertible, and the part of the spectrum $\sigma(A)$ of A in Ω is the set of poles of W. Also some extension of W is analytically equivalent on all of Ω to a fractional linear operator function of the form

$$(0.3)\qquad [\lambda I - (A - BD^{-1}C)](\lambda I - A)^{-1}.$$

Realizations of the form (0.1) appear in several branches of mathematics. In network theory they appear as the transfer functions of a finite network, in system theory as the transfer functions of a linear system [1, 14] and in operator theory as characteristic operator functions [16, 5, 20, 7, 3]. Linearization has its origin in the theory of differential equations. It is a powerful aid in the study of matrix and operator polynomials and holomorphic operator functions ([10], see also [2], Section III.4 and the references given there).

Now let us give a more detailed description of the contents of the different chapters. In the first chapter we discuss the notions of realization and linearization and explain in abstract form their connection. As a first application we derive a result of Brodskii and Švarcman [6]. The concept of minimality of a realization is discussed in Section I and the notion of biminimality is introduced. Further we discuss a general construction for obtaining a minimal realization from an arbitrary one, which will be used in the second chapter.

In the second chapter we derive the linearization theorems appearing in literature by writing the operator function concerned as a realization of the form (0.1), where A, B, C and D are given in an explicit form. For the realizations obtained we construct their (bi)minimal counterparts. Further, we make an explicit connection between linearization and the linearizator introduced by Markus and Macaev [17].

In the third chapter we prove that every meromorphic operator function is analytically equivalent up to extension to a fractional linear operator function of the form (0.3). The proof is given by reducing the problem to the linearization of two holomorphic operator functions in a special way. As an ancillary result it renders a representation of the meromorphic function in the form (0.1). There is, however, a peculiarity involved: whereas for holomorphic operator functions every realization of the form (0.1) gives rise to a linearization of the form (0.2), for the case of meromorphic operator functions not every realization of the form (0.1) yields a fractional linear operator function of the form (0.3).

Finally, a few words about notation and terminology. By an operator we shall mean any linear operator between Banach spaces. The null space and range of an operator T are denoted by Ker T and Im T, respectively. The identity on a Banach space is denoted by I; we shall write I_X if we want to make clear that we mean the identity operator on the space X.

I. LINEARIZATION AND REALIZATION

1. Equivalence, linearization and realization.

Let X and Y be complex Banach spaces. The complex Banach space of all bounded linear operators from X into Y will be denoted by $L(X,Y)$. By $GL(X,Y)$ we mean the set of all invertible operators in $L(X,Y)$. In case $X = Y$ we usually write $L(X)$ and $GL(X)$ instead of $L(X,X)$ and $GL(X,X)$ respectively.

Let Ω be an open subset of the Riemann sphere $\mathbb{C}_\infty$ and X_1, X_2, Y_1 and Y_2 complex Banach spaces. Two holomorphic operator functions $A_1 : \Omega \to L(X_1,Y_1)$ and $A_2 : \Omega \to L(X_2,Y_2)$ are called *equivalent on* Ω, if there exist holomorphic operator functions $E : \Omega \to GL(Y_1,Y_2)$ and $F : \Omega \to GL(X_2,X_1)$, called *equivalence functions*, such that

$$A_2(\lambda) = E(\lambda)A_1(\lambda)F(\lambda), \qquad \lambda \in \Omega. \tag{1.1}$$

Given a holomorphic operator function $A: \Omega \to L(X,Y)$ and a complex Banach space Z, the Z-*extension* of A is the operator function on Ω whose value at $\lambda \in \Omega$ is the operator $A(\lambda) \oplus I_Z \in L(X \oplus Z, Y \oplus Z)$. The Banach space Z is called the *extension space*. Two holomorphic operator functions $A_1: \Omega \to L(X_1,Y_1)$ and $A_2: \Omega \to L(X_2,Y_2)$ are called *equivalent up to extension* on Ω, if there exist complex Banach spaces Z_1 and Z_2 such that the Z_1-extension of A_1 and the Z_2-extension of A_2 are equivalent on Ω.

By a *linearization* of A on Ω we mean an operator pencil $T - \lambda S$ that is equivalent on Ω to some extension of A. If S is the identity operator, we say that the linearization is *monic*, and we call T a *monic linearization*. Similarly, the linearization is said to be *comonic* when T is the identity, and in that case S is called a *comonic linearization.*

In many problems concerning linearization, there is no loss of generality in assuming that $X = Y$. To see this, we argue as follows (cf. [10]). Let $A: \Omega \to L(X,Y)$ be holomorphic. Then the $\ell_1(X \oplus Y)$-extension $\tilde{A}$ of A is a function whose values are operators from $X \oplus \ell_1(X \oplus Y)$ into $Y \oplus \ell_1(X \oplus Y)$. It is clear that $X \oplus \ell_1(X \oplus Y)$ and $Y \oplus \ell_1(X \oplus Y)$ are both isomorphic to $\ell_1(X \oplus Y)$. So $\tilde{A}$ may be viewed as an operator function with values in $L(\ell_1(X \oplus Y))$. This justifies the fact that in the rest of the paper we will always assume that $X = Y$.

The *spectrum* $\Sigma(A)$ of a holomorphic operator function $A: \Omega \to L(X)$ is the subset of Ω consisting of all $\lambda \in \Omega$ for which the operator $A(\lambda)$ is not invertible. Clearly the spectrum of a holomorphic operator function is invariant under extension and equivalence. One can also define the eigenvalues, eigenvectors and associated eigenvectors, and the partial multiplicities of A (cf. [4, 13]). These data of a holomorphic operator function are invariant under extension and equivalence too.

Let X and Y be complex Banach spaces, and let $W: \Omega \to L(Y)$ be a holomorphic operator function. Here Ω is an open subset of $\mathbb{C}$.
A *realization* of W on Ω is a representation of W of the following form:

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \Omega \subset \rho(A). \tag{1.2}$$

Here $\rho(A)$ is the resolvent set of the operator A and $A \in L(X)$, $B \in L(Y,X)$, $C \in L(X,Y)$ and $D \in L(Y)$. The Banach space X on which the *main operator* A is defined is called the *state space* of the realization. If the operator D is invertible, one can define the operator $A^{\times} = A - BD^{-1}C$. This operator is called the *associate (main) operator*. Note that it depends on all of the operators A, B, C and D; still we denote it by $A^{\times}$. If $\lambda \in \Omega \subset \rho(A^{\times})$, then $W(\lambda)$ is invertible while

$$(1.3) \qquad W(\lambda)^{-1} = D^{-1} - D^{-1}C(\lambda I - A^{\times})^{-1}BD^{-1}.$$

Conversely, if $W(\lambda)$ is invertible, then $\lambda \in \rho(A^{\times})$ while

$$(1.4) \qquad (\lambda I - A^{\times})^{-1} = (\lambda I - A)^{-1} - (\lambda I - A)^{-1}BW(\lambda)^{-1}C(\lambda I - A)^{-1}.$$

In particular, $\Sigma(W) = \sigma(A^{\times}) \cap \Omega$.

The realization (1.2) is sometimes called a *monic realization*, because the linear pencil $\lambda I - A$ appearing in it is monic. A *proper comonic realization* of W on Ω is a representation of the form

$$(1.5) \qquad W(\lambda) = D + \lambda C(I - \lambda A)^{-1}B, \quad \lambda \in \Omega.$$

Here the comonic operator pencil $I - \lambda A$ has invertible values for all $\lambda \in \Omega$. In case D is invertible, we define the *associate (main) operator* of (1.5) by $A^{\times} = A - BD^{-1}C$. An *improper comonic realization* of W on Ω is a representation of the form

$$(1.6) \qquad W(\lambda) = D + C(I - \lambda A)^{-1}B, \quad \lambda \in \Omega,$$

where, as in (1.5), the comonic operator pencil $I - \lambda A$ has invertible values for all $\lambda \in \Omega$. Both types of comonic realization will play a role in the sequel. Note that the improper comonic realization (1.6) implies that

$$\begin{aligned} W(\lambda) &= D + CB + \lambda CA(I - \lambda A)^{-1}B \\ &= D + CB + \lambda C(I - \lambda A)^{-1}AB, \quad \lambda \in \Omega, \end{aligned}$$

which are proper comonic realizations for W.

2. The connection between linearization and realization.

In this section we make explicit the connection between realization and linearization. The next theorem, which plays a crucial role in the rest of this paper, is taken from [3], Section II.4.

THEOREM 2.1. *Let*

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \Omega \subset \rho(A), \tag{2.1}$$

be a monic realization on Ω. *Here* $A \in L(X)$, $B \in L(Y,X)$, $C \in L(X,Y)$ *and* $D \in GL(Y)$. *Assume that* B *has a left inverse* B^+, *and put* $Z = \operatorname{Ker} B^+$. *For* $y \in Y$, $z \in Z$ *and* $\lambda \in \Omega$ *define*

$$E(\lambda)(y,z) = BD^{-1}y + z + BD^{-1}C(\lambda I - A)^{-1}z, \tag{2.2a}$$

$$F(\lambda)(y,z) = (\lambda I - A)^{-1}(By + z). \tag{2.2b}$$

Then $E(\lambda), F(\lambda): Y \oplus Z \to X$ *are bijective,* E *and* F *are holomorphic on* Ω *and*

$$E(\lambda)[W(\lambda) \oplus I_Z] = (\lambda I - A^{\times})F(\lambda), \quad \lambda \in \Omega. \tag{2.3}$$

In particular, $A^{\times}$ *is a monic linearization of* W *on* Ω.

REMARK 2.2. (1) If in Theorem 2.1 the assumption that B has a left inverse is replaced by the condition that C has a right inverse C^+, then the final conclusion of the theorem remains true. In fact, in that case one takes $Z = \operatorname{Ker} C$ and proves that

$$E(\lambda)[W(\lambda) \oplus I_Z] = (\lambda I - A^{\times})F(\lambda), \quad \lambda \in \Omega, \tag{2.4}$$

where $E(\lambda), F(\lambda): Y \oplus Z \to X$ are given by

$$E(\lambda)(y,z) = (\lambda I - A)C^+y + (\lambda I - A)z, \tag{2.5a}$$

$$F(\lambda)(y,z) = C^+Dy - (I - C^+C)(\lambda I - A)^{-1}By + z. \tag{2.5b}$$

(2) If we assume that B has a generalized inverse B^+, then the $\operatorname{Ker} B^+$-extension of W and the $\operatorname{Ker} B$ -extension of $\lambda I - A^{\times}$ are equivalent on Ω.

To prove this, we assume without loss of generality that $D = I$.

Taking into account that $Y = \operatorname{Im} B^+ \oplus \operatorname{Ker} B$ and $W(\lambda)|_{\operatorname{Ker} B} = I|_{\operatorname{Ker} B}$, one sees immediately that the matrix representation of $W(\lambda)$ with respect to $Y = \operatorname{Im} B^+ \oplus \operatorname{Ker} B$ has the form

$$(2.6) \qquad W(\lambda) = \begin{bmatrix} W_0(\lambda) & 0 \\ H(\lambda) & I_{\operatorname{Ker} B} \end{bmatrix}, \quad \lambda \in \Omega.$$

If we put $Z = \operatorname{Im} B^+$, we obtain

$$(2.7) \qquad W(\lambda) = \begin{bmatrix} W_0(\lambda) & 0 \\ 0 & I_{\operatorname{Ker} B} \end{bmatrix} \begin{bmatrix} I_Z & 0 \\ H(\lambda) & I_{\operatorname{Ker} B} \end{bmatrix}, \quad \lambda \in \Omega,$$

where the second factor at the right hand side is holomorphic and invertible. Moreover, $W_0(\lambda) = I + C_0(\lambda I - A)^{-1}B_0$, $\lambda \in \Omega$. Here C_0 is an operator from X into Z, and B_0 is a left invertible operator from Z into X one of whose left inverses is B^+, considered as an operator from Y into $Z = \operatorname{Im} B^+$. Since $B_0C_0 = BC$, the desired result is now immediate from Theorem 2.1.

If we assume C to have a generalized inverse C^+, then we can prove in a similar way that $\lambda I - A^\times$ and W are equivalent up to extension on Ω.

(3) Always, irrespective of any invertibility condition on B or C, the functions $W(\lambda) \oplus I_X$ and $(\lambda I - A^\times) \oplus I_Y$ are equivalent on Ω. In fact, we have (cf. [10], Theorem 4.5):

$$(2.8) \qquad E(\lambda)[I_Y \oplus (\ I - A^\times)]F(\lambda) = W(\lambda) \oplus I_X, \quad \lambda \in \Omega,$$

where

$$(2.9) \qquad E(\lambda) = \begin{bmatrix} W(\lambda) & C(\lambda I-A)^{-1} \\ (\lambda I-A)^{-1}B & (\lambda I-A)^{-1} \end{bmatrix}, \quad F(\lambda) = \begin{bmatrix} D^{-1}W(\lambda) & -D^{-1}C \\ -(\lambda I-A)^{-1}B & I_Y \end{bmatrix}$$

Take a outside Ω, for instance $a \in \sigma(A)$. Then it follows from (2.8) that

$$(2.10) \qquad E_1(\lambda)[\lambda I - (aI_Y \oplus A^\times)]F(\lambda) = W(\lambda) \oplus I_X, \quad \lambda \in \Omega.$$

Here the equivalence function F is given by (2.9), while E_1 is defined by

$$E_1(\lambda) = E(\lambda)[(\lambda - a)^{-1}I_Y \oplus I_X], \quad \lambda \in \Omega,$$

where E is given by (2.9). Note that (2.10) shows that $A^\times \oplus aI_Y$ is a monic linearization of W on Ω.

(4) Observe that results analogous to Theorem 2.1 and Remark 2.2, (1)-(3), can be derived for comonic realizations. In case the realization of W on Ω has the form (1.5) with D invertible, we have to consider $I - \lambda A^\times$ instead of $\lambda I - A^\times$. In case the realization of W has the form (1.6) with D invertible, however, we have to consider $I + BD^{-1}C - \lambda A$ instead of $\lambda I - A^\times$.

For $i = 1,2$, let $W_i\colon \Omega \to L(Y)$ be a holomorphic operator function. Here Y is a complex Banach space and Ω is an open subset of $\mathbb{C}$. It is natural to ask the following question. Given monic linearizations T_1 and T_2 of W_1 and W_2 respectively on Ω, is there a simple way to construct a monic linearization for $W = W_1W_2$? Under certain circumstances the answer is positive. Suppose, for instance, that the linearizations T_1 and T_2 appear as the associate operators of realizations of the form

$$(2.11) \qquad W_i(\lambda) = D_i + C_i(\lambda I - A_i)^{-1}B_i, \quad \lambda \in \Omega,$$

where $A_i \in L(X_i)$, $B_i \in L(Y,X_i)$, $C_i \in L(X_i,Y)$ and $D_i \in GL(Y)$, $i = 1,2$. Thus $T_i = A_i - B_iD_i^{-1}C_i$ $(i = 1,2)$. Suppose, in addition, that B_1 and B_2 are left invertible, and write

$$(2.12) \qquad A = \begin{bmatrix} A_1 & B_1C_2 \\ 0 & A_2 \end{bmatrix}, \quad B = \begin{bmatrix} B_1D_2 \\ B_2 \end{bmatrix}, \quad C = [C_1 \quad D_1C_2], \quad D = D_1D_2.$$

Then

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \Omega, \tag{2.13}$$

is a monic realization for $W = W_1W_2$ on Ω (cf. [3]). Since B is left invertible, it follows from Theorem 2.1 that

$$T = \begin{bmatrix} T_1 & T_1 & 0 \\ -B_2D^{-1}C_1 & & T_2 \end{bmatrix} \tag{2.14}$$

is a monic linearization for W on Ω.

