Chapter 1

Introduction

In this document we develop a fairly complete direct and inverse scattering theory for the Schrödinger equation on the line

\[ -\psi''(k,x) + Q(x)\psi(k,x) = k^2\psi(k,x), \quad x \in \mathbb{R}, \tag{1.1} \]

where \( x \in \mathbb{R} \) denotes position, the prime denotes differentiation with respect to \( x \), \( k \) is the spectral variable (\( k \) is the wavenumber and \( k^2 \) is energy) satisfying \( k \in \mathbb{C}^+ \) and \( Q(x) \) is the potential. The potential is a real function belonging to the Faddeev class \( L^1_s(\mathbb{R}) \) for some \( s \geq 1 \), where \( L^1_s(\mathbb{R}) = L^1(\mathbb{R};(1 + |x|)^s dx) \).

The contents of this file is mostly well-known material published in various places \[4, 5, 8, 11\]. In Chapter 2 we introduce the Jost solutions, Faddeev functions, transition coefficients, and scattering coefficients. We prove their continuity and analyticity properties. We prove their Wronskian relations. In Chapter 3 we write the preceding functions as Fourier transforms of suitable \( L^1 \)-functions. We distinguish between the generic case and the exceptional case on the basis of the linear independence or linear dependence of the zero energy Jost solutions. We require \( Q \in L^1_1(\mathbb{R}) \) in the generic case and \( Q \in L^2_1(\mathbb{R}) \) in the exceptional case. In Chapter 4 we extend the results to \( Q \in L^1_1(\mathbb{R}) \), irrespective of the case we are in. In Chapter 5 we derive the Marchenko integral equations and prove their unique solvability. This requires proving that the bound state norming constants are positive. In Chapter 6 we derive the Goursat-type PDE’s for the inverse Fourier transforms of the Jost solutions. In Chapter 7 we discuss the Darboux transformation to replace potentials with potentials having one more or one less bound state eigenvalue.

In the four appendices we discuss Gronwall’s inequality, Marchenko operators, the resolvent of the Schrödinger equation on the (half)line, relating quant-

\(^1\mathbb{C}^+ \) and \( \mathbb{C}^- \) denote the upper and lower open complex half-planes, while \( \mathbb{C}^\pm = \mathbb{C}^\pm \cup \mathbb{R} \). The choice of \( \mathbb{C}^+ \) as the domain of the spectral variable is arbitrary: We could instead have taken \( k \in \mathbb{C}^- \) consistently.
tities pertaining to potentials to quantities pertaining to their fragments, and a few illustrative examples. We also list the steps to solve the Korteweg-de Vries (KdV) equation numerically using the inverse scattering transform (IST) method.
Chapter 2

Jost Solutions and Scattering Coefficients

In this chapter we introduce the Jost solutions, Faddeev functions, and scattering coefficients. We then go on to derive their continuity, analyticity, symmetry, and unitarity properties, as well as their Wronskian relations.

2.1 Basic functions

Let us define the Jost solution from the left \( f_l(k, x) \) and the Jost solution from the right \( f_r(k, x) \) as those solutions of the Schrödinger equation (1.1) which satisfy the asymptotic conditions

\[
\begin{align*}
  f_l(k, x) &= e^{ikx}[1 + o(1)], \quad x \to +\infty, \quad (2.1a) \\
  f_r(k, x) &= e^{-ikx}[1 + o(1)], \quad x \to -\infty. \quad (2.1b)
\end{align*}
\]

Denoting by Faddeev functions the functions \( e^{-ikx}f_l(k, x) \) and \( e^{ikx}f_r(k, x) \), we easily derive for them the Volterra integral equations

\[
\begin{align*}
  e^{-ikx}f_l(k, x) &= 1 + \int_x^\infty dy \frac{e^{2ik(y-x)} - 1}{2ik} Q(y)e^{-iky}f_l(k, y), \quad (2.2a) \\
  e^{ikx}f_r(k, x) &= 1 + \int_{-\infty}^x dy \frac{e^{2ik(x-y)} - 1}{2ik} Q(y)e^{iky}f_r(k, y). \quad (2.2b)
\end{align*}
\]

The Jost solutions themselves satisfy the Volterra integral equations

\[
\begin{align*}
  f_l(k, x) &= e^{ikx} + \int_x^\infty dy \frac{\sin[k(y-x)]}{k} Q(y)f_l(k, y), \quad (2.3a) \\
  f_r(k, x) &= e^{-ikx} + \int_{-\infty}^x dy \frac{\sin[k(x-y)]}{k} Q(y)f_r(k, y). \quad (2.3b)
\end{align*}
\]
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Proposition 2.1 For \( x \in \mathbb{R} \) the Faddeev functions \( e^{-ikx}f_l(k, x) \) and \( e^{ikx}f_r(k, x) \) are continuous in \( k \in \mathbb{C}^+ \), are analytic in \( k \in \mathbb{C}^+ \), and tend to 1 as \( |k| \to +\infty \) from within \( \mathbb{C}^+ \).

Proof. Using the estimate \((y - x) \leq [1 + \max(0, -x)](1 + |y|)\) we derive from (2.2a)

\[
\frac{|e^{-ikx}f_l(k, x)|}{1 + \max(0, -x)} \leq 1 + \int_x^\infty dy (1 + |y|)|Q(y)| \frac{|e^{-iky}f_l(k, y)|}{1 + \max(0, -y)}.
\]

Applying Gronwall’s inequality [Theorem A.1] we obtain

\[
|e^{-ikx}f_l(k, x)| \leq [1 + \max(0, -x)] \exp \left( \int_x^\infty dy (1 + |y|)|Q(y)| \right), \quad (2.4)
\]

uniformly in \((k, x) \in \mathbb{C}^+ \times \mathbb{R}\). The proof for (2.2b) is similar. □

For \( 0 \neq k \in \mathbb{R} \) we can reshuffle (2.2a) and (2.2b) and arrive at the asymptotic expressions

\[
f_l(k, x) = \overline{a}(k)e^{ikx} + b(k)e^{-ikx} + o(1), \quad x \to -\infty, \quad (2.5a)
\]

\[
f_r(k, x) = a(k)e^{-ikx} + b(k)e^{ikx} + o(1), \quad x \to +\infty, \quad (2.5b)
\]

where

\[
\overline{a}(k) = 1 - \frac{1}{2ik} \int_{-\infty}^\infty dy Q(y)e^{-iky}f_l(k, y), \quad (2.6a)
\]

\[
b(k) = \frac{1}{2ik} \int_{-\infty}^\infty dy e^{2iky}Q(y)e^{-iky}f_l(k, y), \quad (2.6b)
\]

\[
a(k) = 1 - \frac{1}{2ik} \int_{-\infty}^\infty dy Q(y)e^{iky}f_r(k, y), \quad (2.6c)
\]

\[
b(k) = \frac{1}{2ik} \int_{-\infty}^\infty dy e^{-2iky}Q(y)e^{iky}f_r(k, y). \quad (2.6d)
\]

Obviously, \( \overline{a}(k) \) and \( a(k) \) are continuous in \( 0 \neq k \in \mathbb{C}^+ \), are analytic in \( k \in \mathbb{C}^+ \), and tend to 1 as \( |k| \to +\infty \) in \( \mathbb{C}^+ \), while \( k[1 - \overline{a}(k)] \) and \( k[1 - a(k)] \) have finite limits as \( k \to 0 \) from within \( \mathbb{C}^+ \). By the same token, \( \overline{b}(k) \) and \( b(k) \) are continuous in \( 0 \neq k \in \mathbb{R} \) and vanish as \( k \to \pm\infty \), while \( kb(k) \) and \( kb(k) \) have finite limits \( k \to 0 \) along the real axis. Equations (2.6) also imply that

\[
a(k) = 1 + O(1/k) \quad \text{and} \quad \overline{a}(k) = 1 + O(1/k), \quad k \to \infty \text{ from within } \mathbb{C}^+; \quad (2.7a)
\]

\[
b(k) = O(1/k) \quad \text{and} \quad \overline{b}(k) = O(1/k), \quad k \to \pm\infty. \quad (2.7b)
\]

We shall prove below [cf. (2.10a)] that \( a(k) \) and \( \overline{a}(k) \) coincide.
2.2 Symmetry and Wronskian relations

The symmetry properties of the Jost solutions and scattering coefficients are based on the following two facts:

a. The spectral parameter $k$ appears in the Schrödinger equation (1.1) as $k^2$. Thus replacing $k$ by $-k$ does not change (1.1).

b. The potential $Q(x)$ is real-valued.

As a result, if $\psi(k,x)$ is a solution with spectral parameter $k$, then $\psi(-k^*, x)^*$ is another solution with spectral parameter $k$. Here and henceforth the asterisk denotes the complex conjugate.

Consequently, for $0 \neq k \in \mathbb{C}^+$ we have

$$ f_l(-k^*, x)^* = f_l(k, x), \quad f_r(-k^*, x)^* = f_r(k, x), \quad a(-k^*)^* = a(k), \quad \overline{a(-k^*)}^* = \overline{a(k)}. \quad (2.8a) $$

For $0 \neq k \in \mathbb{R}$ we have

$$ b(-k)^* = b(k), \quad \overline{b(-k)^*} = \overline{b(k)}. \quad (2.8c) $$

For any two solutions $\psi(k,x)$ and $\phi(k,x)$ of the Schrödinger equation (1.1) we define their Wronskian as follows:

$$ W[\psi, \phi] = \det \begin{pmatrix} \psi(k,x) & \phi(k,x) \\ \psi'(k,x) & \phi'(k,x) \end{pmatrix}. $$

Then

$$ \frac{d}{dx} W[\psi, \phi] = \det \begin{pmatrix} \psi(k,x) & \phi(k,x) \\ \psi''(k,x) & \phi''(k,x) \end{pmatrix} = [Q(x) - k^2] \det \begin{pmatrix} \psi(k,x) & \phi(k,x) \\ \psi(k,x) & \phi(k,x) \end{pmatrix} = 0. $$

Therefore, $W[\psi, \phi]$ does not depend on $k \in \mathbb{R}$ and hence the identity

$$ [W[\psi, \phi]]_{x=+\infty} = [W[\psi, \phi]]_{x=-\infty} $$

leads to useful information. Indeed,

**Proposition 2.2** For $0 \neq k \in \mathbb{R}$ we have the following Wronskian identities:

$$ W[f_l(k, x), f_r(k, x)] = -2ika(k) = -2i\overline{a(k)}, \quad (2.9a) $$

$$ W[f_r(k, x), f_r(-k, x)] = 2ik[a(k)a(-k) - b(k)b(-k)] = 2ik, \quad (2.9b) $$

$$ W[f_l(k, x), f_l(-k, x)] = -2ik = -2i[k\overline{a(k)}\overline{a(-k)} - b(k)b(-k)], \quad (2.9c) $$

$$ W[f_l(k, x), f_r(-k, x)] = -2ikb(-k) = 2ik\overline{b(k)}, \quad (2.9d) $$

$$ W[f_r(k, x), f_l(-k, x)] = -2ikb(k) = 2ik\overline{b(-k)}. \quad (2.9e) $$
Equations (2.9) imply the following:
\[
\begin{align*}
\overline{a}(k) &= a(k), & 0 \neq k \in \mathbb{C}^+, \\
|a(k)|^2 - |b(k)|^2 &= 1, & 0 \neq k \in \mathbb{R}, \\
|\overline{a}(k)|^2 - |\overline{b}(k)|^2 &= 1, & 0 \neq k \in \mathbb{R}.
\end{align*}
\]
(2.10a) (2.10b) (2.10c)

As a result, \(a(k) \neq 0\) for \(0 \neq k \in \mathbb{R}\).

### 2.3 Reflection and transmission coefficients

Let us define the transmission coefficient \(T(k)\), the reflection coefficient from the right \(R(k)\), and the reflection coefficient from the left \(L(k)\) as follows:
\[
\begin{align*}
T(k) &= \frac{1}{a(k)} = \frac{1}{\overline{a}(k)}, & 0 \neq k \in \mathbb{C}^+ \setminus \{i\kappa_s\}_{s=1}^N, \\
R(k) &= \frac{b(k)}{a(k)}, & 0 \neq k \in \mathbb{R}, \\
L(k) &= \frac{\overline{b}(k)}{\overline{a}(k)}, & 0 \neq k \in \mathbb{R},
\end{align*}
\]
(2.11a) (2.11b)

where \(\{i\kappa_s\}_{s=1}^N\) are the at most countably many zeros of \(a(k)\) in \(\mathbb{C}^+\). Then (2.7) implies that
\[
\begin{align*}
T(k) &= 1 + O(1/k), & k \to \infty \text{ from within } \mathbb{C}^+, \\
R(k) &= O(1/k) \text{ and } L(k) = O(1/k), & k \to \pm \infty.
\end{align*}
\]
(2.12a) (2.12b)

Moreover, for \(0 \neq k \in \mathbb{R}\) we can rephrase the asymptotic conditions (2.1) and (2.5) as follows:
\[
\begin{align*}
f_l(k, x) &= \begin{cases} 
  e^{ikx}[1 + o(1)], & x \to +\infty, \\
  \frac{1}{T(k)} e^{ikx} + \frac{b(k)}{T(k)} e^{-ikx} + o(1), & x \to -\infty,
\end{cases}
\]
\[
f_r(k, x) &= \begin{cases} 
  \frac{1}{T(k)} e^{-ikx} + \frac{R(k)}{T(k)} e^{ikx} + o(1), & x \to +\infty, \\
  e^{-ikx}[1 + o(1)], & x \to -\infty.
\end{cases}
\]

Furthermore, (2.8b), (2.8c), and (2.11) imply the symmetry relations
\[
\begin{align*}
T(-k^*)^* &= T(k), & 0 \neq k \in \mathbb{C}^+ \setminus \{i\kappa_s\}_{s=1}^N, \\
R(-k^*)^* &= R(k), & 0 \neq k \in \mathbb{R}, \\
L(-k^*)^* &= L(k), & 0 \neq k \in \mathbb{R}.
\end{align*}
\]
(2.13a) (2.13b) (2.13c)

**Proposition 2.3** For \(0 \neq k \in \mathbb{R}\) the scattering matrix
\[
S(k) = \begin{pmatrix} T(k) & R(k) \\
L(k) & T(k) \end{pmatrix}
\]
is unitary. Moreover, \(\det S(k) = [T(k)/T(-k)]\).
2.3. REFLECTION AND TRANSMISSION COEFFICIENTS

Proof. Using (2.9), (2.11), and (2.13) we easily compute that for $0 \neq k \in \mathbb{R}$

\[
S(k)S(k)^\dagger = \begin{pmatrix} T(k) & R(k) \\ L(k) & T(k) \end{pmatrix} \begin{pmatrix} T(-k) & L(-k) \\ R(-k) & T(-k) \end{pmatrix} = \begin{pmatrix} 1 + b(k)b(-k) & \bar{b}(-k) + b(k) \\ a(k)a(-k) & a(k)\bar{a}(-k) \end{pmatrix} = I_2,
\]

which proves the unitarity of $S(k)$. Moreover,

\[
\det S(k) = T(k)^2 - R(k)L(k) = \frac{1 - b(k)b(-k)}{a(k)^2} = \frac{a(k)a(-k)}{a(k)^2} = \frac{a(-k)}{a(k)} = \frac{T(k)}{T(-k)},
\]

which completes the proof.

The Wronskian relation (2.9a) can be written in the form

\[
W[f_l(k, x), f_r(k, x)] = -2ika(k) = \frac{-2ik}{T(k)},
\]

where the left member is defined for $k \in \mathbb{C}^+$ and the other two members for $0 \neq k \in \mathbb{C}^+$. We can therefore distinguish two situations:

a. generic case: $f_l(0, x)$ and $f_r(0, x)$ are linearly independent. Hence their Wronskian is nonzero and therefore

\[
\lim_{k \to 0, k \in \mathbb{C}^+} 2ika(k) = \lim_{k \to 0, k \in \mathbb{C}^+} \frac{2ik}{T(k)}
\]

is a nonzero real number [cf. (2.8b) and (2.13a)].

b. exceptional case: $f_l(0, x)$ and $f_r(0, x)$ are linearly dependent. Hence their Wronskian vanishes.

Equations (2.6) imply that

\[
\lim_{k \to 0} 2ikb(k) = -\lim_{k \to 0} 2ika(k) = \int_{-\infty}^{\infty} dy Q(y)f_r(0, y), \quad (2.14a)
\]

\[
\lim_{k \to 0} 2ik\bar{b}(k) = -\lim_{k \to 0} 2ik\bar{a}(k) = \int_{-\infty}^{\infty} dy Q(y)f_l(0, y), \quad (2.14b)
\]

where we observe that all of these expressions are identical. Thus in the generic case we have in the sense of continuous limits

\[
T(0) = 0, \quad R(0) = -1, \quad L(0) = -1. \quad (2.15)
\]
In the exceptional case, we have instead
\[ 0 \neq T(0) \in [-1, 1], \quad R(0) = -L(0) = \pm \sqrt{1 - T(0)^2} \in (-1, 1), \quad (2.16) \]
where we have used the existence of \( T(0) \) [yet to be proved] and the determinant relation \( \det S(0) = \frac{T(0)}{T(0)} = 1 \). Thus in the exceptional case
\[
S(0) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix} = \frac{1}{1 + a^2} \begin{pmatrix} 2a & 1 - a^2 \\ a^2 - 1 & 2a \end{pmatrix}, \quad (2.17)
\]
for some \( \phi \in [0, 2\pi) \) such that \( \cos \phi = \frac{2a}{1 + a^2} \) and \( \sin \phi = \frac{(1 - a^2)/(1 + a^2)}{1 + a^2} \).

**Proposition 2.4** Suppose \( Q \in L^1_2(\mathbb{R}) \). Then in the exceptional case the transmission coefficient \( T(k) \) and the reflection coefficients \( R(k) \) and \( L(k) \) are continuous in \( k \in \mathbb{R} \).

We shall prove in Section 4.1 that this proposition holds for \( Q \in L^1_1(\mathbb{R}) \).

**Proof.** Differentiating \((2.2a)\) with respect to \( 0 \neq k \in \mathbb{R} \) we obtain the integral equation
\[
\frac{\partial}{\partial k} \left[ e^{-ikx} f_1(k, x) \right] = \int_x^\infty dy \frac{\partial}{\partial k} \left[ \frac{e^{2ik(y-x)} - 1}{2ik} \right] Q(y) e^{-iky} f_1(k, y) \\
+ \int_x^\infty dy \frac{e^{2ik(y-x)} - 1}{2ik} Q(y) \frac{\partial}{\partial k} \left[ e^{-iky} f_1(k, y) \right]. \quad (2.18)
\]
Using that
\[
\frac{\partial}{\partial k} \left[ \frac{e^{2ik(y-x)} - 1}{2ik} \right] = \frac{\partial}{\partial k} \int_x^y dw e^{2ik(w-x)} \\
= 2i \int_x^y dw (w - x) e^{2ik(w-x)} \leq 2(y - x),
\]
we can bound above the absolute value of the inhomogeneous term in \((2.18)\) as follows [cf. \((2.4)\)]:
\[
2 \int_x^\infty dy (y - x)|Q(y)|[1 + \max(0, -y)] \exp \left( \int_y^\infty dz [1 + z]|Q(z)| \right),
\]
uniformly in \((k, x) \in \mathbb{C}^+ \times \mathbb{R}\). The latter integral converges absolutely if \( Q \in L^1_2(\mathbb{R}) \).
Let us consider the exceptional case, where each of the six members in (2.14) vanishes. Then (2.6) implies that

$$\alpha(k) = 1 - \int_{-\infty}^{\infty} dy Q(y) \frac{e^{-iky} f_i(k, y) - f_i(0, y)}{2ik},$$

$$\beta(k) = \int_{-\infty}^{\infty} dy Q(y) \frac{e^{-iky} f_i(k, y) - f_i(0, y)}{2ik},$$

$$a(k) = 1 - \int_{-\infty}^{\infty} dy Q(y) \frac{e^{-iky} f_r(k, y) - f_r(0, y)}{2ik},$$

$$b(k) = \int_{-\infty}^{\infty} dy Q(y) \frac{e^{iky} f_r(k, y) - f_r(0, y)}{2ik}.$$
Chapter 3

Triangular Representations

In this chapter we write the Jost solutions and the scattering coefficients as Fourier transforms and prove them to belong to the so-called Wiener algebra. Since these Fourier transforms involve integrations over a half-line, such representations will be called triangular.

3.1 Basic triangular representations

Let us write

\[ f_l(k, x) = e^{ikx} + \int_x^\infty dy e^{iky} K(x, y), \quad (3.1a) \]

\[ f_r(k, x) = e^{-ikx} + \int_{-\infty}^x dy e^{-iky} M(x, y), \quad (3.1b) \]

where the auxiliary functions \( K(x, y) \) and \( M(x, y) \) are to be determined. Substituting \((3.1a)\) into \((2.3a)\) we get

\[
1 + \int_0^\infty dw e^{ikw} K(x, x + w) = 1 + \int_x^\infty dy \int_x^y dw e^{2ik(v-x)} Q(y) \left[ 1 + \int_0^\infty dz e^{ikz} K(y, y + z) \right]
\]

\[
= 1 + \frac{1}{2} \int_0^\infty dw e^{ikw} \int_{x + \frac{1}{2} w}^\infty dy Q(y)
\]

\[
+ \frac{1}{2} \int_0^\infty dw e^{ikw} \int_0^w dz \int_{x + \frac{1}{2}(w-z)}^\infty dy Q(y) K(y, y + z).
\]

Stripping off the Fourier transform we get for \( w \geq 0 \)

\[
K(x, x + w) = \frac{1}{2} \int_{x + \frac{1}{2} w}^\infty dy Q(y) + \frac{1}{2} \int_0^w dz \int_{x + \frac{1}{2}(w-z)}^\infty dy Q(y) K(y, y + z),
\]

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and hence for \( w \to 0^+ \) we have
\[
K(x, x) = \frac{1}{2} \int_x^\infty dy \, Q(y). \tag{3.2}
\]

Because \( Q \in L^1(\mathbb{R}) \), the function \( K(x, x) \) is absolutely continuous and hence almost everywhere differentiable. As a result, we have almost everywhere
\[
Q(x) = -2 \frac{d}{dx} K(x, x). \tag{3.3}
\]

In the same way, we substitute (3.1b) into (2.3b) and obtain
\[
1 + \int_0^\infty dw \, e^{ikw} M(x, x - w)
= 1 + \int_{-\infty}^x dy \int_y^x dv \, e^{2ik(x-v)} Q(y) \left[ 1 + \int_0^\infty dz \, e^{ikz} M(y, y - z) \right]
= 1 + \frac{1}{2} \int_0^\infty dw \, e^{ikw} \int_{-\infty}^{x-\frac{1}{2}w} dy \, Q(y)
+ \frac{1}{2} \int_0^\infty dw \, e^{ikw} \int_0^w dz \int_{-\infty}^{x-\frac{1}{2}(w-z)} dy \, Q(y) M(y, y - z).
\]

Stripping off the Fourier transform we get for \( w \geq 0 \)
\[
M(x, x - w) = \frac{1}{2} \int_{-\infty}^{x-\frac{1}{2}w} dy \, Q(y) + \frac{1}{2} \int_0^w dz \int_{-\infty}^{x-\frac{1}{2}(w-z)} dy \, Q(y) M(y, y - z),
\]
and hence for \( w \to 0^+ \) we have
\[
M(x, x) = \frac{1}{2} \int_{-\infty}^x dy \, Q(y).
\]

Because \( Q \in L^1(\mathbb{R}) \), the function \( M(x, x) \) is absolutely continuous and hence almost everywhere differentiable. As a result, we have almost everywhere
\[
Q(x) = 2 \frac{d}{dx} M(x, x). \tag{3.4}
\]

Changing the variables we have derived from (2.3) the following two integral equations:
\[
K(x, y) = \frac{1}{2} \int_{\frac{1}{2}[x+y]}^\infty dw \, Q(w) + \frac{1}{2} \int_y^\infty dz \int_{x+\frac{1}{2}(y-z)}^\infty dw \, Q(w) K(w, w + z - x)
= \frac{1}{2} \int_{\frac{1}{2}[x+y]}^\infty dw \, Q(w) + \frac{1}{2} \int_x^\infty dw \, Q(w) \int_{\max(w,x+y-w)}^{w+y-x} ds \, K(w, s), \tag{3.5a}
\]
\[ M(x, y) = \frac{1}{2} \int_{-\infty}^{\frac{x+y}{2}} dw Q(w) + \frac{1}{2} \int_{y}^{x} dz \int_{-\infty}^{x-\frac{1}{2}z-y} dw Q(w) M(w, w + z - x) \]

\[ = \frac{1}{2} \int_{-\infty}^{\frac{x+y}{2}} dw Q(w) + \frac{1}{2} \int_{-\infty}^{x} dw Q(w) \int_{w+y-x}^{\min(w,x+y-w)} dw M(w, s). \]

(3.5b)

**Theorem 3.1** For each \( Q \in L_1^1(\mathbb{R}) \) the integral equations (3.5) are uniquely solvable. In fact,

\[ \int_{x}^{\infty} dy |K(x, y)| \leq \left( \int_{x}^{\infty} dw (w-x)|Q(w)| \right) \left[ \exp \left( \int_{x}^{\infty} dw (w-x)|Q(w)| \right) \right], \]

(3.6a)

\[ \int_{-\infty}^{x} dy |M(x, y)| \leq \left( \int_{-\infty}^{x} dw (x-w)|Q(w)| \right) \left[ \exp \left( \int_{-\infty}^{x} dw (x-w)|Q(w)| \right) \right]. \]

(3.6b)

**Proof.** For each function \( L(x, y) \) with \( y \geq x \) we define

\[ \| L(x) \|_* = \int_{x}^{\infty} dy |L(x, z)|. \]

Then the integral equation (3.5a) implies the estimate

\[ \| K(x) \|_* \leq \int_{x}^{\infty} dw (w-x)|Q(w)| \{ 1 + \| K(w) \|_* \}. \]

Applying Gronwall’s inequality [Theorem A.1], we find

\[ \| K(x) \|_* \leq \left[ \int_{x}^{\infty} dw (w-x)|Q(w)| \right] \exp \left( \int_{x}^{\infty} dw (w-x)|Q(w)| \right), \]

as claimed. The proof for \( M(x, y) \) is similar. \( \blacksquare \)

Using (2.6a) and (2.6c) we now easily prove the following:

\[ 2ik[1 - \pi(k)] = \int_{-\infty}^{\infty} dy Q(y) + \int_{0}^{\infty} dw e^{ikw} \int_{-\infty}^{\infty} dy Q(y) K(y, y + w), \]

(3.7a)

\[ 2ik[1 - a(k)] = \int_{-\infty}^{\infty} dy Q(y) + \int_{0}^{\infty} dw e^{ikw} \int_{-\infty}^{\infty} dy Q(y) M(y, y - w), \]

(3.7b)

where for each \( Q \in L_1^1(\mathbb{R}) \) we have

\[ \int_{0}^{\infty} dw \left| \int_{-\infty}^{\infty} dy Q(y) K(y, y + w) \right| \leq \int_{-\infty}^{\infty} dy |Q(y)||K(y)||_* < +\infty, \]

\[ \int_{0}^{\infty} dw \left| \int_{-\infty}^{\infty} dy Q(y) M(y, y - w) \right| \leq \int_{-\infty}^{\infty} dy |Q(y)||M(y)||_* < +\infty, \]
where \( \|M(x)\|_\bullet = \int_{-\infty}^{\infty} |M(x, y)| \).

