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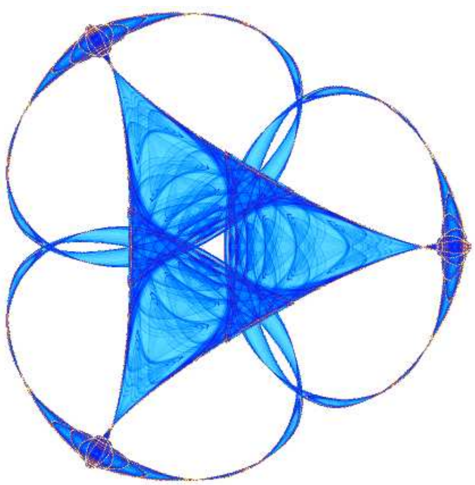

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EXACT SOLUTIONS TO THE FOCUSING NONLINEAR SCHRÖDINGER EQUATION

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Abstract: A method is presented to construct certain explicit solutions to the focusing cubic nonlinear Schrödinger equation on the line. Such solutions involve algebraic combinations of polynomials, trigonometric functions, and exponential functions of x and t. In a particular case, the analytic extensions of such solutions to the entire xt-plane yield soliton solutions where the number of solitons, the multiplicity of poles, and the norming constants are easily specified. In the general case, such solutions are well behaved in those regions in the xt-plane where a key determinant function is nonzero. The compact form of such solutions via matrix exponentials makes their symbolic and numerical evaluations easy.

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Short title: Exact solutions to the NLS equation

1. INTRODUCTION

The cubic nonlinear Schrödinger (NLS) equation in the focusing case given in (3.1) is important for many reasons [3-6,27,32]. It arises in many application areas such as wave propagation in nonlinear media [32], deep water waves [31], and nonlinear optical fibers [20-22]. It was also the second nonlinear partial differential equation (PDE) whose initial value problem was discovered [32] to be solvable via the inverse scattering method.

In this paper we present a method to construct certain explicit solutions to (3.1) in terms of elementary functions. In a particular case our solutions have analytic continuations in x and t to the entire xt-plane, at each fixed $t \in \mathbf{R}$ they decay exponentially as $x \to \pm\infty$, and they yield multisoliton solutions on the full x-axis where the number and multiplicity of the bound-state poles as well as the corresponding norming constants can be chosen at will. In other cases, such solutions are well behaved in those regions in the xt-plane where a key determinant is nonzero. Our explicit formula (5.11), or equivalently (7.9), for such exact solutions allows us to display them in terms of polynomials, trigonometric functions, and exponential functions of x and t and also to animate them using computer software. Our explicit formula (6.6), or equivalently (7.15), for the magnitude of such solutions can also be used to animate them.

The idea behind our method is similar to that used in [7] to generate explicit solutions to the Korteweg-de Vries equation on the half line and is motivated by the use of the inverse scattering transform with rational scattering data. This involves representing the corresponding scattering data in terms of a matrix realization [8], solving algebraically a related Marchenko integral equation with a separable kernel, and observing that the procedure leads to explicit solutions to the NLS equation even when the input to the Marchenko equation does not necessarily come from any scattering data.

Our method has several advantages:

(i) It is generalizable to obtain explicit solutions to other integrable nonlinear partial differential equations where the inverse scattering transform involves the use of a

Marchenko integral equation. For example, a similar method has already been used [7] for the half-line Korteweg-de Vries equation, and it can be applied to other equations such as the defocusing nonlinear Schrödinger equation, the modified Korteweg-de Vries equation, and the sine-Gordon equation.

(ii) It is generalizable to the matrix versions of the aforementioned integrable nonlinear PDEs. For example, a similar method has been applied in the second author's Ph.D. thesis [16] to the matrix NLS in the focusing case with a cubic nonlinearity.

(iii) As seen from (5.11), an explicit solution is displayed in a compact form in terms of a square matrix A, a constant row vector C, and a constant column vector B, where A appears in a matrix exponential. Depending on A, such a solution consists of combinations of exponential, trigonometric, and polynomial functions in x and t. Even though our explicit solutions are written in a simple compact form in terms of A, B, C, it may take many pages to display them in terms of exponential, trigonometric, and polynomial functions. Their compact form using matrix exponentials makes their symbolic and numerical evaluations easy, as evident from the available Mathematica notebooks [34].

(iv) Our method easily deals with nonsimple bound-state poles and the time evolution of the corresponding bound-state norming constants. In the literature, nonsimple bound-state poles are usually avoided due to mathematical complications. We refer the reader to [28], where nonsimple bound-state poles are treated and complications are encountered.