The same conclusion holds, if we suppose that C_1 and C_2 are right invertible. An analogous result can be derived for comonic linearizations, provided proper comonic realizations of W_1 and W_2 on Ω are given and suitable invertibility assumptions hold.

In view of the preceding remark it is of interest to observe that not every linearization can be obtained as the associate operator of a realization. This appears from the following example.

EXAMPLE 2.3. Let Ω be the unit disk and Y a complex Banach space. Let $W: \overline{\Omega} \to L(Y)$ be an operator function, continuous on $\overline{\Omega}$ and holomorphic on Ω such that $W(\lambda)$ is invertible for all $\lambda \in \overline{\Omega}$ and such that there exists at least one point $\lambda_0 \in \partial\Omega$ with the property that W does not admit an analytic continuation to a neighbourhood of λ_0; such a function can be easily constructed. For instance, take

$$W(\lambda) = [1 - (\lambda - 1)^{\frac{1}{2}}]I_Y, \quad \lambda \in \overline{\Omega}. \tag{2.15}$$

Now every operator $T \in L(Y)$ such that $\sigma(T) \cap \overline{\Omega} = \emptyset$ is a monic linearization of W on Ω that does not coincide with the associate operator of a monic realization of W on Ω. Indeed, assume that

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \Omega,$$

is a monic realization of W on Ω such that $D \in GL(Y)$ and $A^{\times} = A - BD^{-1}C = T$. Since

$$W(\lambda)^{-1} = D^{-1} - D^{-1}C(\lambda I - T)^{-1}BD^{-1}, \quad \lambda \in \Omega,$$

W^{-1} has an analytic continuation to some neighbourhood of $\overline{\Omega}$. Since $\Sigma(W) = \emptyset$, it follows that W has an analytic continuation to some neighbourhood of $\overline{\Omega}$, which contradicts the conditions imposed on W.

If T is a monic linearization of a holomorphic Y-valued operator function W on an open subset Ω of $\mathbb{C}$, then $\sigma(T) \cap \Omega = \Sigma(W)$. In particular, $\Sigma(W) \subset \sigma(T)$. In general, the inclusion will be strict, even when $\Sigma(W)$ is a compact subset of Ω. This appears from an example due to H. Bart.

EXAMPLE 2.4. Let Ω be the unit disk and $Y = \mathbb{C}^2$. Define W on Ω by

$$W(\lambda) = \begin{bmatrix} \lambda & 0 \\ 0 & 1 \end{bmatrix}.$$

Let T be a monic linearization of W on Ω. Then there exists a Banach space Z such that $W(\lambda) \oplus I_Z$ is equivalent to $\lambda I - T$ on Ω. It is clear that $\lambda I - T$ is a Fredholm operator for all $\lambda \in \Omega$. Suppose now that $\sigma(T) = \Sigma(W) = \{0\}$. Then $\lambda I - T$ is invertible for all $\lambda \neq 0$. It follows that $\lambda I - T$ is Fredholm for all $\lambda \in \mathbb{C}$. But this can only happen when Z has finite dimension, k say. Consider the function $\det(\lambda I - T)$. This function is a polynomial of degree $k+2$. Since $\sigma(T) = \{0\}$, we actually have

$$\det(\lambda I - T) = \lambda^{k+2}, \quad \lambda \in \mathbb{C}. \tag{2.16}$$

On the other hand, it is obvious from the definition of $W(\lambda)$ and the fact that $\lambda I - T$ and $W(\lambda) \oplus I_Z$ are equivalent on Ω, that 0 is a zero of $\det(\lambda I - T)$ of multiplicity 1. This contradicts (2.16).

A more detailed analysis of the argument given in the preceding example reveals that the following is true: Let Ω be a nonempty open subset of the complex plane, and let W be a holomorphic $n\times n$ matrix function on Ω. Then W admits a monic linearization T on Ω such that $\sigma(T) = \Sigma(W)$ if and only if the number of zeros of $\det W(\lambda)$ counted according to multiplicity is finite and at least equal to n.

Suppose W is of the form (2.1) with D invertible. As we have seen, the spectrum of W is in general a proper subset of that of a monic linearization of W. However, under rather weak assumptions one can construct a monic linearization T of W such that $\sigma(T) \setminus \Sigma(W)$ consists of only one point.

Suppose $\Sigma(W)$ is a compact subset of Ω and take $a \in \mathbb{C} \setminus \Omega$ (for instance $a \in \sigma(A)$). Then $S = A^{\times} \oplus aI_{Y}$ is a monic linearization of W on Ω (see Remark 2.2(3)). In fact the X-extension of W is equivalent to $\lambda I - S$ on Ω. Observe that $\Sigma(W) = \sigma(S) \cap \Omega$. So $\sigma(S) \cap \Omega$ is a spectral subset for S. Let P be the corresponding spectral projection and let S_0 be the restriction of S to $\operatorname{Im} P$. Then $\sigma(S_0) = \sigma(S) \cap \Omega = \Sigma(W)$ and $\lambda I - S$ is equivalent on Ω to the $\operatorname{Ker} P$-extension of $\lambda I - S_0$. Again using that $a \notin \Omega$, we obtain that the $\operatorname{Ker} P$-extension of $\lambda I - S_0$ is equivalent to $(\lambda I - S_0) \oplus (\lambda - a)I_{\operatorname{Ker} P}$ on Ω. Put $T = S_0 \oplus aI_{\operatorname{Ker} P}$. Then it is clear that the X-extension of W is equivalent to $\lambda I - T$ on Ω. So T is a monic linearization of W on Ω. Note that $\sigma(T) = \sigma(S_0) \cup \{a\} = \Sigma(W) \cup \{a\}$.

3. An application.

In this section we shall extend a result of Brodskii and Švarcman concerning characteristic operator functions of Sz-Nagy-Foias type (cf. [6]).

Let λ be a complex number, and let W be an operator function with values in $L(Y_1,Y_2)$ which is holomorphic at λ. Here Y_1 and Y_2 are complex Banach spaces. Let

$$W(z) = \sum_{j=0}^{\infty} (z-\lambda)^j W_j$$

be the Taylor expansion of W near λ. For $n = 1,2,\dots$, we define the operator $T_n(W)_\lambda \in L(Y_1^n, Y_2^n)$ by the Toeplitz matrix

$$T_n(W)_\lambda = \begin{bmatrix} W_0 & \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ W_1 & W_0 & & & & & \cdot \\ \cdot & & \cdot & & & & \cdot \\ \cdot & & & \cdot & & & \cdot \\ \cdot & & & & \cdot & & 0 \\ \cdot & & & & & \cdot & \\ W_{n-1} & \cdot & \cdot & \cdot & \cdot & & W_0 \end{bmatrix} \tag{3.1}$$

When no confusion is possible, we sometimes write $T_n(W)$ instead of $T_n(W)_\lambda$.

Let Y_3 be another Banach space, and let U and V be operator functions, holomorphic on a neighbourhood of λ and with values in $L(Y_1,Y_3)$ and $L(Y_2,Y_3)$, respectively. Suppose that $U(z) = V(z)W(z)$ for z near λ. Then a straightforward calculation shows that

$$T_n(U) = T_n(V)T_n(W), \quad n = 1,2,\dots . \tag{3.2}$$

From this it is seen directly that $W(\lambda) = W_0$ is invertible if and only if $T_n(W)$ is invertible for all n. With the help of the product formula (3.2) we prove the following lemma.

LEMMA 3.1. *Let W and $\widetilde{W}$ be operator functions holomorphic on a neighbourhood of λ, whose values belong to $L(Y_1,Y_2)$ and $L(\widetilde{Y}_1,\widetilde{Y}_2)$ respectively. Suppose that Z and $\widetilde{Z}$ are two complex Banach spaces such that the Z-extension of W and the $\widetilde{Z}$-extension of $\widetilde{W}$ are equivalent on a neighbourhood* *of* λ*. Then, for each* $n \in N$*, the spaces* $\operatorname{Ker} T_n(W)$ *and* $\operatorname{Ker} T_n(\widetilde{W})$ *are isomorphic.*

<u>Proof</u>. Since $W \oplus I_Z$ and $\widetilde{W} \oplus I_{\widetilde{Z}}$ are equivalent on a neighbourhood of λ, there exist operator functions E and F holomorphic on a neighbourhood of λ, whose values belong to $GL(\widetilde{Y}_2 \oplus \widetilde{Z}, Y_2 \oplus Z)$ and $GL(Y_1 \oplus Z, \widetilde{Y}_1 \oplus \widetilde{Z})$ respectively, such that

$$W(z) \oplus I_Z = E(Z)[\widetilde{W}(z) \oplus I_{\widetilde{Z}}]F(z), \quad z \text{ near } \lambda.$$

According to (3.2),

$$T_n(W \oplus I_Z) = T_n(E)T_n(\widetilde{W} \oplus I_{\widetilde{Z}})T_n(F), \quad n \in N.$$

Observing that $T_n(E)$ and $T_n(F)$ are invertible, it follows that the spaces $\operatorname{Ker} T_n(W \oplus I_Z)$ and $\operatorname{Ker} T_n(\widetilde{W} \oplus I_{\widetilde{Z}})$ are isomorphic. It is, however, clear from (3.1) that the latter spaces are isomorphic to $\operatorname{Ker} T_n(W)$ and $\operatorname{Ker} T_n(\widetilde{W})$ respectively. This completes the proof.

We are now in a position to prove the main result of this section.

THEOREM 3.2. *Let* Ω *be an open subset of* $\mathbb{C}$ *and* Y *a complex Banach space. Let* $W: \Omega \to L(Y)$ *be a holomorphic operator function given by the monic realization*

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \Omega \subset \rho(A). \tag{3.3}$$

Here $\rho(A)$ *is the resolvent set of* A*, while* X *is the state space of this realization, and* $D \in GL(Y)$*. Put* $A^{\times} = A - BD^{-1}C$*. Then* $\operatorname{Ker} T_n(W)_\lambda$ *is isomorphic to* $\operatorname{Ker} (\lambda I - A^{\times})^n$ *for* $\lambda \in \Omega$ *and* $n = 1,2,\ldots$ *.*

Proof. According to Remark I.2.2(3), the X-extension of W and the Y-extension of $\lambda I - A^{\times}$ are equivalent on Ω. From Lemma 3.1, it follows that $\operatorname{Ker} T_n(W)_\lambda$ and $\operatorname{Ker} T_n(zI - A^{\times})_\lambda$ are isomorphic ($\lambda \in \Omega$; $n \in N$). To show that the latter space is isomorphic to $\operatorname{Ker} (\lambda I - A^{\times})^n$, note that

$$T_n(zI - A^\times)_\lambda \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} \lambda I - A^\times & 0 & \cdot & \cdot & \cdot & 0 \\ I & \lambda I - A^\times & & & & \cdot \\ 0 & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 0 \\ 0 & \cdot & \cdot & 0 & I & \lambda I - A^\times \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_{n-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{bmatrix},$$

if and only if $(A^\times - \lambda I)x_{n-1} = x_{n-2}, \ldots, (A^\times - \lambda I)x_1 = x_0$, $(A^\times - \lambda I)x_0 = 0$. So the vector $(x_0, x_1, \ldots, x_{n-1}) \in X^n$ with $x_0 \neq 0$ belongs to Ker $T_n(zI - A^\times)_\lambda$ if and only if $(x_0, x_1, \ldots, x_{n-1})$ is a Jordan chain of $A^\times$ of length n in λ. Hence, Ker $T_n(zI - A^\times)_\lambda$ is isomorphic to Ker $(\lambda I - A^\times)^n$. This completes the proof.

The Brodskii-Švarcman theorem [6] is an immediate consequence of Theorem 3.2. A similar result for characteristic operator functions of Livsic-Brodskii type can also be deduced from Theorem 3.2.

4. Minimality.

Let Y be a complex Banach space and W an operator function, holomorphic on an open subset Ω of $\mathbb{C}$, and with values in $L(Y)$. Consider the following monic realization of W on Ω:

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \Omega \subset \rho(A), \tag{4.1}$$

where $A \in L(X)$, $B \in L(Y,X)$, $C \in L(X,Y)$ and $D \in GL(Y)$. Here X denotes the *state space*. The realization is called *controllable*, if the *controllability space* Im $(A|B)$ coincides with X. Here Im $(A|B)$ is the smallest closed A-invariant subspace of X containing Im B. Observe that

$$\operatorname{Im}(A|B) = \operatorname{cl}[\operatorname{span} \bigcup_{n=0}^{+\infty} \operatorname{Im}(A^nB)]. \tag{4.2}$$

The realization (4.1) is called *observable*, if its *observability space* Ker $(C|A)$ is trivial. Here Ker $(C|A)$ is the largest A-invariant subspace of X that is contained in Ker C. Observe that

$$\text{Ker}(C|A) = \bigcap_{n=0}^{+\infty} \text{Ker}(CA^n). \tag{4.3}$$

The realization (4.1) is called *minimal*, if it is both controllable and observable.

The notions of controllability, observability and minimality can be defined for comonic realizations in an analogous way. Both the proper comonic realization (1.5) and the improper comonic realization (1.6) are called controllable if Im $(A|B) = X$. Here the controllability space Im $(A|B)$ is given by (4.2). Observability is defined in a similar way.

The next theorem is well-known. For later use, however, we need the construction described in its proof.

THEOREM 4.1. *Let Ω be a connected open neighbourhood of infinity. Let $W: \Omega \to L(Y)$ be a holomorphic operator function admitting a realization (4.1). Then there exists a minimal realization of* W *on Ω with state space* Im $(A|B)/[\text{Im}(A|B) \cap \text{Ker}\ (C|A)]$.

Proof. (1) Let us first construct a controllable realization of W on Ω, with state space Im $(A|B)$. Define $A_c \in L(\text{Im}\ (A|B))$, $B_c \in L(Y, \text{Im}\ (A|B))$ and $C_c \in L(\text{Im}\ (A|B), Y)$ as follows:

$$\begin{aligned} &A_c x = Ax, \quad C_c x = Cx, \qquad x \in \text{Im}\ (A|B), \\ &B_c y = By, \qquad y \in Y. \end{aligned} \tag{4.4}$$

Then $C_c A_c^n B_c = CA^n B$, $n \geq 0$. For $|\lambda|$ sufficiently large, $\lambda I - A_c$ is invertible, while

$$D + C_c(\lambda I - A_c)^{-1} B_c = D + \sum_{n=0}^{+\infty} \lambda^{-(n+1)} C_c A_c^n B_c = D + \sum_{n=0}^{+\infty} \lambda^{-(n+1)} CA^n B =$$

$$= W(\lambda).$$

With the help of a connectedness argument, it appears that $\lambda I - A_c$ is invertible for all $\lambda \in \Omega$, and

$$W(\lambda) = D + C_c(\lambda I - A_c)^{-1}B_c, \quad \lambda \in \Omega.$$

It is easily seen that this realization is controllable and that its observability space is Im $(A|B) \cap$ Ker $(C|A)$.