Using (2.6b) and (2.6d) we get

\[
2ik\tilde{b}(k) = \frac{1}{2} \int_{-\infty}^{\infty} dw \, e^{ikw} \left[ Q\left(\frac{1}{2}w\right) + \int_{0}^{\infty} dv \, Q\left(\frac{1}{2}[w-v]\right) K\left(\frac{1}{2}[w-v], \frac{1}{2}[w+v]\right) \right],
\]

(3.7c)

\[
2ikb(k) = \frac{1}{2} \int_{-\infty}^{\infty} dw \, e^{ikw} \left[ Q\left(-\frac{1}{2}w\right) + \int_{0}^{\infty} dv \, Q\left(\frac{1}{2}[v-w]\right) M\left(\frac{1}{2}[v-w], -\frac{1}{2}[v+w]\right) \right],
\]

(3.7d)

where for each \( Q \in L^1_1(\mathbb{R}) \) the right-hand sides of the inequalities

\[
\int_{-\infty}^{\infty} dw \left| \int_{0}^{\infty} dv \, Q\left(\frac{1}{2}[w-v]\right) K\left(\frac{1}{2}[w-v], \frac{1}{2}[w+v]\right) \right| \leq \int_{-\infty}^{\infty} dy \, |Q(y)| \|K(y)\|_\bullet,
\]

\[
\int_{-\infty}^{\infty} dw \left| \int_{0}^{\infty} dv \, Q\left(\frac{1}{2}[v-w]\right) M\left(\frac{1}{2}[v-w], -\frac{1}{2}[v+w]\right) \right| \leq \int_{-\infty}^{\infty} dy \, |Q(y)| \|M(y)\|_\bullet,
\]

are finite.

By the Wiener algebra \( W \) we denote the commutative complex Banach algebra of all functions

\[
F(k) = c + \int_{-\infty}^{\infty} dw \, e^{ikw} f(w),
\]

where \( c \in \mathbb{C} \) and \( f \in L^1(\mathbb{R}) \), endowed with the norm

\[
\|F\|_W = |c| + \|f\|_1.
\]

Then \( F \) is an invertible element of \( W \) if and only if \( 0 \neq c \in \mathbb{C} \) and \( F(k) \neq 0 \) for every \( k \in \mathbb{R} \). In that case there exists \( g \in L^1(\mathbb{R}) \) such that

\[
\frac{1}{F(k)} = \frac{1}{c} + \int_{-\infty}^{\infty} dw \, e^{ikw} g(w).
\]

We write \( W_0 = \{ c + \hat{f} \in W : c = 0 \} \). In the same way we denote by \( W^\pm \) the commutative complex Banach algebra of all functions

\[
F(k) = c \pm \int_{0}^{\pm\infty} dw \, e^{ikw} f(w),
\]

where \( c \in \mathbb{C} \) and \( f \in L^1(\mathbb{R}^\pm) \), endowed with the norm

\[
\|F\|_{W^\pm} = |c| + \|f\|_1.
\]
Then $F$ is an invertible element of $\mathcal{W}^\pm$ if and only if $0 \neq c \in \mathbb{C}$ and $F(k) \neq 0$ for every $k \in \mathbb{C}^\pm$. In that case there exists $g \in L^1(\mathbb{R})$ such that

$$\frac{1}{F(k)} = \frac{1}{c} \pm \int_0^{\pm\infty} dw e^{ikw} g(w).$$

Writing $\mathcal{W}_0^\pm = \{F = c + \hat{f} \in \mathcal{W}^\pm : c = 0\}$, we obtain the decomposition

$$\mathcal{W} = \mathcal{W}^+ \oplus \mathcal{W}^- = \mathcal{W}_0^+ \oplus \mathcal{W}_0^-.$$

**Proposition 3.2** Suppose $Q \in L^1_1(\mathbb{R})$. Then the functions $2ik[1 - \bar{a}(k)]$ and $2ik[1 - a(k)]$ belong to $\mathcal{W}^+$, while the functions $2ikb(k)$ and $2ikb(k)$ belong to $\mathcal{W}_0^-$. Moreover,

$$a(k) = 1 - \frac{\int_{-\infty}^{\infty} dy Q(y)}{2ik} + o(1/k), \quad k \to \infty \text{ in } \mathbb{C}^+, \quad (3.8a)$$

$$b(k) = o(1/k), \quad \bar{b}(k) = o(1/k), \quad k \to \pm\infty. \quad (3.8b)$$

**Proof.** The Wiener algebra statements have been proved above. Equations (3.8) are immediate from these statements.

**Corollary 3.3** Suppose $Q \in L^1_1(\mathbb{R})$. Then, in the generic case, the transmission coefficient $T(k)$ and the two reflection coefficients $R(k)$ and $L(k)$ belong to the Wiener algebra $\mathcal{W}$.

**Proof.** Indeed, $1/(k - i)$ belongs to $\mathcal{W}$. Hence,

$$\frac{k}{k - i}a(k) = \frac{k}{k - i} - \frac{2ik[1 - a(k)]}{2i(k - i)}$$

belongs to $\mathcal{W}$. Since $[k/(k - i)]a(k)$ belongs to $\mathcal{W}$ and does not vanish for $k \in \mathbb{R}$ [generic case], we see that its reciprocal $(k - i)[T(k)/k]$ belongs to $\mathcal{W}$. Multiplying by $k/(k - i)$, we see that $T \in \mathcal{W}$. Next,

$$R(k) = \frac{1}{2i(k - i)} \frac{2ikb(k)}{k a(k)}$$

belongs to $\mathcal{W}$. In a similar way we prove that $L \in \mathcal{W}$.

To analyze the exceptional case, we strengthen the estimates (3.6). Putting $\|K(x)\|_\diamond = \int_0^\infty dy (y - x)|K(x, y)|$ and $\|M(x)\|_\diamond = \int_{-\infty}^x dy (x - y)|M(x, y)|$, we first derive

$$\|K(x)\|_\diamond \leq \int_x^\infty dw (w - x)^2|Q(w)| \{1 + \|K(w)\|_\diamond\},$$

$$\|M(x)\|_\diamond \leq \int_{-\infty}^x dw (x - w)^2|Q(w)| \{1 + \|M(w)\|_\diamond\}.$$
We then apply Gronwall’s inequality [Theorem A.1] twice and obtain under the hypothesis $Q \in L^1_2(\mathbb{R})$

$$\|K(x)\|_o \leq \left[ \int_x^\infty dw \,(w-x)^2|Q(w)| \right] \exp \left( \int_x^\infty dw \,(w-x)^2|Q(w)| \right), \quad (3.9a)$$

and

$$\|M(x)\|_o \leq \left[ \int_{-\infty}^x dw \,(x-w)^2|Q(w)| \right] \exp \left( \int_{-\infty}^x dw \,(x-w)^2|Q(w)| \right). \quad (3.9b)$$

**Proposition 3.4** Suppose $Q \in L^1_2(\mathbb{R})$. Then, in the exceptional case, the transmission coefficient $T(k)$ and the two reflection coefficients $R(k)$ and $L(k)$ belong to the Wiener algebra $W$.

We shall prove in Section 4.2 that this proposition holds for $Q \in L^1_2(\mathbb{R})$.

**Proof.** To prove that in the exceptional case the scattering coefficients belong to $W$, we compute

$$a(k) = 1 - \int_0^\infty dw \, e^{ikw} \frac{1}{2ik} M(y, y-w)$$

$$= 1 - \frac{1}{2} \int_0^\infty dw \, \int_0^\infty dv \, e^{ikv} M(y, y-w)$$

$$= 1 - \frac{1}{2} \int_0^\infty dv \, e^{ikv} \int_0^\infty dy \, Q(y) \int_{-\infty}^w dw \, M(y, y-w),$$

where

$$\int_0^\infty dv \, \left| \int_0^\infty dy \, Q(y) \int_{-\infty}^w dw \, M(y, y-w) \right| \leq \int_{-\infty}^\infty dy \, |Q(y)| \int_0^\infty dw \, |M(y, y-w)|$$

is finite [cf. (3.9a)]. Thus $a \in W$. Since $a(k) \neq 0$ for $k \in \mathbb{R}$, also $T = [1/a] \in W$.

Next, we compute

$$b(k) = \int_{-\infty}^\infty dy \, Q(y) \left[ e^{-2iky} - 1 \right] e^{iky} f_r(k, y) + \frac{e^{iky} f_r(k, y) - f_r(0, y)}{2ik}$$

$$= \frac{1}{2} \int_{-\infty}^\infty dw \, e^{ikw} \int_{-\infty}^{\frac{1}{2}w} dy \, Q(y)$$

$$+ \frac{1}{2} \int_{-\infty}^\infty dw \, e^{ikw} \int_0^\infty dv \, \int_{-\infty}^{\frac{1}{2}[v-w]} dy \, Q(y) M(y, y-v),$$

which can be shown to belong to $W$ if $Q \in L^2_2(\mathbb{R})$. Thus $R = (b/a) \in W$. In the same way we prove that $L \in W$. 

In Chapter 4 we shall prove that Proposition 3.4 is true for $Q \in L^1_1(\mathbb{R})$, though by vastly more complicated means.
3.2 Representations for potentials with decay

In this section we prove that $K(x, y)$ and $M(x, y)$ belong to a weighted $L^1$ space in the $y$-variable if the potential $Q(x)$ does.

**Theorem 3.5** Suppose $Q \in L^1(\mathbb{R}; (1 + |x|)^{1+s}dx)$ for some $s \geq 0$. Then for each $x_0 \in \mathbb{R}$ we have

$$
\sup_{x \geq x_0} \int_x^\infty dy \frac{1}{1 + y - x} |K(x, y)| + \sup_{x \leq x_0} \int_{-\infty}^x dy \frac{1}{1 + x - y} |M(x, y)| < +\infty.
$$

(3.10)

**Proof.** Putting $B_l(x, \alpha) = K(x, x+\alpha)$, we write the integral equation for $K(x, y)$ in the form

$$
B_l(x, \alpha) = \frac{1}{2} \int_{x+\frac{1}{2} \alpha}^\infty d\hat{y} Q(\hat{y}) + \frac{1}{2} \int_0^\alpha dz \int_{x+\frac{1}{2} (\alpha - z)}^\infty d\hat{y} Q(\hat{y}) B_l(\hat{y}, z).
$$

(3.11)

We now estimate

$$
\int_0^\infty d\alpha \frac{1}{1 + \alpha} |B_l(x, \alpha)| \leq \frac{1}{2} \int_{x}^\infty d\hat{y} |Q(\hat{y})| \int_0^{2(\hat{y} - x)} d\alpha \frac{1}{1 + \alpha} \leq \int_{x}^\infty d\hat{y} (\hat{y} - x)[1 + 2(\hat{y} - x)]^s |Q(\hat{y})| + \int_{x}^\infty d\hat{y} Q(\hat{y}) \int_{\hat{y}}^\infty dz |B_l(\hat{y}, z)|.
$$

Applying Gronwall’s inequality (A.2), we get the result for $K(x, y)$. The proof for $M(x, y)$ is analogous.

3.3 Potentials with compact support

We now allow $Q(x)$ to be supported on a half-line or on a finite interval.

**Theorem 3.6** If $Q \in L^1(\mathbb{R})$ is supported on $(-\infty, M]$, then the support of $K(x, y)$ is contained in the set

$$
\{(x, y) \in \mathbb{R}^2 : x \leq y \leq 2M - x\}.
$$

If $Q \in L^1(\mathbb{R})$ is supported on $[L, +\infty)$, then the support of $M(x, y)$ is contained in the set

$$
\{(x, y) \in \mathbb{R}^2 : 2L - x \leq y \leq x\}.
Figure 3.1: If the potential $Q(x)$ is supported in $(-\infty, M]$, then the green area indicates the support of $K(x, y)$. On the other hand, if $Q(x)$ is supported on $[L, +\infty)$, then the cyan area indicates the support of $M(x, y)$.

**Proof.** Suppose $x + y > 2M$. Then (3.5a) can be written in the form

$$K(x, y) = \frac{1}{2} \int_{x}^{M} \, dw \, Q(w) \int_{x+y-w}^{w+y-x} \, ds \, K(w, s),$$

so that for $\tilde{L} \leq x \leq M$ and $\tilde{L}$ otherwise arbitrary, we have

$$\int_{2M-x}^{\infty} \, dy \, |K(x, y)| \leq \frac{1}{2} \int_{x}^{M} \, dw \, |Q(w)| \int_{2M-x}^{\infty} \, dy \int_{x+y-w}^{w+y-x} \, ds \, |K(w, s)|$$

$$= \frac{1}{2} \int_{x}^{M} \, dw \, |Q(w)| \int_{w}^{\infty} \, ds \, \int_{\max(2M-x, x+w)}^{-x+w} \, dy \, |K(w, s)|$$

$$\leq \int_{x}^{M} \, dw \, (w-x)|Q(w)| \int_{2M-w}^{\infty} \, ds \, |K(w, s)|$$

$$\leq \int_{x}^{M} \, dw \, (w-\tilde{L})|Q(w)| \int_{2M-w}^{\infty} \, ds \, |K(w, s)|.$$  

By using Gronwall’s inequality, we get $\int_{2M-x}^{\infty} \, dy \, |K(x, y)| = 0$ for $\tilde{L} \leq x \leq M$, whence $K(x, y) = 0$ for $\tilde{L} \leq x \leq M$ and $y \geq x$. Recalling the arbitrariness of $\tilde{L}$, we get $K(x, y) = 0$ for $y \geq x$. The second part of the theorem is proved in an analogous way. 

Now let the potentials be supported on the other half-line.

**Theorem 3.7** If $Q \in L^{1}(\mathbb{R})$ is supported on $[L, +\infty)$, then $K(x, y)$ only depends on $y-x$ if $x+y < 2L$. On the other hand, if $Q \in L^{1}(\mathbb{R})$ is supported on $(-\infty, M]$, then $M(x, y)$ only depends on $x-y$ if $x+y > 2M$. 

Figure 3.2: If the potential $Q(x)$ is supported in $[L, M]$, then the cyan+magenta area indicates the support of $K(x, y)$ [left pane] and $M(x, y)$ [right pane]. In the magenta area, $K(x, y)$ [left pane] and $M(x, y)$ [right pane] only depend on $y - x$.

**Proof.** If $Q(x)$ is supported on $[L, +\infty)$, for $x + y < 2L$ we write (3.5a) as

\[
K(x, y) = \frac{1}{2} \int_{L}^{\infty} dw Q(w) + \frac{1}{2} \int_{L}^{\infty} dw Q(w) \int_{w}^{w+y-x} ds K(w, s),
\]

which proves the first part. On the other hand, if $Q(x)$ is supported on $(-\infty, M]$, we write (3.5b) as

\[
M(x, y) = \frac{1}{2} \int_{-\infty}^{M} dw Q(w) + \frac{1}{2} \int_{-\infty}^{M} dw Q(w) \int_{w}^{w+y-x} dw M(w, s),
\]

which proves the second part. ■

**Corollary 3.8** Suppose $Q \in L^1_{\text{loc}}(\mathbb{R})$ is supported on $[L, M]$. Then $K(x, y)$ is supported on $\{(x, y) \in \mathbb{R}^2 : x \leq y \leq 2M - x\}$ and depends only on $y - x$ if $x + y \leq 2L$. Moreover, $M(x, y)$ is supported on $\{(x, y) \in \mathbb{R}^2 : x \geq y \geq 2L - x\}$ and depends only on $x - y$ if $x + y \geq 2M$.

**Proof.** If $Q(x)$ is supported on $[L, M]$, for $x + y < 2L$ we write (3.5a) as

\[
K(x, y) = \frac{1}{2} \int_{L}^{M} dw Q(w) + \frac{1}{2} \int_{L}^{M} dw Q(w) \int_{w}^{w+y-x} ds K(w, s).
\]

On the other hand, if $Q(x)$ is supported on $[L, M]$, we write (3.5b) as

\[
M(x, y) = \frac{1}{2} \int_{L}^{M} dw Q(w) + \frac{1}{2} \int_{L}^{M} dw Q(w) \int_{w}^{w+y-x} dw M(w, s).
\]

This completes the proof. ■
Chapter 4

Small Energy Asymptotics

The analysis of the small energy asymptotics is elementary in the generic case for \( Q \in L^1_1(\mathbb{R}) \) and in the exceptional case for \( Q \in L^1_2(\mathbb{R}) \). The basic results are due to Faddeev [8] (though with some errors) and have been presented in a corrected way by Deift and Trubowitz [5]. The nontrivial extension of the results in the exceptional case for \( Q \in L^1_1(\mathbb{R}) \) is due to Klaus [11].

4.1 Scattering coefficients

Before discussing the small energy (\( k \to 0 \)) asymptotics in the exceptional case, we study the behavior of \( \psi(k, x) - \psi(0, x) \) for bounded Schrödinger solutions having the same initial conditions at \( x = 0 \). This includes the zero energy Jost solutions in the exceptional case. The solution of the Schrödinger equation (1.1) satisfying

\[
\psi(k, 0) = \alpha, \quad \psi'(k, 0) = \beta,
\]

satisfies the integral equation

\[
\psi(k, x) = \alpha \cos(kx) + \beta \frac{\sin(kx)}{k} + \int_0^x dy \frac{\sin[k(x-y)]}{k} Q(y) \psi(k, y).
\] (4.1)

As \( k \to 0 \) in \( \mathbb{C}^+ \), (4.1) reduces to the following:

\[
\psi(0, x) = \alpha + \beta x + \int_0^x dy (x-y) Q(y) \psi(0, y).
\] (4.2)

If \( \psi(0, x) \) is bounded in \( x \in \mathbb{R} \), then

\[
\beta = - \int_0^\infty dy Q(y) \psi(0, y) = \int_{-\infty}^0 dy Q(y) \psi(0, y)
\] (4.3)
and hence
\[
\psi(0, x) = \alpha - \int_0^\infty dy y Q(y) \psi(0, y) + \int_x^\infty dy (y - x) Q(y) \psi(0, y)
\]
\[
= \alpha - \int_0^\infty dy y Q(y) \psi(0, y) + o(1), \quad x \to +\infty, \quad (4.4a)
\]
\[
\psi(0, x) = \alpha + \int_{-\infty}^0 dy y Q(y) \psi(0, y) + \int_{-\infty}^x dy (x - y) Q(y) \psi(0, y)
\]
\[
= \alpha + \int_{-\infty}^0 dy y Q(y) \psi(0, y) + o(1), \quad x \to -\infty, \quad (4.4b)
\]
because \( Q \in L^1_1(\mathbb{R}) \).

**Proposition 4.1** Let \( \psi(k, x) \) be a solution of the Schrödinger equation \((1.1)\), where the initial conditions at \( x = 0 \) do not depend on \( k \). Suppose that \( \psi(0, x) \) is bounded. Then for some constant \( \tilde{C} \) we have
\[
|\psi(k, x) - \psi(0, x)| \leq \tilde{C} \|\psi(0, \cdot)\|_\infty \frac{k^2 x^2}{1 + k^2 x^2}. \quad (4.5)
\]

**Proof.** Let \( \alpha = \psi(0, 0) \). Using \((4.1)\) and \((4.3)\), we now compute
\[
\psi(k, x) - \psi(0, x) = \alpha \left[ \cos(kx) - 1 \right] - \left[ \sin(kx) \right] \frac{x}{k} \int_x^\infty dy Q(y) \psi(0, y)
\]
\[
+ \sin(kx) \int_0^x dy \cos\left(\frac{ky}{k}\right) - 1 y Q(y) \psi(0, y)
\]
\[
- \left[ \cos(kx) - 1 \right] \int_0^x dy \sin\left(\frac{ky}{k}\right) y Q(y) \psi(0, y)
\]
\[
- \int_0^x dy \left( \frac{\sin(ky)}{ky} - 1 \right) Q(y) y \psi(0, y)
\]
\[
+ \int_0^x dy \frac{\sin[k(x - y)]}{k} Q(y) \left[ \psi(k, y) - \psi(0, y) \right]. \quad (4.6)
\]

Now there exists a positive constant \( C \) such that for \( 0 \neq z \in \mathbb{R} \) we have
\[
|\sin(z)| \leq \frac{C|z|}{\sqrt{1 + z^2}}, \quad |1 - \cos(z)| \leq \frac{C z^2}{1 + z^2}, \quad \left| 1 - \frac{\sin(z)}{z} \right| \leq \frac{C z^2}{1 + z^2}. \quad (4.7)
\]
Since \( Q \in L^1_1(\mathbb{R}) \) and \( \psi(0, x) \) is bounded, it is easy to estimate, using that \( z^2/(1 + z^2) \) is increasing in \( z \in \mathbb{R}^+ \), the \( j \)-th term on the right-hand side of \((4.6)\) by the expression
\[
CD_j \|\psi(0, \cdot)\|_\infty \frac{k^2 x^2}{1 + k^2 x^2},
\]
where \( D_1 = 1, D_2 = \|Q\|_{1,1}, D_3 = C\|Q\|_{1,1}, D_4 = (1 + C)\|Q\|_{1,1}, \) and \( D_5 = \|Q\|_{1,1}. \) We thus obtain, for \( D = \sum_{j=1}^{5} D_j, \) the estimate

\[
|\psi(k, x) - \psi(0, x)| \leq CD\|\psi(0, \cdot)\|_{\infty} \frac{k^2x^2}{1 + k^2x^2} + \frac{C}{|k|} \left| \frac{Q(y)}{\sqrt{1 + k^2y^2}} \int_0^x dy |Q(y)| \right| |\psi(k, y) - \psi(0, y)|.
\]

Writing

\[
w(k, x) = \frac{1 + k^2x^2}{k^2x^2} |\psi(k, x) - \psi(0, x)|,
\]

we obtain

\[
w(k, x) \leq CD\|\psi(0, \cdot)\|_{\infty} + \frac{C}{|k|} \frac{\sqrt{1 + k^2x^2}}{|kx|} \int_0^x dy |Q(y)| \frac{y}{\sqrt{1 + k^2y^2}} w(k, y)
\]

\[
= CD\|\psi(0, \cdot)\|_{\infty} + C \int_0^x dy |Q(y)| \frac{y}{\sqrt{1 + k^2y^2}} w(k, y)
\]

\[
\leq CD\|\psi(0, \cdot)\|_{\infty} + C \int_0^x dy y|Q(y)| w(k, y).
\]

Utilizing (4.7) and Gronwall’s inequality (A.4), we obtain

\[
w(k, x) \leq CD\|\psi(0, \cdot)\|_{\infty} \exp \left( C \int_0^x dy y|Q(y)| \right).
\]

As a result,

\[
|\psi(k, x) - \psi(0, x)| \leq CD\|\psi(0, \cdot)\|_{\infty} \frac{k^2x^2}{1 + k^2x^2} \exp \left( C \int_0^x dy y|Q(y)| \right),
\]

which settles (4.5) for \( \tilde{C} = CD \exp(C\|Q\|_{1,1}). \)

Let \( \theta(k, x) \) and \( \varphi(k, x) \) be the solutions of the Schrödinger equation (1.1) satisfying

\[
\theta(k, 0) = 1, \quad \theta'(k, 0) = 0, \quad (4.8a)
\]

\[
\varphi(k, 0) = 0, \quad \varphi'(k, 0) = 1. \quad (4.8b)
\]

Then (4.1) implies that

\[
\theta(k, x) = \cos(kx) + \int_0^x dy \frac{\sin[k(x - y)]}{k} Q(y)\theta(k, y), \quad (4.9a)
\]

\[
\varphi(k, x) = \frac{\sin(kx)}{k} + \int_0^x dy \frac{\sin[k(x - y)]}{k} Q(y)\varphi(k, y). \quad (4.9b)
\]
Proposition 4.2 Suppose $Q \in L^1_1(\mathbb{R})$. Then the following identities are true:

\begin{align}
  f_i(k, 0) &= 1 + \int_0^\infty dy e^{iky}Q(y)\varphi(k, y), \quad (4.10a) \\
  f_r(k, 0) &= 1 - \int_{-\infty}^0 dy e^{-iky}Q(y)\varphi(k, y), \quad (4.10b) \\
  -f'_i(k, 0) &= -ik + \int_0^\infty dy e^{iky}Q(y)\theta(k, y), \quad (4.10c) \\
  -f'_r(k, 0) &= ik - \int_{-\infty}^0 dy e^{-iky}Q(y)\theta(k, y). \quad (4.10d)
\end{align}

Proof. Inverting the identity

\[
\begin{pmatrix}
  f_i(k, 0) & f'_i(k, 0) \\
  f_r(k, 0) & f'_r(k, 0)
\end{pmatrix}
\begin{pmatrix}
  \theta(k, x) \\
  \varphi(k, x)
\end{pmatrix} =
\begin{pmatrix}
  f_i(k, x) \\
  f_r(k, x)
\end{pmatrix},
\]

we get

\[
\theta(k, x) = \frac{1}{W[f_i(k, x), f_r(k, x)]} \left\{ f'_r(k, 0)f_i(k, x) - f'_i(k, 0)f_r(k, x) \right\},
\]

\[
\varphi(k, x) = \frac{1}{W[f_i(k, x), f_r(k, x)]} \left\{ -f_r(k, 0)f_i(k, x) + f_i(k, 0)f_r(k, x) \right\}.
\]

Next, we write (4.9) in the following form:

\[
\theta(k, x) = \frac{e^{ikx}}{2ik} \left[ ik - \int_x^0 dy e^{-iky}Q(y)\theta(k, y) \right] - \frac{e^{-ikx}}{2ik} \left[ -ik + \int_x^0 dy e^{iky}Q(y)\theta(k, y) \right],
\]

\[
\varphi(k, x) = \frac{e^{ikx}}{2ik} \left[ 1 - \int_x^0 dy e^{-iky}Q(y)\varphi(k, y) \right] - \frac{e^{-ikx}}{2ik} \left[ 1 + \int_x^0 dy e^{iky}Q(y)\varphi(k, y) \right].
\]

In combination with (2.1), (2.5), (2.9a), and (2.11) we now obtain for $0 \neq k \in \mathbb{R}$

\[
\theta(k, x) = \begin{cases} 
  \frac{e^{-ikx}}{2ik} f'_i(k, 0) - \frac{e^{ikx}}{2ik} \left[ T(k) f'_i(k, 0) - R(k) f'_i(k, 0) \right] + o(1), & x \to +\infty, \\
  \frac{e^{ikx}}{2ik} f'_r(k, 0) + \frac{e^{-ikx}}{2ik} \left[ -L(k) f'_r(k, 0) + T(k) f'_i(k, 0) \right] + o(1), & x \to -\infty,
\end{cases}
\]

\[
\varphi(k, x) = \begin{cases} 
  \frac{e^{-ikx}}{2ik} f_i(k, 0) + \frac{e^{ikx}}{2ik} \left[ T(k) f_i(k, 0) - R(k) f_i(k, 0) \right] + o(1), & x \to +\infty, \\
  \frac{e^{ikx}}{2ik} f_r(k, 0) - \frac{e^{-ikx}}{2ik} \left[ T(k) f_i(k, 0) - L(k) f_r(k, 0) \right] + o(1), & x \to -\infty,
\end{cases}
\]

which implies (4.9) for $0 \neq k \in \mathbb{R}$. The extension of (4.9) to $k \in \mathbb{C}^+$ proceeds by analytic continuation. □
We now arrive at the main result of this chapter.

**Theorem 4.3** In the exceptional case we have for \( Q \in L^1_t(\mathbb{R}) \)
\[
S(k) = \frac{1}{1 + a^2} \left( \frac{2a}{a^2 - 1} - \frac{1 - a^2}{2a} \right) + o(1), \quad k \to 0 \text{ from within } \mathbb{C}^+,
\]
where \( a = \lim_{x \to -\infty} f_l(0, x) \).