(v) Our method might be generalizable to the case where the matrix A becomes a linear operator on a separable Hilbert space. Such a generalization we are currently working on would allow us to solve the NLS with initial potentials more general than those considered in our paper.

Our method to produce exact solutions to the cubic focusing NLS equation is based on using the inverse scattering transform [3-6,27,32]. There are also other methods to obtain solutions to (3.1). Such methods include the use of a Darboux transformation [13], the use of a Bäcklund transformation [9,11], the bilinear method of Hirota [24], the use of various other transformations such as the Hasimoto transformation [12,23], and various other techniques [1] based on guessing the form of a solution and adjusting various parameters. The main idea behind using the transformations of Darboux and Bäcklund is to produce new solutions to (3.1) from previously known solutions, and other transformations are used to produce solutions to the NLS equation from solutions to other integrable PDEs. The basic idea behind the method of Hirota is to represent the solution as a ratio of two functions and to determine those two functions by solving the corresponding coupled differential equations. Other techniques may use an ansatz such as determining $\Theta(x,t)$ and $M(x,t)$ by using $u(x,t) = e^{i\Theta(x,t)}M(x,t)$ in (3.1). For example, trying

$$u(x,t) = e^{i(k_1x+k_2t+k_3)}f(k_4x + k_5t + k_6), \tag{1.1}$$

where k_j are constant real parameters and f is a real-valued smooth function, we get an exact solution if we choose $k_2 = 1-k_1^2$, $k_4 = \pm 1$, $k_5 = \mp 2k_1$, and f as the hyperbolic secant. One can also use the fact that if $U(x,t)$ is a solution to (3.1), so is $e^{ic(x-ct)}U(x-2ct,t)$ for any real constant c. Multiplying a solution by a complex constant of unit amplitude yields another solution, and hence such a phase factor can always be omitted from the solution.

There are many references in which some exact solutions to (3.1) are presented. For example, [33] lists five explicit solutions, one is of the form of (1.1) with a constant f, the second and third with f as the hyperbolic secant (those are one-soliton solutions with simple poles), the fourth being periodic in x, and the fifth is the n-soliton solution. Another solution, which is periodic in x, is [1]

$$u(x,t) = ae^{2ia^2t}\left[\frac{2b^2\cosh(2a^2b\sqrt{2-b^2}\,t) + 2ib\sqrt{2-b^2}\,\sinh(2a^2b\sqrt{2-b^2}\,t)}{2\cosh(2a^2b\sqrt{2-b^2}\,t) - \sqrt{2}\sqrt{2-b^2}\cos(\sqrt{2}\,abx)} - 1\right], \tag{1.2}$$

where a and b are arbitrary real parameters. By letting $b \to 0$ in (1.2) we get the solution

$$u(x,t) = a\,e^{2ia^2t}\frac{3 + 16ia^2t - 16a^4t^2 - 4a^2x^2}{1 + 16a^4t^2 + 4a^2x^2}.$$

Another exact solution which is periodic in x is presented [2] in terms of the Jacobi elliptic functions. An exact solution to (3.1) is displayed [18] in the form of a specific matrix realization and is shown to be valid for $t \in [0, \epsilon)$ for some small ϵ and $x \in [0, +\infty)$. In their celebrated paper [32] Zakharov and Shabat list the one- and n-soliton solutions as well as a one-soliton solution with a double pole, which is obtained from a two-soliton solution with simple poles by letting those poles coalesce. In [28] solitons with multiple eigenvalues are analyzed and a one-soliton solution with a double pole and a one-soliton solution with a triple pole are listed with the help of the symbolic software REDUCE, by stating that "in an actual calculation it is very complex to exceed" higher order poles. With our method in this paper we show that such solitons with any number of poles can be easily expressed by using an appropriate representation. Let us also add that some periodic or almost periodic solutions can be obtained in terms of two hyperelliptic theta functions [25,26], and the scattering data for (2.1) can be constructed corresponding to certain initial profiles [29,30].