(2) Next let us construct an observable realization of W on Ω with state space $X/\text{Ker}\ (C|A)$. For $z \in X$, let $[z]$ denote the element of $X/\text{Ker}\ (C|A)$ that contains z. In other words $[z] = z + \text{Ker}\ (C|A)$. Define the operators $A_0 \in L(X/\text{Ker}\ (C|A))$, $B_0 \in L(Y, X/\text{Ker}\ (C|A))$ and $C_0 \in L(X/\text{Ker}\ (C|A), Y)$ by

$$\text{(4.5)} \qquad \begin{aligned} &A_0[x] = [Ax],\ C_0[x] = Cx,\ [x] \in X/\text{Ker}\ (C|A), \\ &B_0 y = [By], \quad y \in Y. \end{aligned}$$

Note that all three operators are well-defined. Moreover, $C_0A_0^nB_0 = CA^nB$, $n \geq 0$. In the same way as before, we prove that

$$W(\lambda) = D + C_0(\lambda I - A_0)^{-1}B_0, \quad \lambda \in \Omega.$$

Furthermore, this realization is observable.

(3) A minimal realization of W on Ω can now be constructed by applying the procedures of the first (second) paragraph to the observable (controllable) realization obtained in the second (first) one. This completes the proof.

Two realizations $W(\lambda) = D_i + C_i(\lambda I - A_i)^{-1}B_i$, $\lambda \in \Omega \subset \rho(A_i)$, with state space X_i $(i = 1,2)$ are called *quasisimilar*, if there exists a densily defined injective linear operator $S(X_1 \to X_2)$ with a dense range, called a *quasisimilarity*, such that

$$\text{(4.6)} \qquad SA_1 = A_2S, \quad SB_1 = B_2, \quad C_2S = C_1, \quad D_1 = D_2.$$

It is clear that the relation called quasisimilarity is reflexive and symmetric. Any two minimal realizations of an operator function W on a neighbourhood of infinity are quasisimilar (cf. [5, 14]). The quasisimilarity S is given by

$$(4.7) \qquad D(S) = \operatorname{span} \bigcup_{n=0}^{+\infty} \operatorname{Im}(A_1^n B_1), \quad S\Big(\sum_{n=0}^{\ell} A_1^n B_1 x_n\Big) = \sum_{n=0}^{\ell} A_2^n B_2 x_n .$$

Note that

$$\operatorname{Im} S = \operatorname{span} \bigcup_{n=0}^{+\infty} \operatorname{Im}(A_2^n B_2).$$

The operator S is always closable. To see this, let $(z^{(\kappa)})_{\kappa=1}^{+\infty}$ be a sequence in D(S) converging to 0 while $Sz^{(\kappa)}$ tends to some $x \in X$ as $\kappa \to +\infty$. We have to prove that $x = 0$. Write $z^{(\kappa)}$ in the form

$$\sum_{n=0}^{\ell_\kappa} A_1^n B_1 y_n^{(\kappa)} .$$

Then

$$\sum_{n=0}^{\ell_\kappa} A_1^n B_1 y_n^{(\kappa)} \to 0, \quad \sum_{n=0}^{\ell_\kappa} A_2^n B_2 y_n^{(\kappa)} \to x \qquad (\kappa \to +\infty).$$

For all $m \geq 0$ we have

$$C_2 A_2^m x = \lim_{\kappa\to+\infty} \sum_{n=0}^{\ell_\kappa} C_2 A_2^{m+n} B_2 y_n^{(\kappa)} = \lim_{\kappa\to+\infty} \sum_{n=0}^{\ell_\kappa} C_1 A_1^{m+n} B_1 y_n^{(\kappa)} =$$

$$= C_1 A_1^n \Big[\lim_{\kappa\to+\infty} \sum_{n=0}^{\ell_\kappa} A_1^n B_1 y_n^{(\kappa)}\Big] = 0.$$

Since $\operatorname{Ker}(C_2|A_2) = \{0\}$, it follows that $x = 0$. This completes the proof.

Two realizations $W(\lambda) = D_i + C_i(\lambda I - A_i)^{-1} B_i$, $\lambda \in \Omega \subset \rho(A_i)$, with state space X_i, $i = 1,2$, are called *similar*, if there exists an operator $S \in GL(X_1, X_2)$ such that (4.6) holds. Obviously, similar realizations are quasisimilar.

The minimalization construction obtained in Theorem 4.1 can be applied only to monic realizations on a neighbourhood of infinity, or to comonic realizations on a neighbourhood of 0. This is rather unfortunate, because in the next chapter we shall deal mainly with monic realizations on a neighbourhood of 0. The main operator of such a realization is invertible. If we write

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \quad \lambda \in \Omega \subset \rho(A), \tag{4.8}$$

we can exploit the invertibility of A to obtain the improper comonic realization

$$W(\lambda) = D - CA^{-1}(I - \lambda A^{-1})^{-1}B, \quad \lambda \in \Omega,$$

and construct a minimal comonic realization of W on Ω. In this way, however, the connection of realization with a monic linearization given by Theorem 2.1 is lost completely. To circumvent this difficulty, and thereby retaining the possibility to obtain a monic linearization of W on Ω, we introduce the notion of biminimality. The realization (4.8) with invertible main operator A is called *bicontrollable*, if its *bicontrollability space* $\mathrm{Im}\,(A|B)_{-\infty}^{+\infty}$ coincides with the state space X of the realization. Here

$$\mathrm{Im}\,(A|B)_{-\infty}^{+\infty} = \mathrm{cl}[\mathrm{span} \bigcup_{n=-\infty}^{+\infty} \mathrm{Im}\,(A^nB)]$$

is the smallest closed subspace of X that is invariant under A and A^{-1} and contains Im B. The realization (4.8) is called *biobservable*, if the *biobservability space* $\mathrm{Ker}\,(C|A)_{-\infty}^{+\infty}$ is trivial. Here

$$\mathrm{Ker}\,(C|A)_{-\infty}^{+\infty} = \bigcap_{n=-\infty}^{+\infty} \mathrm{Ker}\,(CA^n)$$

is the largest subspace of X that is invariant under A and A^{-1} and is contained in Ker C. The realization (4.8) is called *biminimal*, if it is both bicontrollable and biobservable.

It is clear that minimality implies biminimality. The converse is true in the finite-dimensional case. This is due to the fact that A is then an algebraic operator.

As to biminimality, results analogous to the ones obtained earlier for minimality can be derived. For instance, to every monic realization of a holomorphic operator function W on a connected neighbourhood Ω of 0, whose main operator is invertible, a biminimal realization can be constructed on Ω. The construction is completely analogous to that given in the proof of Theorem 4.1. Also any two biminimal realizations of W on a neighbourhood of both 0 and ∞ are quasisimilar.

II. SPECIAL CLASSES

In this chapter we shall apply the theory developed in the first chapter to several classes of holomorphic operator functions. These classes correspond to the linearization results appearing in literature. For an arbitrary operator function in each class, we shall describe a linearization, a realization, and a (bi)minimal realization.

1. Operator polynomials

Let Y be a complex Banach space and let $L(\lambda) = \lambda^{\ell} I + \lambda^{\ell-1} A_{\ell-1} + \dots + \lambda A_1 + A_0$ be a monic operator polynomial whose coefficients are operators from $L(Y)$. Then it is well-known (cf. [11]) that a monic linearization of L on $\mathbb{C}$ is given by the so-called first companion operator of L, i.e.

$$
(1.1) \qquad C_1 = \begin{bmatrix} 0 & I & 0 & \cdots & & 0 \\ \vdots & 0 & I & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & I & 0 \\ 0 & & & & 0 & I \\ -A_0 & \cdots & \cdots & \cdots & \cdots & -A_{\ell-1} \end{bmatrix}
$$

Further, $[L(\lambda) \oplus I_{Y^{\ell-1}}]F(\lambda) = E(\lambda)(\lambda I - C_1)$, $\lambda \in \mathbb{C}$, where the equivalence functions E and F have the form

$$
(1.2) \qquad E(\lambda) = \begin{bmatrix} B_{\ell-1}(\lambda) & \cdots & \cdots & \cdots & B_0(\lambda) \\ -I & 0 & & & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -I & 0 \end{bmatrix}, \quad F(\lambda) = \begin{bmatrix} I & 0 & \cdots & \cdots & 0 \\ -\lambda I & I & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -\lambda I & I \end{bmatrix}
$$

Here $B_0(\lambda) = I$ and $B_n(\lambda) = \lambda^n I + \lambda^{n-1} A_{\ell-1} + \dots + A_{\ell-n}$, $1 \leq n \leq \ell-1$.

The spectrum of L coincides with the spectrum of the operator C_1.

To obtain a realization of L^{-1}, we write $L(\lambda)^{-1} \oplus I = F(\lambda)(\lambda I - C_1)^{-1}E(\lambda)^{-1}$, $\lambda \in \mathbb{C} \setminus \Sigma(L)$. Let the operators $Q \in L(Y^{\ell},Y)$ and $R \in L(Y,Y^{\ell})$ be given by

$$(1.3) \qquad Q = \operatorname{row}\,(\delta_{i1}I)_{i=1}^{\ell}, \quad R = \operatorname{col}\,(\delta_{i\ell}I)_{i=1}^{\ell}.$$

Since the first row of $F(\lambda)$ is the operator Q and the first column of $E(\lambda)^{-1}$ is the operator R, we have $L(\lambda)^{-1} = Q[L(\lambda)^{-1} \oplus I]R = Q(\lambda I - C_1)^{-1}R$, $\lambda \in \mathbb{C} \setminus \Sigma(L)$. This is a *monic realization* of L^{-1} on $\mathbb{C} \setminus \Sigma(L)$, called the *resolvent form* of L (cf. [11]). Because of the equations

$$\begin{bmatrix} Q \\ QC_1 \\ \vdots \\ QC_1^{\ell-1} \end{bmatrix} = I, \quad [C_1^{\ell-1}R \;\; C_1^{\ell-2}R \;\; \dots \;\; R] = I ,$$

this realization is a minimal realization of L^{-1} on $\mathbb{C} \setminus \Sigma(L)$.

In a similar way, a linearization of the polynomial L can be obtained with the help of the second companion operator C_2 of L. For details, we refer to [11].

Let $M(\lambda) = I + \lambda A_{\ell-1} + \dots + \lambda^{\ell-1}A_1 + \lambda^{\ell}A_0$ be a comonic operator polynomial whose coefficients are operators from $L(Y)$. We associate with M a monic operator polynomial L given by $L(\lambda) = \lambda^{\ell}I + \lambda^{\ell-1}A_{\ell-1} + \dots$ $\dots + \lambda A_1 + A_0$. Then $\lambda^{\ell}L(\lambda^{-1}) = M(\lambda)$, $\lambda \neq 0$. A comonic linearization of M on $\mathbb{C} \setminus \{0\}$ is given by

$$(1.4) \qquad M(\lambda) \oplus I = \lambda^{\ell-1}(I \oplus \lambda^{-\ell}I)E(\lambda^{-1})(I - \lambda C_1)F(\lambda^{-1})^{-1}, \qquad \lambda \in \mathbb{C} \setminus \{0\},$$

where the equivalence functions E and F are defined by (1.2) and the comonic linearization C_1 by (1.1). A proper comonic realization of M on $\mathbb{C}$ is given by

$$M(\lambda) = I + \lambda[A_0 \; \dots \; A_{\ell-1}](I - \lambda S)^{-1}R, \quad \lambda \in \mathbb{C}, \tag{1.5}$$

where S and R are given by

$$S = (\delta_{i,j-1}I)_{i,j=1}^{\ell}, \quad R = \mathrm{col}\ (\delta_{i\ell}I)_{i=1}^{\ell}. \tag{1.6}$$

Note that R is left invertible. Further, C_1 is the associate operator of the realization (1.5). Hence, C_1 is a comonic linearization of M on $\mathbb{C}$. Both comonic linearizations appearing in this section have been obtained earlier by Gohberg and Rodman (cf. [12]).

2. Holomorphic operator functions with continuous boundary values.

In this section Ω will be a bounded Cauchy domain, i.e. a bounded open set whose boundary $\partial\Omega$ is composed of a finite number of disjoint closed rectifiable Jordan curves. We assume that $\partial\Omega$ is oriented in the positive sense.

By $C(\partial\Omega, Y)$ we denote the Banach space of all Y-valued continuous functions on $\partial\Omega$ endowed with the supremum norm. Here Y is an arbitrary complex Banach space. For the sake of simplicity we shall suppose that $0 \in \Omega$.

Let A be an operator function, holomorphic on Ω, continuous on the closure $\overline{\Omega}$, and with values in $L(Y)$. Let $V \in L(C(\partial\Omega,Y))$, $\tau \in L(Y, C(\partial\Omega,Y))$, $\omega \in L(C(\partial\Omega,Y),Y)$ and $M \in L(C(\partial\Omega,Y))$ be the bounded linear operators defined by

$$(Vf)(z) = zf(z), \quad (\tau y)(z) = y, \tag{2.1a}$$

$$(Mf)(z) = A(z)f(z), \quad \omega f = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta}\, d\zeta. \tag{2.1b}$$

Then it follows that for each $y \in Y$ and $\lambda \notin \partial\Omega$

$$(2.2) \qquad \omega(I - M)V(\lambda I - V)^{-1}\tau y = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{A(\zeta) - I}{\zeta-\lambda}\, y\, d\zeta,$$

so that we obtain the following monic realization of A on Ω:

$$(2.3) \qquad A(\lambda) = I + \omega(I - M)V(\lambda I - V)^{-1}\tau, \qquad \lambda \in \Omega.$$

As 0 belongs to Ω, we have $\omega\tau = I$. Since, in addition, the operator $T \in \mathcal{L}(\mathcal{C}(\partial\Omega, Y))$, defined by

$$(2.4) \qquad (Tf)(z) = zf(z) - (2\pi i)^{-1} \int_{\partial\Omega} [I - A(\zeta)]f(\zeta)d\zeta,$$

is the associate operator of the realization (2.3), the operator T is a monic linearization of A on Ω. In fact we know from Theorem I.2.1 that $E(\lambda)[A(\lambda) \oplus I_{\mathrm{Ker}\,\omega}] = (\lambda I - T)F(\lambda)$, where the equivalence functions are given by

$$[E(\lambda)(y,f)](z) = y + f(z) + \frac{1}{2\pi i}\int_{\partial\Omega} \frac{[A(\zeta) - I]f(\zeta)}{\zeta-\lambda}\, d\zeta,$$

$$[F(\lambda)(y,f)](z) = (\lambda-z)^{-1}(y + f(z)).$$

The monic linearization (2.4) has been obtained earlier by Gohberg, Kaashoek and Lay (cf. [10]). However, they did not make use of the connection between linearization and realization explicitly. They also proved that $\sigma(T) = \Sigma(A) \cup \partial\Omega$. To obtain a shorter proof of this, we argue as follows. As a consequence of (2.2),

$$I + \omega(I - M)V(\lambda I - V)^{-1}\tau = \begin{cases} A(\lambda), & \lambda \in \Omega \\ I, & \lambda \in \mathbb{C} \setminus \overline{\Omega}. \end{cases}$$

So T is a monic linearization of A on Ω and of I on $\mathbb{C} \setminus \overline{\Omega}$. Therefore, $\sigma(T) \cap \Omega = \Sigma(A) \cap \Omega$ and $\sigma(T) \subset \overline{\Omega}$. It remains to prove that $\partial\Omega \subset \sigma(T)$. Suppose this is not the case. Then there exists a point $\lambda_0 \in \partial\Omega$ such that $\lambda_0 I - T$ is invertible. Notice that

$$(2.5) \qquad I - \omega(I - M)V(\lambda I - T)^{-1}\tau = \begin{cases} A(\lambda)^{-1}, & \lambda \in \Omega \setminus \Sigma(A) \\ I, & \lambda \in \mathbb{C} \setminus \overline{\Omega} \end{cases}$$

So there exists a neighbourhood $\mathcal{U}$ of λ_0 such that $\mathcal{U} \cap \Sigma(A) = \emptyset$. From (2.5) it is clear that the restriction of A^{-1} to $\mathcal{U} \cap \Omega$ has an analytic continuation to $\mathcal{U}$ with value I. Since V is the associate operator of the monic realization (2.5), it follows that $\lambda_0 \in \rho(V)$, which contradicts the fact that $\sigma(V) = \partial\Omega$. Hence, $\partial\Omega \subset \sigma(T)$.