**Proof.** Let \( \psi(k, x) \) the solution of the Schrödinger equation \((1.1)\) which has the same initial conditions at \( x = 0 \) as \( f_l(0, x) \). Then
\[
\psi(k, x) = f_l(0, 0) \theta(k, x) + f'_l(0, 0) \varphi(k, x). \tag{4.11}
\]

We now compute
\[
f_l(0, 0) W[f_l(k, x), f_r(k, x)] = f_l(0, 0) W[f_l(k, 0), f_r(k, 0)]
\]
\[
= f_r(k, 0) W[f_l(k, 0), \psi(k, 0)] - f_l(k, 0) W[f_r(k, 0), \psi(k, 0)]
\]
\[
= f_l(k, 0) \{-f_l(0, 0) f'_l(k, 0) + f'_l(0, 0) f_l(k, 0)\}
\]
\[
- f_l(k, 0) \{-f_l(0, 0) f'_r(k, 0) + f'_r(0, 0) f_l(k, 0)\}
\]
\[
= f_l(k, 0) \left\{ -i k f_l(0, 0) + f'_l(0, 0) + \int_0^\infty dy e^{iky} Q(y) \psi(k, y) \right\}
\]
\[
- f_l(k, 0) \left\{ i k f_l(0, 0) + f'_l(0, 0) - \int_{-\infty}^0 dy e^{-iky} Q(y) \psi(k, y) \right\}, \tag{4.12}
\]
where we have used \((4.2)\), \((4.10)\) and \((4.11)\). Let us now estimate the two integrals in the last member, where we assume the exceptional case and hence the boundedness of \( \psi(0, x) = f_l(0, x) \). Write\footnote{We have \( O(k^2) \) if \( Q \in L^1_t(\mathbb{R}^+) \).}
\[
\int_0^\infty dy e^{iky} Q(y) \psi(k, y) = \int_0^\infty dy Q(y) \psi(0, y) + ik \int_0^\infty dy y Q(y) \psi(0, y)
\]
\[
+ \int_0^\infty dy \left[ e^{iky} - 1 - iky \right] Q(y) \psi(0, y)
\]
\[
+ \int_0^\infty dy e^{iky} Q(y) [\psi(k, y) - \psi(0, y)]] = -f'_l(0, 0) + ik [f_l(0, 0) - 1] + o(k),
\]
where we have used \((4.3)\) and \((4.4a)\). In the same way we write\footnote{We have \( O(k^2) \) if \( Q \in L^1_t(\mathbb{R}^-) \).}
\[
\int_{-\infty}^0 dy e^{-iky} Q(y) \psi(k, y) = \int_{-\infty}^0 dy Q(y) \psi(0, y) - ik \int_{-\infty}^0 dy y Q(y) \psi(0, y)
\]
\[
+ \int_{-\infty}^0 dy \left[ e^{-iky} - 1 + iky \right] Q(y) \psi(0, y).
\]
\[ f_l(0, 0) = \frac{1}{2} f_r(0, 0)[1 + o(1)] + \frac{1}{2} f_l(0, 0)[a + o(1)] = \frac{1 + a^2}{2a} f_l(0, 0) + o(1), \]

where \( a = \lim_{x \to -\infty} f_l(0, x) \) and we have used (4.3) and (4.4b). Using that \( a = [f_l(0, x)/f_r(0, x)] \), we now obtain

\[ f_l(0, 0)a(k) = \frac{1}{2} f_r(0, 0)[1 + o(1)] + \frac{1}{2} f_l(0, 0)[a + o(1)] = \frac{1 + a^2}{2a} f_l(0, 0) + o(1), \]

because \( f_l(k, 0) - f_l(0, 0) = O(k) \) and similarly for \( f_r \). If \( f_l(0, 0) \neq 0 \), we get

\[ a(k) = \frac{1 + a^2}{2a} + o(1). \]  

(4.13)

Next we compute

\[ f_l(0, 0)W[f_l(k, x), f_r(-k, x)] = f_l(0, 0)W[f_l(k, 0), f_r(-k, 0)] \]

\[ = f_r(-k, 0)W[f_l(k, 0), \psi(k, 0)] - f_l(k, 0)W[f_r(-k, 0), \psi(k, 0)] \]

\[ = f_r(-k, 0) \left\{ -ik f_l(0, 0) + f_l'(0, 0) + \int_0^\infty dy e^{iky} Q(y) \psi(k, y) \right\} \]

\[ - f_l(k, 0) \left\{ f_l'(0, 0) - ik f_l(0, 0) - \int_{-\infty}^0 dy e^{iky} Q(y) \psi(-k, y) \right\}, \]  

(4.14)

where we have used (4.2), (4.10) and (4.11). Let us now estimate the two integrals in the last member, assuming the exceptional case and hence the boundedness of \( \psi(0, x) = f_l(0, x) \). Writing

\[ \int_0^\infty dy e^{iky} Q(y) \psi(k, y) = \int_0^\infty dy Q(y) \psi(0, y) + ik \int_0^\infty dy y Q(y) \psi(0, y) \]

\[ + \int_0^\infty dy [e^{iky} - 1 - iky] Q(y) \psi(0, y) \]

\[ + \int_0^\infty dy e^{iky} Q(y)[\psi(k, y) - \psi(0, y)] \]

\[ = f_l'(0, 0) + ik[f_l(0, 0) - 1] + o(k) \]

and

\[ \int_{-\infty}^0 dy e^{iky} Q(y) \psi(-k, y) = \int_{-\infty}^0 dy Q(y) \psi(0, y) + ik \int_{-\infty}^0 dy y Q(y) \psi(0, y) \]

\[ + \int_{-\infty}^0 dy [e^{iky} - 1 - iky] Q(y) \psi(0, y) \]

\[ + \int_{-\infty}^0 dy e^{iky} Q(y)[\psi(-k, y) - \psi(0, y)] \]

\[ = f_l'(0, 0) - ik[f_l(0, 0) - a] + o(k), \]
we obtain
\[
\bar{b}(k) = \frac{a^2 - 1}{2a} + o(1),
\] (4.15)
provided \( f_i(0,0) \neq 0 \).

If \( f_i(0,0) = 0 \), we apply a translation \( x \mapsto x + x_0 \) for some \( x_0 \in \mathbb{R} \) where \( f_i(0,x_0) \neq 0 \). Using tildes to denote the quantities pertaining to the potential \( \tilde{Q}(x) = Q(x + x_0) \), we get
\[
\tilde{f}_i(k,x) = e^{-ikx_0} f_i(k,x + x_0), \quad \tilde{f}_r(k,x) = e^{ikx_0} f_r(k,x + x_0),
\] (4.16a)
\[
\tilde{f}_i(0,x) = f_i(0,x), \quad \tilde{f}_r(0,x) = f_r(0,x),
\] (4.16b)
\[
\tilde{a}(k) = a(k), \quad \tilde{\alpha}(k) = \alpha(k),
\] (4.16c)
\[
\tilde{b}(k) = e^{2ikx_0} b(k), \quad \tilde{\tilde{b}}(k) = e^{-2ikx_0} \tilde{b}(k).
\] (4.16d)

Thus the translation carries the exceptional case into the exceptional case. Moreover,
\[
\tilde{a} = \lim_{x \to -\infty} \tilde{f}_i(0,x) = \lim_{x \to -\infty} f_i(0,x) = a.
\]

Consequently,
\[
a(k) = \tilde{a}(k) = \frac{1 + \tilde{a}^2}{2\tilde{a}} + o(1) = \frac{1 + a^2}{2a} + o(1),
\]
\[
\bar{b}(k) = e^{2ikx_0} \bar{b}(k) = e^{2ikx_0} \frac{\tilde{a}^2 - 1}{2\tilde{a}} + o(1) = e^{2ikx_0} \frac{a^2 - 1}{2a} + o(1) = \frac{a^2 - 1}{2a} + o(1).
\]

As a result, (4.13) and (4.15) are also true if \( f_i(0,0) = 0 \). Also,
\[
b(k) = -\bar{b}(-k) = \frac{1 - a^2}{2a} + o(1).
\] (4.17)

Finally,
\[
T(k) = \frac{2a}{1 + a^2}, \quad R(k) = \frac{a^2 - 1}{1 + a^2}, \quad L(k) = \frac{1 - a^2}{1 + a^2},
\] (4.18)
which completes the proof.

\[\square\]

### 4.2 Fourier representations

In this section we prove that the scattering coefficients belong to the Wiener algebra \( \mathcal{W} \). This has been established before in the generic case for \( Q \in L^1_1(\mathbb{R}) \) [cf. Proposition 3.2] and in the exceptional case for \( Q \in L^2_1(\mathbb{R}) \) [cf. Proposition 3.4]. Here we establish this result in the exceptional case for \( Q \in L^1_1(\mathbb{R}) \).
Equations (4.12) and (4.14) can be written in the form

$$-2ikf_1(0,0)a(k) = f_r(k,0) \left\{ -ikf_1(0,0) + f'_1(0,0) + \int_0^\infty dy e^{iky}Q(y)\psi(k,y) \right\}$$

$$-f_i(k,0) \left\{ ikf_1(0,0) + f'_1(0,0) - \int_0^\infty dy e^{-iky}Q(y)\psi(k,y) \right\},$$

$$2ikf_1(0,0)b(k) = f_r(-k,0) \left\{ -ikf_1(0,0) + f'_1(0,0) + \int_0^\infty dy e^{iky}Q(y)\psi(k,y) \right\}$$

$$-f_i(k,0) \left\{ f'_1(0,0) - ikf_1(0,0) - \int_0^\infty dy e^{iky}Q(y)\psi(-k,y) \right\},$$

where $Q \in L^1_1(\mathbb{R})$ and $\psi(k,x)$ as defined by (4.11) belongs to the Wiener algebra $W$. Using (2.9a) and (2.9d), these equation imply

$$f_i(0,0)a(k) = \frac{1}{2}f_r(k,0) \left\{ 1 - \int_0^\infty dy \frac{e^{iky} - 1 - iky}{ik}Q(y)\psi(k,y) \right\}$$

$$+ i \int_0^\infty dy e^{iky}Q(y)\frac{\psi(k,y) - \psi(k,0)}{ik}$$

$$+ \frac{1}{2}f_i(k,0) \left\{ a + \int_0^\infty dy \frac{1 - e^{-iky} - iky}{ik}Q(y)\psi(0,y) \right\}$$

$$- \int_0^\infty dy e^{iky}Q(y)\frac{\psi(k,y) - \psi(0,y)}{ik},$$

(4.19a)

$$f_i(0,0)b(k) = \frac{1}{2}f_r(-k,0) \left\{ -1 + \int_0^\infty dy \frac{e^{iky} - 1 - iky}{ik}Q(y)\psi(0,y) \right\}$$

$$- i \int_0^\infty dy e^{iky}Q(y)\frac{\psi(k,y) - \psi(0,y)}{ik}Q(y)\psi(0,y)$$

$$+ \frac{1}{2}f_i(k,0) \left\{ a + \int_0^\infty dy \frac{e^{iky} - 1 - iky}{ik}Q(y)\psi(0,y) \right\}$$

$$- \int_{-\infty}^0 dy e^{iky}Q(y)\frac{\psi(0,y) - \psi(-k,y)}{ik},$$

(4.19b)

**Lemma 4.4** Let $\psi(k,x)$ be a solution of the Schrödinger equation (1.1), where the initial conditions at $x = 0$ do not depend on $k$. Suppose that $\psi(0,x)$ is bounded and $Q \in L^1_1(\mathbb{R})$. Then

$$F(k) = \int_0^\infty dy e^{iky}Q(y)\frac{\psi(k,y) - \psi(0,y)}{ik}$$

belongs to the Wiener class $W$. 
4.2. FOURIER REPRESENTATIONS

Proof. It suffices to prove the existence of \( P(y; \cdot) \in L^1(\mathbb{R}) \) such that

\[
\int_{-\infty}^{\infty} dz \, e^{ikz} P(y, z) = \frac{\psi(k, y) - \psi(0, y)}{ik},
\]
uniformly in \( y \in \mathbb{R}^+ \). Indeed, in that case

\[
F(k) = \int_{-\infty}^{\infty} dw \, e^{ikw} \int_{-\infty}^{w} dz \, Q(w - z) P(y; z),
\]
which is the Fourier transform of a function with \( L^1 \)-norm bounded above by \( \|Q\|_1 \sup_{y \geq 0} \|P(y; \cdot)\|_1 \).

It is easily verified that, when divided by \( ik \), the first four terms in the right-hand side of (4.6) are Fourier transform of \( L^1 \) functions. Also, when divided by \( ik \), the last term on the right-hand side of (4.6) can be written as

\[
\int_{-\infty}^{\infty} dv \, e^{ikv} \int_{0}^{x} dy \, Q(y) \int_{v-2(x-y)}^{v} dz \, P(y; z),
\]
where

\[
\int_{-\infty}^{\infty} dv \, \left| \int_{0}^{x} dy \, Q(y) \int_{v-2(x-y)}^{v} dz \, P(y; z) \right| \leq 2 \sup_{y \geq 0} \|P(y; \cdot)\|_1 \sup_{x \geq 0} \int_{0}^{x} dy \, (x - y)|Q(y)|.
\]

By stripping off the Fourier transform, we now convert (4.6) into the integral equation

\[
P(x; v) = \Pi(x; v) + \int_{0}^{x} dy \, Q(y) \int_{v-2(x-y)}^{v} dz \, P(y; z)
\]

\[
= \Pi(x; v) + \int_{v-2x}^{v} dz \int_{0}^{x-\frac{1}{2}[v-z]} dy \, Q(y) P(y; z),
\]
where \( \sup_{x \geq 0} \|\Pi(x; \cdot)\|_1 \) is finite and the integral operator is bounded (and in fact quasinilpotent) on the complex Banach space of measurable functions \( P(x; v) \) for which \( \sup_{x \geq 0} \|P(x; \cdot)\|_1 \) is finite. As a result, \( F(k) \) belongs to \( \mathcal{W} \), which completes the proof.

Theorem 4.5 Suppose \( Q \in L^1_1(\mathbb{R}) \). Then the transmission coefficient \( T(k) \) and the reflection coefficients \( R(k) \) and \( L(k) \) belong to the Wiener class \( \mathcal{W} \).

Proof. It remains to prove Theorem 4.5 in the exceptional case. We investigate the Wiener algebra behavior of the various components of (4.19). First of all, \( f_l(k, 0) \), \( f_r(k, 0) \), and \( f_r(-k, 0) \) belong to \( \mathcal{W} \). Secondly,

\[
\int_{0}^{\infty} dz \, \frac{e^{iky} - 1}{ik} Q(y) \psi(0, y) = \int_{0}^{\infty} dw \, e^{ikw} \int_{0}^{\infty} dz \, Q(z) \psi(0, z)
\]
belongs to $\mathcal{W}$, because
\[
\int_0^\infty dw \left| \int_w^\infty dz Q(z) \psi(0, z) \right| \leq \|\psi(0, \cdot)\|_\infty \int_0^\infty dz |Q(z)| < +\infty.
\]

Finally, using Lemma 4.4, we now see that $f_l(0, 0)a(k)$ and $f_l(0, 0)b(k)$ belong to $\mathcal{W}$. Hence, $a(k)$ and $b(k)$ belong to $\mathcal{W}$ if $f_l(0, 0) \neq 0$. Using the translation (4.16) whenever $f_l(0, x_0) \neq 0$, we see that $\tilde{a}(k)$ and $\tilde{b}(k)$ belong to $\mathcal{W}$. Hence, $a(k)$ and $b(k)$ belong to $\mathcal{W}$ irrespective of the value of $f_l(0, 0)$.

Since $a(k) \neq 0$ for $k \in \mathbb{R}$, we conclude that $T(k)$, $R(k)$, and $L(k)$ belong to $\mathcal{W}$ [cf. (2.11)], as claimed. \hfill \blacksquare
Chapter 5

Marchenko Equations

In this chapter we derive the Marchenko integral equations and prove their unique solvability. We also study the discrete spectrum of the Schrödinger equation.

5.1 Riemann-Hilbert problem

From the Wronskian relation (2.9a) it is clear that the Jost solutions \( f_l(k, x) \) and \( f_r(k, x) \) are linearly independent for \( k \in \mathbb{R} \) in the generic case and for \( 0 \neq k \in \mathbb{R} \) in the exceptional case. For these values of \( k \) there exists a basis transformation

\[
\begin{pmatrix}
  f_l(-k, x) \\
  f_r(-k, x)
\end{pmatrix}
= \begin{pmatrix}
  \alpha_{11}(k) & \alpha_{12}(k) \\
  \alpha_{21}(k) & \alpha_{22}(k)
\end{pmatrix}
\begin{pmatrix}
  f_r(k, x) \\
  f_l(k, x)
\end{pmatrix},
\]

where the coefficients \( \alpha_{11}(k) \), \( \alpha_{12}(k) \), \( \alpha_{21}(k) \), and \( \alpha_{22}(k) \) are to be determined. Using the asymptotic properties of the Jost solutions as \( x \to +\infty \), we have

\[
\begin{align*}
1 &= \alpha_{11}(k)a(k), \\
0 &= \alpha_{11}(k)b(k) + \alpha_{12}(k), \\
b(-k) &= \alpha_{21}(k)a(k), \\
a(-k) &= \alpha_{21}(k)b(k) + \alpha_{22}(k).
\end{align*}
\]

Thus \( \alpha_{11}(k) = \frac{1}{a(k)} = T(k), \) \( \alpha_{12}(k) = -\frac{b(k)}{a(k)} = -R(k), \) \( \alpha_{21}(k) = \frac{b(-k)}{a(k)} = -\frac{\overline{b}(k)}{\overline{a}(k)} = -L(k), \) \( \alpha_{22}(k) = \frac{a(k) - b(-k)b(k)}{a(k)} = \frac{|a(k)|^2 - |b(k)|^2}{a(k)} = 1/a(k) = T(k). \) Consequently,

\[
\begin{pmatrix}
  f_l(-k, x) \\
  f_r(-k, x)
\end{pmatrix}
= \begin{pmatrix}
  T(k) & -R(k) \\
  -L(k) & T(k)
\end{pmatrix}
\begin{pmatrix}
  f_r(k, x) \\
  f_l(k, x)
\end{pmatrix},
\]
Taking into account the asymptotic behavior of the Jost solutions as $k \to \infty$, we have the so-called Riemann-Hilbert problem

$$\begin{pmatrix} e^{ikx} f_l(-k,x) \\ e^{-ikx} f_r(-k,x) \end{pmatrix} = \begin{pmatrix} T(k) & -e^{2ikx} R(k) \\ -e^{-2ikx} L(k) & T(k) \end{pmatrix} \begin{pmatrix} e^{ikx} f_r(k,x) \\ e^{-ikx} f_l(k,x) \end{pmatrix},$$

where $\sigma_3 = \text{diag}(1, -1)$ and

$$S(k, x) = \begin{pmatrix} e^{ikx} & 0 \\ 0 & e^{-ikx} \end{pmatrix} S(k) \begin{pmatrix} e^{-ikx} & 0 \\ 0 & e^{ikx} \end{pmatrix}.$$  

In the generic case the reflection and transmission coefficients belong to the Wiener algebra $W$ for $Q \in L^1_1(\mathbb{R})$ [cf. Corollary 3.3]. This result also holds in the exceptional case [cf. Proposition 3.4 for $Q \in L^2_2(\mathbb{R})$ and Chapter 4 for $Q \in L^1_1(\mathbb{R})$]. In other words,

$$R(k) = \int_{-\infty}^{\infty} d\alpha e^{-ik\alpha \hat{R}(\alpha)}, \quad L(k) = \int_{-\infty}^{\infty} d\alpha e^{ik\alpha \hat{L}(\alpha)},$$

where $\hat{R}, \hat{L} \in L^1(\mathbb{R})$.

### 5.2 Discrete eigenvalues

Let us now study the discrete eigenvalues of the Schrödinger equation (1.1).

**Theorem 5.1** The zeros of $a(k)$ are positive imaginary, simple, and finite in number. These zeros can be denoted as $ik_s$ ($s = 1, 2, \ldots, N$), where $\kappa_1 > \ldots > \kappa_N > 0$.

**Proof.** Let $a(k) = 0$ for $k = \gamma + ik \in \mathbb{C}^+$, where $\kappa > 0$. Since $f_l(k, x) = Cf_r(k, x)$ for a nonzero constant $C$, then

$$\begin{cases} f_l(k, x) \sim e^{i\gamma x} e^{-\kappa x}, & x \to +\infty, \\ f_l(k, x) = C e^{-i\gamma x} e^{\kappa x}, & x \to \infty, \end{cases}$$

and hence $f_l(k, x)$ decays exponentially as $x \to \pm \infty$. But then the Schrödinger (1.1) has a discrete eigenvalue $k^2$ which, due to its selfadjointness, must necessarily be negative. Hence $k$ is positive imaginary. The (geometric) simplicity of the eigenvalues follows from the fact that the Wronskian of two eigenfunctions corresponding to the same negative eigenvalue must necessarily vanish.
Observe that
\[
\left[ \frac{\partial}{\partial k} W[f_t(k, x), f_r(k, x)] \right]_{k = i\kappa_j} = \left[ \frac{\partial}{\partial k} [-2iak(k)] \right]_{k = i\kappa_j} = 2\kappa_j a'(i\kappa_j) - 2ia(i\kappa_j) = 2\kappa_j a'(i\kappa_j).
\]

Denoting partial differentiation with respect to \( k \) by an overdot, we get from the Schrödinger equation (1.1) the differential equation
\[
-\dot{\psi}''(k, x) + Q(x)\psi(k, x) = k^2 \dot{\psi}(k, x) + 2k\psi(k, x).
\]

Using this identity for \( \psi = f_t, r \) and \( k = i\kappa_j \), we obtain
\[
\left[ \frac{\partial}{\partial k} W[f_t(k, x), f_r(k, x)] \right]_{k = i\kappa_j} = W[f_t(i\kappa_j, x), f_r(i\kappa_j, x)] + W[f_l(i\kappa_j, x), \dot{f}_r(i\kappa_j, x)]
= -\int_x^\infty dy \frac{d}{dy} W[\dot{f}_l(i\kappa_j, y), f_r(i\kappa_j, y)] + \int_{-\infty}^x dy \frac{d}{dy} W[f_t(i\kappa_j, y), \dot{f}_r(i\kappa_j, y)]
= -\int_x^\infty dy (\dot{f}_l f'_r - f'_l \dot{f}_r) + \int_{-\infty}^x dy (f_l \dot{f}_r' - f'_l \dot{f}_r')
= -\int_x^\infty dy (\dot{f}_l f''_r - f''_l \dot{f}_r) + \int_{-\infty}^x dy (f_l \dot{f}_r'' - f''_l \dot{f}_r)
= -\int_x^\infty dy \left\{ f_l (Q + \kappa_j^2) f_r - [(Q + \kappa_j^2) \dot{f}_l - 2i\kappa_j f_l] f_r \right\}
+ \int_{-\infty}^x dy \left\{ f_l [(Q + \kappa_j^2) \dot{f}_r - 2i\kappa_j f_r] - (Q + \kappa_j^2) f_l \dot{f}_r \right\}
= -2i\kappa_j \left( \int_x^\infty + \int_{-\infty}^x \right) dy f_l f_r = -2i\kappa_j \int_{-\infty}^\infty dy f_l(i\kappa_j, x) f_r(i\kappa_j, x) \neq 0,
\]

because \( f_l(i\kappa_j, x) \) and \( f_r(i\kappa_j, x) \) are proportional and nontrivial functions belonging to \( L^2(\mathbb{R}) \). Consequently, \( a(i\kappa_j) = 0 \) and \( a'(i\kappa_j) \neq 0 \), so that \( i\kappa_j \) is a simple zero of \( a(k) \). Further,
\[
\int_{-\infty}^\infty dx f_l(i\kappa_j, x) f_r(i\kappa_j, x) = i a'(i\kappa_j) = \frac{i}{\tau_j}, \quad (5.3)
\]

where \( \tau_j \) is the residue of \( T(k) \) at \( k = i\kappa_j \).

Because \( a(k) \) is analytic in \( k \in \mathbb{C}^+ \), the zeros of \( a(k) \) can only be finite in number or accumulate at \( k = 0 \). In the generic case, where \( \lim_{k \to 0} ka(k) \neq 0 \), the number of zeros of \( a(k) \) must be finite. In the exceptional case we reach the same conclusion, because in this case \( \lim_{k \to 0} a(k) = [1/T(0)] \neq 0 \).
Theorem 5.1 implies that the transmission coefficient has the following representation:

\[ T(k) = T_0(k) + \sum_{s=1}^{N} \frac{\tau_s}{k - i\kappa_s}, \]  

(5.4)

where \( T_0(k) \) is continuous in \( k \in \mathbb{C}^+ \), is analytic in \( k \in \mathbb{C}^+ \), and tends to 1 as \( k \to \infty \) in \( \mathbb{C}^+ \).

There are no discrete eigenvalues if the potential is repulsive, i.e., if \( Q(x) \geq 0 \) for each \( x \in \mathbb{R} \). Indeed, if \( 0 \not\equiv \psi \in L^2(\mathbb{R}) \) is an eigenfunction corresponding to the bound state pole \( k = i\kappa \in \mathbb{C}^+ \), then

\[ \int_{-\infty}^{\infty} dx \, Q(x) |\psi(x)|^2 = -\kappa^2 \|\psi\|_2^2 - \|\psi'\|_2^2. \]

We arrive at a contradiction, because the left-hand side is nonnegative and the right-hand side is negative.

**Theorem 5.2 (Levinson)** Put \( T(k) = |T(k)|e^{i\delta(k)} \), where \( \delta(k) \) is continuous in \( 0 \not= k \in \mathbb{R} \). Then

\[ \delta(0^+) - \delta(+\infty) = \begin{cases} (N - \frac{1}{2})\pi, & \text{generic case}, \\ N\pi, & \text{exceptional case}, \end{cases} \]

where \( N \) is the number of discrete eigenvalues.

**Proof.** Given \( R > \varepsilon > 0 \), let \( \Gamma(R,\varepsilon) \) be the positively oriented contour consisting of \([-R,-\varepsilon], \{z \in \mathbb{C}^+: |z| = \varepsilon\}, [\varepsilon,R], \) and \( \{z \in \mathbb{C}^+: |z| = R\} \), where the poles \( i\kappa_s \) of the transmission coefficients all belong to \((i\varepsilon,iR)\). Then

\[ N = \frac{1}{2\pi i} \oint_{\Gamma(R,\varepsilon)} \frac{\dot{T}(k)}{T(k)} dk. \]

Because of (2.12a), the contribution to the contour integral of the arc \( \{z \in \mathbb{C}^+: |z| = R\} \) vanishes as \( R \to +\infty \), whereas as \( \varepsilon \to 0^+ \) that of \( \{z \in \mathbb{C}^+: |z| = \varepsilon\} \) vanishes in the exceptional case and tends to \(-\frac{1}{2}\) in the generic case. Using that \( T(-k) = T(k)^* \) for \( k \in \mathbb{R} \), we immediately obtain Levinson’s theorem.