Our paper is organized as follows. In Section 2 we outline the Marchenko method to solve the inverse scattering problem for the Zakharov-Shabat system given in (2.1). In Section 3 we summarize the inverse scattering transform for the NLS equation given in (3.1), and we list in (3.3) the time evolution of the norming constants in a compact form [10], which is valid even when bound-state poles may have multiplicities greater than one. In Section 4 we consider (2.1) with some rational scattering data, which in turn we express in terms of the matrices A, B, C given in (4.6)-(4.8), respectively. In Section 5, via (5.11) we present our explicit solutions to (3.1) in terms of A, B, C, and we show that such solutions have analytic extensions to the whole xt-plane when the real parts of the eigenvalues of A are positive. In Section 6 we show that (5.11) is a solution to (3.1) as long as the matrix $\Gamma(x;t)$ given in (5.7) is invertible, which is assured on the whole xt-plane when the real parts of the eigenvalues of A are positive. In Section 6 we also show that $|u(x,t)|^2$ can be expressed in terms of the logarithmic derivative of the determinant of $\Gamma(x;t)$. In Section 7 we analyze our explicit solutions given in (5.1) when the eigenvalues of A are not necessarily confined to the right half complex plane. Finally, in Section 8 we show that $u(x,t)$ given in (5.11) and $|u(x,t)|^2$ given in (6.6) can easily be displayed in terms of exponential, trigonometric, and polynomial functions by using a symbolic computation package such as Mathematica, that the "n-soliton" solution to (3.1) is obtained in the easiest case where A is diagonal with positive eigenvalues, and that various Mathematica notebooks are available [34], in which the user can produce various explicit solutions to (3.1) and their animations by specifying A, B, C.

2. PRELIMINARIES

Consider the Zakharov-Shabat system on the full line

$$\begin{bmatrix} \xi \\ \eta \end{bmatrix}' = \begin{bmatrix} -i\lambda & q(x) \\ -\overline{q(x)} & i\lambda \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}, \qquad x \in \mathbf{R}, \tag{2.1}$$

where the prime denotes the x-derivative, λ is the complex-valued spectral parameter, q is a complex-valued integrable potential, and the bar denotes complex conjugation. There are two linearly independent vector solutions to (2.1) denoted by $\psi(\lambda,x)$ and $\phi(\lambda,x)$, which are usually known as the Jost solutions and are uniquely obtained by imposing the respective asymptotic conditions

$$\psi(\lambda,x) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + o(1), \qquad x \to +\infty, \tag{2.2}$$

$$\phi(\lambda,x) = \begin{bmatrix} e^{-i\lambda x} \\ 0 \end{bmatrix} + o(1), \qquad x \to -\infty.$$

The transmission coefficient T, the left reflection coefficient L, and the right reflection coefficient R are then obtained through the asymptotics

$$\psi(\lambda,x) = \begin{bmatrix} e^{-i\lambda x} L(\lambda)/T(\lambda) \\ e^{i\lambda x}/T(\lambda) \end{bmatrix} + o(1), \qquad x \to -\infty, \tag{2.3}$$

$$\phi(\lambda,x) = \begin{bmatrix} e^{-i\lambda x}/T(\lambda) \\ e^{i\lambda x} R(\lambda)/T(\lambda) \end{bmatrix} + o(1), \qquad x \to +\infty. \tag{2.4}$$

For further information on these scattering solutions to (2.1) we refer the reader to [3-6,27,32] and the references therein.

Besides scattering solutions to (2.1), we have so-called bound-state solutions, which are square-integrable solutions to (2.1). They occur at the poles of T in the upper half complex plane $\mathbf{C}^+$. Let us denote the (distinct) bound-state poles of T by λ_j for $j = m+1, \dots, m+n$, and suppose that the multiplicity of the pole at λ_j is given by n_j. The reason to start indexing the bound states with $j = m+1$ instead of $j = 1$ is for notational convenience. It is known [3-6,27,32] that there is only one linearly independent square-integrable vector solution to (2.1) when $\lambda = \lambda_j$ for $j = m+1, \dots, m+n$. Associated with each such λ_j, we have n_j bound-state norming constants c_{js} for $s = 0, \dots, n_j - 1$.

The inverse scattering problem for (2.1) consists of recovery of $q(x)$ for $x \in \mathbf{R}$ from an appropriate set of scattering data such as the one consisting of the reflection coefficient $R(\lambda)$ for $\lambda \in \mathbf{R}$ and the bound-state information $\{\lambda_j, \{c_{js}\}_{s=0}^{n_j-1}\}_{j=m+1}^{m+n}$. This problem can be solved via the Marchenko method as follows [3-6,27,32]:

a) From the scattering data $\{R(\lambda), \{\lambda_j\}, \{c_{js}\}\}$, form the Marchenko kernel Ω as

$$\Omega(y) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda \, R(\lambda) \, e^{i\lambda y} + \sum_{j=m+1}^{m+n} \sum_{s=0}^{n_j-1} c_{js} \frac{y^s}{s!} \, e^{i\lambda_j y}. \tag{2.5}$$

b) Solve the Marchenko equation

$$K(x,y) - \overline{\Omega(x+y)} + \int_x^{\infty} dz \int_x^{\infty} ds \, K(x,s) \, \Omega(s+z) \, \overline{\Omega(z+y)} = 0, \quad y > x. \tag{2.6}$$

c) Recover the potential q from the solution $K(x,y)$ to the Marchenko equation via

$$q(x) = -2K(x,x). \tag{2.7}$$

d) Having determined $K(x,y)$, also determine

$$G(x,y) := -\int_x^{\infty} dz \, \overline{K(x,z)} \, \overline{\Omega(z+y)}. \tag{2.8}$$