We suppose that $\Sigma(A) \cap \Omega$ is a compact subset of Ω. Then $\sigma(T) \cap \Omega$ and $\partial\Omega$ are spectral subsets for T. We shall construct an operator V_A such that A and $\lambda I - V_A$ are equivalent up to extension on Ω, while $\sigma(V_A) = \Sigma(A) \cap \Omega = \sigma(T) \cap \Omega$. Such an operator V_A is not difficult to construct. Let P be the Riesz projection of T corresponding to the spectral subset $\Sigma(A)$. Then V_A is defined as the restriction of T to Im P. Since T is a monic linearization of A on Ω with extension space Ker ω, it follows from the decomposition $\lambda I - T = (\lambda I - V_A) \oplus (\lambda I - T)|_{\text{Ker } P}$ that the Ker ω-extension of A and the Ker P-extension of $\lambda I - V_A$ are equivalent on Ω.

To obtain an intrinsic characterization of Im P and V_A, we introduce the following notation. Let M_A be the subspace of $C(\partial\Omega, Y)$ consisting of all f that have an extension to a vector function holomorphic on a neighbourhood of $\mathbb{C}_\infty \setminus \Omega$ and vanishing at ∞, while Mf has an analytic continuation to Ω. Since we assumed that $\Sigma(A) \cap \Omega$ is a compact subset of Ω, it follows that each $f \in M_A$ has an analytic continuation to $\mathbb{C}_\infty \setminus [\Sigma(A) \cap \Omega]$, also denoted by f.

THEOREM 2.1. *Let $\Sigma(A) \cap \Omega$ be a compact subset of Ω and* P *the Riesz projection of* T *corresponding to the spectral subset $\Sigma(A) \cap \Omega$ of* T. *Then*

$$\operatorname{Im} P = M_A,$$

while the restriction V_A of T *to* Im P *is given by*

$$(V_A f)(z) = zf(z) - \frac{1}{2\pi i} \int_{\partial\Omega} f(\zeta)d\zeta.$$

Proof. As a consequence of (2.3) and I (1.4), we have for all $\lambda \in \Omega \setminus \Sigma(A)$:

$$[(\lambda I - T)^{-1}f](z) = \frac{f(z)}{z-\lambda} - \frac{A(\lambda)^{-1}}{\lambda-z} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{[A(\zeta) - I]f(\zeta)}{\zeta-\lambda} d\zeta.$$

Let Δ be a Cauchy domain such that $\Sigma(A) \cap \Omega \subset \Delta \subset \overline{\Delta} \subset \Omega$. Put $\Gamma = \partial\Delta$. Then the Riesz projection P of T corresponding to the spectral subset $\Sigma(A) \cap \Omega$ of T is given by

$$(Pf)(z) = \frac{-1}{2\pi i} \int_{\Gamma} \frac{A(\lambda)^{-1}}{\lambda-z} \frac{1}{2\pi i} \int_{\partial\Omega} \frac{[A(\zeta) - I]f(\zeta)}{\zeta-\lambda} d\zeta\, d\lambda. \tag{2.6}$$

Note that Pf has an analytic continuation (again denoted by Pf) to $\mathbb{C}_\infty \setminus \overline{\Delta}$ that vanishes at ∞. To prove that MPf has an analytic continuation to Ω, it suffices to prove that

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{(MPf)(z)}{z-z_0} dz = A(z_0)(Pf)(z_0), \qquad z_0 \in \Omega \setminus \overline{\Delta}. \tag{2.7}$$

From (2.6) we have

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{(MPf)(z)}{z-z_0} dz = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{A(z)}{z-z_0} \left\{ \frac{1}{2\pi i} \int_{\Gamma} \frac{A(\lambda)^{-1}}{z-\lambda} \cdot \right.$$

$$\left. \cdot \left[\frac{1}{2\pi i} \int_{\partial\Omega} \frac{A(\zeta) - I}{\zeta-\lambda} f(\zeta)d\zeta \right] d\lambda \right\} dz.$$

As the vector function under the integral sign is a continuous function in

(z,λ,ζ) on the compact set $\partial\Omega \times \Gamma \times \partial\Omega$, we may apply Fubini's theorem and change the order of integration. At first, we evaluate the integral over z, and obtain

$$\text{(2.8)} \qquad \frac{1}{2\pi i}\int_{\partial\Omega} \frac{A(z)}{(z-z_0)(z-\lambda)}\,dz = \frac{A(z_0) - A(\lambda)}{z_0-\lambda}, \qquad \lambda \in \Gamma.$$

By substitution of (2.8), we get

$$\frac{1}{2\pi i}\int_{\partial\Omega} \frac{(MPf)(z)}{z-z_0}\,dz = \frac{1}{2\pi i}\int_{\Gamma} \frac{A(z_0) - A(\lambda)}{z_0-\lambda} A(\lambda)^{-1}\,\cdot$$
$$\cdot \left[\frac{1}{2\pi i}\int_{\partial\Omega} \frac{A(\zeta) - I}{\zeta-\lambda} f(\zeta)d\zeta\right] d\lambda.$$

We split the integral into two terms. From (2.6) it is clear that the first term is equal to $A(z_0)(Pf)(z_0)$. The second one has the form

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{1}{\lambda-z_0}\left[\frac{1}{2\pi i}\int_{\partial\Omega} \frac{A(\zeta) - I}{\zeta-\lambda} f(\zeta)d\zeta\right] d\lambda = 0.$$

This completes the proof of (2.7). So MPf has an analytic continuation to Ω. In particular, we proved that Im $P \subset M_A$.

To prove that $M_A \subset$ Im P, take $f \in M_A$. Then f has an extension that is holomorphic in $\mathbb{C}_\infty \setminus \overline{\Delta}$, continuous on $\mathbb{C}_\infty \setminus \Delta$ and vanishes at ∞, while Mf has an analytic continuation to Ω. From (2.6) we have

$$(Pf)(z) = \frac{-1}{2\pi i}\int_{\Gamma} \frac{A(\lambda)^{-1}}{\lambda-z}\left[\frac{1}{2\pi i}\int_{\partial\Omega} \frac{A(\zeta)f(\zeta)}{\zeta-\lambda}\,d\zeta\right] d\lambda =$$
$$= \frac{-1}{2\pi i}\int_{\Gamma} \frac{A(\lambda)^{-1}}{\lambda-z} A(\lambda)f(\lambda)d\lambda = f(z), \qquad z \in \partial\Omega.$$

In particular, we proved that $M_A \subset$ Im P. Hence,

$$\text{Im } P = M_A.$$

From (2.4) and the fact that Mf has an analytic continuation to Ω for every $f \in M_A$, we obtain

$$(V_A f)(z) = zf(z) - \frac{1}{2\pi i}\int_{\partial\Omega} f(\zeta)d\zeta, \qquad z \in \partial\Omega.$$

This completes the proof.

In the results obtained above the state space $C(\partial\Omega,Y)$ of the monic realization (2.3) can be replaced by other Banach spaces of Y-valued functions on $\partial\Omega$ where the operators are defined by the same formulas. For instance, if A is Hölder continuous of index $\alpha \in (0,1)$ on $\overline{\Omega}$, where Ω is a simply connected Cauchy region with a piecewise smooth boundary curve, we may take the Banach space $H_\alpha(\partial\Omega,Y)$ of Y-valued Hölder continuous functions of index α on $\partial\Omega$ endowed with the norm

$$||f|| = \sup_{z\in\partial\Omega} ||f(z)|| + \sup_{z_1\neq z_2\in\partial\Omega} \frac{||f(z_1) - f(z_2)||}{|z_1-z_2|^\alpha}$$

If Y is a separable Hilbert space, we may take the Banach space $L_2(\partial\Omega,Y)$ of all Y-valued strongly measurable vector functions on $\partial\Omega$ that are square integrable. For this case, the space M_A has been introduced earlier by Markus and Macaev in [17]. The operator function considered in [17] is defined on the closed unit disk. For this case, Theorem 2.1 has been obtained by A. Ran, who also observed that the Ker ω-extension of A and the Ker P-extension of $\lambda I - V_A$ are equivalent on Ω (cf. [19]).

In the above considerations the particular form of the state space in the realization (2.3) did not play a very significant role. This is different when we try to construct (bi)minimal realizations. Let us restrict ourselves to the case when Ω is simply connected. Suppose that Y is a separable Hilbert space. Then the Hilbert space $L_2(\partial\Omega,Y)$ admits the decomposition

$$(2.9) \qquad L_2(\partial\Omega,Y) = L_2^+(\partial\Omega,Y) \oplus L_2^-(\partial\Omega,Y).$$

Here $L_2^+(\partial\Omega,Y)$ is the subspace of $L_2(\partial\Omega,Y)$ consisting of all functions that admit an analytic continuation to Ω, while $L_2^-(\partial\Omega,Y)$ is the subspace of all functions that have an extension to a function holomorphic on $\mathbb{C}_\infty \setminus \overline{\Omega}$ and vanishing at ∞. As in (2.1) we can define operators $V: L_2(\partial\Omega, Y) \to L_2(\partial\Omega, Y)$, $\tau: Y \to L_2(\partial\Omega, Y)$, $M: L_2(\partial\Omega, Y) \to L_2(\partial\Omega, Y)$ and $\omega: L_2(\partial\Omega, Y) \to Y$. These operators appear to be bounded while $\sigma(V) = \partial\Omega$. In the same way as before we obtain the monic realization (2.3) of A on Ω. Observe that $VM = MV$. Then we rewrite (2.3) and obtain

$$\text{(2.10a)} \qquad A(\lambda) = I + \omega V(\lambda I - V)^{-1}(I - M)\tau, \quad \lambda \in \Omega.$$

To obtain a minimal improper comonic realization of A on Ω, we observe that V is invertible and rewrite (2.10a) as

$$\text{(2.10b)} \qquad A(\lambda) = I + \omega(I - \lambda V^{-1})^{-1} (M - I)\tau, \quad \lambda \in \Omega.$$

The observability space $\mathrm{Ker}\,(\omega|V^{-1}) = \bigcap_{n=0}^{+\infty} \mathrm{Ker}\,(\omega V^{-n})$ of (2.10b) consists of all $f \in L_2(\partial\Omega, Y)$ such that

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta = 0 \qquad (n = 0,1,2,\ldots).$$

From (2.9) it follows that $\mathrm{Ker}\,(\omega|V^{-1}) = L_2^-(\partial\Omega, Y)$. Identifying the quotient space $L_2(\partial\Omega, Y)/\mathrm{Ker}\,(\omega|V^{-1})$ with $L_2^+(\partial\Omega, Y)$, we obtain from (the proof of) Theorem I.4.1 the following minimal improper comonic realization of A on Ω:

$$\text{(2.11)} \qquad A(\lambda) = I + \omega_1(I - \lambda V_1)^{-1} B_1, \quad \lambda \in \Omega.$$

Its state space N_1 is the closure in $L_2^+(\partial\Omega, Y)$ of all functions g given by $g = P_+ h$, where $h(z) = z(A(z) - I)f(z)$, and f belongs to $L_2^-(\partial\Omega, Y)$. Here P_+ is the projection of $L_2(\partial\Omega, Y)$ onto $L_2^+(\partial\Omega, Y)$ along $L_2^-(\partial\Omega, Y)$. The operators $\omega_1 \in L(N_1, Y)$, $V_1 \in L(N_1)$ and $B_1 \in L(Y, N_1)$ are given by

$$\omega_1 f = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\zeta)}{\zeta}\, d\zeta, \quad (B_1 y)(z) = (A(z) - I)y,$$

$$(V_1 f)(z) = z^{-1}[f(z) - \omega_1 f].$$

The construction of N_1 is performed by direct computation using the fact that the Y-valued polynomials in z^{-1} span a dense linear subspace of $VL_2^-(\partial\Omega,Y)$.

To obtain a minimal proper comonic realization of A on Ω, we note that we have, in addition to (2.3),

$$(2.12) \qquad A(\lambda) = A(0) + \lambda\omega(V - \lambda I)^{-1} VM_0\tau, \quad \lambda \in \Omega.$$

Here $M_0 \in L(C(\partial\Omega,Y))$ is defined by

$$(M_0 f)(z) = z^{-1}(A(z) - A(0))f(z), \quad z \in \partial\Omega.$$

This operator can also be defined as a bounded linear operator on $L_2(\partial\Omega,Y)$, provided Y is a separable Hilbert space. We rewrite (2.12) as

$$A(\lambda) = A(0) + \lambda\omega(I - \lambda V^{-1})^{-1} M_0\tau, \quad \lambda \in \Omega.$$

With the help of (the proof of) Theorem I.4.1 and the fact that the Y-valued polynomials in z^{-1} span a dense linear subspace of $VL_2^-(\partial\Omega,Y)$, we obtain the following minimal proper comonic realization of A on Ω:

$$(2.13) \qquad A(\lambda) = A(0) + \lambda\omega_2(I - \lambda V_2)^{-1} B_2, \quad \lambda \in \Omega.$$

Its state space N_2 is the closure in $L_2^+(\partial\Omega,Y)$ of all functions g given by $g = P_+h$, where $h(z) = (A(z) - I)f(z)$, and $f \in L_2^-(\partial\Omega,Y)$. The operators $\omega_2 \in L(N_2,Y)$, $V_2 \in L(N_2)$ and $B_2 \in L(Y,N_2)$ are given by

$$\omega_2 f = \frac{1}{2\pi i}\int_{\partial\Omega} \frac{f(\zeta)}{\zeta}\, d\zeta, \quad (B_2 y)(z) = \frac{A(z) - A(0)}{z}\, y,$$

$$(V_2 f)(z) = z^{-1}[f(z) - \omega_2 f].$$

To obtain a biminimal monic realization of A on Ω, we compute the biobservability space $\mathrm{Ker}\,(\omega|V)_{-\infty}^{+\infty}$ of the monic realization (2.10a).