It easily follows from the unitarity of the scattering matrix that

\[ |T(k)| = \sqrt{1 - |R(k)|^2} = \sqrt{1 - |L(k)|^2}, \quad k \in \mathbb{R}. \]

As a result \([4, XVII.1.63]\),

\[ T(k) = \left( \prod_{s=1}^{N} \frac{k + i\kappa_s}{k - i\kappa_s} \right) \lim_{\varepsilon \to 0^+} \exp \left\{ \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk}{k - k - i\varepsilon} \left[ \sqrt{1 - |R(k)|^2} \ln \left( \frac{k - k - i\varepsilon}{k - k + i\varepsilon} \right) \right] \right\}. \]
5.3 Marchenko equations: derivation

The Riemann-Hilbert problem (5.1) implies that
\[
\begin{align*}
 e^{ikx} f_l(-k, x) &= T(k) e^{ikx} f_r(k, x) - e^{2ikx} R(k) [e^{-ikx} f_l(k, x)] \\
 &= T_0(k) e^{ikx} f_r(k, x) + \sum_{s=1}^{N} \tau_s \frac{e^{ikx} f_r(k, x) - e^{-\kappa_s x} f_r(i\kappa_s, x)}{k - i\kappa_s} \\
 &\quad + i \sum_{s=1}^{N} N_s e^{-2\kappa_s x} [e^{\kappa_s x} f_l(i\kappa_s, x)] - e^{2ikx} R(k) [e^{-ikx} f_l(k, x)],
\end{align*}
\]

where we have employed (5.4) and \(N_s\) follows from the proportionality relation
\[
\tau_s f_r(i\kappa_s, x) = i N_s f_l(i\kappa_s, x) \quad (5.5)
\]

We call \(N_s (s = 1, 2, \ldots, N)\) the norming constants. Substituting (3.1) and (5.2) into the Riemann-Hilbert problem (5.1) and observing that the second line of the right-hand side allows the Fourier representation
\[
1 + \int_{-\infty}^{0} dw e^{-ikw} f(x, w)
\]
for a suitable function \(f(x, w)\), we have
\[
1 + \int_{-\infty}^{\infty} dw e^{-ikw} K(x, x + w) = 1 + \int_{-\infty}^{0} dw e^{-ikw} f(x, w)
\]
\[
= -\int_{-\infty}^{0} dw e^{-ikw} \sum_{s=1}^{N} N_s \left[ e^{-\kappa_s (w+2x)} + \int_{0}^{\infty} dz e^{-\kappa_s (w+z+2x)} K(x, x + z) \right]
\]
\[
= -\int_{-\infty}^{0} dw e^{-ikw} \left[ \hat{R}(w + 2x) + \int_{0}^{\infty} dz \hat{R}(w + z + 2x) K(x, x + z) \right].
\]

Restricting oneself to the terms \(\int_{0}^{\infty} dw e^{ikw} \ldots\) and stripping off the Fourier transforms, we obtain for \(w \geq 0\) the integral equation
\[
K(x, x + w) = -\sum_{s=1}^{N} N_s e^{-\kappa_s (w+2x)} + \int_{0}^{\infty} dz \sum_{s=1}^{N} N_s e^{-\kappa_s (w+z+2x)} K(x, x + z)
\]
\[- \hat{R}(w + 2x) - \int_{0}^{\infty} dz \hat{R}(w + z + 2x) K(x, x + z).
\]

Introducing the Marchenko kernel
\[
\Omega_l(w) = \hat{R}(w) + \sum_{s=1}^{N} N_s e^{-\kappa_s w}, \quad (5.6)
\]
we arrive at the Marchenko integral equation
\[
K(x, y) + \Omega_l(x + y) + \int_{x}^{\infty} dz K(x, z) \Omega_l(z + y) = 0, \quad y > x. \quad (5.7)
\]
The Riemann-Hilbert problem (5.1) also implies that
\[ e^{-ikx} f_r(-k, x) = T(k)[e^{-ikx} f_l(k, x)] - e^{-2ikx} L(k)[e^{ikx} f_r(k, x)] \]
\[ = T_0(k)[e^{-ikx} f_l(k, x)] + \sum_{s=1}^{N} \frac{e^{-ikx} f_l(k, x) - e^{\kappa_s x} f_l(i\kappa_s, x)}{k - i\kappa_s} \]
\[ + i \sum_{s=1}^{N} \tilde{N}_s e^{2\kappa_s x} \frac{e^{-\kappa_s x} f_r(i\kappa_s, x)}{k - i\kappa_s} - \frac{e^{-2ikx}}{L(k)}[e^{ikx} f_r(k, x)], \]
where
\[ \tau_s f_l(i\kappa_s, x) = i\tilde{N}_s f_r(i\kappa_s, x) \quad (5.8) \]
for certain norming constants \( \tilde{N}_s \) \((s = 1, 2, \ldots, N)\). Hence, \( \tilde{N}_s = -i[\tau_s^2/N_s] \).
Substituting (3.1) and (5.2) into the Riemann-Hilbert problem (5.1) and observing that the second line of the right-hand side allows the Fourier representation
\[ 1 + \int_{-\infty}^{0} dw e^{-ikw} g(x, w) \]
for a suitable function \( g(x, w) \), we have
\[ 1 + \int_{0}^{\infty} dw e^{-ikw} M(x, x - w) = 1 + \int_{-\infty}^{0} dw e^{-ikw} g(x, w) \]
\[ = -\int_{0}^{\infty} dw e^{-ikw} \sum_{s=1}^{N} \tilde{N}_s \left[ e^{-\kappa_s (w - 2x)} + \int_{0}^{\infty} dz e^{-\kappa_s (w + z - 2x)} M(x, x - z) \right] \]
\[ = -\int_{-\infty}^{0} dw e^{-ikw} \left[ \hat{L}(w - 2x) + \int_{0}^{\infty} dz \hat{L}(w + z - 2x) M(x, x - z) \right]. \]
Restricting oneself to the terms \( \int_{0}^{\infty} dw e^{ikw}[\ldots] \) and stripping off the Fourier transforms, we obtain for \( w \geq 0 \) the integral equation
\[ M(x, x - w) = -\sum_{s=1}^{N} \tilde{N}_s e^{-\kappa_s (w - 2x)} + \int_{0}^{\infty} dz \sum_{s=1}^{N} \tilde{N}_s e^{-\kappa_s (w + z - 2x)} M(x, x - z) \]
\[ - \hat{L}(w - 2x) - \int_{0}^{\infty} dz \hat{L}(w + z - 2x) M(x, x - z). \]
Introducing the Marchenko kernel
\[ \Omega_r(w) = \hat{L}(w) + \sum_{s=1}^{N} \tilde{N}_s e^{-\kappa_s w}, \quad (5.9) \]
we arrive at the Marchenko integral equation
\[ M(x, y) + \Omega_r(x + y) + \int_{-\infty}^{x} dz M(x, z) \Omega_r(z + y) = 0, \quad y < x. \quad (5.10) \]
We now derive an important auxiliary result.
Proposition 5.3  The norming constants $N_s$ and $\tilde{N}_s$ are positive. In fact,

$$
\int_{-\infty}^{\infty} dx f_l(i\kappa_s, x)^2 = \frac{1}{N_s}, \quad \int_{-\infty}^{\infty} dx f_r(i\kappa_s, x)^2 = \frac{1}{\tilde{N}_s}.
$$

(5.11)

Proof.  We easily see that

$$
-i\tau_s = -i\text{Res}_{k=\kappa_s} T(k) = \lim_{s \to \kappa_s} (s - \kappa_s)T(is)
$$

are real, because $T(k)$ is real-valued on the imaginary axis. Since the poles of $T(k)$ are sign changes and $T(is) \to 1$ as $s \to +\infty$, we get $(-1)^{s-1}[-i\tau_s] > 0$ ($s = 1, 2, \ldots, N$).

Next, the real-valued functions $f_l(i\kappa_s, x)$ are the eigenfunctions of the Schrödinger operator $-\partial_x^2 + Q$ at the consecutive eigenvalues $\kappa^2_1 < \kappa^2_2 < \ldots < \kappa^2_N < 0$. Thus $f_l(i\kappa_s, x)$ has precisely $s - 1$ zeros for $x \in \mathbb{R}$, all of them simple. Since $f_l(i\kappa_s, x) > 0$ as $x \to +\infty$ and $f_r(i\kappa_s, x) = C_s f_l(i\kappa_s, x) > 0$ as $x \to -\infty$, we see that $(-1)^{s-1}C_s > 0$ ($s = 1, 2, \ldots, N$). Consequently,

$$
N_s = [-i\tau_s]C_s > 0, \quad s = 1, 2, \ldots, N. \quad (5.12)
$$

Since $\tilde{N}_s = [-\tau^2_s]/N_s$ and $\tau^2_s < 0$, we get $\tilde{N}_s > 0$ ($s = 1, 2, \ldots, N$), as claimed.

Equations (5.11) are immediate from (5.3), (5.5), and (5.8), and provide an alternative proof of the positivity of the norming constants.  

Equation (5.3) implies that $\int_{-\infty}^{\infty} dx f_l(i\kappa_s, x)f_r(i\kappa_s, x)$ is a nonzero real number of sign $(-1)^{N-s}$. This is easily understood by taking into account that $T(k)$ is real for positive imaginary $k$, positive for $(-ik) > \kappa_N$ [because $T(\infty) = 1$], and changes its sign at each of the values $i\kappa_s$ ($s = 1, 2, \ldots, N$). Since all zeros of $\psi_l(i\kappa_s, x)$ are simple, we see that $\psi_l(i\kappa_s, x)$ has an even [odd] number of zeros if $N-s$ is even [odd]. This is the only oscillation property we need in order to prove the positivity of the norming constants.

We have utilized an important result originating from Sturm [1803-1855]:

Proposition 5.4  Suppose $\kappa_s > \kappa_\sigma > 0$. Then between any two consecutive zeros of $f_l(i\kappa_s, x)$ there is at least one zero of $f_l(i\kappa_\sigma, x)$. Furthermore, to the left [right] of the smallest [largest] zero of $f_l(i\kappa_s, x)$ there is at least one zero of $f_l(i\kappa_\sigma, x)$.

Proof.  To prove this fact, we let $\psi_s(x)$ and $\psi_\sigma(x)$ be two (real-valued) solutions to the Schrödinger equations

$$
-\psi''_s + Q\psi_s = -\kappa^2_s\psi_s, \quad -\psi''_\sigma + Q\psi_\sigma = -\kappa^2_\sigma\psi_\sigma.
$$
CHAPTER 5. MARCHENKO EQUATIONS

Multiplying the first equation by $-\psi$ and the second by $\psi$ and adding the resulting equations we obtain

$$\left(\psi'\psi' - \psi\psi'\right)' = (\kappa^2_\sigma - \kappa^2)\psi\psi',$$

which is our basic identity. Now assume that $x_1$ and $x_2$ are consecutive zeros of $\psi$; without loss of generality, $\psi(x_1) = \psi(x_2) = 0$ and $\psi(x) > 0$ for $x_1 < x < x_2$. Then $\psi'(x_1) > 0$ and $\psi'(x_2) < 0$. If $\psi$ does not change sign for $x \in (x_1, x_2)$ [i.e., if $\psi(x) \geq 0$ for $x \in [x_1, x_2]$, without loss of generality], then

$$0 < (\kappa^2 - \kappa^2_x) \int_{x_1}^{x_2} dx \psi \psi' = [\psi\psi' - \psi\psi']_{x_1} = [\psi\psi']_{x_1}^x = \psi(x_2)\psi'(x_2) - \psi(x_1)\psi'(x_1) \leq 0,$$

which is a contradiction. Thus $\psi$ changes sign on $[x_1, x_2]$; as a result, it must have at least one zero in $(x_1, x_2)$, as claimed. The proof also works if $x_1 = -\infty$ or if $x_2 = +\infty$.

5.4 Marchenko equations: unique solvability

We now prove the unique solvability of the solution of the Marchenko equations (5.7) and (5.10).

**Theorem 5.5** Suppose $\hat{R} \in L^1(\mathbb{R})$ and that $R(k)$ given by (5.2) either satisfies (a) $|R(k)| < 1$ for $k \in \mathbb{R}$, or (b) $|R(k)| < 1$ for $0 \neq k \in \mathbb{R}$ and $R(0) = -1$. Define $\Omega(x)$ by (5.6), where the norming constants $N_s$ ($s = 1, 2, \ldots, N$) are positive. Then for each $x_0 \in \mathbb{R}$ the Marchenko integral equation (5.7) has a unique solution $K(x, y)$ satisfying

$$\sup_{x \geq x_0} \int_x^\infty dy |K(x, y)| < +\infty.$$

**Proof.** According to Proposition B.1, it suffices to prove the unique solvability on $L^2(x, +\infty)$. In the absence of discrete eigenvalues, where the Marchenko kernel is $\hat{R}(x + y)$, this is immediate from (B.4a). Indeed, in the exceptional case we have

$$\sup_{k \in \mathbb{R}} |R(k)e^{2ikx}| < 1,$$

whereas in the generic case there exists a sufficiently small $\epsilon > 0$ such that

$$\sup_{k \in \mathbb{R}} |R(k)e^{2ikx} + \epsilon| < 1.$$

In the presence of discrete eigenvalues, we use the positivity of the norming constants to prove that the Marchenko operator has its eigenvalues within $(-1, +\infty)$, so that the Marchenko equation is uniquely solvable. ▫
Theorem 5.6 Suppose \( \hat{L} \in L^1(\mathbb{R}) \) and that \( L(k) \) given by (5.2) either satisfies (a) \( |L(k)| < 1 \) for \( k \in \mathbb{R} \), or (b) \( |L(k)| < 1 \) for \( 0 \neq k \in \mathbb{R} \) and \( L(0) = -1 \). Define \( \Omega_{r}(x) \) by (5.9), where the norming constants \( \tilde{N}_s \) \((s = 1, 2, \ldots, N)\) are positive. Then for each \( x_0 \in \mathbb{R} \) the Marchenko integral equation (5.10) has a unique solution \( M(x, y) \) satisfying

\[
\sup_{x \leq x_0} \int_{-\infty}^{x} dy |M(x, y)| < +\infty.
\]

Proof. According to Proposition B.1 it suffices to prove the unique solvability on \( L^2(-\infty, x) \). In the absence of discrete eigenvalues, where the Marchenko kernel is \( \hat{L}(x + y) \), this is immediate from (B.4b). Indeed, in the exceptional case we have

\[
\sup_{k \in \mathbb{R}} |L(k)e^{-2ikx}| < 1,
\]

whereas in the generic case there exists a sufficiently small \( \varepsilon > 0 \) such that

\[
\sup_{k \in \mathbb{R}} |L(k)e^{-2ikx} + \varepsilon| < 1.
\]

In the presence of discrete eigenvalues, we use the positivity of the norming constants to prove that the Marchenko operator has its eigenvalues within \((-1, +\infty)\), so that the Marchenko equation is uniquely solvable. \( \blacksquare \)
Chapter 6

PDE’s for Auxiliary Functions

In this section we derive smoothness properties of the auxiliary function $K(x,y)$ under the assumption of smoothness of the potential. We then go on to derive the time evolution of the scattering data if the potential is time dependent and satisfies the Korteweg-de Vries (KdV) equation.

6.1 Smoothness of auxiliary functions

Recall the integral equation [cf. (3.11)]

\[
B_l(x, \alpha) = \frac{1}{2} \int_{x + \frac{1}{2} \alpha}^{\infty} d\hat{y} Q(\hat{y}) + \frac{1}{2} \int_0^\alpha dz \int_{x + \frac{1}{2} (\alpha - z)}^{\infty} d\hat{y} Q(\hat{y}) B_l(\hat{y}, z).
\] (6.1)

It is known [cf. Theorem 3.1] that \(\sup_{x \geq x_0} \int_0^\infty d\alpha |B_l(x, \alpha)| < +\infty\) whenever \(Q \in L_1^1(\mathbb{R})\). It is also known that the solution \(B_r(x, \alpha) = M(x, x - \alpha)\) of the integral equation

\[
B_r(x, \alpha) = \frac{1}{2} \int_{-\infty}^{x - \frac{1}{2} \alpha} d\hat{y} Q(\hat{y}) + \frac{1}{2} \int_0^\alpha dz \int_{-\infty}^{x - \frac{1}{2} (\alpha - z)} d\hat{y} Q(\hat{y}) B_r(\hat{y}, z)
\] (6.2)

satisfies \(\sup_{x \leq x_0} \int_0^\infty d\alpha |B_r(x, \alpha)| < +\infty\) whenever \(Q \in L_1^1(\mathbb{R})\).

Following Tanaka [20], put

\[
B_l^{(j,0)}(x, \alpha) = \frac{\partial^j}{\partial x^j} \frac{\partial}{\partial \alpha^l} B_l(x, \alpha),
\]

where the derivative may be distributional. Then (6.1) implies that

\[
B_l^{(1,0)}(x, \alpha) = -\frac{1}{2} Q(x + \frac{1}{2} \alpha) - \frac{1}{2} \int_0^\alpha dz Q(x + \frac{1}{2} (\alpha - z)) B_l(x + \frac{1}{2} (\alpha - z), z),
\] (6.3a)
\[ B_i^{(0,1)}(x, \alpha) = -\frac{1}{8}Q(x + \frac{1}{2}\alpha) + \frac{1}{2} \int_x^{\infty} d\hat{y} Q(\hat{y}) B_i(\hat{y}, \alpha) \]

\[ \quad - \frac{1}{4} \int_0^x dz Q(x + \frac{1}{2}(\alpha - z)) B_i(x + \frac{1}{2}(\alpha - z), z), \quad (6.3b) \]

so that

\[ B_i^{(1,0)}(x, \alpha) - 2B_i^{(0,1)}(x, \alpha) = -\int_x^{\infty} d\hat{y} Q(\hat{y}) B_i(\hat{y}, \alpha), \quad (6.3c) \]

where we point out the weak \(x\)-differentiability of the right-hand side of \((6.3c)\). It is easily verified that the right-hand sides of \((6.3a)-(6.3c)\) are integrable with respect to \(\alpha \in \mathbb{R}^+\) uniformly in \(x \geq x_0\) if \(Q \in L^1_1(\mathbb{R})\). Indeed, \((6.3a)\) implies that

\[ \|B_i^{(1,0)}(x, \cdot)\|_1 \leq \int_x^{\infty} dy |Q(y)| \{1 + \|B_i(x, \cdot)\|_1\}, \]

which, in combination with \((3.6a)\), leads to the above conclusion. In the same way we get

\[ \|B_i^{(0,1)}(x, \cdot)\|_1 \leq \frac{1}{2} \int_x^{\infty} dy |Q(y)| \{1 + 2\|B_i(y, \cdot)\|_1\}. \]

Similar results hold for \(B_i(x, \alpha)\).

Assuming \(Q \in L^1_1(\mathbb{R})\) and \(Q' \in L^1(\mathbb{R})\), we obtain

\[ B_i^{(2,0)}(x, \alpha) = -\frac{1}{8}Q'(x + \frac{1}{2}\alpha)\]

\[ \quad - \frac{1}{2} \int_0^x dz \left[ Q'(x + \frac{1}{2}(\alpha - z)) B_i(x + \frac{1}{2}(\alpha - z), z) \right. \]

\[ \left. + Q(x + \frac{1}{2}(\alpha - z)) B_i^{(1,0)}(x + \frac{1}{2}(\alpha - z), z) \right], \]

\[ B_i^{(1,1)}(x, \alpha) = -\frac{1}{8}Q'(x + \frac{1}{2}\alpha) - \frac{1}{2}Q(x) B_i(x, \alpha) \]

\[ \quad - \frac{1}{4} \int_0^x dz \left[ Q'(x + \frac{1}{2}(\alpha - z)) B_i(x + \frac{1}{2}(\alpha - z), z) \right. \]

\[ \left. + Q(x + \frac{1}{2}(\alpha - z)) B_i^{(1,0)}(x + \frac{1}{2}(\alpha - z), z) \right], \]

\[ B_i^{(0,2)}(x, \alpha) = -\frac{1}{8}Q'(x + \frac{1}{2}\alpha) - \frac{1}{2}Q(x) B_i(x, \alpha) - \frac{1}{4}\int_x^{\infty} d\hat{y} Q(\hat{y}) \]

\[ \quad - \frac{1}{8} \int_0^x dz \left[ Q'(x + \frac{1}{2}(\alpha - z)) B_i(x + \frac{1}{2}(\alpha - z), z) \right. \]

\[ \left. + Q(x + \frac{1}{2}(\alpha - z)) B_i^{(1,0)}(x + \frac{1}{2}(\alpha - z), z) \right], \]

so that

\[ B_i^{(2,0)}(x, \alpha) - 2B_i^{(1,1)}(x, \alpha) = Q(x) B_i(x, \alpha), \quad (6.4a) \]

\[ B_i^{(1,1)}(x, \alpha) - 2B_i^{(0,2)}(x, \alpha) = \frac{1}{2} B_i^{(0,1)}(x, \alpha) \int_x^{\infty} d\hat{y} Q(\hat{y}). \quad (6.4b) \]
6.1. SMOOTHNESS OF AUXILIARY FUNCTIONS

It is easily verified that for \( Q \in L^1_1(\mathbb{R}) \) and \( Q' \in L^1(\mathbb{R}) \) we have

\[
\int_0^\infty d\alpha \left( |B_1^{(2,0)}(x,\alpha)| + |B_1^{(1,1)}(x,\alpha)| + |B_1^{(0,2)}(x,\alpha)| \right) < +\infty. \tag{6.5}
\]

Similar results hold for \( B_1(x,\alpha) \).

Differentiating (6.4), we get

\[
\begin{align*}
B_1^{(3,0)}(x,\alpha) - 2B_1^{(2,1)}(x,\alpha) &= Q'(x)B_1(x,\alpha) + Q(x)B_1^{(1,0)}(x,\alpha), \tag{6.6a} \\
B_1^{(2,1)}(x,\alpha) - 2B_1^{(1,2)}(x,\alpha) &= \frac{1}{2} B_1^{(1,1)}(x,\alpha) \int_x^\infty dy Q(y) \\
&\quad - \frac{1}{2} Q(x)B_1^{(0,1)}(x,\alpha), \tag{6.6b} \\
B_1^{(1,2)}(x,\alpha) - 2B_1^{(0,3)}(x,\alpha) &= \frac{1}{2} B_1^{(0,2)}(x,\alpha) \int_x^\infty dy Q(y), \tag{6.6c}
\end{align*}
\]

which implies (6.7) below for \( N = 2 \). The separate terms \( B_1^{(j,3-j)}(x,\alpha) \) require that \( Q \in L^1_1(\mathbb{R}) \) and \( Q', Q'' \in L^1(\mathbb{R}) \). In fact, we have for \( B_1^{(3,0)}(x,\alpha) \)

\[
B_1^{(3,0)}(x,\alpha) = -\frac{1}{2} Q''(x + \frac{1}{2} \alpha) \\
- \frac{1}{2} \sum_{j=0}^2 \binom{2}{j} \int_0^\alpha dz Q^j(x + \frac{1}{2}(\alpha - z)) B_1^{(2-j,0)}(x + \frac{1}{2}(\alpha - z), z).
\]

The following result is due to Tanaka [20, Lemma 1], although he did not give the details of the induction proof.

**Theorem 6.1** Suppose \( Q \in L^1_1(\mathbb{R}) \) and \( Q^{(j)} \in L^1(\mathbb{R}) \) for \( j = 1, 2, \ldots, N \). Then

\[
\sum_{j=0}^{N+1} \int_0^\infty d\alpha \left( |B_1^{(j,N+1-j)}(x,\alpha)| + |B_1^{(j,N-j)}(x,\alpha)| \right) < +\infty. \tag{6.7}
\]

**Proof.** Suppose

\[
\begin{align*}
B_1^{(N,0)}(x,\alpha) &= -\frac{1}{2} Q^{(N-1)}(x + \frac{1}{2} \alpha) \tag{6.8a} \\
&\quad - \frac{1}{2} \sum_{j=0}^{N-1} \binom{N-1}{j} \int_0^\alpha dz Q^j(x + \frac{1}{2}(\alpha - z)) B_1^{(2-j,0)}(x + \frac{1}{2}(\alpha - z), z), \\
B_1^{(N-1-j,j)}(x,\alpha) - 2B_1^{(N-2-j,j+1)}(x,\alpha) &= C_{N-1-j,j}(x,\alpha), \tag{6.8b}
\end{align*}
\]

where \( j = 0, 1, \ldots, N - 1 \). For \( N = 2, 3 \) the highest order derivative of \( Q(x) \) appearing in the right-hand side of (6.8b) is \( Q^{(N-1)}(x) \). Differentiating (6.8a)
with respect to \( x \), we get (6.8a) with \( N \) replaced by \( N + 1 \). Differentiating (6.8b) with respect to \( x \) we find the recurrence relation

\[
C_{N-j,j}(x, \alpha) = \frac{\partial}{\partial x} C_{N-1-j,j}(x, \alpha),
\]

where the highest order derivative of \( Q(x) \) appearing in the right-hand side is \( Q^{(N)}(x) \). Differentiating (6.8b) with respect to \( \alpha \) we obtain

\[
C_{N-1-j,j+1}(x, \alpha) = \frac{\partial}{\partial \alpha} C_{N-1-j,j}(x, \alpha).
\]

A similar proof holds for \( B_r(x, \alpha) \).

**Corollary 6.2** Suppose \( Q \in L^1_1(\mathbb{R}) \). Then for each \( x_0 \in \mathbb{R} \) the derivatives of the Marchenko kernels \( \Omega_l \) and \( \Omega_r \) belong to \( L^1(x_0, +\infty) \) and \( L^1(-\infty, x_0) \), respectively. In other words, the Marchenko kernels are absolutely continuous.

**Proof.** Differentiating (5.7) with respect to \( x \) and using (3.2) we get

\[
\frac{\partial K}{\partial x}(x, y) + \Omega_l'(x+y) + \int_x^\infty dz \frac{\partial K}{\partial x}(x, z) \Omega_l(z+y) - \frac{1}{2} \int_x^\infty dw Q(w) \Omega_l(x+y) = 0.
\]

As \( (\partial K/\partial x)(x, \cdot) \in L^1(x, +\infty) \) and \( \Omega_l \in L^1(2x, +\infty) \), we get \( \Omega_l' \in L^1(2x, +\infty) \), as claimed. The proof for \( \Omega_r' \) is similar.

### 6.2 Supports of auxiliary functions

In Sec. 3.3 we have discussed the special properties of the auxiliary functions \( K(x, y) \) and \( M(x, y) \) in the case where the potential \( Q(x) \) is compactly supported. This situation has been described in a concised way by Fig. 3.2 and Corollary 3.8. The support of \( K(x, y) \) is the set \( \{(x, y) \in \mathbb{R}^2 : x \leq y \leq 2M - x\} \) and \( K(x, y) \) only depends on \( y - x \) if \( x \leq y \leq 2L \). Similarly, the support of \( M(x, y) \) is the set \( \{(x, y) \in \mathbb{R}^2 : 2L - x \leq y \leq x\} \) and \( M(x, y) \) only depends on \( x - y \) if \( 2M - x \leq y \leq x \). In this section we apply (6.3c) to dramatically improve these results. We shall use the functions \( B_l(x, \alpha) = K(x, x + \alpha) \) and \( B_r(x, \alpha) = M(x, x - \alpha) \) instead of \( K(x, y) \) and \( M(x, y) \).

**Theorem 6.3** Suppose \( Q \in L^1_{\text{loc}}(\mathbb{R}) \) is supported on \([L, M]\). Then \( K(x, y) \) is supported on the set

\[
\{(x, y) : x \leq L, x \leq y \leq x + 2(M - L)\} \cup \{(x, y) : L \leq x \leq M, x \leq y \leq 2M - x\},
\]

while \( K(x, y) \) only depends on \( y - x \) if \( x \leq L \). On the other hand, \( M(x, y) \) is supported on the set

\[
\{(x, y) : x \geq M, x + 2(L - M) \leq y \leq x\} \cup \{(x, y) : L \leq x \leq M, 2L - x \leq y \leq x\},
\]

while \( M(x, y) \) only depends on \( x - y \) if \( x \geq M \).
Figure 6.1: If the potential $Q(x)$ is supported in $[L, M]$, then the cyan+magenta area indicates the support of $K(x, y)$ [left pane] and $M(x, y)$ [right pane]. In the magenta area, $K(x, y)$ [left pane] and $M(x, y)$ [right pane] only depend on $y - x$.

**Proof.** Assume that the real potential $Q \in L^1_1(\mathbb{R})$ is supported on $[L, M]$. Then (6.3c) implies that

$$
\frac{\partial}{\partial x} B_l(x, \alpha) - 2 \frac{\partial}{\partial \alpha} B_l(x, \alpha) = \begin{cases} 
0, & x \geq M, \\
- \int_x^M dz Q(z) B_l(z, \alpha), & L \leq x \leq M, \\
H(\alpha), & x \leq L,
\end{cases}
$$

(6.9)

where $H(\alpha) = - \int_L^M dz Q(z) B_l(z, \alpha)$. Then, for $x \geq M$, $B_l(x, \alpha)$ is an absolutely continuous function of $2x + \alpha$ which vanishes identically [see Theorem 3.6]. For $x \leq L$ we get

$$
B_l(x, \alpha) = F(2x + \alpha) + \frac{1}{2} \int_\alpha^\infty d\beta H(\beta),
$$

(6.10)

where $F$ is an absolutely continuous one-variable function and

$$
\int_0^\infty d\alpha |H(\alpha)| \leq \int_L^M dz |Q(z)| \int_0^\infty d\alpha |B_l(z, \alpha)| \\
\leq \int_L^M dz |Q(z)| \int_z^M dw (w - z) |Q(w)| e^{\int_z^M dv (v-z)|Q(v)|},
$$

which is finite. Thus $H \in L^1(\mathbb{R}^+)$. Moreover,

$$
\int_0^\infty d\alpha \left| \frac{1}{2} \int_\alpha^\infty d\beta H(\beta) \right| \leq \frac{1}{2} \int_0^\infty d\beta \int_0^\beta d\alpha |H(\beta)| = \frac{1}{2} \int_0^\infty d\beta |H(\beta)| \\
\leq \frac{1}{2} \int_L^M dz |Q(z)| \int_0^\infty d\beta |B_l(z, \beta)| < +\infty,
$$
where the last inequality follows directly from the estimates contained in the proof of Theorem 3.5.