Then, obtain the Jost solution $\psi(\lambda, x)$ to the Zakharov-Shabat system (2.1)-(2.2) via

$$\psi(\lambda, x) = \begin{bmatrix} 0 \\ e^{i\lambda x} \end{bmatrix} + \int_x^\infty dy \begin{bmatrix} K(x,y) \\ G(x,y) \end{bmatrix} e^{i\lambda y}. \tag{2.9}$$

Note that $|q(x)|^2$ can be calculated from (2.7) or equivalently by using [32]

$$\int_x^\infty dz\, |q(z)|^2 = -2\, G(x,x), \quad |q(x)|^2 = 2\, \frac{dG(x,x)}{dx}. \tag{2.10}$$

Let us also mention that it is possible to use the left reflection coefficient $L(\lambda)$ instead of $R(\lambda)$ in our scattering data. A Marchenko integral equation exists whose kernel uses the scattering data $\{L(\lambda), \{\lambda_j\}, \{c_{js}\}\}$, and one can recover $q(x)$, $|q(x)|^2$, and $\psi(\lambda, x)$ from the solution to that Marchenko equation by a similar procedure.

3. THE FOCUSING NLS EQUATION WITH CUBIC NONLINEARITY

Consider the focusing cubic NLS equation

$$iu_t + u_{xx} + 2|u|^2 u = 0, \tag{3.1}$$

where the subscripts denote the appropriate partial derivatives. The initial-value problem for (3.1) consists of recovery of $u(x,t)$ for $t > 0$ when $u(x,0)$ is available. When $u(x,0) = q(x)$, where q is the potential appearing in (2.1), it is known that such an initial-value problem can be solved [3-6,27,32] by the method of inverse scattering transform as indicated in the following diagram:

$$\begin{array}{ccc} u(x,0) & \xrightarrow{\text{direct scattering}} & \{R(\lambda), \{\lambda_j\}, \{c_{js}\}\} \\ \text{solution to NLS} \downarrow & & \downarrow \text{time evolution} \\ u(x,t) & \xleftarrow[\text{inverse scattering}]{} & \{R(\lambda;t), \{\lambda_j\}, \{c_{js}(t)\}\} \end{array}$$

The application of the inverse scattering transform involves three steps:

i) Corresponding to the initial potential $q(x)$, obtain the scattering data at $t = 0$; namely, the reflection coefficient $R(\lambda)$, the bound-state poles λ_j of $T(\lambda)$, and the norming constants c_{js}.

ii) Let the initial scattering data evolve in time. The time-evolved reflection coefficient $R(\lambda;t)$ is obtained from the reflection coefficient $R(\lambda)$ via

$$R(\lambda;t) = R(\lambda)\, e^{4i\lambda^2 t}. \tag{3.2}$$

The bound-state poles λ_j and $T(\lambda)$ do not change in time. The time evolution of the bound-state norming constants $c_{js}(t)$ has been known when $s = 0$ as

$$c_{j0}(t) = c_{j0}\, e^{4i\lambda_j^2 t}, \qquad j = n+1, \dots, m+n.$$

The time evolution of the remaining terms has recently been analyzed in a systematic way [10], and the evolution of $c_{js}(t)$ is described by the product of $e^{4i\lambda_j^2 t}$ and a polynomial in t of order s; we have [10]

$$[\, c_{j(n_j-1)}(t) \quad \dots \quad c_{j0}(t)\,] = [\, c_{j(n_j-1)} \quad \dots \quad c_{j0}\,]\, e^{-4iA_j^2 t}, \tag{3.3}$$

where A_j is the matrix defined in (4.3). See also [28], where a more complicated procedure is given to obtain $c_{js}(t)$.

iii) Solve the inverse scattering problem for (2.1) with the time-evolved scattering data $\{R(\lambda;t), \{\lambda_j, \{c_{js}(t)\}_{s=0}^{n_j-1}\}_{j=m+1}^{m+n}\}$ in order to obtain the time-evolved potential. It turns out that the resulting time-evolved potential $u(x,t)$ is a solution to (3.1) and reduces to $q(x)$ at $t = 0$. This inverse problem can be solved by the Marchenko method as outlined in Section 5 by replacing the kernel $\Omega(y)$ with its time-evolved version $\Omega(y;t)$, which is obtained by replacing in (2.5) $R(\lambda)$ by $R(\lambda;t)$ and c_{js} by $c_{js}(t)$.