This space consists of all functions f from the state space $L_2(\partial\Omega,Y)$ such that

$$\frac{1}{2\pi i}\int_{\partial\Omega}\zeta^m f(\zeta)d\zeta = 0 \qquad (m = 0,1,2,\dots).$$

Since we assumed that Ω is simply connected, it follows that the realization (2.10a) is biobservable. A biminimal monic realization of A on Ω is then given by

$$A(\lambda) = I - \omega_0(\lambda I - V_0)^{-1}B_0, \quad \lambda \in \Omega. \tag{2.14}$$

Its state space N_0 is the closure in $L_2(\partial\Omega,Y)$ of all functions g of the form $g(z) = (A(z) - I)f(z)$, $f \in L_2(\partial\Omega,Y)$. This follows from (the proof of) the analogue of Theorem I.4.1 for constructing biminimal realizations, and the fact that the Y-valued polynomials in z and z^{-1} span a dense linear subspace of $L_2(\partial\Omega,Y)$. The operators $\omega_0 \in L(N_0,Y)$, $V_0 \in L(N_0)$ and $B_0 \in L(Y,N_0)$ are given by

$$\omega_0 f = \frac{1}{2\pi i}\int_{\partial\Omega} f(\zeta)d\zeta, \quad (B_0 y)(z) = (A(z) - I)y,$$

$$(V_0 f)(z) = zf(z).$$

The computation of the biminimal realization (2.14) can be repeated for any monic realization of the form (2.10a), provided the state space is a Banach space of Y-valued functions on $\partial\Omega$ such that the Y-valued polynomials in z and z^{-1} span a dense linear subspace of it. Further the operators used to construct (2.10a) have to be bounded.

To repeat the computation of the minimal comonic realizations (2.11) and (2.13), the state space under consideration has to be a Banach space of Y-valued functions on $\partial\Omega$ that admits a decomposition analogous to (2.9). This can be done, for instance, if Y is a separable Hilbert space and the state space under consideration is $L_p(\partial\Omega,Y)$, $1 \le p < +\infty$. Then (2.13) coincides with the so-called restricted shift realization obtained by Fuhrmann (cf. [9]).

3. Arbitrary holomorphic operator functions.

In this section Ω will be an arbitrary open subset of $\mathbb{C}$ containing 0. Let Y be a complex Banach space and let $A: \Omega \to L(Y)$ be a holomorphic operator function. By F we denote the Banach space of all Y-valued holomorphic functions f on Ω that are bounded with respect to the norm

$$(3.1) \qquad \|f\| = \sup_{z\in\Omega} b(z)\|f(z)\|.$$

Here $b(z) = [\max(1,\|A(z)\|)]^{-1}$, $z \in \Omega$. Note that b is continuous and that

$$(3.2) \qquad \sup_{z\in\Omega} b(z)\|A(z)\| < +\infty.$$

Define $V: F \to F$ by

$$(3.3) \qquad (Vf)(z) = \begin{cases} z^{-1}(f(z) - f(0)), & z \in \Omega \setminus \{0\}. \\ f'(0) & , \quad z = 0. \end{cases}$$

Further, let $\omega: F \to Y$, $\tau: Y \to F$ and $P: F \to F$ be given by

$$(3.4) \qquad \omega f = f(0), \quad (\tau y)(z) = y, \quad (Pf)(z) = f(0).$$

Then all these operators are bounded. Moreover, for all $\lambda \in \Omega$ the operator $I - \lambda V$ is invertible and

$$(3.5a) \qquad \omega(I - \lambda V)^{-1} f = f(\lambda), \quad f \in F$$

(cf. [8]). Put

$$(3.5b) \qquad (Mf)(z) = A(z)f(0), \quad f \in F,\ z \in \Omega.$$

It follows from (3.2) that M is a bounded linear operator on F. A straightforward calculation, using (3.5), shows that

$$(3.6a) \qquad A(\lambda) = I + \omega(I - \lambda V)^{-1}(M - I)\tau, \quad \lambda \in \Omega.$$

$$(3.6b) \qquad A(\lambda) = A(0) + \lambda\omega(I - \lambda V)^{-1} VM\tau, \quad \lambda \in \Omega.$$

Now $\omega\tau = I$. Hence, putting $S = I + M - P$ and applying Remark I.2.2(4), we obtain the following linearization of A on Ω:

$$E(\lambda)[A(\lambda) \oplus I_{\mathrm{Ker}\,\omega}] = (S - \lambda V)F(\lambda), \qquad \lambda \in \Omega.$$

The equivalence functions E and F are given by

$$(E(\lambda)(y,f))(z) = \begin{cases} y + (1 - \frac{\lambda}{z})f(z), & z \in \Omega \setminus \{0\}, \\ y - \lambda f'(0) \quad , & z = 0, \end{cases}$$

and

$$(F(\lambda)(y,f))(z) = \begin{cases} y + f(z) - (A(z) - A(\lambda))y \\ \qquad\qquad -\lambda \dfrac{A(z) - A(\lambda)}{z-\lambda} y, & z \in \Omega \setminus \{\lambda\}. \\ y + f(\lambda) - \lambda A'(\lambda)y \quad , & z = \lambda. \end{cases}$$

This linearization has been obtained by Den Boer without the help of a realization (cf. [8]). If A(0) is an invertible operator, we can obtain from equation (3.6b) the following comonic linearization of A on Ω:

$$(T_0 f)(z) = \begin{cases} \dfrac{f(z) - A(z)A(0)^{-1}f(0)}{z}, & z \in \Omega \setminus \{0\}. \\ f'(0) - A'(0)A(0)^{-1}f(0), & z = 0. \end{cases}$$

We now prove a theorem that will be needed in the next section.

THEOREM 3.1. *Let λ_0 be an isolated point of $\mathbb{C} \setminus \Omega$. Then the following statements are equivalent:*

(1) *The operator function* A *has a removable singularity in* λ_0.

(2) *Every vector function* $f \in F$ *has a removable singularity in* λ_0.

(3) $I - \lambda_0 V$ *is an invertible operator.*

Proof. (1) $\Rightarrow$ (2) Suppose λ_0 is a removable singularity for A. Because $b(z) = [\max(1,\|A(z)\|)]^{-1}$, $z \in \Omega$, the function b is bounded away from zero in a deleted neighbourhood $\mathcal{U}$ of λ_0. It follows that every $f \in F$

is bounded on $\mathcal{U}$. But then λ_0 is a removable singularity of f too.

(2) $\Rightarrow$ (1) For every $y \in Y$, the vector function $M\tau y$ has an analytic continuation to $\Omega \cup \{\lambda_0\}$. Observe that $(M\tau y)(z) = A(z)y$, $z \in \Omega$. Since $y \in Y$ has been chosen arbitrarily, A has an analytic continuation to $\Omega \cup \{\lambda_0\}$.

(1) $\Rightarrow$ (3) Suppose that A has an analytic continuation to $\Omega \cup \{\lambda_0\}$. Then we already know that every $f \in F$ has an analytic continuation to $\Omega \cup \{\lambda_0\}$ too. Put $A(\lambda_0) = \lim\limits_{\lambda\to\lambda_0} A(\lambda)$, $f(\lambda_0) = \lim\limits_{\lambda\to\lambda_0} f(\lambda)$ and

$$b(\lambda_0) = [\max(1, ||A(\lambda_0)||)]^{-1}.$$

Then there is a natural norm isomorphism between F and the Banach space $\tilde{F}$ of all Y-valued holomorphic vector functions f on $\Omega \cup \{\lambda_0\}$ that are bounded with respect to the norm

$$||f|| = \sup_{z\in\Omega\cup\{\lambda_0\}} b(z)||f(z)||. \tag{3.7}$$

Since $V \in L(F)$ is similar to an analogous operator on $\tilde{F}$, it follows that $I - \lambda_0 V$ is invertible.

(3) $\Rightarrow$ (1) This is clear from (3.6a).

The realizations (3.6a) and (3.6b) have as their observability space the subspace consisting of all $f \in F$ such that $\omega V^n f = 0$, $n \geq 0$. Since

$$f(\lambda) = \omega(I - \lambda V)^{-1} f = \sum_{n=0}^{+\infty} \lambda^n \omega V^n f$$

in a neighbourhood of 0, it follows that the observability space $\operatorname{Ker}(\omega|V)$ is the subspace of all $f \in F$ that vanish on the connected component of Ω containing 0. So the realizations (3.6a) and (3.6b) are observable if and only if Ω is connected.

To obtain a minimal improper comonic realization of A, let us suppose that Ω is connected. Then the realizations (3.6a) and (3.6b) are observable. Putting $C_0(z) = A(z) - I$, $C_{n+1}(z) = z^{-1}(C_n(z) - C_n(0))$, $z \in \Omega \setminus \{0\}$, while $C_{n+1}(0) = C_n'(0)$, we obtain the following minimal improper comonic realization of A on Ω:

$$A(\lambda) = I + \omega_0(I - \lambda V_0)^{-1}B_0, \quad \lambda \in \Omega. \tag{3.8}$$

Its state space X_0 is the closure in F of all functions $\sum_{n=0}^{\ell} C_n(.)y_n$, where $y_0,\dots,y_\ell$ range over Y independently and ℓ ranges over the nonnegative integers. The operators ω_0: $X_0 \to Y$, B_0: $Y \to X_0$ and V_0: $X_0 \to X_0$ are given by

$$\omega_0 f = f(0), \quad (B_0 y)(z) = (A(z) - I)y,$$

$$(V_0 f)(z) = \begin{cases} z^{-1}(f(z) - f(0)), & z \in \Omega \setminus \{0\}. \\ f'(0), & z = 0. \end{cases}$$

Observe that ω_0, V_0 and B_0 are bounded while $I - \lambda V_0$ is invertible for all $\lambda \in \Omega$. The proof of (3.8) rests upon the construction given in the proof of Theorem I.4.1.

4. Holomorphic operator functions on a bounded domain.

In this section Ω will be a bounded open subset of $\mathbb{C}$ containing 0. Let Y be a complex Banach space, and A: $\Omega \to L(Y)$ a holomorphic operator function. In the previous section we introduced the Banach space F of all Y-valued holomorphic functions on Ω that are bounded with respect to the norm (3.1). The weight function b in this norm is, as previously, given by $b(z) = [\max(1, ||A(z)||)]^{-1}$, $z \in \Omega$. We adopt all notations from Section 3. In addition, we introduce the Banach space G of all bounded Y-valued holomorphic functions on $\mathbb{C}_\infty \setminus \overline{\Omega}$ that vanish at ∞, endowed with the supremum norm. Let X be the direct sum of F and G.

Consider the operators $\widetilde{\omega}: X \to Y$, $\widetilde{V}: X \to X$, $\widetilde{M}: X \to X$ and $\widetilde{\tau}: Y \to X$ defined by

$$(4.1) \qquad \widetilde{\omega}(f,g) = \lim_{\omega\to\infty} \omega g(\omega), \quad \widetilde{\tau}y = (\tau y, 0), \quad \widetilde{M} = M \oplus I_G,$$

$$(4.2) \qquad \widetilde{V}(f,g) = (\widetilde{f},\widetilde{g}),$$

$$\widetilde{f}(z) = zf(z) + \lim_{\omega\to\infty} \omega g(\omega), \qquad z \in \Omega,$$

$$\widetilde{g}(z) = zg(z) - \lim_{\omega\to\infty} \omega g(\omega), \qquad z \in \mathbb{C}_\infty \setminus \overline{\Omega}.$$

Here the operators M and τ are given by (3.5b) and (3.4) respectively. The operators $\widetilde{\omega}$, $\widetilde{V}$, $\widetilde{M}$ and $\widetilde{\tau}$ are bounded (cf. [8]). By direct computation, it follows that $\sigma(\widetilde{V}) \subset \partial\Omega$ and for $\lambda \in \Omega$ we have

$$(4.3) \qquad (\widetilde{V} - \lambda I)^{-1}(f,g) = (\check{f},\check{g}),$$

$$\check{f}(z) = \begin{cases} (z-\lambda)^{-1}(f(z) - f(\lambda)), & z \in \Omega \setminus \{\lambda\}, \\ f'(\lambda) & , \quad z = \lambda, \end{cases}$$

$$\check{g}(z) = (z-\lambda)^{-1}(g(z) + f(\lambda)), \qquad z \in \mathbb{C}_\infty \setminus \overline{\Omega}.$$

From this, we obtain the following monic realization of A on Ω:

$$(4.4) \qquad A(\lambda) = I - \widetilde{\omega}(\lambda I - \widetilde{V})^{-1}(\widetilde{M} - I)\widetilde{\tau}, \qquad \lambda \in \Omega.$$

Next we want to give a (partial) description of the spectrum of $\widetilde{V}$. By V_+ we denote the restriction of $\widetilde{V}$ to F. Thus $V_+: F \to F$ is given by

$$(4.5) \qquad (V_+f)(z) = zf(z), \qquad z \in \Omega.$$

Note that $\sigma(V_+) = \overline{\Omega}$.

THEOREM 4.1. *All limit points of the boundary of $\overline{\Omega}$ belong to $\sigma(\widetilde{V})$. An isolated point of the boundary of Ω belongs to $\sigma(\widetilde{V})$, if and only if it is not a removable singularity of the operator function* A.

<u>Proof</u>. Let λ_0 be a limit point of $\partial\overline{\Omega}$. Suppose that $\lambda_0 \in \rho(\widetilde{V})$. Then

there exists a connected neighbourhood $\mathcal{U}$ of λ_0 such that $\mathcal{U} \subset \rho(\widetilde{V})$. Since the operator V_+ given by (4.5) is the restriction of $\widetilde{V}$ to $\mathcal{F}$, either $\mathcal{U} \subset \rho(V_+)$ or $\mathcal{U} \subset \sigma(V_+)$. Since $\mathcal{U}$ intersects both Ω and $\mathbb{C} \setminus \overline{\Omega}$, and $\sigma(V_+) = \overline{\Omega}$, we obtain a contradiction. Hence, $\lambda_0 \in \sigma(\widetilde{V})$.

Now let λ_0 be an isolated point of $\partial\Omega$. If $\lambda_0 \in \rho(\widetilde{V})$, then it is a direct consequence of (4.4) that A has a removable singularity in λ_0.

Conversely, let λ_0 be a removable singularity of A. Then it follows from Theorem 3.1 that all $f \in \mathcal{F}$ have an analytic continuation to $\Omega \cup \{\lambda_0\}$. Recall that there is a natural norm isomophism from $\mathcal{F}$ into the Banach space $\widetilde{\mathcal{F}}$ of all Y-valued holomorphic functions on $\Omega \cup \{\lambda_0\}$ that are bounded with respect to the norm

$$||f|| = \sup_{z \in \Omega \cup \{\lambda_0\}} b(z)||f(z)||.$$

Here $b(\lambda_0) = \lim_{\lambda \to \lambda_0} b(\lambda)$. Observe that $\widetilde{V}$ is similar to an analogous operator from $L(\widetilde{\mathcal{F}} \oplus \mathcal{G})$. But then $\sigma(\widetilde{V}) \subset \partial(\Omega \cup \{\lambda_0\})$ so that $\lambda_0 \in \rho(\widetilde{V})$. This completes the proof.