Using the proof of Corollary 3.8 we get for \( x + y < 2L \)

\[
K(x, y) = \begin{cases} 
\frac{1}{2} \int_{L}^{M} dw Q(w) \left[ 1 + \int_{w}^{w+y-x} ds K(w, s) \right] \\
F(x + y) - \frac{1}{2} \int_{L}^{M} dw Q(w) \int_{w+y-x}^{\infty} ds K(w, s),
\end{cases}
\]

which implies that

\[
F(x + y) = \frac{1}{2} \int_{L}^{M} dw Q(w) \left[ 1 + \int_{w}^{\infty} ds K(w, s) \right]
\]

is a constant function. Returning to the domain \( \{(x, y) \in \mathbb{R}^2 : x \leq L, y \geq x\} \) we obtain

\[
K(x, y) = \frac{1}{2} \int_{L}^{M} dw Q(w) \left[ 1 + \int_{w}^{\infty} ds K(w, s) \right] \\
- \frac{1}{2} \int_{L}^{M} dw Q(w) \int_{w+y-x}^{\infty} ds K(w, s) \\
= \frac{1}{2} \int_{L}^{M} dw Q(w) \left[ 1 + \int_{w}^{w+y-x} ds K(w, s) \right],
\]

which only depends on \( y - x \), as claimed. A similar proof holds for \( M(x, y) \). \( \blacksquare \)

### 6.3 Time evolution according to the KdV

Let \( Q_N \) be the real Banach space of potentials \( Q \in L^1(\mathbb{R}) \) for which the functions \( Q', Q'', \ldots, Q^{(N)} \in L^1(\mathbb{R}) \), endowed with the norm

\[
\|Q\|_{Q_N} = \sum_{j=0}^{N} \int_{-\infty}^{\infty} dx |Q^{(j)}(x)|.
\]

For \( Q \in Q_N \) we consider the linear operators

\[
L_Q = -\partial_x^2 + Q, \quad B_Q = -4\partial_x^3 + 3Q\partial_x + 3\partial_x Q,
\]

where \( \partial_x = (d/dx) \). Then we easily derive the Lax pair equation

\[
\frac{d}{dt} L_Q = [B_Q, L_Q], \quad (6.11)
\]
6.3. TIME EVOLUTION ACCORDING TO THE KDV

because \([B_Q, L_Q]\) is the multiplication by \(6QQ_x - Q_{xxx}\). Equation \((6.11)\) is equivalent to the KdV equation

\[ Q_t - 6QQ_x + Q_{xxx} = 0. \tag{6.12} \]

In this equation the time derivative is to be computed in the strong sense in the real Banach space \(L^1(\mathbb{R})\), while \(Q \in Q_3\). In that case \(Q_x\) is continuous and vanishes as \(x \to \pm \infty\), so that the left-hand side of \((6.12)\) is a continuous function of \(t\) with values in \(L^1(\mathbb{R})\).

Consider the identity [cf. (5.1), with the help of (2.11)]

\[ f_r(k, x; t) = a(k; t) f_i(-k, x; t) + b(k; t) f_i(k, x; t), \tag{6.13} \]

where

\[ 2ika(k; t) = W[f_r(k, x; t), f_i(k, x; t)], \quad 2ikb(k; t) = W[f_i(-k, x; t), f_r(k, x; t)], \]

(cf. (2.9a) and (2.9e)). On the other hand,

\[ L_Q f_i(k, \cdot) = k^2 f_i(k, \cdot), \quad L_Q f_r(k, \cdot) = k^2 f_r(k, \cdot), \]

so that

\[ L_Q \left( \frac{df_i}{dt} - B_Q f_i \right) = \frac{d}{dt} (L_Q f_i) - [B_Q, L_Q] f_i - L_Q B_Q f_i \]

\[ = \frac{d}{dt} (L_Q f_i) - B_Q L_Q f_i \]

\[ = \frac{d}{dt} (k^2 f_i) - B_Q (k^2 f_i) = k^2 \left( \frac{df_i}{dt} - B_Q f_i \right), \]

and similarly for \(f_r\). Since \(f_i(k, x; t) = e^{ikx}[1 + o(1)] = \) as \(x \to +\infty\) and \(f_r(k, x; t) = e^{-ikx}[1 + o(1)] \) as \(x \to -\infty\), we see that

\[ \frac{d}{dt} f_i(k, x; t) - (B_Q f_i)(k, x; t) = -4ik^3 f_i(k, x; t), \tag{6.14a} \]

\[ \frac{d}{dt} f_r(k, x; t) - (B_Q f_r)(k, x; t) = +4ik^3 f_r(k, x; t), \tag{6.14b} \]

we obtain by differentiation \((6.11)\) with respect to \(t\)

\[ \frac{d}{dt} f_r(k, x; t) = B_Q \left( a(k; t) f_i(-k, x; t) + b(k; t) f_i(k, x; t) \right) \]

\[ + 4ik^3 \left( a(k; t) f_i(-k, x; t) + b(k; t) f_i(k, x; t) \right) \]

\[ + \frac{da}{dt}(k; t) f_i(-k, x; t) + \left( \frac{db}{dt}(k; t) - 8ik^3 b(k; t) \right) f_i(k, x; t). \]
Comparing the right-hand side of the latter equation with the right-hand side of (6.14b) and using (6.11), we get
\[
\frac{da}{dt}(k; t) = 0, \quad \frac{db}{dt}(k; t) = 8ik^3b(k; t).
\]
(6.15)

In other words,
\[
T(k; t) = T(k; 0), \quad R(k; t) = e^{8ik^3t}R(k; 0), \quad L(k; t) = e^{-8ik^3t}L(k; 0).
\]
(6.16)

Equations (6.16) can be written in the form
\[
S(k; t) = e^{4ik^3t\sigma_3}S(k; 0)e^{-4ik^3t\sigma_3},
\]
where \(\sigma_3 = \text{diag}(1, -1)\). Then
\[
S_t = 4ik^3\{\sigma_3S - S\sigma_3\},
\]
so that \((S(k; t), 4ik^3\sigma_3)\) is a Lax pair.

Let us now derive the time evolution of the norming constants. First of all, since \(f_l(i\kappa_s, x; t)\) and \(f_r(i\kappa_s, x; t)\) are real-valued and \(B_Q\) is a skew-selfadjoint operator on \(L^2(\mathbb{R})\), we get
\[
\int_{-\infty}^{\infty} dx f_l(i\kappa_s, x; t)(B_Qf_l)(i\kappa_s, x; t) = \langle f_l, B_Qf_l \rangle = -\langle B_Qf_l, f_l \rangle = -\langle f_l, B_Qf_l \rangle,
\]
which must vanish. Differentiating (5.11) with respect to \(t\), using (6.14), and employing \(\langle f_l, B_Qf_l \rangle = \langle f_r, B_Qf_r \rangle = 0\), we obtain
\[
N_s(t) = e^{8\kappa_3^3t}N_s(0), \quad \tilde{N}_s(t) = e^{-8\kappa_3^3t}\tilde{N}_s(0).
\]
(6.17)

Let us return to (6.16). Using (5.2) we get
\[
\tilde{R}(\alpha; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ika}R(k; t)
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ika}e^{8ik^3t}R(k; 0)
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ika}e^{8ik^3t} \int_{-\infty}^{\infty} d\beta e^{-ik\beta}\tilde{R}(\beta; 0)
\]
\[
= \int_{-\infty}^{\infty} d\beta \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik(\alpha - \beta) + 8ik^3t} \right] \tilde{R}(\beta; 0)
\]
\[
= \int_{-\infty}^{\infty} d\beta \left[ \frac{1}{\pi} \int_{0}^{\infty} dk \cos[k(\alpha - \beta) + 8k^3t] \right] \tilde{R}(\beta; 0)
\]
\[
= \int_{-\infty}^{\infty} d\beta \left[ (24t)^{-1/3}Ai[(24t)^{-1/3}(\alpha - \beta)] \right] \tilde{R}(\beta; 0),
\]
where \( t > 0 \). In the same way we compute

\[
\hat{L}(\alpha; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{-ika} L(k; t)
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{-ika} e^{-8ik^3t} L(k; 0)
\]
\[
= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{-ika} e^{-8ik^3t} \int_{-\infty}^{\infty} d\beta \ e^{ik\beta} \hat{L}(\beta; 0)
\]
\[
= \int_{-\infty}^{\infty} d\beta \left[ \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \ e^{-ik(\alpha - \beta) - 8ik^3t} \right] \hat{L}(\beta; 0)
\]
\[
= \int_{-\infty}^{\infty} d\beta \left[ \frac{1}{\pi} \int_{0}^{\infty} dk \ \cos[k(\alpha - \beta) + 8k^3t] \right] \hat{L}(\beta; 0)
\]
\[
= \int_{-\infty}^{\infty} d\beta \left[ (24t)^{-1/3} \text{Ai}[(24t)^{-1/3}(\alpha - \beta)] \right] \hat{L}(\beta; 0),
\]

where \( t > 0 \). Here a well-known identity for the Airy functions [1, 10.4.33] has been used. In fact, the time evolution of \( \hat{R}(\alpha; t) \) and \( \hat{L}(\alpha; t) \) proceeds by applying an Airy transform [21, Eq. (4.16) for \( \alpha = (24t)^{1/3} \)].
Chapter 7

Darboux transformations

Let \( Q(x) \) be a Faddeev class potential, i.e., a real potential belonging to \( L^1_1(\mathbb{R}) \). Then there exist zero or finitely many bound states which correspond to the (positive imaginary and simple) poles of \( T(k) \). These poles we denote by \( i\kappa_j \) \( (j = 1, \ldots, N) \), where \( 0 < \kappa_1 < \ldots < \kappa_N \). In this chapter we develop a procedure to add or delete bound state poles called the Darboux transformation. We roughly follow [5].

**Proposition 7.1** Suppose \( f \) and \( u \) are solutions of the Schrödinger equations

\[
-f'' + Qf = k^2 f, \quad -u'' + Qu = \mu^2 u,
\]

where \( \mu \in \mathbb{C}^+ \) satisfies \((-i\mu) > (-i\kappa_N)\). Then

\[
f^T = f' - \left(\frac{u'}{u}\right)f
\]

is a solution of the transformed Schrödinger equation

\[
-(F^T)'' + \tilde{Q}f^T = k^2 f^T,
\]

where

\[
\tilde{Q} = Q - 2 \left(\frac{u'}{u}\right)'
\]

is a Faddeev class potential having one more bound state pole, namely at \( k = \mu \).
Proof. We compute

\[(F^T)' = f''' - u'u''f' - 2\left[\frac{u'}{u} - \frac{(u')^2}{u^2}\right] f' - \left[\frac{u''}{u} - 3\frac{u'u''}{u^2} + \frac{2(u')^3}{u^3}\right] f\]

\[= -(k^2 - Q)f' + Qf + \frac{u'}{u} (k^2 - Q)f - 2\left[-(\mu^2 - Q) - \frac{(u')^2}{u^2}\right] f'\]

\[= \left[-(\mu^2 - Q)\frac{u'}{u} + Q' + 3\frac{u'}{u} (\mu^2 - Q) + 2\frac{(u')^3}{u^3}\right] f\]

\[= \left[2(\mu^2 - Q) - (k^2 - Q) + 2\frac{(u')^2}{u^2}\right] \left\{ f' - \frac{u'}{u} f\right\}\]

\[= \left[-k^2 + Q - 2\left(\frac{u'}{u}\right)\right] \left\{ f' - \frac{u'}{u} f\right\} = -(k^2 - \tilde{Q}) f^T,\]

as claimed.
Appendix A

Gronwall’s Inequality

In this section we prove two versions of Gronwall’s inequality. The proof of Theorem A.2 can be found in wikipedia under “Gronwall’s inequality.”

**Theorem A.1 (Gronwall)** Suppose \( \alpha(x) \) and \( \beta(x) \) are nonnegative functions, \( \alpha(x) \) is nonincreasing, and

\[
0 \leq u(x) \leq \alpha(x) + \int_{x}^{\infty} dy \beta(y)u(y). \tag{A.1}
\]

Then

\[
0 \leq u(x) \leq \alpha(x)e^{\int_{x}^{\infty} dz \beta(z)}. \tag{A.2}
\]

**Proof.** Put

\[
v(y) = \left(e^{-\int_{y}^{x} dz \beta(z)}\right)\int_{y}^{\infty} dz \beta(z)u(z).
\]

Then

\[
-v'(y) = \beta(y) \left(u(y) - \int_{y}^{\infty} dz \beta(z)u(z)\right)e^{-\int_{y}^{x} dz \beta(z)} \leq \alpha(y)\beta(y)e^{-\int_{y}^{x} dz \beta(z)}.
\]

Thus

\[
v(x) = -\int_{x}^{\infty} dy v'(y) \leq \int_{x}^{\infty} dy \alpha(y)\beta(y)e^{-\int_{y}^{x} dz \beta(z)}.
\]

Substituting this inequality into (A.1) we obtain

\[
0 \leq u(x) \leq \alpha(x) + \int_{x}^{\infty} dy \beta(y)u(y)
\]

\[
\leq \alpha(x) + e^{\int_{x}^{\infty} dz \beta(z)}v(x)
\]

\[
\leq \alpha(x) + e^{\int_{x}^{\infty} dz \beta(z)}\int_{x}^{\infty} dy \alpha(y)\beta(y)e^{-\int_{y}^{x} dz \beta(z)}
\]

\[
\leq \alpha(x) \left\{1 + \int_{x}^{\infty} dy \beta(y)e^{-\int_{y}^{x} dz \beta(z)}\right\}
\]

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\[= \alpha(x) \left\{ 1 + \left[ e^{\int_y^x dz \beta(z)} \right]_y^\infty \right\} = \alpha(x)e^{\int_x^\infty dz \beta(z)}, \]

as claimed.

The proof of the next theorem can be found in Wikipedia under “Gronwall’s inequality.”

**Theorem A.2 (Gronwall)** Suppose \( \alpha(x) \) and \( \beta(x) \) are nonnegative functions, \( \alpha(x) \) is nondecreasing, and

\[0 \leq u(x) \leq \alpha(x) + \int_0^x dy \beta(y) u(y). \tag{A.3}\]

Then

\[0 \leq u(x) \leq \alpha(x)e^{\int_0^x dz \beta(z)}. \tag{A.4}\]

**Proof.** Put

\[v(y) = \left( e^{-\int_0^y dz \beta(z)} \right) \int_0^y dz \beta(z) u(z). \]

Then

\[v'(y) = \beta(y) \left\{ u(y) - \int_0^y dz \beta(z) u(z) \right\} e^{-\int_0^y dz \beta(z)} \leq \alpha(y)\beta(y)e^{\int_0^y dz \beta(z)}. \]

Thus

\[v(x) = \int_0^x dy v'(y) \leq \int_0^x dy \alpha(y)\beta(y)e^{\int_0^y dz \beta(z)}. \]

Substituting this inequality into (A.3) we obtain

\[0 \leq u(x) \leq \alpha(x) + \int_0^x dy \beta(y) u(y) \leq \alpha(x) + e^{\int_0^x dz \beta(z)} v(x) \leq \alpha(x) + e^{\int_0^x dz \beta(z)} \int_0^x dy \alpha(y)\beta(y)e^{\int_0^y dz \beta(z)} \leq \alpha(x) \left\{ 1 + \int_0^x dy \beta(y)e^{\int_0^x dz \beta(z)} \right\} \]

\[= \alpha(x) \left\{ 1 - \left[ e^{\int_y^x dz \beta(z)} \right]_y^\infty \right\} = \alpha(x)e^{\int_0^x dz \beta(z)}, \]

as claimed.
Gronwall’s inequality is applied as follows. Consider the integral equation
\[ U(x) = A(x) + \int_{x}^{\infty} dy \, B(x, y) U(y), \] (A.5)
where
\[ \|A(x)\| \leq \alpha(x), \quad \|B(x, y)\| \leq \beta(y), \]
and \( \alpha(x) \) and \( \beta(y) \) satisfy the conditions of Theorem A.1. Consider the iterates
\[ U_0(x) = A(x), \quad U_{n+1}(x) = A(x) + \int_{x}^{\infty} dy \, B(x, y) U_n(y). \]
Then
\[ \|U_n(x)\| \leq \alpha(x) \sum_{j=0}^{n} \frac{1}{j!} \left( \int_{x}^{\infty} dz \, \beta(z) \right)^j. \]
Indeed, the estimate is obviously true for \( n = 0 \). Assuming it to be true for a certain \( n \), we perform the following estimate:
\[
\|U_{n+1}(x)\| \leq \|A(x)\| + \int_{x}^{\infty} dy \, \|B(x, y)\| \|U_n(y)\|
\leq \alpha(x) + \int_{x}^{\infty} dy \, \beta(y) \sum_{j=0}^{n} \frac{\alpha(y)}{j!} \left( \int_{y}^{\infty} dz \, \beta(z) \right)^j
\leq \alpha(x) + \alpha(x) \int_{x}^{\infty} dy \sum_{j=0}^{n} \frac{\beta(y)}{j!} \left( \int_{y}^{\infty} dz \, \beta(z) \right)^j
= \alpha(x) + \alpha(x) \sum_{j=0}^{n} \left[ \frac{-1}{(j+1)!} \left( \int_{y}^{\infty} dz \, \beta(z) \right)^{j+1} \right]_{y=x}^{y=\infty}
= \alpha(x) \sum_{j=0}^{n+1} \frac{1}{j!} \left( \int_{x}^{\infty} dz \, \beta(z) \right)^j,
\]
as claimed. Note that we have used that \( \alpha(x) \) is nonincreasing. As a result, the integral equation (A.5) is uniquely solvable (by iteration) and
\[ \|U(x)\| \leq \alpha(x) \exp \left( \int_{x}^{\infty} dz \, \beta(z) \right). \]
In the same way we prove, under the conditions of Theorem A.2, that the integral equation
\[ U(x) = A(x) + \int_{0}^{x} dy \, B(x, y) U(y), \]
where

$$\|A(x)\| \leq \alpha(x), \quad \|B(x, y)\| \leq \beta(y),$$

is uniquely solvable (by iteration) and

$$\|U(x)\| \leq \alpha(x) \exp \left( \int_0^x dz \beta(z) \right).$$
Appendix B

Marchenko operators

Given \( x \in \mathbb{R} \), we define by \( E_l[x] \) a suitable complex Banach space of functions \( f : (x, +\infty) \to \mathbb{C} \). We are particularly thinking of the spaces \( L^p(x, +\infty) \) (\( 1 \leq p \leq +\infty \)), \( BC[x, +\infty) \), and \( C_0[x, +\infty) \), equipped with their natural norms. By \( AC[x, +\infty) \) we denote the complex Banach space of all continuous functions \( f : [x, +\infty) \to \mathbb{C} \) whose almost everywhere existing derivative \( f' \) belongs to \( L^1(x, +\infty) \), endowed with the norm

\[
\|f\|_{AC} = \sup_{y \geq x} |f(y)| + \int_x^\infty dy |f'(y)|.
\]

In the same way, given \( x \in \mathbb{R} \), we define by \( E_r[x] \) a suitable complex Banach space of functions \( f : (-\infty, x) \to \mathbb{C} \). Then \( AC(-\infty, x] \) is the complex Banach space of all continuous functions \( f : (-\infty, x] \to \mathbb{C} \) whose almost everywhere existing derivative \( f' \) belongs to \( L^1(-\infty, x] \), endowed with the norm

\[
\|f\|_{AC} = \sup_{y \leq x} |f(y)| + \int_{-\infty}^x dy |f'(y)|.
\]

**Proposition B.1** Suppose \( \omega \in L^1(2x, +\infty) \) and let the space identifier \( E \) stand for one of \( L^p \) (\( 1 \leq p \leq +\infty \)), \( BC \), \( C_0 \), or \( AC \). Then the Marchenko integral operator \( K^x_\omega \) defined by

\[
(K^x_\omega f)(y) = \int_x^\infty dz \omega(y + z)f(z)
\]

is compact on \( E_l[x] \) and its nonzero spectrum and the Jordan structure of each nonzero eigenvalue do not depend on the choice of function space. Analogously, if \( \omega \in L^1(-\infty, 2x) \), then the Marchenko integral operator \( M^x_\omega \) defined by

\[
(M^x_\omega f)(y) = \int_{-\infty}^x dz \omega(y + z)f(z)
\]
APPENDIX B. MARCHENKO OPERATORS

is compact on \( E_r[x] \) and its nonzero spectrum and the Jordan structure of each nonzero eigenvalue do not depend on the choice of function space.

Proof. It is easy to bound the operator norm of \( K_\omega^{[x]} \) on \( L^1(x, +\infty) \) and \( L^\infty(x, +\infty) \) above by \( \int_{2x}^\infty dz |\omega(z)| \) and to bound the operator norm of \( M_\omega^{[x]} \) on \( L^1(-\infty, x) \) and \( L^\infty(-\infty, x) \) above by \( \int_{-\infty}^{2x} dz |\omega(z)| \). In fact, these expressions for the operator norms on \( L^1 \) and \( L^\infty \) are exact\(^1\) By the M. Riesz interpolation theorem [13], we get the same norm upper bound of \( K_\omega^{[x]} \) for the operator norms on \( L^p(-\infty, x) \), but for \( 1 < p < +\infty \) the norm estimate is usually strict. Next, for each \( f \in L^\infty(x, +\infty) \) we have

\[
|\( (K_\omega^{[x]} f)(y_1) - (K_\omega^{[x]} f)(y_2) \) | \leq \int_x^\infty dz |\omega(y_1 + z) - \omega(y_2 + z)| |f(z)| \leq \|f\|_\infty \int_x^\infty dz |\omega(y_1 + z) - \omega(y_2 + z)|;
\]

a simple dominated convergence argument implies that \( K_\omega^{[x]} \) maps \( L^\infty(x, +\infty) \) into \( C_0[x, +\infty) \). In a similar way we prove that \( M_\omega^{[x]} \) maps \( L^\infty(-\infty, x) \) into \( C_0(-\infty, x) \). In the case of \( E = L^2 \), the norm upper bounds can be improved to

\[
\|K_\omega^{[x]}\| \leq \sup_{\lambda \in \mathbb{R}} |\hat{\omega}(\lambda)|, \quad \|M_\omega^{[x]}\| \leq \sup_{\lambda \in \mathbb{R}} |\hat{\omega}(\lambda)|, \tag{B.1}
\]

by using the Fourier transform \( \hat{\omega}(\lambda) = \int_{-\infty}^\infty dz e^{iz\lambda}\omega(z) \).

Now let \( f \in AC[x, +\infty) \). Then for almost every \( y \in [x, +\infty) \) we have \( f' \in L^1(x, +\infty) \), \( K_\omega^{[x]} f \in BC[x, +\infty) \) and \( K_\omega^{[x]} f' \in L^1(x, +\infty) \). Also, for almost every \( y \in [x, +\infty) \) we have

\[
(K_\omega^{[x]} f)'(y) = -\omega(x + y) f(x) - \int_x^\infty dz \omega(y + z) f'(z).
\]

Therefore,

\[
\|(K_\omega^{[x]} f)'\|_1 \leq \int_{2x}^\infty dy |\omega(y)| \{\|f\|_{BC} + \|f'\|_1\} = \left( \int_{2x}^\infty dy |\omega(y)| \right) \|f\|_{AC},
\]

which proves the boundedness of \( K_\omega^{[x]} \) on \( AC[x, +\infty) \).

It is easily verified that

\[
\int_x^\infty dy \int_x^\infty dz |\omega(y + z)|^2 = \int_{2x}^\infty dw (w - 2x) |\omega(w)|^2. \tag{B.2}
\]

\(^1\)If \( (Kf)(x,y) = \int_E d\mu(y) k(x,y) f(y) \), then its norms on \( L^1(E, d\mu) \) and \( L^\infty(E, d\mu) \) are given by \( \text{ess sup}_{y \in E} \int_E d\mu(x) |k(x,y)| \) and \( \text{ess sup}_{x \in E} \int_E d\mu(y) |k(x,y)| \), respectively [cf. [7] VI.9.54].
So, if this integral is finite, the operator $K_\omega^{[x]}$ is Hilbert-Schmidt and hence compact on $L^2(x, +\infty)$. By approximating $K_\omega^{[x]}$ with arbitrary $L^1$ kernel by such Hilbert-Schmidt operators, we prove that $K_\omega^{[x]}$ is a compact operator on $L^2(x, +\infty)$. In a similar way we prove that $M_\omega^{[x]}$ is a compact operator on $L^2(-\infty, x)$. By compact interpolation \[12\], we then prove that, for $1 < p < +\infty$, $K_\omega^{[x]}$ is compact on $L^p(x, +\infty)$ and $M_\omega^{[x]}$ is compact on $L^p(-\infty, x)$.

Next, it is obvious that $K_\omega^{[x]}$ has separated variables if
\[\omega(y + z) = Ce^{-(y+z)A}B\] (B.3)

for some complex $p \times p$ matrix $A$ having only eigenvalues with positive real part, some complex $p \times 1$ matrix $B$, and some complex $1 \times p$ matrix $C$. Thus in this case $K_\omega^{[x]}$ is a compact operator on $L^p(x, +\infty)$ ($1 \leq p < +\infty$) and, because the Banach dual of $K_\omega^{[x]}$ is $K_\omega^{[x]}$ itself, also on $L^\infty(x, +\infty)$; by taking restrictions such $K_\omega^{[x]}$ is compact on $BC[x, +\infty)$ and $C_0[x, +\infty)$ as well. Since functions of the type (B.3) are dense in $L^1(2x, +\infty)$ [17 Sec. 7.3.2], a simple approximation argument yields that, for general $L^1$ kernels, $K_\omega^{[x]}$ is a compact operator on any of the above $E_\lambda[x]$. In the same way we prove that $M_\omega^{[x]}$ is a compact operator on any of the above $E_\lambda[x]$. The final statement of the proposition can be based on the following:

Let $E_1$ and $E_2$ be two complex Banach spaces, where $E_2$ is continuously and densely imbedded in $E_1$. Suppose $F$ is a Fredholm operator of index zero on $E_1$ such that $F[E_2] \subseteq E_2$. Suppose also that the restriction of $F$ to $E_2$ is a Fredholm operator of index zero on $E_2$. Then these two operators $F$ are both invertible or both noninvertible.

For $m = 1, 2, 3, \ldots$ and $\lambda \in \mathbb{C}$, this property can now be applied to $F = [I - \lambda K_\omega^{[x]}]^m$ and $F = [I - \lambda M_\omega^{[x]}]^m$, where the class of space identifiers contains $L^p$ ($1 \leq p < +\infty$), $BC$, $C_0$, $AC$ and the intersection of any two such spaces (endowed with the sum of the norms). These operators $F$ are obviously Fredholm of index zero on $E_\lambda[x]$ and $E_\lambda[x]$ for $E$ as in the statement of this proposition, but also on the intersection of any two such spaces. To extend this result to $E = L^\infty$ we use the fact that $F$ maps $L^\infty$ continuously into $C_0$. \[\square\]

**Corollary B.2** The Marchenko operator $K_\omega^{[x]}$ is Hilbert-Schmidt iff the integral $[B.2]$ converges.

To compare $K_\omega^{[x]}$ for various $x \in \mathbb{R}$ we apply shift operators to define the Marchenko operators on $E_\lambda[0]$ and $E_\lambda[0]$. In that case $K_\omega^{[x]}$ and $M_\omega^{[x]}$ are to be

\[2\text{According to Kronecker’s theorem [19], this is the necessary and sufficient condition for } K_\omega^{[x]} \text{ to have finite rank.}\]
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replaced by

\[
\tilde{K}_x^{[x]} f(y) = \int_0^\infty dz \omega(2x + y + z)f(z),
\]

\[
\tilde{M}_x^{[x]} f(y) = \int_{-\infty}^0 dz \omega(2x + y + z)f(z).
\]

but the translational similarity applied to modify the operators does not affect their boundedness and compactness properties nor does it affect their spectra. It is now quite obvious that \( \tilde{K}_x^{[x]} \) depends on \( x \in \mathbb{R} \) in the operator norm (on each \( E_0(0) \)) and vanishes in the operator norm as \( x \to +\infty \). Analogously, \( \tilde{M}_x^{[x]} \) depends on \( x \in \mathbb{R} \) in the operator norm (on each \( E_r(0) \)) and vanishes in the operator norm as \( x \to -\infty \). Defining the singular values (or: approximation numbers or s-numbers \([10]\)) in the usual way for \( E = L^2 \), we see that the \( N \)-th singular values of \( K_x^{[x]} \) and \( M_x^{[x]} \) depend continuously on \( x \in \mathbb{R} \) and vanish as \( x \to +\infty \) and \( x \to -\infty \), respectively.