4. USE OF SCATTERING DATA

We are interested in obtaining explicit solutions to (3.1) when the reflection coefficient $R(\lambda)$ appearing in (2.4) is a rational function of λ with poles occurring in $\mathbf{C}^+$. For this purpose we will use a method similar to the one developed in [7] and already applied to the half-line Korteweg-de Vries equation. We will first represent our scattering data in terms of a constant square matrix A, a constant column vector B, and a constant row vector C. We will then rewrite the Marchenko kernel $\Omega(y)$ given in (2.5) in terms of A, B, C. It will turn out that the time-evolved kernel $\Omega(y;t)$ will be related to $\Omega(y)$ in an easy manner. By solving the Marchenko equation (2.6) with the time-evolved kernel $\Omega(y;t)$, we will obtain the time-evolved solution $K(x,y;t)$, from which we will recover the time-evolved potential $u(x,t)$ in a manner analogous to (2.7).

In this section we show how to construct A, B, C from some rational scattering data associated with the Zakharov-Shabat system. We show that our exact solutions can be obtained by choosing our triplet A, B, C as in (4.6)-(4.8), where λ_j are distinct and $c_{j(n_j-1)} \neq 0$ for $j=1,\dots,m+n$.

When the rational $R(\lambda)$ has poles at λ_j in $\mathbf{C}^+$ with multiplicity n_j for $j=1,\dots,m$, since $R(\lambda)\to 0$ as $\lambda\to\infty$, the partial fraction expansion of $R(\lambda)$ can be written as

$$R(\lambda)=\sum_{j=1}^{m}\sum_{s=1}^{n_j}\frac{(-i)^s r_{js}}{(\lambda-\lambda_j)^s}, \tag{4.1}$$

for some complex coefficients r_{js}. Note that we can represent the inner summation in (4.1) in the form

$$\sum_{s=1}^{n_j}\frac{(-i)^s r_{js}}{(\lambda-\lambda_j)^s}=-iC_j(\lambda-iA_j)^{-1}B_j, \tag{4.2}$$

where, for $j=1,\dots,m$, we have defined

$$A_j:=\begin{bmatrix} -i\lambda_j & -1 & 0 & \dots & 0 & 0\\ 0 & -i\lambda_j & -1 & \dots & 0 & 0\\ 0 & 0 & -i\lambda_j & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & -i\lambda_j & -1\\ 0 & 0 & 0 & \dots & 0 & -i\lambda_j \end{bmatrix}, \quad B_j:=\begin{bmatrix}0\\0\\0\\\vdots\\0\\1\end{bmatrix}, \tag{4.3}$$

$$C_j:=\begin{bmatrix} r_{jn_j} & \dots & r_{j1}\end{bmatrix},$$

so that

$$\lambda-iA_j=\begin{bmatrix} \lambda-\lambda_j & i & 0 & \dots & 0 & 0\\ 0 & \lambda-\lambda_j & i & \dots & 0 & 0\\ 0 & 0 & \lambda-\lambda_j & \dots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & \lambda-\lambda_j & i\\ 0 & 0 & 0 & \dots & 0 & \lambda-\lambda_j \end{bmatrix},$$

$$
(\lambda - iA_j)^{-1} = \begin{bmatrix} \frac{1}{\lambda-\lambda_j} & \frac{-i}{(\lambda-\lambda_j)^2} & \frac{(-i)^2}{(\lambda-\lambda_j)^3} & \dots & \frac{(-i)^{n_j-2}}{(\lambda-\lambda_j)^{n_j-1}} & \frac{(-i)^{n_j-1}}{(\lambda-\lambda_j)^{n_j}} \\ 0 & \frac{1}{\lambda-\lambda_j} & \frac{-i}{(\lambda-\lambda_j)^2} & \dots & \frac{(-i)^{n_j-3}}{(\lambda-\lambda_j)^{n_j-2}} & \frac{(-i)^{n_j-2}}{(\lambda-\lambda_j)^{n_j-1}} \\ 0 & 0 & \frac{1}{\lambda-\lambda_j} & \dots & \frac{(-i)^{n_j-4}}{(\lambda-\lambda_j)^{n_j-3}} & \frac{(-i)^{n_j-3}}{(\lambda-\lambda_j)^{n_j-4}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \frac{1}{\lambda-\lambda_j} & \frac{-i}{(\lambda-\lambda_j)^2} \\ 0 & 0 & 0 & \dots & 0 & \frac{1}{\lambda-\lambda_j} \end{bmatrix}.
$$

We remark that the row vector C_j contains n_j entries, the column vector B_j contains n_j entries, and A_j is an $n_j \times n_j$ square matrix, $(-A_j)$ is in a Jordan canonical form, and that $(\lambda - iA_j)^{-1}$ is an upper triangular Toeplitz matrix.