To obtain a monic linearization of A on Ω, we define the operator $\widetilde{T}: X \to X$ by

$$\widetilde{T}(f,g) = (\hat{f},\hat{g})$$

$$\hat{f}(z) = zf(z) + A(z) \lim_{\omega\to\infty} \omega g(\omega), \qquad z \in \Omega,$$

$$\hat{g}(z) = zg(z) - \lim_{\omega\to\infty} \omega g(\omega), \qquad z \in \mathbb{C}_\infty \setminus \overline{\Omega}.$$

Then $\widetilde{T}$ is bounded. Since $\widetilde{T} = \widetilde{V} + (\widetilde{M} - I)\widetilde{\tau\omega}$, it follows from Remark I.2.2(1) that $\widetilde{T}$ is a monic linearization of A on Ω. In particular,

$$\widetilde{E}(\lambda)[A(\lambda) \oplus I_{\mathrm{Ker}\,\widetilde{\omega}}] = (\lambda I - \widetilde{T})\widetilde{F}(\lambda), \qquad \lambda \in \Omega, \tag{4.6a}$$

where the equivalence functions $\widetilde{E}$ and $\widetilde{F}$ are given by

$$(4.6b) \qquad \tilde{E}(\lambda)(y,(f,g)) = (f_E, g_E),$$

$$f_E(z) = -y + (\lambda - z)f(z) - \lim_{\omega\to\infty} \omega g(\omega), \qquad z \in \Omega,$$

$$g_E(z) = (\lambda - z)g(z) + \lim_{\omega\to\infty} \omega g(\omega) + \lambda z^{-1} y, \qquad z \in \mathbb{C}_\infty \setminus \overline{\Omega},$$

and

$$(4.6c) \qquad \tilde{F}(\lambda)(y,(f,g)) = (f_F, g_F),$$

$$f_F(z) = f(z) - (z-\lambda)^{-1}(A(z) - A(\lambda))y, \qquad z \in \Omega \setminus \{\lambda\},$$

$$f_F(\lambda) = g(\lambda) - A'(\lambda)y,$$

$$g_F(z) = g(z) - (z-\lambda)^{-1}(A(\lambda) - I)y + z^{-1}A(\lambda)y, \qquad z \in \mathbb{C}_\infty \setminus \overline{\Omega}.$$

This linearization has been obtained by Den Boer without using the connection between linearization and realization explicitly (cf. [8]). A similar result has been obtained earlier by Mitiagin (cf. [18]). We note that the assertion apperaring in both [8] and [18] that $\sigma(\tilde{T})$ and $\Sigma(A) \cup \partial\Omega$ coincide is not correct in general. A (partial) description of the spectrum of $\tilde{T}$ is given by the following theorem:

THEOREM 4.2. *The spectrum of* $\tilde{T}$ *has the following properties:*

1. $\sigma(\tilde{T}) \cap \Omega = \Sigma(A)$, $\sigma(\tilde{T}) \subset \overline{\Omega}$.
2. *All limit points of the boundary of* $\overline{\Omega}$ *are contained in* $\sigma(\tilde{T})$,
3. *An isolated point* λ_0 *of* $\partial\Omega$ *does not belong to* $\sigma(\tilde{T})$, *if and only if* λ_0 *has a deleted neighbourhood on which* $A(\lambda)$ *is invertible, while both* A *and* A^{-1} *have a removable singularity in* λ_0.

Proof. For the proof of the first property, we refer to [8].

Since the operator V_+ given by (4.5) is the restriction of $\tilde{T}$ to F and $\sigma(V_+) = \overline{\Omega}$, we prove as in Theorem 4.1 that all limit points of $\partial\overline{\Omega}$ belong to $\sigma(\tilde{T})$.

Let λ_0 be an isolated point of $\partial\Omega$. If $\lambda_0 \in \rho(\widetilde{T})$, then clearly λ_0 is a removable singularity of A^{-1}. Therefore, we may assume that λ_0 has a deleted neighbourhood $\mathcal{U}$ on which $A(\lambda)$ is invertible. Let Γ be a positively oriented circle in $\mathcal{U}$ with centre λ_0. Note that for $\lambda \in \Omega \setminus \Sigma(A)$

$$(\widetilde{T} - \lambda I)^{-1}(f,g) = (f_\lambda, g_\lambda),$$

$$f_\lambda(z) = \begin{cases} (z-\lambda)^{-1}(f(z) - f(\lambda)), & z \in \Omega \setminus \{\lambda\}, \\ f'(\lambda), & z = \lambda. \end{cases}$$

$$g_\lambda(z) = (z-\lambda)^{-1}(g(z) + A(\lambda)^{-1}f(\lambda)), \quad z \in \mathbb{C}_\infty \setminus \overline{\Omega}.$$

Take a fixed $z \in \Omega$ in the outer domain of Γ. Since $\lambda \to f_\lambda(z)$ is a holomorphic vector function on Ω, it is clear that

$$\frac{1}{2\pi i}\int_\Gamma \frac{f(\lambda)}{\lambda - z}\, d\lambda = \frac{1}{2\pi i}\int_\Gamma f_\lambda(z)\, d\lambda = 0.$$

Because z is an arbitrary point from Ω in the outer domain of Γ, it follows that f has an analytic continuation to $\Omega \cup \{\lambda_0\}$. From Theorem 3.1 it follows that λ_0 is a removable singularity of A. That λ_0 is a removable singularity of A^{-1} too, is clear from the fact that

$$A(\lambda)^{-1} = I + \widetilde{\omega}(\lambda I - \widetilde{T})^{-1}(\widetilde{M} - I)\widetilde{\tau}, \quad \lambda \in \Omega \setminus \Sigma(A).$$

This identity is obtained from (4.4) with the help of (I.1.3).

Let λ_0 be a removable singularity of both A and A^{-1}. From (the proof of) Theorem 3.1 it is clear that there exists a natural norm isomorphism between $\mathcal{F}$ and the Banach space $\widetilde{\mathcal{F}}$ of Y-valued holomorphic functions on $\Omega \cup \{\lambda_0\}$ that are bounded with respect to the norm introduced in (3.7). Observe that $\widetilde{T}$ is similar to an analogous operator on $\widetilde{\mathcal{F}} \oplus \mathcal{G}$. By virtue of the first part of the present theorem, $\lambda_0 \in \rho(\widetilde{T})$. This completes the proof.

The biobservability space of the monic realization (4.4) is the subspace of $X = F \oplus G$ consisting of all $(\phi,\psi) \in F \oplus G$ for which $\widetilde{\omega}\widetilde{V}^n(\phi,\psi) = 0$, $n \in \mathbb{Z}$. To determine this subspace, we compute $\widetilde{V}^n(\phi,\psi)$, $n \in \mathbb{Z}$. For all non-negative integers n, we have

(4.7a)
$$\widetilde{V}^n(\phi,\psi) = (\hat{\phi},\hat{\psi}),$$
$$\hat{\phi}(z) = z^n\phi(z) + (z^{n-1}\psi_{-1} + z^{n-2}\psi_{-2} + \dots + \psi_{-n}), \quad z \in \Omega,$$
$$\hat{\psi}(z) = z^n\psi(z) - (z^{n-1}\psi_{-1} + z^{n-2}\psi_{-2} + \dots + \psi_{-n}), \quad z \in \mathbb{C}_\infty \setminus \bar{\Omega},$$

(4.7b)
$$\widetilde{V}^{-(n+1)}(\phi,\psi) = (\check{\phi},\check{\psi}),$$
$$\check{\phi}(z) = \begin{cases} z^{-(n+1)}\{\phi(z) - (\phi_0 + z\phi_1 + \dots + z^n\phi_n)\}, & z \in \Omega \setminus \{0\}, \\ \phi_{n+1}, & z = 0, \end{cases}$$
$$\check{\psi}(z) = z^{-(n+1)}\{\psi(z) + (\phi_0 + z\phi_1 + \dots + z^n\phi_n)\}, \quad z \in \mathbb{C}_\infty \setminus \bar{\Omega}.$$

Here the Laurent series of ϕ in 0 and ψ in ∞ are given by

(4.8)
$$\phi(z) = \sum_{n=0}^{+\infty} z^n\phi_n, \quad \psi(z) = \sum_{n=1}^{+\infty} z^{-n}\psi_{-n}.$$

From (4.7a) and (4.7b), we have $\widetilde{\omega}\widetilde{V}^n(\phi,\psi) = \psi_{-(n+1)}$ and $\widetilde{\omega}\widetilde{V}^{-(n+1)}(\phi,\psi) = \phi_n$ $(n = 0,1,2,\dots)$. With the help of (4.8), it is clear now that the biobservability space of the monic realization (4.4) consists of all (ϕ,ψ) such that ϕ vanishes at the connected component of Ω containing 0 and ψ vanishes at the unbounded component of $\mathbb{C}_\infty \setminus \bar{\Omega}$. So the realization (4.4) is biobservable, if and only if Ω is simply connected.

To obtain a biminimal monic realization of A, we suppose that Ω is simply connected. If $A(z) - I = \sum_{n=0}^{+\infty} z^n A_n$ near 0, then D_{k+1} is the restriction to Ω of the Y-valued polynomial given by $D_{k+1}(z) = A_0 + zA_1 + \dots + z^k A_k$, $k \geq 0$. Let us define $C_1, C_2, C_3, \dots$ as in Section 3. Then we prove with the help of the equations (4.7) that the bicontrollability space $\mathrm{Im}(\widetilde{V}|(\widetilde{M} - I)\widetilde{T})_{-\infty}^{+\infty}$ consists of the closure in $X = F \oplus G$ of all ordered pairs (f,g) of the form $(f,g) = (f_1 + f_2, g)$ where f_1, f_2 and g have the form

$$(4.9)\quad \begin{cases} f_1(z) = \sum_{n=0}^{r} C_{n+1}(z)x_n, & f_2(z) = (A(z) - I)\sum_{n=0}^{\ell} z^n y_n, \quad z \in \Omega, \\ g(z) = \sum_{n=0}^{r} z^{-(n+1)} D_{n+1}(z)x_n, & z \in \mathbb{C}_\infty \setminus \overline{\Omega}. \end{cases}$$

Here $x_0,\dots,x_r$ and $y_0,\dots,y_\ell$ ranges over Y independently, while each of the numbers r and ℓ range over the non-negative integers. Thus we obtain the following biminimal monic realization of A on Ω:

$$A(\lambda) = I + \tilde{\omega}_1(\lambda I - \tilde{V}_1)^{-1}\tilde{B}_1, \quad \lambda \in \Omega.$$

Its state space is the bicontrollability space X_1 of the realization (4.4) which consists of the closure in $X = F \oplus G$ of all ordered pairs (f,g) of the form $(f,g) = (f_1 + f_2, g)$, where f_1, f_2 and g are given by (4.9). The operators $\tilde{\omega}_1 : X_1 \to Y$ and $\tilde{V}_1 : X_1 \to X_1$ are the restrictions of $\tilde{\omega}$ and $\tilde{V}$ to X_1 respectively. The operator $\tilde{B}_1 : Y \to X_1$ is defined by $\tilde{B}_1 y = (\phi, 0)$, where $\phi(z) = (A(z) - I)y$, $z \in \Omega$. The operators $\tilde{\omega}_1$, $\tilde{V}_1$ and $\tilde{B}_1$ appear to be bounded.

5. Entire operator functions

Let Y be a complex Banach space and $\ell_1\{Y\}$ the Banach space of all Y-valued sequences $(y_n)_{n=1}^{+\infty}$ that are bounded with respect to the norm

$$||(y_1,y_2,y_3,\dots)|| = \sum_{n=1}^{+\infty} ||y_n||.$$

Suppose that A is an entire operator function whose values are operators from $L(Y)$. Let

$$A(\lambda) = \sum_{n=0}^{+\infty} \lambda^n A_n, \quad \lambda \in \mathbb{C},$$

be the Taylor series of A in 0. Abbreviate $Y \oplus \ell_1\{Y\}$ by X. Put $\alpha_k = \sup_{n \geq k} ||A_n||^{1/n}$. Then $\alpha_1 \geq \alpha_2 \geq \dots \geq 0$ and $\lim_{k\to+\infty} \alpha_k = 0$. Let $\beta_0 = 1$ and

$\beta_\kappa = \alpha_1 \ldots \alpha_\kappa$, $\kappa = 1,2,3,\ldots$. If A is not a polynomial, then $\beta_\kappa \neq 0$, $\kappa = 1,2,3,\ldots$, while

$$\lim_{\kappa\to+\infty} \beta_\kappa \beta_{\kappa-1}^{-1} = \lim_{\kappa\to+\infty} \alpha_\kappa = 0.$$

From this it follows that $\lim_{\kappa\to+\infty} \beta_\kappa^{1/\kappa} = 0$. If A is a polynomial of degree ℓ, then $\beta_\kappa = 0$, $\kappa \geq \ell+1$, and hence trivially $\lim_{\kappa\to+\infty} \beta_\kappa^{1/\kappa} = 0$.