The norm estimates (B.1) can easily be optimized by using Nehari's theorem [18]. Working with the operators \( \tilde{K}_x^{[x]} \) on \( L^2(0, +\infty) \) and \( \tilde{M}_x^{[x]} \) on \( L^2(-\infty, 0) \), we get the following two commutative diagrams:

\[
\begin{array}{cccccccc}
L^2(0, +\infty) & \xrightarrow{\pi_+^1} & L^2(\mathbb{R}) & \xrightarrow{\sigma} & L^2(\mathbb{R}) & \xrightarrow{\omega(2x+\ldots)^*} & L^2(\mathbb{R}) & \xrightarrow{\pi_+} & L^2(0, +\infty) \\
\downarrow F & & \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F \\
H^2(\mathbb{C}^+) & \xrightarrow{h_+^1} & L^2(\mathbb{R}) & \xrightarrow{\sigma} & L^2(\mathbb{R}) & \xrightarrow{e^{-2(\cdot)\tau \omega(\cdot)}} & L^2(\mathbb{R}) & \xrightarrow{h_+} & H^2(\mathbb{C}^+)
\end{array}
\]

and

\[
\begin{array}{cccccccc}
L^2(-\infty, 0) & \xrightarrow{\pi_+^1} & L^2(\mathbb{R}) & \xrightarrow{\sigma} & L^2(\mathbb{R}) & \xrightarrow{\omega(2x+\ldots)^*} & L^2(\mathbb{R}) & \xrightarrow{\pi_-} & L^2(-\infty, 0) \\
\downarrow F & & \downarrow F & & \downarrow F & & \downarrow F & & \downarrow F \\
H^2(\mathbb{C}^-) & \xrightarrow{h_+^1} & L^2(\mathbb{R}) & \xrightarrow{\sigma} & L^2(\mathbb{R}) & \xrightarrow{e^{-2(\cdot)\tau \omega(\cdot)}} & L^2(\mathbb{R}) & \xrightarrow{h_-} & H^2(\mathbb{C}^-)
\end{array}
\]

Here \( \tilde{K}_x^{[x]} \) is represented by the top line of the first diagram and \( \tilde{M}_x^{[x]} \) is represented by the top line of the second diagram. The Fourier transform is denoted by \( F \), \( H^2(\mathbb{C}^\pm) = F[L^2(\mathbb{R}^\pm)] \) are the Hardy spaces, \( \pi_\pm : L^2(\mathbb{R}) \to L^2(\mathbb{R}^\pm) \) and \( h_\pm : L^2(\mathbb{R}) \to H^2(\mathbb{C}^\pm) \) are natural projections, \( \pi_+^1 : L^2(\mathbb{R}^+) \to L^2(\mathbb{R}) \) and \( h_+^1 : H^2(\mathbb{C}^+) \to L^2(\mathbb{R}) \) are natural imbeddings, \( \sigma \) is sign inversion [i.e., \( (\sigma \phi)(x) = \phi(-x) \)], the asterisk denotes convolution, and the superscript of the third arrow in the bottom line of either diagram the premultiplication by
\( e^{-2i(x\hat{\omega}(\cdot))} \). Since \( \pi_{\pm}, h_{\pm}, \pi_{\pm}^*, h_{\pm}^* \), \( \sigma \), and \((2\pi)^{-1/2}F\) have unit norm, we immediately have (B.1). The diagrams allow us to sharpen the norm estimates on \( L^2(\mathbb{R}^\pm) \) to the exact expressions

\[
\|K^{[\overline{x}]_{\omega}}\|_{L^2(\mathbb{R}^+)} = \|\tilde{K}^{[\overline{x}]_{\omega}}\|_{L^2(\mathbb{R}^+)} = \text{dist} \left( e^{-2i(x\hat{\omega}(\cdot)}, H_{\infty}^- \right), \quad (B.4a)
\]

\[
\|M^{[\overline{x}]_{\omega}}\|_{L^2(\mathbb{R}^-)} = \|\tilde{M}^{[\overline{x}]_{\omega}}\|_{L^2(\mathbb{R}^-)} = \text{dist} \left( e^{-2i(x\hat{\omega}(\cdot)}, H_{\infty}^+ \right), \quad (B.4b)
\]

where \( H_{\infty}^\pm \) are the complex Banach spaces of those functions belonging to \( L^\infty(\mathbb{R}) \) which are a.e. boundary values of bounded analytic functions on \( \mathbb{C}^\pm \). In (B.4) the distances are measured in the supremum norm.
Appendix C

Resolvent formulas

In this chapter we compute the resolvent operator of $-\partial_x^2 + Q$ on the complex Hilbert space $L^2(\mathbb{R})$. For later use, we let $\theta(k, x)$ and $\varphi(k, x)$ be the solutions of the Schrödinger equation which satisfy the initial conditions

$$
\begin{align*}
\theta(k, 0) &= 1, & \theta'(k, 0) &= 0, \\
\varphi(k, 0) &= 0, & \varphi'(k, 0) &= 1.
\end{align*}
$$

1. Full line. Given $g \in L^2(\mathbb{R})$ we consider the inhomogeneous Schrödinger equation

$$
-\psi''(k, x) + Q(x)\psi(k, x) = k^2 \psi(k, x) - g(x), \quad (C.1)
$$

where $k \in \mathbb{C}^+$. Using the method of variation of constants we write

$$
\psi(k, x) = c_1(x)f_l(k, x) + c_2(x)f_r(k, x),
$$

where $k$ is not a pole of the transmission coefficient. Then

$$
\begin{pmatrix}
  f_l(k, x) & f_r(k, x) \\
  f'_l(k, x) & f'_r(k, x)
\end{pmatrix}
\begin{pmatrix}
  c'_1(x) \\
  c'_2(x)
\end{pmatrix}
= \begin{pmatrix} 0 \\ g(x) \end{pmatrix}.
$$

Thus

$$
\begin{pmatrix}
  c'_1(x) \\
  c'_2(x)
\end{pmatrix}
= \frac{1}{-2ika(k)}
\begin{pmatrix}
  f'_r(k, x) & -f_r(k, x) \\
  -f'_l(k, x) & f_l(k, x)
\end{pmatrix}
\begin{pmatrix} 0 \\ g(x) \end{pmatrix}
= \frac{g(x)}{-2ika(k)}
\begin{pmatrix}
  -f_r(k, x) \\
  f_l(k, x)
\end{pmatrix}.
$$

Therefore,

$$
\begin{align*}
c_1(x) &= c_1 + \frac{1}{2ika(k)} \int_{-\infty}^{x} dy f_r(k, y) g(y), \\
c_2(x) &= c_2 + \frac{1}{2ika(k)} \int_{x}^{\infty} dy f_l(k, y) g(y).
\end{align*}
$$
Choosing $c_1 = c_2 = 0$ to arrive at a function $\psi(k, \cdot) \in L^2(\mathbb{R})$, we get

$$\psi(k, x) = \int_{-\infty}^{\infty} dy \mathcal{G}(k; x, y) g(y),$$  \hspace{1cm} (C.2)

where the Green’s function is given by

$$\mathcal{G}(k; x, y) = \frac{1}{2ika(k)} \begin{cases} f_l(k, x) f_r(k, y), & y \leq x, \\ f_r(k, x) f_l(k, y), & y \geq x. \end{cases} \hspace{1cm} (C.3)$$

Letting $i\kappa_1, \ldots, i\kappa_N$ stand for the (simple) poles of the transmission coefficient, we see that

$$\text{Res}_{k=i\kappa_s} \mathcal{G}(k; x, y) = -\frac{1}{2\kappa_s a(i\kappa_s)} \gamma_s f_l(i\kappa_s, x) f_l(i\kappa_s, y),$$

where $\gamma_s = [f_r(i\kappa_s, x)/f_l(i\kappa_s, x)]$. Using (5.5) and (5.8) we get

$$\text{Res}_{k=i\kappa_s} \mathcal{G}(k; x, y) = \frac{1}{2i\kappa_s} N_s f_l(i\kappa_s, x) f_l(i\kappa_s, y) = \frac{1}{2i\kappa_s} \tilde{N}_s f_r(i\kappa_s, x) f_r(i\kappa_s, y).$$

Consequently,

$$\text{Res}_{k^2=-\kappa^2_s} \mathcal{G}(k; x, y) = \lim_{k^2 \to -\kappa^2_s} (k^2 + \kappa^2_s) \mathcal{G}(k; x, y)$$

$$= N_s f_l(i\kappa_s, x) f_l(i\kappa_s, y)$$

$$= \tilde{N}_s f_r(i\kappa_s, x) f_r(i\kappa_s, y).$$  \hspace{1cm} (C.4)

Hence, the (orthogonal) projections onto the eigenspaces are given by

$$(Ps g)(x) = N_s f_l(i\kappa_s, x) \int_{-\infty}^{\infty} dy f_l(i\kappa_s, y) g(y)$$

$$= \tilde{N}_s f_r(i\kappa_s, x) \int_{-\infty}^{\infty} dy f_r(i\kappa_s, y) g(y),$$

where $\sqrt{N_s} f_l(i\kappa_s, x) = (-1)^{s-1} \sqrt{\tilde{N}_s} f_r(i\kappa_s, x)$ are normalized eigenfunctions.

2. **Positive half-line.** Given $g \in L^2(\mathbb{R}^+)$ we consider the following boundary value problem for the inhomogeneous Schrödinger equation

$$\begin{cases} -\psi''(k, x) + Q(x) \psi(k, x) = k^2 \psi(k, x) - g(x), \\ [\cos \alpha] \psi(k, 0) = [\sin \alpha] \psi'(k, 0), \end{cases}$$  \hspace{1cm} (C.5)

\[\text{Note that } k^2 \text{ is the spectral variable in (1.1).}\]
where $k \in \mathbb{C}^+$. Using the method of variation of constants we write

$$
\psi(k, x) = c_1(x) f_1(k, x) + c_2(x) [(\sin \alpha) \theta(k, x) + (\cos \alpha) \varphi(k, x)],
$$

where the Wronskian of $f_1(k, x)$ and $(\sin \alpha) \theta(k, x) + (\cos \alpha) \varphi(k, x)$ does not vanish. Then

$$
\begin{pmatrix}
  f_1(k, x) & (\sin \alpha) \theta(k, x) + (\cos \alpha) \varphi(k, x) \\
  f'_1(k, x) & (\sin \alpha) \theta'(k, x) + (\cos \alpha) \varphi'(k, x)
\end{pmatrix}
\begin{pmatrix}
  c'_1(x) \\
  c'_2(x)
\end{pmatrix}
= \begin{pmatrix}
  0 \\
  g(x)
\end{pmatrix},
$$

where

$$
\det
\begin{pmatrix}
  f_1(k, x) & (\sin \alpha) \theta(k, x) + (\cos \alpha) \varphi(k, x) \\
  f'_1(k, x) & (\sin \alpha) \theta'(k, x) + (\cos \alpha) \varphi'(k, x)
\end{pmatrix}
= f_1(k, 0) (\cos \alpha - f'_1(k, 0) \sin \alpha).
$$

Then

$$
\begin{aligned}
&\begin{pmatrix}
  c'_1(x) \\
  c'_2(x)
\end{pmatrix}
= \frac{1}{f_1(k, 0) \cos \alpha - f'_1(k, 0) \sin \alpha} \\
&\times \begin{pmatrix}
  (\sin \alpha) \theta'(k, x) + (\cos \alpha) \varphi'(k, x) & - (\sin \alpha) \theta(k, x) - (\cos \alpha) \varphi(k, x) \\
  - f'_1(k, x) & f_1(k, x)
\end{pmatrix}
\begin{pmatrix}
  0 \\
  g(x)
\end{pmatrix}
\end{aligned}
= \frac{g(x)}{f_1(k, 0) \cos \alpha - f'_1(k, 0) \sin \alpha} \begin{pmatrix}
  - (\sin \alpha) \theta(k, x) - (\cos \alpha) \varphi(k, x) \\
  f_1(k, x)
\end{pmatrix}.
$$

Therefore,

$$
c_1(x) = c_1 - \int_x^\infty dy \frac{([\sin \alpha] \theta(k, y) + (\cos \alpha) \varphi(k, y)] g(y)}{f_1(k, 0) \cos \alpha - f'_1(k, 0) \sin \alpha},
$$

$$
c_2(x) = c_2 - \int_x^\infty dy \frac{f_1(k, y) g(y)}{f_1(k, 0) \cos \alpha - f'_1(k, 0) \sin \alpha}.
$$

We then easily derive that

$$
\psi(k, 0) \cos \alpha - \psi'(k, 0) \sin \alpha = c_1 [f_1(k, 0) \cos \alpha - f'_1(k, 0) \sin \alpha],
$$

so that $c_1 = 0$. In order that $\psi(k,) \in L^2(\mathbb{R}^+)$ for those $k \in \mathbb{C}^+$ for which $f_1(k, 0) \cos \alpha - f'_1(k, 0) \sin \alpha \neq 0$, we require $c_2 = 0$. As a result,

$$
\psi(k, x) = \int_0^\infty dy \mathcal{G}_+(k; x, y) g(y),
$$

where the Green’s function is given by

$$
\mathcal{G}_+(k; x, y) = \frac{-1}{f_1(k, 0) \cos \alpha - f'_1(k, 0) \sin \alpha} \times \begin{cases}
  f_1(k, x) [\theta(k, y) \sin \alpha + \varphi(k, y) \cos \alpha], & 0 \leq y \leq x, \\
  [\theta(k, x) \sin \alpha + \varphi(k, x) \cos \alpha] f_1(k, y), & 0 \leq x \leq y.
\end{cases}
$$
The function

\[ S_\alpha(k) = f_l(k, 0) \cos \alpha - f'_l(k, 0) \sin \alpha \]

is called the \( S \)-matrix. Its zeros correspond to the discrete eigenvalues.

**3. Negative half-line.** Given \( g \in L^2(\mathbb{R}^-) \) we consider the following boundary value problem for the inhomogeneous Schrödinger equation

\[
\begin{cases}
-\psi''(k, x) + Q(x)\psi(k, x) = k^2 \psi(k, x) - g(x), \\
[\cos \alpha] \psi(k, 0) = [\sin \alpha] \psi'(k, 0),
\end{cases}
\]

where \( k \in \mathbb{C}^+ \). Using the method of variation of constants we write

\[
\psi(k, x) = c_1(x) f_r(k, x) + c_2(x) [(\sin \alpha) \theta(k, x) + (\cos \alpha) \varphi(k, x)],
\]

where the Wronskian of \( f_r(k, x) \) and \((\sin \alpha) \theta(k, x) + (\cos \alpha) \varphi(k, x)\) does not vanish. Then

\[
\begin{pmatrix}
 f_r(k, x) \\
 f'_r(k, x)
\end{pmatrix} \begin{pmatrix}
 (\sin \alpha) \theta(k, x) + (\cos \alpha) \varphi(k, x) \\
 (\sin \alpha) \theta'(k, x) + (\cos \alpha) \varphi'(k, x)
\end{pmatrix} \begin{pmatrix}
 c_1(x) \\
 c_2(x)
\end{pmatrix} = \begin{pmatrix}
 0 \\
 g(x)
\end{pmatrix},
\]

where

\[
det \begin{pmatrix}
 f_r(k, x) & (\sin \alpha) \theta(k, x) + (\cos \alpha) \varphi(k, x) \\
 f'_r(k, x) & (\sin \alpha) \theta'(k, x) + (\cos \alpha) \varphi'(k, x)
\end{pmatrix} = f_r(k, 0) \cos \alpha - f'_r(k, 0) \sin \alpha.
\]

Then

\[
\begin{pmatrix}
 c'_1(x) \\
 c'_2(x)
\end{pmatrix} = \frac{1}{f_r(k, 0) \cos \alpha - f'_r(k, 0) \sin \alpha} \times
\begin{pmatrix}
 (\sin \alpha) \theta'(k, x) + (\cos \alpha) \varphi'(k, x) & -(\sin \alpha) \theta(k, x) - (\cos \alpha) \varphi(k, x) \\
 -f'_r(k, x) & f_r(k, x)
\end{pmatrix} \begin{pmatrix}
 0 \\
 g(x)
\end{pmatrix}
\]

\[
= \frac{g(x)}{f_r(k, 0) \cos \alpha - f'_r(k, 0) \sin \alpha} \begin{pmatrix}
 -(\sin \alpha) \theta(k, x) - (\cos \alpha) \varphi(k, x) \\
 f_r(k, x)
\end{pmatrix}.
\]

Therefore,

\[
c_1(x) = c_1 + \frac{\int_x^0 dy \, [(\sin \alpha) \theta(k, y) + (\cos \alpha) \varphi(k, y)] g(y)}{f_r(k, 0) \cos \alpha - f'_r(k, 0) \sin \alpha},
\]

\[
c_2(x) = c_2 + \frac{\int_{-\infty}^x dy \, f_r(k, y) g(y)}{f_l(k, 0) \cos \alpha - f'_l(k, 0) \sin \alpha}.
\]

We then easily derive that

\[
\psi(k, 0) \cos \alpha - \psi'(k, 0) \sin \alpha = c_1 f_r(k, 0) \cos \alpha - f'_r(k, 0) \sin \alpha,
\]
so that $c_1 = 0$. In order that $\psi(k, \cdot) \in L^2(\mathbb{R}^-)$ for those $k \in \mathbb{C}^+$ for which $f_r(k, 0) \cos \alpha - f'_r(k, 0) \sin \alpha \neq 0$, we require $c_2 = 0$. As a result,

$$
\psi(k, x) = \int_{-\infty}^{0} dy \mathcal{G}_-(k; x, y) g(y),
$$

where the Green’s function is given by

$$
\mathcal{G}_-(k; x, y) = \frac{+1}{f_r(k, 0) \cos \alpha - f'_r(k, 0) \sin \alpha}
\times \begin{cases} 
    f_r(k, x) \left[ \theta(k, y) \sin \alpha + \varphi(k, y) \cos \alpha \right], & x \leq y \leq 0, \\
    [\theta(k, x) \sin \alpha + \varphi(k, x) \cos \alpha] f_r(k, y), & y \leq x \leq 0.
\end{cases}
$$

The function

$$
S_\alpha(k) = f_r(k, 0) \cos \alpha - f'_r(k, 0) \sin \alpha
$$

is called the $S$-matrix. Its zeros correspond to the discrete eigenvalues.
Appendix D

Fragmentation and Examples

In this appendix we construct the Jost solutions and scattering coefficients from the Jost solutions and scattering coefficients for the same potential supported on a subinterval, using a partition of the real line into finitely many so-called fragments. We also discuss some examples.

D.1 Fragmentation

In this section we construct the Jost solutions and scattering coefficients from the Jost solutions and scattering coefficients for the same potential supported on a subinterval, using a partition of the real line into finitely many so-called fragments. Here we follow [2].

Consider a subdivision of the real line into \( n + 1 \) “fragments” by defining the division points

\[-\infty = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} = +\infty\]

and putting \( I_j = (x_{j-1}, x_j) \) \((j = 1, 2, \ldots, n + 1)\). Consider a real potential \( Q \in L^1(\mathbb{R}; (1 + |x|)dx) \) and define its fragments \( Q^{[j]}(x) \) as follows:

\[Q^{[j]}(x) = \begin{cases} Q(x), & x \in I_j, \\ 0, & x \in \mathbb{R} \setminus I_j. \end{cases}\]

Quantities pertaining to the potential \( Q^{[j]}(x) \) are indicated by attaching the superscript \([j]\) to the corresponding quantities pertaining to \( Q(x) \).

On \( I_j \) and for \( k \in \mathbb{R} \) in the generic case and for \( 0 \neq k \in \mathbb{R} \) in the exceptional case, the Jost solutions are linearly independent. Let us define the Wronskian matrix

\[W(k, x) = \begin{pmatrix} f_l(k, x) & f_r(k, x) \\ f'_l(k, x) & f'_r(k, x) \end{pmatrix},\]
so that \( \det W(k, x) = [-2ik/T(k)] \). Then there exist constants \( c^{[j]}_1(k), c^{[j]}_2(k), c^{[j]}_3(k), \) and \( c^{[j]}_4(k) \) such that for \( x \in I_j \)

\[
\begin{align*}
f^{[j]}_1(k, x) &= c^{[j]}_1(k) f_1(k, x) + c^{[j]}_3(k) f_r(k, x), \\
f^{[j]}_r(k, x) &= c^{[j]}_2(k) f_1(k, x) + c^{[j]}_4(k) f_r(k, x).
\end{align*}
\]

Putting

\[
\mathbf{C}^{[j]}(k) = \begin{pmatrix} c^{[j]}_1(k) & c^{[j]}_2(k) \\ c^{[j]}_3(k) & c^{[j]}_4(k) \end{pmatrix}
\]

we obtain

\[
W^{[j]}(k, x) = W(k, x) \mathbf{C}^{[j]}(k), \quad x \in I_j,
\]

where \( \mathbf{C}^{[j]}(k) \) is nonsingular if \( 0 \neq k \in \mathbb{R} \). We obviously have

\[
\begin{align*}
W^{[1]}(k, x) &= \begin{pmatrix} e^{ikx} & f_r(k, x) \\ ike^{ikx} & f_r^t(k, x) \end{pmatrix}, & W^{[n+1]}(k, x) &= \begin{pmatrix} f_1(k, x) & e^{-ikx} \\ f_r^t(k, x) & -ike^{-ikx} \end{pmatrix}, \\
\mathbf{C}^{[1]}(k) &= \begin{pmatrix} c^{[1]}_1(k) & 0 \\ c^{[1]}_3(k) & 1 \end{pmatrix}, & \mathbf{C}^{[n+1]}(k) &= \begin{pmatrix} 1 & c^{[n+1]}_2(k) \\ 0 & c^{[n+1]}_4(k) \end{pmatrix}.
\end{align*}
\]

Further, for \( 0 \neq k \in \mathbb{R} \) we have \( 0 \notin \{ c^{[1]}_1(k), c^{[n+1]}_4(k) \} \), while

\[
W(k, x) = \begin{pmatrix} e^{ikx} & f_r(k, x) \\ ike^{ikx} & f_r^t(k, x) \end{pmatrix} \begin{pmatrix} 1 \\ c^{[1]}_3(k) \end{pmatrix} = \begin{pmatrix} f_1(k, x) & e^{-ikx} \\ f_r^t(k, x) & -ike^{-ikx} \end{pmatrix} \begin{pmatrix} 1 \\ c^{[n+1]}_4(k) \end{pmatrix}.
\]

Writing the first row elements \( f_i(k, x) \) and \( f_r(k, x) \) of \( W(k, x) \) as linear combinations of \( e^{-ikx} \) and \( e^{ikx} \) (apart from an \( o(1) \) term) as \( x \rightarrow \pm \infty \), we get

\[
T(k) = c^{[1]}_1(k) = c^{[n+1]}_4(k), \quad R(k) = -c^{[n+1]}_2(k), \quad L(k) = -c^{[1]}_3(k).
\]

Therefore,

\[
\mathbf{C}^{[1]}(k) = \begin{pmatrix} T(k) & 0 \\ -L(k) & 1 \end{pmatrix}, \quad \mathbf{C}^{[n+1]}(k) = \begin{pmatrix} 1 & -R(k) \\ 0 & T(k) \end{pmatrix}.
\]

Since \( W(k, x) \) is continuous in \( x \in \mathbb{R} \) for every \( 0 \neq k \in \mathbb{R} \), we obtain by virtue of the continuity of \( W(k, x) \) in \( x \in \mathbb{R} \)

\[
W^{[j+1]}(k, x_j) \mathbf{C}^{[j+1]}(k)^{-1} = W^{[j]}(k, x_j) \mathbf{C}^{[j]}(k)^{-1}, \quad j = 1, \ldots, n. \quad (D.1)
\]
Consequently,
\[
\mathcal{C}^{[1]}(k)^{-1} \mathcal{C}^{[n+1]}(k) = \prod_{j=1}^{n} W^{[j]}(k, x_j)^{-1} W^{[j+1]}(k, x_j),
\]
where the factors in the matrix product are arranged from left to right as \( j \) runs from 1 to \( n \). We now compute
\[
\mathcal{C}^{[1]}(k)^{-1} \mathcal{C}^{[n+1]}(k) = \left( \begin{array}{cc} \frac{1}{T(k)} & 0 \\ \frac{L(k)}{T(k)} & 1 \end{array} \right) \left( \begin{array}{cc} 1 & -R(k) \\ 0 & T(k) \end{array} \right) = \left( \begin{array}{cc} \frac{1}{T(k)} & \frac{R(k)}{T(k)} \\ \frac{L(k)}{T(k)} & \frac{1}{T(-k)} \end{array} \right),
\]
where we have used Proposition 2.3. As a result \[ \mathcal{C}^{[1]}(k)^{-1} \mathcal{C}^{[n+1]}(k) = \prod_{j=1}^{n} W^{[j]}(k, x_j)^{-1} W^{[j+1]}(k, x_j), \] (D.2)
where the factors in the matrix product are arranged from left to right as \( j \) runs from 1 to \( n \).

Next, put
\[
M(k, x) = \begin{pmatrix} e^{ikx} & e^{-ikx} \\ ike^{ikx} & -ike^{-ikx} \end{pmatrix}.
\]
Then for \( j = 2, \ldots, n \) we have
\[
\begin{align*}
\bar{f}^{[j]}(k, x) &= \begin{cases} e^{ikx}, & x \geq x_j, \\
\frac{1}{T[j](k)} e^{ikx} + \frac{L[j](k)}{T[j](k)} e^{-ikx}, & x \leq x_{j-1}, \end{cases} \\
\bar{f}^{[j]}(k, x) &= \begin{cases} e^{-ikx}, & x \geq x_j, \\
\frac{1}{T[j](k)} e^{-ikx} + \frac{R[j](k)}{T[j](k)} e^{ikx}, & x \leq x_{j-1}. \end{cases}
\end{align*}
\]
Therefore,
\[
W^{[j]}(k, x_j) = M(k, x_j) \begin{pmatrix} \frac{1}{T[j](k)} & \frac{R[j](k)}{T[j](k)} \\ 0 & 1 \end{pmatrix},
\]
\[
W^{[j+1]}(k, x_j) = M(k, x_j) \begin{pmatrix} \frac{1}{T[j+1](k)} & \frac{R[j+1](k)}{T[j+1](k)} \\ 0 & 1 \end{pmatrix}.
\]
Consequently, we can write \[ (D.2) \] in the form \[ (D.3) \]
\[
\left( \begin{array}{cc} \frac{1}{T(k)} & \frac{R(k)}{T(k)} \\ \frac{L(k)}{T(k)} & \frac{1}{T(-k)} \end{array} \right) = \prod_{j=1}^{n} \left( \begin{array}{cc} \frac{1}{T[j](k)} & \frac{R[j](k)}{T[j](k)} \\ \frac{L[j](k)}{T[j](k)} & \frac{1}{T[j+1](k)} \end{array} \right),
\]
where the factors in the matrix product are arranged from left to right as \( j \) runs from 1 to \( n \).
D.2 Examples

In this section we discuss a few examples. For these examples we compute the Jost solutions and scattering coefficients.