As for the bound states, for $j = m+1, \dots, m+n$, let us use (4.3) to define the $n_j \times n_j$ matrix A_j and the column n_j-vector B_j, and let C_j be the row n_j-vector defined as

$$
C_j := [\, c_{j(n_j-1)} \quad \dots \quad c_{j0} \,], \tag{4.4}
$$

so that the summation term in (2.5) is obtained as

$$
\sum_{s=0}^{n_j-1} c_{js} \frac{y^s}{s!} \, e^{i\lambda_j y} = -\frac{i}{2\pi} \int_{-\infty}^{\infty} d\lambda \, C_j (\lambda - iA_j)^{-1} B_j e^{i\lambda y}, \qquad y > 0. \tag{4.5}
$$

Now let us define the $p \times p$ block diagonal matrix A as

$$
A := \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{m+n} \end{bmatrix}, \tag{4.6}
$$

where p is the integer given by

$$
p := \sum_{j=1}^{m+n} n_j.
$$

Similarly, let us define the column p-vector B as

$$
B := \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_{m+n} \end{bmatrix}, \tag{4.7}
$$

and the row p-vector C as

$$C := [\,C_1 \quad C_2 \quad \dots \quad C_{m+n}\,]\,. \tag{4.8}$$

Without loss of generality we can assume that λ_j for $j = 1, \dots, m+n$ are all distinct; in case one of λ_j for $j = 1, \dots, m$ coincides with one of λ_j for $j = m+1, \dots, m+n$, we can simply combine the corresponding blocks in (4.6) to reduce the number of blocks in A by one. In case more such λ_j coincide, we can proceed in a similar way so that each block in (4.6) will be associated with a distinct λ_j. Similarly, we can combine the corresponding blocks in each of (4.7) and (4.8) so that the sizes of B and C will be compatible with the size of A.

Consider the function $P(\lambda)$ defined as

$$P(\lambda) := -iC(\lambda - iA)^{-1}B, \qquad \lambda \in \mathbf{C}, \tag{4.9}$$

with the triplet A, B, C, where the constant matrices A, B, C have sizes $p \times p$, $p \times 1$, and $1 \times p$, respectively, and the singularities of $P(\lambda)$ occur at the eigenvalues of iA. Such a representation is called *minimal* [8] if there do not exist constant matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ with sizes $\tilde{p} \times \tilde{p}$, $\tilde{p} \times 1$, and $1 \times \tilde{p}$, respectively, such that $P(\lambda) = -i\tilde{C}(\lambda - i\tilde{A})^{-1}\tilde{B}$ and $\tilde{p} < p$. There always exists a triplet corresponding to a minimal representation. It is known [8] that the realization with the triplet A, B, C is minimal if and only if the two $p \times p$ matrices defined as

$$\operatorname{col}_p(C, A) := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{p-1} \end{bmatrix}, \quad \operatorname{row}_p(A, B) := [\,B \quad AB \quad \dots \quad A^{p-1}B\,], \tag{4.10}$$

both have rank p.

The following theorem shows that, for the sake of constructing exact solutions to (3.1), it is sufficient to consider only the triplet A, B, C given in (4.6)-(4.8) with distinct λ_j for $j = 1, \dots, m+n$ because any other triplet $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ with sizes $p \times p$, $p \times 1$, and $1 \times p$, respectively, can be equivalently expressed in terms of A, B, C.

Theorem 4.1 *Consider the triplet A, B, C given in (4.6)-(4.8) with distinct λ_j for $j = 1, \ldots, m{+}n$. Given any arbitrary triplet $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ with sizes $p{\times}p$, $p{\times}1$, and $1{\times}p$, respectively, there exists a triplet A, B, C having the form given in (4.6)-(4.8), respectively, which yields the same exact solution to (3.1). The construction of A, B, C can be achieved by using*

$$\tilde{A} = MAM^{-1}, \quad \tilde{B} = MSB, \quad C = \tilde{C}MS, \tag{4.11}$$

where M is an invertible matrix whose columns consist of the generalized eigenvectors of $\tilde{A}$, the matrix S is an upper triangular Toeplitz matrix commuting with A, and the complex entries of C are chosen as in (4.11).