Define the operators $\omega \in L(X,Y)$, $\tau \in L(Y,X)$ and $V \in L(X)$ by

$$\omega(y_0;\ y_1,y_2,y_3,\ldots) = y_0,\quad \tau y = (y;\ 0,0,0,\ldots), \tag{5.1a}$$

$$V(y_0;\ y_1,y_2,y_3,\ldots) = (0;\ \alpha_1 y_0,\alpha_2 y_1,\alpha_3 y_2,\ldots). \tag{5.1b}$$

Then $||V^\kappa|| = \beta_\kappa$, $\kappa \geq 0$. So $\sigma(V) = \{0\}$. Therefore, $I - \lambda V \in GL(X)$, $\lambda \in \mathbb{C}$, while

$$(I - \lambda V)^{-1}\tau y = (\beta_0 y;\ \lambda\beta_1 y,\lambda^2\beta_2 y,\lambda^3\beta_3 y,\ldots). \tag{5.2}$$

Put $\ell = \sup\{\kappa\colon A_\kappa \neq 0\}$. Note that ℓ is finite, if and only if A is a polynomial. Define the operator $M\colon X \to X$ by

$$M(y_0;\ y_1,y_2,y_3,\ldots) = (\sum_{j=0}^{\ell} \beta_j^{-1}A_j y_j;\ 0,0,0,\ldots). \tag{5.3a}$$

This operator is well-defined and bounded. The boundedness of M is clear from the estimate

$$||A_j|| = [||A_j||^{1/j}]^j \leq \alpha_j^j \leq \beta_j \qquad (j = 1,2,3,\ldots). \tag{5.3b}$$

From (5.1a), (5.2) and (5.3) we obtain the following improper comonic realization of A on $\mathbb{C}$:

$$A(\lambda) = I + \omega(M - I)(I - \lambda V)^{-1}\tau,\qquad \lambda \in \mathbb{C}. \tag{5.4}$$

Define $T \in L(X)$ by

$$T(y_0;\ y_1,y_2,y_3,\ldots) = (\sum_{j=0}^{\ell} \beta_j^{-1}A_j\ y_j;\ y_1,y_2,y_3,\ldots). \tag{5.5}$$

Since $\omega\tau = I$, $\operatorname{Ker}\omega = \{0\} \oplus \ell_1\{Y\}$ and $(I - \lambda V) + \tau\omega(M - I) = T - \lambda V$, it follows that the $\ell_1\{Y\}$-extension of A is equivalent on $\mathbb{C}$ to the linear operator pencil $T - \lambda V$. In fact,

$$E(\lambda)[A(\lambda) \oplus I_{\ell_1\{Y\}}] = (T - \lambda V)F(\lambda), \quad \lambda \in \mathbb{C}, \tag{5.6}$$

where the equivalence functions E and F are given by

$$E(\lambda) = I + \tau\omega M(I - \lambda V)^{-1}(I - \tau\omega),$$
$$F(\lambda) = (I - \lambda V)^{-1}.$$

A straightforward calculation shows that

$$E(\lambda)(y_0;\ y_1, y_2, y_3, \ldots) = \left(y_0 + \sum_{\kappa=0}^{\ell-1} \lambda^{\kappa} \left[\sum_{j=1}^{\ell-\kappa} \beta_j^{-1} A_{j+\kappa} y_j\right]; y_1, y_2, \ldots\right),$$
$$F(\lambda)(y_0;\ y_1, y_2, y_3, \ldots) = \left(\sum_{\kappa=0}^{n} \lambda^{\kappa} \beta_{n-\kappa}^{-1} \beta_n y_{n-\kappa}\right)_{n=0}^{+\infty}.$$

Here we read $\ell-1$ and $\ell-\kappa$ as ∞ when $\ell = \infty$.

The linearization (5.6) has been derived earlier by Den Boer. It is a generalization of a linearization obtained in [10] for entire operator functions of the form

$$A(\lambda) = \sum_{n=0}^{+\infty} \lambda^n A_n, \quad \lambda \in \mathbb{C},$$

for which $||A_n|| \leq \gamma(n!)^{-1}$ $(n = 0, 1, 2, \ldots)$.

The controllability space of the realization (5.4) coincides with X, unless A is a polynomial. In case A is a polynomial of degree ℓ the controllability space of the realization (5.4) consists of all sequences $(y_n)_n$ from X such that $y_n = 0$ $(n \geq \ell+1)$. So the realization (5.4) is controllable if and only if A is not a polynomial. The observability space of (5.4), however, is not easy to compute.

Instead of $\ell_1\{Y\}$, we can take as the extension space the Banach space $\ell_p\{Y\}$ of all Y-valued sequences $(y_n)_{n=1}^{+\infty}$ for which $(||y_n||)_n$ belongs to ℓ_p. Here $1 < p \leq +\infty$. The norm of such a sequence will be the norm of $(||y_n||)_n$ in ℓ_p. It is clear that the operators ω, τ and V can be defined as in (5.1). In order to ensure that the operator M given by (5.3) is well-defined and bounded we now take

$$\alpha_\kappa = 2 \sup_{n\geq\kappa} ||A_n||^{1/n}, \quad \kappa = 1,2,3,\ldots .$$

One then has the inequality

$$||A_j|| = [||A_j||^{1/j}]^j \leq 2^{-j}\alpha_j^j \leq 2^{-j}\beta_j \quad (j = 1,2,3,\ldots), \tag{5.7}$$

and hence

$$\sum_{j=0}^{\ell} [\beta_j^{-1} \; ||A_j||]^q < +\infty.$$

Here $p^{-1} + q^{-1} = 1$. In the same way as before, we derive the realization (5.4) and the linearization (5.6) of A on $\mathbb{C}$. Further, if $1 \leq p < +\infty$, the realization (5.4) is controllable, if and only if A is not a polynomial. If $p = +\infty$, then the realization (5.4) is not controllable, regardless of the choice of A. In fact, if $p = +\infty$ and A is not a polynomial, the controllability space of the realization (5.4) is the Banach space $Y \oplus c_0\{Y\}$. Here $c_0\{Y\}$ is the closed subspace of $\ell_\infty\{Y\}$ consisting of all sequences $(y_n)_n$ that tend to zero.

III. EQUIVALENCE WITH A FRACTIONAL LINEAR FUNCTION

1. Equivalence and realization

In the first chapter we discussed the notion of equivalence for holomorphic operator functions. Here we shall (mainly) deal with meromorphic functions. Therefore it is necessary to extend the notion of equivalence.

Let Ω be an open subset of the Riemann sphere $\mathbb{C}_\infty$ and X_1, X_2, Y_1 and Y_2 complex Banach spaces. Suppose that for $i = 1,2$, A_i is an $L(X_i,Y_i)$-valued operator function on Ω that is holomorphic on Ω except for isolated singularities. Then A_1 and A_2 are called *equivalent* on Ω, if there exist operator functions $E\colon \Omega \to GL(Y_1,Y_2)$ and $F\colon \Omega \to GL(X_2,X_1)$, holomorphic on all of Ω, such that

$$A_2(\lambda) = E(\lambda)A_1(\lambda)F(\lambda)$$

for all $\lambda \in \Omega$ outside the singularities of A_1 and A_2. As before, E and F are called *equivalence functions*. The operator functions A_1 and A_2 are called *equivalent up to extension* on Ω, if there exist complex Banach spaces Z_1 and Z_2, called *extension spaces*, such that the Z_1-extension of A_1 and the Z_2-extension of A_2 are equivalent on Ω. It is clear that the notion of equivalence (up to extension) is defined for any pair of meromorphic operator functions on Ω. Two meromorphic operator functions that are equivalent up to extension have the same poles, pole multiplicities and partial multiplicities (cf. [4, 13]).

To generalize the notion of realization, let Ω be an open subset of $\mathbb{C}$, and Y a complex Banach space. Suppose that W is an $L(Y)$-valued operator function on Ω that is holomorphic except for isolated singularities. A *monic realization* of W on Ω is a representation of W in the form

$$W(\lambda) = D + C(\lambda I - A)^{-1}B, \qquad \lambda \in \Omega \setminus \Sigma,$$

where $A \in L(X)$, $B \in L(Y,X)$, $C \in L(X,Y)$ and $D \in L(Y)$, and the set $\Sigma = \sigma(A) \cap \Omega$ is discrete in Ω. The Banach space X on which the *main operator* A is defined is called the *state space* of the realization. If D is invertible, the *associate (main) operator* $A^{\times} = A - BD^{-1}C$ is defined as in Section I.1. Note that Σ contains the singularities of the the operator function W, and that the notion of realization defined above generalizes both the corresponding notion in Section I.1 and the notion of (monic) realization current in system theory (cf. [1, 14]).

2. Fractional linear operator functions

Earlier we proved that up to extension every holomorphic operator function is equivalent to a linear operator polynomial of degree (at most) one. Now that the notion of equivalence has been generalized to meromorphic operator functions, we want to establish a similar result for meromorphic operator functions. In view of the fact that a meromorphic function can always be written as the quotient of two holomorphic functions, it is reasonable to expect that the role of the linear operator pencil will be played by a function of the form $(\lambda I - \tilde{A})(\lambda I - A)^{-1}$ or $(\lambda I - A)^{-1}(\lambda I - \tilde{A})$.

It is convenient to introduce the following notions. Let Ω be an open subset of $\mathbb{C}$ and Y a complex Banach space. Suppose that A and $\tilde{A}$ are bounded linear operators on Y, and let $\sigma(A) \cap \Omega$ be a discrete subset of Ω. Then the operator function

$$(\lambda I - \widetilde{A})(\lambda I - A)^{-1}$$

is called a *left fractional linear operator function (left f.l.o.f.)* on Ω. Similarly, the operator function

$$(\lambda I - A)^{-1}(\lambda I - \widetilde{A})$$

is called a *right fractional linear operator function (right f.l.o.f.)* on Ω. For quotients of monic operator polynomials, such as rational matrix functions, we have the following result.

THEOREM 2.1. *Let* W *be an operator function on* Y *that can be written as the quotient* $W(\lambda) = P(\lambda)Q(\lambda)^{-1}$ *of the monic operator polynomials* P *and* Q*, while the spectrum of* Q *is a discrete set. Suppose that* $P(\lambda) = \lambda^{\ell}I + \lambda^{\ell-1}P_{\ell-1} + \dots + P_0$ *and* $Q(\lambda) = \lambda^{\ell}I + \lambda^{\ell-1}Q_{\ell-1} + \dots + Q_0$. *Then*

$$W(\lambda) \oplus I_{Y(\ell-1)} = E_P(\lambda)(\lambda I - C_{1,P})(\lambda I - C_{1,Q})^{-1}E_Q(\lambda)^{-1}, \quad \lambda \in \Sigma(Q).$$

Here $C_{1,P}$ *and* $C_{1,Q}$ *are the first companion operators of the monic polynomials* P *and* Q *respectively, while the equivalence functions* E_P *and* E_Q *have the form*

$$E_P(\lambda) = \begin{bmatrix} B^{(P)}_{\ell-1}(\lambda) & \cdots\cdots & & B^{(P)}_0(\lambda) \\ -I & 0 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 \quad -I & 0 \end{bmatrix}, \quad E_Q(\lambda) = \begin{bmatrix} B^{(Q)}_{\ell-1}(\lambda) & \cdots\cdots & & B^{(Q)}_0(\lambda) \\ -I & 0 & & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 \quad -I & 0 \end{bmatrix}.$$

Here $B^{(P)}_0(\lambda) = B^{(Q)}_0(\lambda) = I$, *while for* $n = 1,2,\dots,\ell-1$

$$B^{(P)}_n(\lambda) = \lambda^n I + \lambda^{n-1}P_{\ell-1} + \dots + P_{\ell-n},$$

$$B^{(Q)}_n(\lambda) = \lambda^n I + \lambda^{n-1}Q_{\ell-1} + \dots + Q_{\ell-n}.$$

Proof. By virtue of the linearization result for monic operator polynomials (cf. Section II.1), there exists one single equivalence function F such that

$$P(\lambda) \oplus I_{Y^{\ell-1}} = E_P(\lambda)(\lambda I - C_{1,P})F(\lambda)^{-1},$$
$$Q(\lambda) \oplus I_{Y^{\ell-1}} = E_Q(\lambda)(\lambda I - C_{1,Q})F(\lambda)^{-1}.$$

From this the desired result is clear.

From the preceding theorem it is clear that, up to extension, every rational matrix function W such that $W(\infty) = I$ is equivalent on $\mathbb{C}$ to a left f.l.o.f. Similarly, we prove that, up to extension, W is equivalent on $\mathbb{C}$ to a right f.l.o.f. To see this, write $W(\lambda) = \tilde{Q}(\lambda)^{-1}\tilde{P}(\lambda)$ as the quotient of two monic matrix polynomials $\tilde{P}$ and $\tilde{Q}$. Then, up to extension, W is equivalent on $\mathbb{C}$ to $(\lambda I - C_{2,\tilde{Q}})^{-1}(\lambda I - C_{2,\tilde{P}})$. Here $C_{2,\tilde{P}}$ and $C_{2,\tilde{Q}}$ denote the second companion operators of $\tilde{P}$ and $\tilde{Q}$ respectively.

The idea to write a meromorphic operator function as the quotient of two holomorphic operator functions suggests a general procedure to make a given meromorphic operator function equivalent up to extension to a left (right) f.l.o.f. This procedure is described by the next theorem. Although at first sight the theorem may look rather restrictive, it nevertheless renders the most general result.

THEOREM 2.2. *Let Ω be an open subset of $\mathbb{C}$. Suppose that W_1 and W_2 are two $L(Y)$-valued holomorphic operator functions given by the monic realizations*

$$W_i(\lambda) = I + C_i(\lambda I - A)^{-1}B, \quad \lambda \in \Omega \subset \rho(A),\ i = 1,2,$$

where $A \in L(X)$, $C_1 \in L(X,Y)$, $C_2 \in L(X,Y)$ and $B \in L(Y,X)$. Let B have a left inverse B^+ and $A_i^\times = A - BC_i$ $(i = 1,2)$. If $\sigma(A_2^\times) \cap \Omega$ is a discrete *subset* Σ *of* Ω *and* $W(\lambda) = W_1(\lambda)W_2(\lambda)^{-1}$, $\lambda \in \Omega \setminus \Sigma$, *then a monic realization of* W *on* Ω *is given by*

$$W(\lambda) = I + (C_1 - C_2)(\lambda I - A_2^{\times})^{-1}B, \quad \lambda \in \Omega \setminus \Sigma. \tag{2.1}$$

Further, the $\operatorname{Ker} B^{+}$*-extension of* W *is equivalent on* Ω *to a left fractional linear operator function. In fact,*

$$E_1(\lambda)[W(\lambda) \oplus I_{\operatorname{Ker} B^{+}}]E_2(\lambda)^{-1} = (\lambda I - A_1^{\times})(\lambda I - A_2^{\times})^{-1}, \quad \lambda \in \Omega \setminus \Sigma, \tag{2.2}$$

where the equivalence functions E_1 *and* E_2 *are given by*

$$E_1(\lambda)(y,z) = By + z + BC_1(\lambda I - A)^{-1}z, \quad \lambda \in \Omega,$$
$$E_2(\lambda)(y,z) = By + z + BC_2(\lambda I - A)^{-1}z, \quad \lambda \in \Omega.$$

<u>Proof</u>. A straightforward computation, using that $BC_2 = (\lambda I - A_2^{\times}) - (\lambda I - A)$, yields

$$\begin{aligned} W(\lambda) &= [I + C_1(\lambda I - A)^{-1}B].[I - C_2(\lambda I - A_2^{\times})^{-1}B] \\ &= I + (C_1 - C_2)(\lambda I - A_2^{\times})^{-1}B. \end{aligned}$$

From Theorem I.2.1, we know that for $i = 1,2$

$$E_i(\lambda)(y,z) = By + z + BC_i(\lambda I - A)^{-1}z, \quad \lambda \in \Omega,$$

where E_1 and E_2 are as above, and

$$F_1(\lambda)(y,z) = F_2(\lambda)(y,z) = (\lambda I - A)^{-1}(By + z).$$

From this (2.2) is clear.