Esempio D.1 (Square well potential) Consider the square well potential

\[ Q(x) = \begin{cases} 
0, & x < 0, \\
-Q_0, & 0 < x < L, \\
0, & x > L,
\end{cases} \]

where \( Q_0 > 0 \). Introduce the conformal mapping \( \beta = \sqrt{k^2 + Q_0} \) from the complex \( k \)-plane cut along \([-i\sqrt{Q_0}, i\sqrt{Q_0}]\) onto the complex \( \beta \)-plane such that \( \beta \sim k \) as \( k \to \infty \). Then \( \beta \) belongs to the upper (lower) half-plane iff \( k \) belongs to the upper (lower) half-plane minus the branch cut. The conformal mapping maps the real \( k \)-line plus the left and right edges of the branch cut into the real \( \beta \)-line. It is now easily verified that

\[
f_i(k, x) = \begin{cases} 
0, & x < 0, \\
-Q_0 e^{-i\beta x} + Q_0 e^{i\beta x}, & 0 \leq x \leq L, \\
\beta e^{-i\beta x} + Q_0 e^{i\beta x}, & x \geq L,
\end{cases}
\]

where we have used that \( \beta^2 - k^2 = Q_0 \). Then

\[
\frac{1}{T(k)} = \frac{(\beta + k)^2}{4\beta k} e^{i(\beta - k)L} - \frac{(\beta - k)^2}{4\beta k} e^{i(\beta + k)L} \cos(\beta L) - i \frac{\beta^2 + k^2}{2\beta k} \sin(\beta L),
\]

\[
L(k) = -\frac{Q_0}{4\beta k} e^{i(\beta - k)L} + \frac{Q_0}{4\beta k} e^{i(\beta + k)L} = i e^{ikL} \frac{Q_0}{2\beta k} \sin(\beta L).
\]

In the same way we obtain

\[
f_r(k, x) = \begin{cases} 
0, & x < 0, \\
\left(\frac{\beta + k}{2\beta k} e^{-i\beta x} + \frac{\beta - k}{2\beta k} e^{i\beta x} - \frac{(\beta - k)^2}{4\beta k} e^{i(\beta + k)L} \right) e^{-ikx}, & 0 \leq x \leq L, \\
\left(\frac{\beta + k}{2\beta k} e^{-i\beta x} + \frac{\beta - k}{2\beta k} e^{i\beta x} - \frac{(\beta + k)^2}{4\beta k} e^{i(\beta - k)L} \right) e^{ikx}, & x \geq L,
\end{cases}
\]
Moreover, we are in the exceptional case iff 

\[ 1 \over T(k) = \frac{(\beta + k)^2}{4\beta k} e^{i(\beta + k)L} - \frac{(\beta - k)^2}{4\beta k} e^{i(\beta - k)L} \]

\[ = e^{ikL} \left\{ \cos(\beta L) - i \frac{\beta^2 + k^2}{2\beta k} \sin(\beta L) \right\}, \]

\[ R(k) \over T(k) = -\frac{Q_0}{4\beta k} e^{-i(\beta + k)L} + \frac{Q_0}{4\beta k} e^{-i(\beta - k)L} = i e^{-ikL} \frac{Q_0}{2\beta k} \sin(\beta L), \]

where the discrete eigenvalues \( k = i\kappa \) occur for \( \kappa \in (0, \sqrt{Q_0}) \). Using that for \( \beta^2 - \kappa^2 = Q_0 \) we have

\[ \frac{\partial \beta^2 - \kappa^2}{\partial \beta} = \frac{Q_0^2}{2\beta^2 \kappa^3} > 0, \]

we see that there is always at least one discrete eigenvalue and that their number equals the positive integer \( n \) for which

\[ (n - 1)\pi < L\sqrt{Q_0} \leq n\pi. \]

Moreover, we are in the exceptional case if \( L\sqrt{Q_0} \) is an integer multiple of \( \pi \); otherwise we are in the generic case. The proof proceeds by drawing the graphs of \( \cotan(\beta L) \) and \( [(\beta^2 - \kappa^2)/2\beta \kappa] \) for \( 0 < \beta < \sqrt{Q_0} \) in one figure.

Let \( i\kappa \), with \( \kappa > 0 \), be a discrete eigenvalue and let \( C = [f_r(i\kappa, x)/f_l(i\kappa, x)] \) be the corresponding dependency constant. Computing \( C \) for \( x \geq L \) and substituting \( k = i\kappa \), we get

\[ C = \left[ \frac{(\beta + i\kappa)^2}{4i\beta \kappa} e^{-\beta L} - \frac{(\beta - i\kappa)^2}{4i\beta \kappa} e^{i\beta L} \right] e^{i(2x - \beta L)} + \frac{Q_0}{4i\beta \kappa} e^{i\beta L} [e^{i\beta L} - e^{-i\beta L}] \]

\[ = e^{i(2x - \beta L)} \left[ \cos(\beta L) - \frac{\beta^2 - \kappa^2}{2\beta \kappa} \sin(\beta L) \right] + \frac{Q_0}{2\beta \kappa} e^{i\beta L} \sin(\beta L) \]

\[ = \frac{Q_0}{2\beta \kappa} e^{i\beta L} \sin(\beta L). \]

Next, we compute the residue \( \tau \) of \( T(k) \) at the pole \( k = i\kappa \). We compute

\[ \frac{d}{dk} \frac{1}{T(k)} = \frac{iL}{T(k)} + e^{ikL} \frac{kL}{\beta} \left[ -\sin(\beta L) - i \frac{\beta^2 + k^2}{2\beta k} \cos(\beta L) \right] + i e^{ikL} \frac{Q_0^2}{2\beta^3 \kappa^2} \sin(\beta L). \]

Substituting \( k = i\kappa \) we get

\[ \frac{i}{\tau} = e^{-\kappa L} \left[ \frac{\kappa L}{\beta} \cos(\beta L) + \frac{\kappa L \beta^2 - \kappa^2}{\beta^3 \kappa} \cos(\beta L) + \frac{Q_0^2}{2\beta^3 \kappa^2} \sin(\beta L) \right] \]

\[ = \frac{\kappa L}{\beta} e^{-\kappa L} \sin(\beta L) \left[ 1 + \frac{(\beta^2 - \kappa^2)^2}{4\beta^2 \kappa^2} + \frac{Q_0^2}{2\beta^2 \kappa^3 L} \right] \]

\[ = \frac{\kappa L Q_0^2}{\beta} e^{-\kappa L} \sin(\beta L) \left[ \frac{1}{4\beta^2 \kappa^2} + \frac{1}{2\beta^2 \kappa^3 L} \right] = \frac{Q_0^2}{2\beta^3 \kappa^2} e^{-\kappa L} \sin(\beta L) \left[ 1 + \frac{1}{2} \kappa L \right], \]
where we have used that \( \cos(\beta L) = \frac{(\beta^2 - \kappa^2)/2\beta \kappa}{\sin(\beta L)} \). Using (5.12) and \( \tilde{N} = -\tau^2/N \), we obtain for the norming constants

\[
N = \frac{C}{i/\tau} = \frac{\beta^2 \kappa e^{2\kappa L}}{Q_0(1 + \frac{1}{2} \kappa L)}, \quad \tilde{N} = \frac{\beta^2 \kappa}{Q_0(1 + \frac{1}{2} \kappa L)} = Ne^{-2\kappa L}.
\]

The auxiliary functions \( K(x, y) \) and \( M(x, y) \) are difficult to compute. \( \square \)

**Esempio D.2 (One-soliton potential)** Consider the Marchenko kernel

\[
\Omega_l(x + y) = Ne^{-\kappa(x+y)},
\]

where \( \kappa \) and \( N \) are positive constants. Then the unique solution of the Marchenko integral equation (5.7) is given by

\[
K(x, y) = -\frac{Ne^{-\kappa(x+y)}}{1 + \frac{N}{2\kappa} e^{-2\kappa x}}.
\]

Using (3.3) we get the one-soliton potential

\[
Q(x) = -4\kappa Ne^{-2\kappa x} \left[1 + \frac{N}{2\kappa} e^{-2\kappa x}\right]^2.
\]

The Jost solution from the right is given by

\[
f_l(k, x) = e^{ikx} \left\{1 - i \frac{Ne^{-2\kappa x}}{1 + \frac{N}{2\kappa} e^{-2\kappa x}} \frac{1}{k + i\kappa}\right\} = e^{ikx} \left\{1 - i \frac{2\kappa}{1 + \frac{N}{2\kappa} e^{2\kappa x}} \frac{1}{k + i\kappa}\right\}.
\]

As a result,

\[
T(k) = \frac{k + i\kappa}{k - i\kappa}, \quad R(k) = L(k) = 0.
\]

Let us consider the Marchenko integral kernel

\[
\Omega_r(x + y) = \tilde{N} e^{\kappa(x+y)},
\]

where \( \tilde{N} \) is a positive constant chosen as to lead to the same potential \( Q(x) \). Solving the Marchenko integral equation (5.10) we get

\[
M(x, y) = -\frac{\tilde{N} e^{\kappa(x+y)}}{1 + \frac{N}{2\kappa} e^{2\kappa x}}.
\]

Using (3.4) we obtain

\[
Q(x) = -4\kappa \tilde{N} e^{2\kappa x} \left[1 + \frac{N}{2\kappa} e^{2\kappa x}\right]^2 = -4\kappa \frac{4\kappa^2}{N} e^{-2\kappa x} \left[1 + \frac{N}{2\kappa} e^{-2\kappa x}\right]^2.
\]
Thus we get the same potential iff
\[ N \tilde{N} = 4\kappa^2. \]

For the Jost solution from the left we get
\[ f_r(k, x) = e^{-ikx} \left\{ 1 - i \frac{\tilde{N} e^{2\kappa x}}{1 + \frac{N}{2\kappa} e^{2\kappa x} k + i\kappa} \right\} = e^{-ikx} \left\{ 1 - i \frac{2\kappa}{1 + \frac{2\kappa}{N} e^{-2\kappa x} k + i\kappa} \right\}. \]

Consequently,
\[ T(k) = \frac{k + i\kappa}{k - i\kappa}, \quad R(k) = L(k) = 0. \]

Let us now derive the corresponding KdV solution. We need to replace \( N \) and \( \tilde{N} \) by \( e^{8\kappa^3 t} N \) and \( e^{-8\kappa^3 t} \tilde{N} \), respectively. We get
\[ Q(x, t) = -\frac{4\kappa N e^{-2\kappa(x-4\kappa^2 t)}}{1 + \frac{N}{2\kappa} e^{-2\kappa(x-4\kappa^2 t)}}^2 = -\frac{4\kappa \tilde{N} e^{2\kappa(x-4\kappa^2 t)}}{1 + \frac{\tilde{N}}{2\kappa} e^{2\kappa(x-4\kappa^2 t)}}^2. \]

Putting \( e^{2\kappa x_0} = (N/2\kappa) \), we can write these expressions in the traditional form
\[ Q(x, t) = \frac{-4\kappa^2}{\cosh^2[\kappa(x - x_0 - 4\kappa^2 t)]}, \]
which is a wave of amplitude \( 4\kappa^2 \) travelling to the right with velocity \( 4\kappa^2 \).

**Esempio D.3 (**\( N \)-soliton potential\)**) Consider the Marchenko integral kernel
\[ \Omega_l(x + y) = \sum_{j=1}^{N} N_j e^{-\kappa_j(x+y)}, \]
where \( N_j \) (\( j = 1, 2, \ldots, N \)) are positive constants and \( \kappa_1 > \ldots > \kappa_N > 0 \). Let \( \mathbf{d} \) and \( \mathbf{d}^T \) be the column vector and the row vector, respectively, with entries \( \sqrt{N_j} \) (\( j = 1, 2, \ldots, N \)). Then
\[ \Omega_l(x + y) = \sum_{j=1}^{N} N_j e^{-\kappa_j x} = \mathbf{d}^T e^{-(x+y)A} \mathbf{d}, \]
where \( A = \text{diag}(\kappa_1, \ldots, \kappa_N) \). Then the solution of the Marchenko integral equation is given by
\[ K(x, y) = -\mathbf{d}^T e^{-xA} \left[ I_N + e^{-xA} \mathbf{q} e^{-xA} \right]^{-1} e^{-yA} \mathbf{d} = -\mathbf{d}^T \left[ e^{2xA} + \mathbf{q} \right]^{-1} e^{-(y-x)A} \mathbf{d}, \]
where $q$ is the Cauchy matrix given by

$$q = \int_0^\infty dw \, e^{-wA}dd^T e^{-wA} = \left\{ \sqrt{N_j N_l} \right\}_{j,l=1}^N$$

is a positive real symmetric $N \times N$ matrix. In particular, $q$ is the unique solution of the Lyapunov equation $Aq + qA = dd^T$. The Jost solution from the right is then given by

$$f_l(k, x) = e^{ikx} \left\{ 1 - i d^T e^{-xA} \left[ I_N + e^{-xA} q e^{-xA} \right]^{-1} e^{-xA} (kI_N + iA)^{-1} d \right\}$$

Thus $R(k) = L(k) = 0$, while

$$\frac{1}{T(k)} = 1 - i d^T q^{-1} (kI_N + iA)^{-1} d.$$

As a result,

$$T(k) = 1 + i d^T q^{-1} (kI_N + i[A - ddq^{-1}])^{-1} d = 1 + i d^T (kI_N - iA)^{-1} q^{-1} d.$$

This function is quite clearly unimodular for $k \in \mathbb{R}$. We are also quite clearly in the exceptional case because $T(0) \neq 0$. Using (3.3) we get for the potential

$$Q(x) = 2 \frac{d}{dx} \left[ d^T (e^{2xA} + q)^{-1} d \right] = -4d^T (e^{2xA} + q)^{-1} A e^{2xA} (e^{2xA} + q)^{-1} d$$

Now consider the right Marchenko kernel

$$\Omega_r(x + y) = \sum_{j=1}^N \tilde{N}_j e^{\kappa_j (x+y)},$$

where $\tilde{N}_j$ ($j = 1, 2, \ldots, N$) are positive constants. Letting $\tilde{d}$ and $\tilde{d}^T$ stand for the column vector and row vector with entries $\sqrt{\tilde{N}_j}$ ($j = 1, 2, \ldots, N$), we get

$$\Omega_r(x + y) = \sum_{j=1}^N \tilde{N}_j e^{\kappa_j (x+y)} = \tilde{d}^T e^{(x+y)A} \tilde{d},$$
where, as before, \( A = \text{diag}(\kappa_1, \ldots, \kappa_N) \). Then the solution of the Marchenko integral equation (5.10) is given by

\[
M(x, y) = -\tilde{d}^T e^{xA} \left[ I_N + e^{xA} \tilde{q} e^{xA} \right]^{-1} e^{yA} \tilde{d}
\]

\[
= -\tilde{d}^T \left[ e^{-2xA} + \tilde{q} \right]^{-1} e^{(y-x)A} \tilde{d}.
\]

Thus the potential is given by

\[
Q(x) = -2 \frac{d}{dx} \left\{ 1 - i \tilde{d}^T e^{xA} \left[ I_N + e^{xA} \tilde{q} e^{xA} \right]^{-1} e^{xA} (kI_N + iA)^{-1} \tilde{d} \right\}.
\]

For the Jost solution from the left we obtain

\[
f_\nu(k, x) = e^{-ikx} \left\{ 1 - i \tilde{d}^T e^{xA} \left[ I_N + e^{xA} \tilde{q} e^{xA} \right]^{-1} e^{xA} (kI_N + iA)^{-1} \tilde{d} \right\}.
\]

We also get \( R(k) = L(k) = 0 \), while

\[
\frac{1}{T(k)} = 1 - i \tilde{d}^T \tilde{q}^{-1} (kI_N + iA)^{-1} \tilde{d}
\]

\[
T(k) = 1 + i \tilde{d}^T (kI_N - iA)^{-1} \tilde{q}^{-1} \tilde{d}.
\]

It remains to relate \( d \) and \( \tilde{d} \) in such a way that we get the same potential \( Q(x) \). Since \( K(x, x) + M(x, x) = \text{constant} \) for \( x \in \mathbb{R} \), we have

\[
d^T \left[ e^{2xA} + \tilde{q} \right]^{-1} d + \tilde{d}^T \left[ e^{-2xA} + \tilde{q} \right]^{-1} \tilde{d} = d^T q^{-1} d = \tilde{d}^T \tilde{q}^{-1} \tilde{d}.
\]

The natural way to do so is to define \( \tilde{d} \) in terms of \( d \) as follows:

\[
\tilde{d} = q^{-1} d,
\]

where \( d \) is the unique solution of the Lyapunov equation

\[
Aq + qA = dd^T.
\]

Then \( q^{-1} \) is the unique solution of the Lyapunov equation

\[
Aq^{-1} + q^{-1} A = q^{-1} dd^T q^{-1} = \tilde{d} \tilde{d}^T.
\]
as is \( \tilde{q} \). We thus have \( \tilde{q} = q^{-1} \). As a result, using that \( q^T = q \),

\[
\begin{align*}
    d^T \left[ e^{2xA} + q \right]^{-1} d + \tilde{d}^T \left[ e^{-2xA} + \tilde{q} \right]^{-1} \tilde{d} \\
    = d^T \left[ e^{2xA} + q \right]^{-1} d + d^T \left[ e^{-2xA} + q \right]^{-1} \left\{ e^{2xA} + q \right\}^{-1} q^{-1} d \\
    = d^T \left[ e^{2xA} + q \right]^{-1} d + d^T q^{-1} e^{2xA} \left[ e^{2xA} + q \right]^{-1} q^{-1} d \\
    = d^T q^{-1} \left[ e^{2xA} + q \right]^{-1} d + d^T q^{-1} e^{2xA} \left[ e^{2xA} + q \right]^{-1} d = d^T q^{-1} d,
\end{align*}
\]
as claimed.

To derive the corresponding KdV solution, we apply (6.16) and (6.17) and replace \( d, \tilde{d}, q, \) and \( \tilde{q} \) by \( K(t)d, K(-t)\tilde{d}, K(t)qK(t), \) and \( K(-t)\tilde{q}K(-t) \), respectively, where

\[
K(t) = \text{diag}(e^{4x_j^2t})_{j=1}^N.
\]

We then get

\[
Q(x, t) = -4d^T K(t)e^{-xA} \left[ I_N + e^{-xA} K(t)qK(t)e^{-xA} \right]^{-1} \times \\
\times A \left[ I_N + e^{-xA} K(t)qK(t)e^{-xA} \right]^{-1} e^{-xA} K(t)d \\
= -4d^T K(-t)e^{xA} \left[ I_N + e^{xA} K(-t)\tilde{q}K(-t)e^{xA} \right]^{-1} \times \\
\times A \left[ I_N + e^{xA} K(-t)\tilde{q}K(-t)e^{xA} \right]^{-1} e^{xA} K(-t)\tilde{d},
\]

which is the usual \( N \)-soliton solution.

**Esempio D.4 \((-Q_0/\cosh^2(x)\) potential)** Consider the potential \([6]\)

\[
Q(x) = \frac{-Q_0}{\cosh^2(x)},
\]

where \( Q_0 > 0 \). Then the substitution \( \xi = \tanh(x) \) converts the Schrödinger equation \(-\psi'' + Q\psi = k^2\psi\) into the associated Legendre differential equation

\[
\frac{d}{d\xi} \left( (1 - \xi^2) \frac{d\psi}{d\xi} \right) + \left( Q_0 + \frac{k^2}{1 - \xi^2} \right) \psi = 0,
\]

where \( \xi \in I = (-1, 1) \).

First suppose that \( Q_0 = L(L + 1) \) for some \( L = 1, 2, 3, \ldots \). Then the discrete eigenvalues occur for \( k = i\kappa \), where \( \kappa \in \{1, 2, \ldots, L\} \); the corresponding eigenfunctions are proportional to the associated Legendre functions \( (1 - \xi^2)^{\kappa/2} \left( \frac{d}{d\xi} \right)^\kappa P_L(\xi) \), where \( P_l(\xi) = (2^l \cdot l!)^{-1} \frac{d^l}{d\xi^l} (\xi^2 - 1)^l \) denotes the \( l \)-th Legendre polynomial.
The transformation \( \psi = 2^ik(1 - \xi^2)^{-\frac{1}{2}ik}\phi \) converts the above associated Legendre equation into the differential equation
\[
(1 - \xi^2)\frac{d^2\phi}{d\xi^2} + (2ik - 2)\xi \frac{d\phi}{d\xi} + (Q_0 + ik + k^2)\phi = 0.
\]
The further substitution \( s = \frac{1}{2}(1 + \xi) \) converts the latter equation into the hypergeometric differential equation
\[
s(1 - s)\frac{d^2\phi}{ds^2} + \{\gamma - (\alpha + \beta + 1)s\} \frac{d\phi}{ds} - \alpha \beta \phi = 0,
\]
where \( \alpha + \beta = 1 - 2ik, \alpha \beta = -(Q_0 + ik + k^2), \) and \( \gamma = 1 - ik. \) We then have
\[
\alpha, \beta = \frac{1}{2} - ik \pm \sqrt{Q_0 + \frac{1}{4}}, \text{ and } \gamma = 1 - ik.
\]
Using that \( 2^ik(1 - \xi^2)^{-\frac{1}{2}ik} = e^{-ikx}[1 + e^{2x}]^{ik} \), it is then easily verified that
\[
2^ik(1 - \xi^2)^{-\frac{1}{2}ik}F_1(\frac{1}{2} - ik + \sqrt{Q_0 + \frac{1}{4}}, \frac{1}{2} - ik - \sqrt{Q_0 + \frac{1}{4}}, 1 - ik; s) \sim e^{-ikx}, \quad x \to -\infty,
\]
where we have used that \( 2F_1(\alpha, \beta, \gamma; 0) = 1 \). Therefore,
\[
f_r(k, x) = [2 \cosh(x)]^{ik} \times 2 F_1(\frac{1}{2} - ik + \sqrt{Q_0 + \frac{1}{4}}, \frac{1}{2} - ik - \sqrt{Q_0 + \frac{1}{4}}, 1 - ik; 1 + \tanh(x)).
\]
Using the identity \([8, (8.3.6)]\)
\[
2F_1(\alpha, \beta, \gamma; s) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}2F_1(\alpha, \beta, \alpha + \beta + 1 - \gamma; 1 - s) + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1 - s)^{\gamma - \alpha - \beta}2F_1(\gamma - \alpha, \gamma - \beta, 1 + \gamma - \alpha - \beta; 1 - s),
\]
as well as \( 2^ik(1 - \xi^2)^{-\frac{1}{2}ik} = e^{ikx}[1 + e^{-2x}]^{ik} \) and \( 2^ik(1 - \xi^2)^{-\frac{1}{2}ik}(1 - \xi)\xi^{ik} = e^{-ikx}, \)
we obtain as \( x \to +\infty \)
\[
f_r(k, x) = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}e^{ikx} + \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}e^{-ikx} + o(1)
\]
\[
= \frac{1}{T(k)}e^{-ikx} + \frac{R(k)}{T(k)}e^{ikx} + o(1),
\]
so that
\[
\frac{1}{T(k)} = \frac{\Gamma(\gamma)\Gamma(\alpha + \beta - \gamma)}{\Gamma(\alpha)\Gamma(\beta)}, \quad \frac{R(k)}{T(k)} = \frac{\Gamma(\gamma)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)}.
\]
Let us substitute the values of $\alpha$, $\beta$, and $\gamma$ to evaluate $T(k)$ and $R(k)$. We then evaluate $L(k)$ by using the identity

\[
L(k) = -R(-k) \frac{T(k)}{T(-k)} = -\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma - \alpha - \beta)}{\Gamma(\alpha + \gamma)\Gamma(\gamma - \alpha)\Gamma(\gamma - \beta)} \Gamma(\gamma),
\]

where $\tilde{\alpha}$, $\tilde{\beta}$, and $\tilde{\gamma}$ are obtained from $\alpha$, $\beta$, and $\gamma$ by replacing $k$ with $-k$. Consequently,

\[
T(k) = \frac{\Gamma(\frac{1}{2} - ik + \sqrt{Q_0 + \frac{1}{4}})\Gamma(\frac{1}{2} - ik - \sqrt{Q_0 + \frac{1}{4}})}{\Gamma(1 - ik)\Gamma(-ik)},
\]

\[
R(k) = \frac{\Gamma(\frac{1}{2} - ik + \sqrt{Q_0 + \frac{1}{4}})\Gamma(\frac{1}{2} - ik - \sqrt{Q_0 - \frac{1}{4}})\Gamma(ik)}{\Gamma(\frac{1}{2} - \sqrt{Q_0 + \frac{1}{4}})\Gamma(\frac{1}{2} + \sqrt{Q_0 + \frac{1}{4}})\Gamma(-ik)}
\]

\[
= \cos\left(\frac{\pi}{\sqrt{Q_0 + \frac{1}{4}}}\right)\frac{\Gamma(\frac{1}{2} - ik + \sqrt{Q_0 + \frac{1}{4}})\Gamma(\frac{1}{2} - ik - \sqrt{Q_0 - \frac{1}{4}})\Gamma(ik)}{\Gamma(\frac{1}{2} - \sqrt{Q_0 + \frac{1}{4}})\Gamma(\frac{1}{2} + \sqrt{Q_0 + \frac{1}{4}})\Gamma(-ik)}, \]

\[
L(k) = \frac{\Gamma(1 + ik)}{\Gamma(1 - ik)} = \frac{-\Gamma(ik)}{-\Gamma(-ik)} = \frac{-\Gamma(ik)}{\Gamma(-ik)}.
\]

In retrospect it is quite obvious that $R(k) = L(k)$, because $f_l(k, x) = f_r(k, -x)$ as a result of the fact that the potential is an even function. Therefore, using that $\cosh(-x) = \cosh(x)$ and $\tanh(-x) = -\tanh(x)$ we get

\[
f_l(k, x) = [2 \cosh(x)]^{ik} \times \sum_{n=-\infty}^{\infty} F_1\left(\frac{1}{2} - ik + \sqrt{Q_0 + \frac{1}{4}}, \frac{1}{2} - ik - \sqrt{Q_0 + \frac{1}{4}}, 1 - ik; \frac{1}{2} [1 - \tanh(x)]\right).
\]

Using $[3] (2.5.8)$, it is easily verified that as $k \to \pm\infty$

\[
|R(k)| = |L(k)| = 2 \left| \cos\left(\frac{\pi}{\sqrt{Q_0 + \frac{1}{4}}}ight) e^{-\pi|k|} [1 + O(\frac{1}{k})] \right|
\]

\[\text{We shall use that } \Gamma(z)\Gamma(1 - z) = [\pi/\sin(\pi z)]. \text{ See } [3] (2.2.7).\]
The discrete eigenvalues are the poles of $T(k)$ for $k \in \mathbb{C}^+$. Since the gamma function does not have any zeros but has simple poles at $0, -1, -2, \ldots$, we need to single out those values of $k \in \mathbb{C}^+$ for which

$$\frac{1}{2} - ik \pm \sqrt{Q_0 + \frac{1}{4}} = -m \in \{0, -1, -2, \ldots\}.$$ 

In other words, find those $k \in \mathbb{C}^+$ such that $k = i \left( \pm \sqrt{Q_0 + \frac{1}{4}} - (m + \frac{1}{2}) \right)$. We thus take plus the square root and those finitely many nonnegative integers $m$ that are smaller than the positive number $-\frac{1}{2} + \sqrt{Q_0 + \frac{1}{4}}$. Note that $Q_0 = L(L + 1)$ for some positive integer $L$ leads to the condition $0 \leq m < L$. Clearly, there is always at least one discrete eigenvalue, while, for $L = 1, 2, 3, \ldots$, there are $L$ (necessarily simple) eigenvalues iff

$$L - \frac{1}{2} < \sqrt{Q_0 + \frac{1}{4}} \leq L + \frac{1}{2},$$

in other words iff $L(L - 1) < Q_0 \leq L(L + 1)$.