PROOF: Since $(-A)$ is in the Jordan canonical form, any given $\tilde{A}$ can be converted to A by using $\tilde{A} = MAM^{-1}$, where M is a matrix whose columns are formed by using the generalized eigenvectors of $(-\tilde{A})$. Next, consider all matrices S commuting with A. Any such matrix has the block diagonal form

$$S := \begin{bmatrix} S_1 & 0 & \ldots & 0 \\ 0 & S_2 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & S_{m+n} \end{bmatrix}, \quad S_j := \begin{bmatrix} \alpha_{jn_j} & \alpha_{j(n_j-1)} & \cdots & \alpha_{j1} \\ 0 & \alpha_{jn_j} & \cdots & \alpha_{j2} \\ 0 & 0 & \cdots & \alpha_{j3} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{jn_j} \end{bmatrix}, \tag{4.12}$$

where n_j is the order of the pole λ_j for $j = 1, \ldots, m{+}n$, and the constants α_{js} are arbitrary. We will determine such α_{js} and hence S itself by using $M^{-1}\tilde{B} = SB$. Note that SB is the column p-vector consisting of $m + n$ column blocks, where the jth block has entries $\alpha_{j1}, \ldots, \alpha_{jn_j}$. Thus, S is unambiguously constructed from M and $\tilde{B}$. Having constructed M and S from $\tilde{A}$ and $\tilde{B}$, we finally choose the complex entries in the matrix C appearing in (4.8) so that $C = \tilde{C}MS$. Let us now show the equivalence of the representation with the triplet $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and that with the triplet A, B, C. From (4.9) we see that we must show

$$-iC(\lambda - iA)^{-1}B = -i\tilde{C}(\lambda - i\tilde{A})^{-1}\tilde{B}. \tag{4.13}$$

Since $MA = \tilde{A}M$ and $SA = AS$, we also have

$$S(\lambda - iA)^{-1} = (\lambda - iA)^{-1}S, \quad M(\lambda - iA)^{-1} = (\lambda - i\tilde{A})^{-1}M. \tag{4.14}$$

Replacing C by $\tilde{C}MS$ on the left hand side of (4.13) and using (4.14), we establish the equality in (4.13). Similarly, replacing C by $\tilde{C}MS$ on the right hand side of (5.2) and using $MA = \tilde{A}M$ and $SA = AS$ and (4.14), we prove that $\Omega(y;t)$ remains unchanged if A, B, C are replaced with $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, respectively, in (5.2). Hence the triplet A, B, C and the triplet $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ yield the same solution to (5.4). ∎

Note that the invertibility of S is not needed in Theorem 4.1. On the other hand, from (4.12) it is seen that S is invertible if and only if $\alpha_{jn_j} \neq 0$ for $j = 1, \dots, m+n$. In the rest of this section we will give a characterization for the minimality of the representation in (4.9) with the triplet A, B, C given in (4.6)-(4.8). We will show that as long as λ_j are distinct and $c_{j(n_j-1)} \neq 0$ in (4.8) for $j = 1, \dots, m+n$, the triplet A, B, C given in (4.6)-(4.8) can be used to recover in the form of (5.11) our exact solutions to (3.1). First, we need a result needed in the proof of Theorem 4.3.

Proposition 4.2 *The matrix $\text{row}_p(A,B)$ defined in (4.10) is invertible if and only if λ_j for $j = 1, \dots, m+n$ appearing in (4.6) are distinct.*

PROOF: It is enough to prove that the rows of $\text{row}_p(A,B)$ are linearly independent if and only if λ_j for $j = 1, \dots, m+n$ are distinct. We will give the proof by showing that a row-echelon equivalent matrix T defined below has linearly independent rows. Using (4.6) and (4.7) we get

$$\text{row}_p(A,B) = \begin{bmatrix} \text{row}_p(A_1,B_1) \\ \text{row}_p(A_2,B_2) \\ \vdots \\ \text{row}_p(A_{m+n},B_{m+n}) \end{bmatrix}.$$

With the help of (4.3) we see that the $n_j \times p$ matrix $\text{row}_p(A_j,B_j)$ is given by

$$\begin{bmatrix} 0 & 0 & 0 & \dots & 0 & (-1)^{p-1} \\ 0 & 0 & 0 & \dots & (-1)^{p-2} & (-1)^{p-1}(p-1)(i\lambda_j) \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots \\ 0 & -1 & 2i\lambda_j & \dots & (-1)^{p-2}(p-2)(i\lambda_j)^{p-3} & (-1)^{p-1}(p-1)(i\lambda_j)^{p-2} \\ 1 & -i\lambda_j & (i\lambda_j)^2 & \dots & (-1)^{p-2}(i\lambda_j)^{p-2} & (-1)^{p-1}(i\lambda_j)^{p-1} \end{bmatrix},$$

where we observe the binomial expansion of $(-i\lambda_j - 1)^s$ in the $(s-1)$st column. Put $\sigma(k) := \#\{j : \ n_j \geq k\}$, i.e. the number of Jordan blocks of A of size at least k. Then,