If $W(\lambda) = W_2(\lambda)^{-1}W_1(\lambda)$, $\lambda \in \Omega \setminus \Sigma$, and W_1 and W_2 are given by $W_i(\lambda) = I + C(\lambda I - A)^{-1}B_i$, $\lambda \in \Omega \subset \rho(A)$, $i = 1,2$, while C has a right inverse C^{+}, then W is equivalent up to extension to the right f.l.o.f. $(\lambda I - A_2^{\times})^{-1}(\lambda I - A_1^{\times})$. Here $A_i^{\times} = A - B_iC$ $(i = 1,2)$. In fact,

$$W(\lambda) \oplus I_{\operatorname{Ker} C} = F_2(\lambda)^{-1}(\lambda I - A_2^{\times})^{-1}(\lambda I - A_1^{\times})F_1(\lambda), \quad \lambda \in \Omega \setminus \Sigma,$$

where the equivalence functions F_1 and F_2 are given by

$$F_i(\lambda)(y,z) = C^+y + z - (I-C^+C)(\lambda I - A)^{-1}B_iy, \quad \lambda \in \Omega,\ i = 1,2.$$

This follows from Remark I.2.2(1). Further, an easy computation shows that

$$W(\lambda) = I + C(\lambda I - A_2^\times)^{-1}(B_1 - B_2), \quad \lambda \in \Omega \setminus \Sigma.$$

With the help of the remark following Theorem 2.2, we prove

THEOREM 2.3. *Let Ω be a bounded open subset of $\mathbb{C}$ and $W: \Omega \to L(Y)$ a meromorphic operator function. Then W is equivalent up to extension on Ω to a right fractional linear operator function.*

Proof. It is clear from the Weierstrass product theorem that there exist holomorphic operator functions W_1, W_2: $\Omega \to L(Y)$ such that $\Sigma(W_2)$ coincides with the poles of W and $W(\lambda) = W_2(\lambda)^{-1}W_1(\lambda)$, $\lambda \in \Omega \setminus \Sigma(W_2)$. Let G be the Banach space of all Y-valued bounded holomorphic functions on $\mathbb{C}_\infty \setminus \overline{\Omega}$ that vanish at ∞, endowed with the supremum norm, and F the Banach space of all Y-valued holomorphic functions on Ω that are bounded with respect to the norm

$$||f|| = \sup_{z\in\Omega} b(z)\ ||f(z)||,$$

where

$$b(z) = [\max(1, ||W_1(z)||, ||W_2(z)||)]^{-1}.$$

Define the operators M_1: $F \to F$ and M_2: $F \to F$ by

$$(M_1f)(z) = W_1(z)f(0), \quad (M_2f)(z) = W_2(z)f(0), \quad z \in \Omega.$$

Since $\sup_{z\in\Omega} b(z)||W_i(z)|| < +\infty$ $(i = 1,2)$, the operators M_1 and M_2 are bounded (cf. Section II.3). Define $\widetilde{M}_i \in L(F \oplus G)$ by $\widetilde{M}_i = M_i \oplus I$ $(i = 1,2)$. The operators $\widetilde{\omega} \in L(F \oplus G, Y)$, $\widetilde{V} \in L(F \oplus G)$ and $\widetilde{\tau} \in L(Y, F \oplus G)$ are defined as in Section II.4. For $i = 1,2$, put $\widetilde{T}_i = \widetilde{V} + (\widetilde{M}_i - I)\widetilde{\tau\omega}$.

Then $\tilde{T}_i$ is the associate operator of the monic realization

$$W_i(\lambda) = I - \tilde{\omega}(\lambda I - \tilde{V})^{-1}(\tilde{M}_i - I)\tilde{\tau}, \quad \lambda \in \Omega,$$

of W_i on Ω. Note that $\tilde{\omega}$ is right invertible with right inverse $\tilde{\omega}^+$ given by

$$\tilde{\omega}^+ y = (0,g], g(z) = z^{-1}y, \quad z \in \mathbb{C}_\infty \setminus \overline{\Omega}.$$

From the remark made after Theorem 2.2, we have for $Z = \operatorname{Ker} \tilde{\omega}^+$

$$W(\lambda) \oplus I_Z = F_2(\lambda)^{-1}(\lambda I - \tilde{T}_2)^{-1}(\lambda I - \tilde{T}_1)F_1(\lambda), \quad \lambda \in \Omega \setminus \Sigma(W_2).$$

The equivalence functions F_1 and F_2 can be obtained from (II.4.6) by replacing A by W_1 and W_2 respectively. This completes the proof.

With the help of Den Boers linearization result (cf. Section II.4) we proved above that every meromorphic operator function on a bounded open set is equivalent up to extension to a right f.l.o.f. The analogous result involving equivalence up to extension to a left f.l.o.f. is also true and follows from a linearization result of Mitiagin (cf. [18]).

As a final application of Theorem 2.2, we derive a monic realization for rational matrix functions that is well-known in system theory (cf. [1]). Let W be a rational matrix function on $\mathbb{C}^n$ such that $W(\infty) = I$. Put $W(\lambda) = P(\lambda)Q(\lambda)^{-1}$, where P and Q are two monic matrix polynomials of the form $P(\lambda) = \lambda^{\ell}I + \lambda^{\ell-1}P_{\ell-1} + \dots + P_0$ and $Q(\lambda) = \lambda^{\ell}I + \lambda^{\ell-1}Q_{\ell-1} + \dots$ $\dots + Q_0$. Let

$$P(\lambda) - Q(\lambda) = \lambda^{\ell-1}G_{\ell-1} + \dots + \lambda G_1 + G_0, \quad \lambda \in \mathbb{C}.$$

Then for $i = 0,1,\dots,\ell-1$, we have $G_i = P_i - Q_i$. According to Section II.1,

$$\lambda^{-\ell}P(\lambda) = I + [P_0 \ \dots \ P_{\ell-1}](\lambda I - S)^{-1}R, \quad \lambda \in \mathbb{C} \setminus \{0\},$$
$$\lambda^{-\ell}Q(\lambda) = I + [Q_0 \ \dots \ Q_{\ell-1}](\lambda I - S)^{-1}R, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$

Here S and R are given by (II.1.6). From (2.1), we obtain

(2.3) $$W(\lambda) = I + [G_0 \ \dots \ G_{\ell-1}](\lambda I - C_{1,Q})^{-1}R,$$

where $C_{1,Q}$ is the first companion operator of Q. Further, the $\mathbb{C}^{n(\ell-1)}$-extension of W is equivalent on $\mathbb{C} \setminus \{0\}$ to $(\lambda I - C_{1,P})(\lambda I - C_{1,Q})^{-1}$. Hence, in this particular case Theorem 2.2 gives a less general result than Theorem 2.1 insofar as the equivalence up to extension of W and $(\lambda I - C_{1,P})(\lambda I - C_{1,Q})^{-1}$ is obtained on a proper subset of $\mathbb{C}$ instead of on the whole of $\mathbb{C}$.

3. Fractional linear operator functions on a finite-dimensional space

In the previous section we proved that, up to extension, every rational matrix function W such that $W(\lambda) = I$ is equivalent on $\mathbb{C}$ to a left [right] f.l.o.f. This f.l.o.f. has the form $(\lambda I - A^{\times})(\lambda I - A)^{-1}$ $[(\lambda I - A)^{-1}(\lambda I - A^{\times})]$, where A and $A^{\times}$ are the main and associate operator of a monic realization of W on $\mathbb{C}$ (cf. Theorem 2.1 and Equation (2.3)). On the other hand, there exist rational matrix functions W that have a monic realization on $\mathbb{C}$ with main operator A and associate operator $A^{\times}$ such that W is neither equivalent up to extension to $(\lambda I - A^{\times})(\lambda I - A)^{-1}$ nor equivalent up to extension to $(\lambda I - A)^{-1}(\lambda I - A^{\times})$ on $\mathbb{C}$. As an example, consider the matrix function

$$W(\lambda) = \begin{bmatrix} 1 & \lambda^{-1} \\ 0 & 1 \end{bmatrix} = I + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \lambda^{-1} [0 \quad 1].$$

Because the main and associate operator coincide, W cannot be equivalent up to extension to either $(\lambda I - A^{\times})(\lambda I - A)^{-1}$ or $(\lambda I - A)^{-1}(\lambda I - A^{\times})$.

These considerations lead to the question as to which monic realizations of a rational matrix function W of the form

$$W(\lambda) = I + C(\lambda I - A)^{-1}B, \quad A^{\times} = A - BC, \tag{3.1}$$

are equivalent up to extension on $\mathbb{C}$ to the left [right] f.l.o.f. $(\lambda I - A^{\times})(\lambda I - A)^{-1}$ $[(\lambda I - A)^{-1}(\lambda I - A^{\times})]$. This question is partially answered by the next theorem.

THEOREM 3.1. *Let* W *be a rational matrix function with monic realization* (3.1). *Then, if any pair of the statements below is true, the third one is also correct.*

1. W *is equivalent up to extension on* $\mathbb{C}$ *to the left fractional linear operator function* $(\lambda I - A^{\times})(\lambda I - A)^{-1}$.
2. *The realization* (3.1) *is minimal.*
3. *The realization* $I + BC(\lambda I - A)^{-1}$ *of* $(\lambda I - A^{\times})(\lambda I - A)^{-1}$ *is minimal.*

Proof. Let the first statement be true. From system theory we know that the McMillan degree of a rational matrix function coincides with its minimal state space dimension (cf. [21, 3] for the definition and the main properties of the McMillan degree). Further, the McMillan degree is invariant under equivalence and extension. Because the monic realizations (3.1) and $I + BC(\lambda I - A)^{-1}$ have the same state space, the second statement is true, if and only if the third one is true.

Let the second and the third statement be correct. Recall that the partial pole (zero) multiplicities in the point $\lambda_0 \in \mathbb{C}$ of a rational matrix matrix function Z with $Z(\infty) = I$ coincide with the partial multiplicities in λ_0 of the main (associate) operator of a minimal monic realization of Z on $\mathbb{C}$ (cf. [3], Section 2.1). Note that the realizations (3.1) and $I + BC(\lambda I - A)^{-1}$ have the same main and associate operator, and are both minimal. So, at each point λ_0 in $\mathbb{C}$, the partial multiplicities of W and $(\lambda I - A^{\times})(\lambda I - A)^{-1}$ coincide, except possibly one, which is equal to the first zero multiplicity of $(\lambda-\lambda_0)W(\lambda)$ and $(\lambda-\lambda_0)(\lambda I - A^\times)(\lambda I - A)^{-1}$ in λ_0. Recall that, at each point λ_0 in $\mathbb{C}$, the sum of all partial multiplicities of W and $(\lambda I - A^\times)(\lambda I - A)^{-1}$ is equal to n and m respectively, where n is the size of $W(\lambda)$ and m the dimension of the state space of the realization (3.1). Therefore, at each point λ_0 in $\mathbb{C}$, all partial multiplicities of the $\mathbb{C}^{\max(m-n,0)}$-extension of W and the $\mathbb{C}^{\max(n-m,0)}$-extension of $(\lambda I - A^\times)(\lambda I - A)^{-1}$ coincide. From Gohberg and Sigal [13] it is clear that W and $(\lambda I - A^\times)(\lambda I - A)^{-1}$ are locally equivalent up to extension on $\mathbb{C}$. By virtue of a well-known result of Leiterer (cf. [15]), W and $(\lambda I - A^\times)(\lambda I - A)^{-1}$ are globally equivalent up to extension on $\mathbb{C}$. This completes the proof.

Similarly, we get the following result.

THEOREM 3.2. *Let* W *be a rational matrix function with monic realization* (3.1). *Then, if any pair of the statements below is true, the third one is also correct.*

1. W *is equivalent up to extension on* $\mathbb{C}$ *to the right fractional linear operator function* $(\lambda I - A)^{-1}(\lambda I - A^\times)$.
2. *The realization* (3.1) *is minimal.*
3. *The realization* $I + (\lambda I - A)^{-1}BC$ *of* $(\lambda I - A)^{-1}(\lambda I - A^\times)$ *is minimal.*

As to the minimality of a monic realization of the form $I + BC(\lambda I - A)^{-1}$, the following can be mentioned.

PROPOSITION 3.3. *Let* W *be a rational matrix function with a monic realization of the form* (3.1) *and suppose that the state space of* (3.1) *is finite dimensional. Then the following statements are equivalent:*

1. *The monic realization* $I + BC(\lambda I - A)^{-1}$ *is minimal.*
2. *The monic realization* $I + BC(\lambda I - A)^{-1}$ *is observable.*

3. *The subspace* $\bigcap_{n=0}^{+\infty} \mathrm{Ker}(A^n - A^{\times n})$ *is trivial.*
4. *For each* $\lambda_0 \in \mathbb{C}$, $\mathrm{Ker}(\lambda_0 I - A) \cap \mathrm{Ker}(\lambda_0 I - A^{\times}) = \{0\}$.
5. *The pencils* $\lambda I - A$ *and* $\lambda I - A^{\times}$ *are left coprime.*

Proof. Obviously the first three statements are equivalent, while the third statement implies the fourth one. The equivalence of the fourth and the fifth statement is well-known in system theory (cf. [1]). It remains to prove that the fourth statement implies the third one. To see this, note that $M = \bigcap_{n=0}^{+\infty} \mathrm{Ker}(A^n - A^{\times n})$ is an A- and $A^{\times}$-invariant subspace of the state space of the realization (3.1) such that the restrictions of A and $A^{\times}$ to M coincide. Since for all $\lambda_0 \in \mathbb{C}$ the spaces $\mathrm{Ker}(\lambda_0 I - A)$ and $\mathrm{Ker}(\lambda_0 I - A^{\times})$ have a trivial intersection, and the state space of the realization (3.1) is finite dimensional, it follows that $M = (0)$. This completes the proof.

The dual of Proposition 3.3 is given by the next proposition:

PROPOSITION 3.4. *Let* W *be a rational matrix function with a monic realization of the form* (3.1) *and suppose that the state space of* (3.1) *is finite dimensional. Then the following statements are equivalent.*

1. *The monic realization* $I + (\lambda I - A)^{-1}BC$ *is minimal.*
2. *The monic realization* $I + (\lambda I - A)^{-1}BC$ *is controllable.*
3. *The subspace* span $\bigcup_{n=0}^{+\infty} \mathrm{Im}(A^n - A^{\times n})$ *is the whole state space.*
4. *For each* $\lambda_0 \in \mathbb{C}$, $\mathrm{Im}(\lambda_0 I - A) + \mathrm{Im}(\lambda_0 I - A^{\times})$ *is the whole state space.*
5. *The pencils* $\lambda I - A$ *and* $\lambda I - A^{\times}$ *are right coprime.*

Theorems 3.1 and 3.2 only give a partial answer to the question asked above. A related question is the following. If a rational matrix function W with a monic realization of the form (3.1) is equivalent up to extension on $\mathbb{C}$ to a left f.l.o.f. $(\lambda I - A^{\times})(\lambda I - A)^{-1}$, is it also equivalent up to extension to the right f.l.o.f. $(\lambda I - A)^{-1}(\lambda I - A^{\times})$?

This question is answered in the negative by the following example:

$$W(\lambda) = 1 - \frac{1}{\lambda} = I + [0 \quad -1]\left(\lambda I - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Then

$$(\lambda I - A^{\times})(\lambda I - A)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1-\lambda^{-1} \end{bmatrix},$$

$$(\lambda I - A)^{-1}(\lambda I - A^{\times}) = \begin{bmatrix} 1 & \lambda^{-2} \\ 0 & 1-\lambda^{-1} \end{bmatrix}.$$

So $(\lambda I - A^{\times})(\lambda I - A)^{-1}$ and $(\lambda I - A)^{-1}(\lambda I - A^{\times})$ cannot be equivalent up to extension on $\mathbb{C}$, because their pole orders in $\lambda = 0$ differ.

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