The reflection function $R(k)$ vanishes identically iff one of the gamma functions in the denominator has a pole. This occurs iff

$$\sqrt{Q_0 + \frac{1}{4}} = L + \frac{1}{2}$$

for some positive integer $L$, i.e., iff $Q_0 = L(L + 1)$ for some positive integer $L$. In this reflectionless situation there are exactly $L$ discrete eigenvalues. Using the expressions for the Jost solutions and the identity $\frac{1}{2}[1 \pm \tanh(x)] = [e^{\pm x}/\cosh(x)]$, we obtain

$$f_{l,r}(i\kappa_m, x) = [2 \cosh(x)]^{m-L} \sum_{j=0}^{m} \frac{(2L + 1 - m)_j(-m)_j}{j!(L + 1 - m)_j} \left( \frac{1}{2}[1 \pm \tanh(x)] \right)^j$$

$$= \sum_{j=0}^{m} \frac{(2L + 1 - m)_j(-m)_j}{j!(L + 1 - m)_j} \frac{e^{\pm jx}}{2^{L-m}[\cosh(x)]^{L-m+j}},$$

where $m = 0, 1, \ldots, L - 1$. If we are not in a reflectionless situation [i.e., if $Q_0 \neq L(L + 1)$ for some positive integer $L$], then $[\Gamma(ik)/\Gamma(-ik)] \to -1$ as $k \to 0$ implies that $T(0) = 0$ and $R(0) = L(0) = -1$; we are therefore in the generic case.

It remains to compute the norming constants. Suppose there is a discrete eigenvalue at $k = i\kappa_m = i \left( \sqrt{Q_0 + \frac{1}{4}} - (m + \frac{1}{2}) \right)$. Then $[1/T(i\kappa_m)] = 0$. Using (D.4) we obtain

$$f_r(i\kappa_m, x) = \frac{\Gamma(\gamma_m)\Gamma(\gamma_m - \alpha_m - \beta_m)}{\Gamma(\gamma_m - \alpha_m)\Gamma(\gamma_m - \beta_m)} f_l(i\kappa_m, x),$$
where \( \alpha_m = -m + 2\sqrt{Q_0 + \frac{1}{4}}, \beta_m = -m, \gamma_m = 1 + \kappa_m, \gamma_m - \alpha_m - \beta_m = -\kappa_m, \gamma_m - \alpha_m = \frac{1}{2} - \sqrt{Q_0 + \frac{1}{4}}, \) and \( \gamma_m - \beta_m = \frac{1}{2} + \sqrt{Q_0 + \frac{1}{4}}. \) Hence,

\[
f_r(i\kappa_m, x) = \frac{\cos\left(\pi\sqrt{Q_0 + \frac{1}{4}}\right)}{\pi} \Gamma(1 + \kappa_m) \Gamma(-\kappa_m) f_1(i\kappa_m, x),
\]

where \( L(L - 1) < Q_0 < L(L + 1) \). It is then clear that, for \( L(L - 1) < Q_0 < L(L + 1) \), \( \Gamma(-\kappa_m) \) and \( (-1)^{L-m} \) have the same sign and \( \cos(\pi\sqrt{Q_0 + \frac{1}{4}}) \) and \( (-1)^L \) have the same sign. Thus the proportionality constant between \( f_1(i\kappa_m, x) \) and \( f_r(i\kappa_m, x) \) has the same sign as \( (-1)^m \).

Computing the residue

\[
\tau_m = \lim_{k \to i\kappa_m} (k - i\kappa_m) T(k)
= \frac{\Gamma\left(\frac{1}{2} + \kappa_m + \sqrt{Q_0 + \frac{1}{4}}\right)}{\Gamma(1 + \kappa_m) \Gamma(\kappa_m)} \lim_{k \to i\kappa_m} (k - i\kappa_m) \Gamma\left(\frac{1}{2} - ik - \sqrt{Q_0 + \frac{1}{4}}\right)
= i \frac{\Gamma\left(\frac{1}{2} + \kappa_m + \sqrt{Q_0 + \frac{1}{4}}\right)}{\Gamma(1 + \kappa_m) \Gamma(\kappa_m)} \lim_{z \to 0} z \Gamma(z - m) = i \frac{\Gamma\left(\frac{1}{2} + \kappa_m + \sqrt{Q_0 + \frac{1}{4}}\right)}{\Gamma(1 + \kappa_m) \Gamma(\kappa_m)} (-1)^m \frac{1}{m!}
\]

we get a number which is positive apart from the factor \( (-1)^m \). Using (5.5), we get the positive norming constant

\[
N_m = \frac{\Gamma\left(\frac{1}{2} + \kappa_m + \sqrt{Q_0 + \frac{1}{4}}\right)}{\Gamma(1 + \kappa_m) \Gamma(\kappa_m)} (-1)^m \frac{\Gamma(\gamma_m - \alpha_m - \beta_m)}{m! \Gamma(\gamma_m - \alpha_m) \Gamma(\gamma_m - \beta_m)}
= \frac{(-1)^m \cos\left(\pi\sqrt{Q_0 + \frac{1}{4}}\right)}{m!} \frac{\Gamma(-\kappa_m)}{\Gamma(\kappa_m)} \Gamma\left(\frac{1}{2} + \kappa_m + \sqrt{Q_0 + \frac{1}{4}}\right).
\]

It remains to compute the norming constant for \( Q_0 = L(L + 1) \). First,

\[
\lim_{Q_0 \to [L(L+1)]^{-}} \frac{\cos\left(\pi\sqrt{Q_0 + \frac{1}{4}}\right)}{L + \frac{1}{2} - \sqrt{Q_0 + \frac{1}{4}}} = \lim_{Q_0 \to [L(L+1)]^{-}} \frac{\pi \sin\left(\pi\sqrt{Q_0 + \frac{1}{4}}\right)}{L + \frac{1}{2} - \sqrt{Q_0 + \frac{1}{4}}} = (-1)^L \pi.
\]

Next, for \( z = L + \frac{1}{2} - \sqrt{Q_0 + \frac{1}{4}} = 0 \) we have

\[
(L + \frac{1}{2} - \sqrt{Q_0 + \frac{1}{4}}) \Gamma(m + \frac{1}{2} - \sqrt{Q_0 + \frac{1}{4}}) = \Gamma(z + m - L)
= \frac{\Gamma(z + 1)}{(z + m - L)(z + m + 1 - L)\ldots(z + 1)} = (-1)^{L-m} \frac{1}{(L - m)!}.
\]
Therefore,

\[ N_m = \frac{(-1)^m}{m!} (-1)^L \frac{(-1)^{L-m} \Gamma(m + 1)}{(L - m)! \Gamma(L - m)} = \frac{1}{(L - m)!(L - m - 1)!}. \]
Appendix E

IST for the KdV equation

PROBLEM: Given a real function $Q(x; 0)$ belonging to $L^1(\mathbb{R}; (1 + |x|)dx)$, find a real function $Q(x; t)$ belonging to $L^1(\mathbb{R}; (1 + |x|)dx)$ for each $t > 0$, which satisfies the Korteweg-de Vries (KdV) equation

$$Q_t - 6QQ_x + Q_{xxx} = 0. \quad (E.1)$$

E.1 Consecutive steps of the IST

For the sake of convenience we write

$$B_l(x, w; t) = K(x, x + w; t), \quad B_r(x, w; t) = M(x, x - w; t). \quad (E.2)$$

STEP 1: Starting from the initial potential $Q(x; 0)$, we solve the right-hand Volterra integral equations

$$B_l(x, w; 0) = \frac{1}{2} \int_{x+\frac{1}{2}w}^{\infty} dy Q(y; 0)$$

$$+ \frac{1}{2} \int_{0}^{w} dz \int_{x+\frac{1}{2}(w-z)}^{\infty} dy Q(y; 0) B_l(y, z; 0), \quad (E.3)$$

where $w \geq 0$. Next, we compute

$$\frac{\partial}{\partial x} B_l(x, w; 0) = -\frac{1}{2} Q(x + \frac{1}{2}w; 0)$$

$$- \frac{1}{2} \int_{0}^{w} dz Q(x + \frac{1}{2}(w - z); 0) B_l(x + \frac{1}{2}(w - z), z; 0). \quad (E.4)$$
Alternatively, starting from the initial potential $Q(x;0)$, we solve the left-hand Volterra integral equations

\begin{equation}
B_r(x, w; 0) = \frac{1}{2} \int_{-\infty}^{x-\frac{1}{2}w} dy Q(y; 0) + \frac{1}{2} \int_0^w dz \int_{-\infty}^{x-\frac{1}{2}(w-z)} dy Q(y; 0) B_r(y, z; 0),
\end{equation}

where $w \geq 0$. Next, we compute

\begin{equation}
\frac{\partial}{\partial x} B_r(x, w; 0) = \frac{1}{2} Q(x - \frac{1}{2}w; 0) + \frac{1}{2} \int_0^w dz Q(x - \frac{1}{2}(w-z); 0) B_r(x - \frac{1}{2}(w-z), z; 0),
\end{equation}

where $w \geq 0$.

**STEP 2:** Find the right Marchenko kernel $\Omega_l(x;0)$ by solving the integral equation

\begin{equation}
B_l(x, w; 0) + \Omega_l(2x + w; 0) + \int_0^\infty d\hat{w} B_l(x, \hat{w}; 0) \Omega_l(2x + \hat{w} + w; 0) = 0,
\end{equation}

where $w \geq 0$. In the same way we solve the integral equation

\begin{equation}
\frac{\partial}{\partial x} B_l(x, w; 0) + 2 [\Omega_l]_x (2x + w; 0) + \int_0^\infty d\hat{w} \frac{\partial}{\partial x} B_l(x, \hat{w}; 0) \Omega_l(2x + \hat{w} + w; 0) = 0,
\end{equation}

where $w \geq 0$.

Alternatively, find the left Marchenko kernel $\Omega_r(x;0)$ by solving the integral equation

\begin{equation}
B_r(x, w; 0) + \Omega_r(2x - w; 0) + \int_0^\infty d\hat{w} B_r(x, \hat{w}; 0) \Omega_r(2x - \hat{w} - w; 0) = 0,
\end{equation}

where $w \geq 0$. In the same way we solve the integral equation

\begin{equation}
\frac{\partial}{\partial x} B_r(x, w; 0) + 2 [\Omega_r]_x (2x - w; 0) + \int_0^\infty d\hat{w} \frac{\partial}{\partial x} B_r(x, \hat{w}; 0) \Omega_r(2x - \hat{w} - w; 0) = 0,
\end{equation}

\textsuperscript{1}Here $\frac{\partial}{\partial x} \Omega_l(2x + w; 0) = 2 [\Omega_l]_x (2x + w; 0)$. 

\textsuperscript{1}Here $\frac{\partial}{\partial x} \Omega_l(2x + w; 0) = 2 [\Omega_l]_x (2x + w; 0)$. 

\textsuperscript{1}Here $\frac{\partial}{\partial x} \Omega_l(2x + w; 0) = 2 [\Omega_l]_x (2x + w; 0)$.
where \( w \geq 0 \).

**STEP 3**: Implement the time evolution of the right Marchenko kernel by solving the PDE
\[
[\Omega^l](x; t) + 8[\Omega^l]_{xxx}(x; t) = 0,
\]
where \( \Omega^l(x; t) \) is integrable on each right half-line. In the same way we solve the PDE
\[
[2[\Omega^l]_x](x; t) + 8[2[\Omega^l]_x]_{xxx}(x; t) = 0,
\]
where \( [\Omega^l]_x(x; t) \) is integrable on each right half-line. Analogously, implement the time evolution of the left Marchenko kernel by solving the PDE
\[
[\Omega^r](x; t) + 8[\Omega^r]_{xxx}(x; t) = 0,
\]
where \( \Omega^r(x; t) \) is integrable on each left half-line. In the same way
\[
[2[\Omega^r]_x](x; t) + 8[2[\Omega^r]_x]_{xxx}(x; t) = 0,
\]
where \( [\Omega^r]_x(x; t) \) is integrable on each left half-line.

We shall give more details of Step 3 in Sec. E.2

**STEP 4**: Solve the right Marchenko equation
\[
B^l(x, w; t) + \Omega^l(2x + w; t) + \int_0^\infty dw \, B^l(x, w; t)\Omega^l(2x + w + t) = 0,
\]
where \( w \geq 0 \). Analogously, solve the left Marchenko equation
\[
B^r(x, w; t) + \Omega^r(2x - w; t) + \int_0^\infty dw \, B^r(x, w; t)\Omega^r(2x - w - t) = 0,
\]
where \( w \geq 0 \).

**STEP 5**: Solve the right Marchenko equation
\[
\frac{\partial}{\partial x} B^l(x, w; t) + \frac{\partial}{\partial x} \Omega^l(2x + w; t) \\
+ \int_0^\infty dw \, \frac{\partial}{\partial x} B^l(x, w; t)\Omega^l(2x + w + t) \\
+ \int_0^\infty dw \, B^l(x, w; t)2[\Omega^l]_x(2x + w + t) = 0,
\]
where \( w \geq 0 \). Then put
\[
Q(x; t) = -2\frac{\partial}{\partial x} B^l(x, 0^+; t).
\]
Analogously, solve the left Marchenko equation
\[
\frac{\partial}{\partial x} B_r(x, w; t) + \frac{\partial}{\partial x} \Omega_r(2x - w; t) \\
+ \int_0^\infty d\hat{w} \frac{\partial}{\partial x} B_r(x, \hat{w}; t) \Omega_r(2x - \hat{w} - w; t) \\
+ \int_0^\infty d\hat{w} B_r(x, \hat{w}; t) 2[\Omega_r]_x (2x - \hat{w} - w; t) = 0, \tag{E.19}
\]
where \( w \geq 0 \). Then put
\[
Q(x; t) = +2 \frac{\partial}{\partial x} B_r(x, 0^+; t). \tag{E.20}
\]

### E.2 Separating reflection and bound states

The third step is not easily implemented if \( \Omega_l(x; 0) \) and \( \Omega_r(x; 0) \) are integrable as \( x \to \pm \infty \) and blow up exponentially as \( x \to \mp \infty \). In this case we present an alternative to step 3. We need to do it separately for the generic and exceptional cases.

Let us not write the time dependence and start from the initial potential \( Q(x) \). Equations (3.7) imply in the generic case
\[
2i k [1 - a(k)] = \int_{-\infty}^{\infty} dy Q(y) + \int_0^\infty dw e^{ikw} \int_{-\infty}^\infty dy Q(y) B_l(y, w), \tag{E.21a}
\]
\[
2i k [1 - a(k)] = \int_{-\infty}^{\infty} dy Q(y) + \int_0^\infty dw e^{ikw} \int_{-\infty}^\infty dy Q(y) B_r(y, w), \tag{E.21b}
\]
\[
2 i k \overline{b}(k) = \int_{-\infty}^{\infty} dw e^{ikw} \left[ \frac{1}{2} Q \left( \frac{1}{2} w \right) + \int_{-\infty}^{\frac{1}{2} w} dy Q(y) B_l(y, w - 2y) \right], \tag{E.21c}
\]
\[
2 i k b(k) = \int_{-\infty}^{\infty} dw e^{ikw} \left[ \frac{1}{2} Q \left( -\frac{1}{2} w \right) + \int_{-\infty}^{\frac{1}{2} w} dy Q(y) B_r(y, w + 2y) \right]. \tag{E.21d}
\]

Hence if the potential is supported on \([L, M]\), then \( \overline{b}(k) \) and \( b(k) \) are Fourier transforms of functions in \( L^1(2L, 2M) \) and \( L^1(-2M, -2L) \), respectively, and thus entire functions. Moreover, without any support assumptions, the reflection

\[\text{Equations (E.21) are also valid in the exceptional case, but then (E.22) reduce to 0/0 divisions.}\]
and transmission coefficients are given by

\[ T(k) = \frac{2i k}{2ik - 2ik[1 - a(k)]}, \]  
\[ R(k) = \frac{2ikb(k)}{2ik - 2ik[1 - a(k)]}, \]  
\[ L(k) = \frac{2ik\tilde{b}(k)}{2ik - 2ik[1 - a(k)]}. \]

(E.22a)
(E.22b)
(E.22c)

where \( k \in \mathbb{C}^+ \) in (E.22a) and \( k \in \mathbb{R} \) in (E.22b) and (E.22c). Applying (E.22) to compute the reflection coefficients may be numerically unstable. We then compute

\[ \hat{R}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{ik\alpha} R(k), \]  
\[ \hat{L}(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-ik\alpha} L(k). \]

(E.23a)
(E.23b)

Then we apply the Prony method to one of the expressions

\[ \sum_{s=1}^{N} N_s e^{-\kappa_s x} = \Omega_l(x) - \hat{R}(x), \]  
\[ \sum_{s=1}^{N} \tilde{N}_s e^{+\kappa_s x} = \Omega_r(x) - \hat{L}(x), \]

(E.24a)
(E.24b)

and compute the positive constants \( \kappa_s \) and \( N_s \) [or: \( \kappa_s \) and \( \tilde{N}_s \)] for \( s = 1, 2, \ldots, N \).

In the exceptional case the four left-hand sides of (E.21) all vanish as \( k \to 0 \). Thus we may alter the right-hand sides of (E.21) by subtracting such right-hand sides for \( k = 0 \). If we then also divide by \( 2ik \), we get

\[ 1 - a(k) = \int_{0}^{\infty} dw \, \frac{e^{ikw} - 1}{2ik} \int_{-\infty}^{\infty} dy \, Q(y) B_l(y, w), \]  
\[ 1 - a(k) = \int_{0}^{\infty} dw \, \frac{e^{ikw} - 1}{2ik} \int_{-\infty}^{\infty} dy \, Q(y) B_r(y, w), \]

(E.25a)
(E.25b)

\[ \tilde{b}(k) = \int_{-\infty}^{\infty} dw \, \frac{e^{ikw} - 1}{2ik} \left[ \frac{1}{2} Q\left(\frac{1}{2}w\right) + \int_{-\infty}^{\frac{1}{2}w} dy \, Q(y) B_l(y, w - 2y) \right], \]  
\[ b(k) = \int_{-\infty}^{\infty} dw \, \frac{e^{ikw} - 1}{2ik} \left[ \frac{1}{2} Q\left(-\frac{1}{2}w\right) + \int_{-\infty}^{\frac{1}{2}w} dy \, Q(y) B_r(y, w + 2y) \right]. \]

(E.25c)
(E.25d)
where we can use \( \frac{e^{ikw} - 1}{(2ik)} = \frac{1}{2} \int_{-w}^{w} dv \, e^{ikv} = \frac{1}{2} \int_{0}^{0} dv \, e^{-ikv} \) to reduce the four right-hand sides of (E.25) to Fourier transforms. Hence if the potential is supported on \([L, M]\), then \( b(k) \) and \( b(k) \) are Fourier transforms of functions in \( L^1(2L, 2M) \) and \( L^1(-2M, -2L) \), respectively, and thus entire functions. Moreover, without any support assumptions, the reflection and transmission coefficients are given by

\[
T(k) = \frac{1}{1 - [1 - a(k)]}, \quad (E.26a)
\]

\[
R(k) = \frac{b(k)}{1 - [1 - a(k)]}, \quad (E.26b)
\]

\[
L(k) = \frac{\bar{b}(k)}{1 - [1 - a(k)]}. \quad (E.26c)
\]

where \( k \in \mathbb{C}^+ \) in (E.26a) and \( k \in \mathbb{R} \) in (E.26b) and (E.26c). Applying (E.26) to compute the reflection coefficients may be numerically instable. We then compute \( \hat{R}(\alpha) \) and \( \hat{L}(\alpha) \) by using (E.23) and apply Prony’s method to compute the bound states and norming constants [cf. (E.24)].

At this point, irrespective of whether we are in the generic case or in the exceptional case, we implement time evolution and hence write time dependence explicitly. We now have the following time evolution:

\[
\Omega_l(x; t) = \hat{R}(x, t) + \sum_{s=1}^{N} N_s e^{i\kappa_s t} e^{-\kappa_s x}, \quad (E.27a)
\]

\[
\Omega_r(x; t) = \hat{L}(x, t) + \sum_{s=1}^{N} \tilde{N}_s e^{-i\kappa_s t} e^{\kappa_s x}, \quad (E.27b)
\]

where

\[
\hat{R}(\alpha; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ik\alpha} e^{i\kappa^3 t} R(k), \quad (E.28a)
\]

\[
\hat{L}(\alpha; t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{-ik\alpha} e^{-i\kappa^3 t} L(k). \quad (E.28b)
\]

A more direct way of computing the time evolved Marchenko kernels is as follows:

\[
\Omega_l(x; t) = \int_{-\infty}^{\infty} db \left[(24t)^{-1/3} \text{Ai}[(24t)^{-1/3}(\alpha - b)]\right] \hat{R}(b; 0) + \sum_{s=1}^{N} N_s e^{i\kappa_s t} e^{-\kappa_s x}, \quad (E.29a)
\]
\( \Omega_r(x; t) = \int_{-\infty}^{\infty} db \left[ (24t)^{-1/3} \text{Ai}[(24t)^{-1/3}(\alpha - b)] \right] \hat{\tilde{L}}(b; 0) + \sum_{s=1}^{N} \tilde{N}_s e^{-8e^{2t}e^{+\kappa_s x}}, \)

(E.29b)

in terms of Airy transforms. Unfortunately, the difficulty of computing the Airy function for negative arguments makes it hard to apply \(\text{(E.29)}\).

### E.3 Computing reflection coefficients

A numerically stable method to compute the reflection coefficients consists of converting \(\text{(E.22b)}, \text{(E.22c)}, \text{(E.26b)}, \text{and (E.26c)}\) into integral equations for \(\hat{\tilde{R}}(\alpha)\) and \(\hat{\tilde{L}}(\alpha)\). This is most easily done in the exceptional case. Writing

\[
a(k) = 1 - \int_0^\infty dv e^{ikv} \hat{\tilde{a}}(v),
\]

(E.30a)

\[
b(k) = \int_{-\infty}^\infty dv e^{-ikv} \hat{\tilde{b}}(v),
\]

(E.30b)

\[
\hat{\tilde{b}}(k) = \int_{-\infty}^\infty dv e^{ikv} \hat{\tilde{b}}(v),
\]

(E.30c)

where

\[
\hat{\tilde{a}}(v) = \frac{1}{2} \int_{-\infty}^\infty dy Q(y) \int_v^\infty dw B_l(y, w) = \frac{1}{2} \int_{-\infty}^\infty dy Q(y) \int_v^\infty dw B_r(y, w),
\]

\[
\hat{\tilde{b}}(v) = \begin{cases} 
-\int_{-\infty}^{-v} dw \left[ \frac{1}{2} Q(-\frac{1}{2} w) + \int_{-\frac{1}{2} w}^{\infty} dy Q(y) B_r(y, w + 2y) \right], & v > 0, \\
+\int_{-\infty}^\infty dw \left[ \frac{1}{2} Q(-\frac{1}{2} w) + \int_{-\frac{1}{2} w}^{\infty} dy Q(y) B_r(y, w + 2y) \right], & v < 0,
\end{cases}
\]

\[
\hat{\tilde{b}}(v) = \begin{cases} 
+\int_{v}^\infty dw \left[ \frac{1}{2} Q(\frac{1}{2} w) + \int_{-\infty}^{\frac{1}{2} w} dy Q(y) B_l(y, w - 2y) \right], & v > 0, \\
-\int_{-\infty}^{-v} dw \left[ \frac{1}{2} Q(\frac{1}{2} w) + \int_{-\infty}^{\frac{1}{2} w} dy Q(y) B_l(y, w - 2y) \right], & v < 0,
\end{cases}
\]

we obtain

\[
\hat{\tilde{R}}(\alpha) - \int_0^\infty dv \hat{\tilde{R}}(\alpha + v) \hat{\tilde{a}}(v) = \hat{\tilde{b}}(\alpha), \quad \text{(E.31a)}
\]

\[
\hat{\tilde{L}}(\alpha) - \int_{-\infty}^0 dv \hat{\tilde{L}}(\alpha + v) \hat{\tilde{a}}(v) = \hat{\tilde{b}}(\alpha). \quad \text{(E.31b)}
\]
In the generic case the situation is more involved. Putting

\[ 2ik[1 - a(k)] = <Q> + \int_0^\infty dw e^{ikw} \hat{A}(w), \]  
\[ \text{(E.32a)} \]

\[ 2ikb(k) = \int_{-\infty}^{\infty} dw e^{-ikw} \hat{B}(w), \]  
\[ \text{(E.32b)} \]

\[ 2ikb(k) = \int_{-\infty}^{\infty} dw e^{ikw} \hat{B}(w), \]  
\[ \text{(E.32c)} \]

where \(<Q> = \int_{-\infty}^{\infty} dy Q(y)\) and

\[ \hat{A}(w) = \int_{-\infty}^{\infty} dy Q(y)B_l(y, w) = \int_{-\infty}^{\infty} dy Q(y)B_r(y, w), \]

\[ \hat{B}(w) = \left[ \frac{1}{2}Q(\frac{1}{2}w) + \int_{1/2}^{\infty} dy Q(y)B_r(y, -w + 2y) \right], \]

\[ \hat{B}(w) = \left[ \frac{1}{2}Q(\frac{1}{2}w) + \int_{-\infty}^{1/2} dy Q(y)B_l(y, w - 2y) \right], \]

and using that

\[ 2ikR(k) = 2 \int_{-\infty}^{\infty} d\alpha e^{-ik\alpha} \hat{R}'(\alpha), \]  
\[ \text{(E.33a)} \]

\[ 2ikL(k) = -2 \int_{-\infty}^{\infty} d\alpha e^{ik\alpha} \hat{L}'(\alpha), \]  
\[ \text{(E.33b)} \]

we obtain

\[ 2\hat{R}'(\alpha) = <Q> \hat{R}(\alpha) + \int_0^\infty dw \hat{R}(\alpha + w)\hat{A}(w) + \hat{B}(\alpha), \]  
\[ \text{(E.34a)} \]

\[ -2\hat{L}'(\alpha) = <Q> \hat{L}(\alpha) + \int_{-\infty}^0 dw \hat{L}(\alpha + w)\hat{A}(-w) + \hat{B}(\alpha). \]  
\[ \text{(E.34b)} \]

Integrating \((E.34a)\) on \([\alpha, +\infty)\) and \((E.34b)\) on \((-\infty, \alpha]\), we finally get

\[ \hat{R}(\alpha) = -\frac{1}{2} \int_{\alpha}^{\infty} d\beta e^{-\frac{1}{2}(\beta - \alpha)<Q>} \int_0^\infty dw \hat{R}(\beta + w)\hat{A}(w) \]

\[ - \frac{1}{2} \int_{\alpha}^{\infty} d\beta e^{-\frac{1}{2}(\beta - \alpha)<Q>} \hat{B}(\beta), \]  
\[ \text{(E.35a)} \]

\[ \hat{L}(\alpha) = -\frac{1}{2} \int_{-\infty}^{\alpha} d\beta e^{-\frac{1}{2}(\alpha - \beta)<Q>} \int_{-\infty}^0 dw \hat{L}(\beta + w)\hat{A}(-w) \]

\[ - \frac{1}{2} \int_{-\infty}^{\alpha} d\beta e^{-\frac{1}{2}(\alpha - \beta)<Q>} \hat{B}(\beta), \]  
\[ \text{(E.35b)} \]
for $<Q> \geq 0$, and

$$\hat{R}(\alpha) = \frac{1}{2} \int_{-\infty}^{\alpha} d\beta e^{\frac{1}{2}(\alpha-\beta)<Q>} \int_{0}^{\infty} dw \, \hat{R}(\beta + w) \hat{A}(w)$$

$$+ \frac{1}{2} \int_{-\infty}^{\alpha} d\beta e^{\frac{1}{2}(\alpha-\beta)<Q>} \hat{B}(\beta), \quad (E.35c)$$

$$\hat{L}(\alpha) = \frac{1}{2} \int_{\alpha}^{\infty} d\beta e^{\frac{1}{2}(\beta-\alpha)<Q>} \int_{-\infty}^{0} dw \, \hat{L}(\beta + w) \hat{A}(-w)$$

$$+ \frac{1}{2} \int_{\alpha}^{\infty} d\beta e^{\frac{1}{2}(\beta-\alpha)<Q>} \hat{B}(\beta), \quad (E.35d)$$

for $<Q> \leq 0$. 
Bibliography