$m+n=\sigma(1)\geq\sigma(2)\geq\sigma(3)\geq ....$ By reordering the rows of $\text{row}_p(A,B)$ we obtain a row-equivalent $p\times p$ echelon matrix T such that $T_{r1}=0$ for $r>\sigma(1)$, $T_{r2}=0$ for $r>\sigma(1)+\sigma(2)$, $T_{r3}=0$ for $r>\sigma(1)+\sigma(2)+\sigma(3)$, etc., while the submatrices consisting of the elements T_{rs} for $r=\sigma(1)+\cdots+\sigma(k-1)+1,\ldots,\sigma(1)+\cdots+\sigma(k)$ and $s=k,k+1,\ldots,p$ have the form

$$\begin{bmatrix} 1 & a_{k1}\mu_1 & a_{k2}\mu_1^2 & \ldots & a_{k(k-1)}\mu_1^{p-k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_{k1}\mu_{\sigma(k)} & a_{k2}\mu_{\sigma(k)}^2 & \ldots & a_{k(k-1)}\mu_{\sigma(k)}^{p-k-1} \end{bmatrix}, \tag{4.15}$$

where apart from a sign, the coefficients a_{ks} are the binomial coefficients and hence nonzero, and the constants $\mu_1,\ldots,\mu_{\sigma(k)}$ correspond to a rearrangement of those of $-i\lambda_j$ for which $n_j\geq k$. Since the matrix given in (4.15) can be written as the product of a Vandermonde matrix and a nonsingular diagonal matrix, its rows are linearly independent if and only if λ_j with $n_j\geq k$ are distinct. From the echelon structure of the matrix T it then follows that all the rows of T, and hence the rows of $\text{row}_p(A,B)$ are linearly independent. ∎

Theorem 4.3 *The triplet A, B, C given in (4.6)-(4.8) corresponds to a minimal representation in (4.9) if and only if λ_j are all distinct and $c_{j(n_j-1)}\neq 0$ for $j=1,\ldots,m+n$.*

PROOF: Note that the matrix S defined in (4.12) commute with A, and we have $SA=AS$ and $S_jA_j=A_jS_j$ for $j=1,\ldots,m+n$. Let us use a particular choice for S_j by letting $\alpha_{j1}=c_{j0}$, $\alpha_{j2}=c_{j1},\ldots,\alpha_{jn_j}=c_{j(n_j-1)}$. Thus, S is invertible if and only if $c_{j(n_j-1)}\neq 0$ for $j=1,\ldots,m+n$. Let us define the column p-vector $\hat{B}$ and the row p-vector $\hat{C}$ via $\hat{B}=SB$ and $\hat{C}S=C$. As in the proof of (4.13) in Theorem 4.1 we obtain

$$-iC(\lambda-iA)^{-1}B=-i\hat{C}(\lambda-iA)^{-1}\hat{B}.$$

and hence the representation in (4.9) with the triplet A, B, C is equivalent to that with A, $\hat{B}$, $\hat{C}$. From the statement containing (4.10) it then follows that our theorem is proved if we can show that $\text{row}_p(A,\hat{B})$ and $\text{col}_p(\hat{C},A)$ are both invertible if and only if λ_j are all distinct and $c_{j(n_j-1)}\neq 0$ for $j=1,\ldots,m+n$. Below we will prove that $\text{row}_p(A,\hat{B})$ and $\text{col}_p(\hat{C},A)$ are invertible if and only if $\text{row}_p(A,B)$ and S are invertible. Our theorem then follows from Proposition 4.2 and the fact that S is invertible if and only if $c_{j(n_j-1)}\neq 0$ for

$j=1,\dots,m+n$. Since $SA=AS$ and $SB=\hat B$, from (4.10) we obtain

$$S\,\text{row}_p(A,B)=\text{row}_p(A,SB)=\text{row}_p(A,\hat B),$$

and hence $\text{row}_p(A,\hat B)$ is invertible if and only if $\text{row}_p(A,B)$ and S are invertible. We complete the proof by showing that $\text{col}_p(\hat C,A)$ is invertible if and only if $\text{row}_p(A,B)$ is invertible. Define the $n_j\times n_j$ matrix J_j and the $p\times p$ matrix J as

$$J_j:=\begin{bmatrix}0&0&\dots&0&1\\0&0&\dots&1&0\\ \vdots&\vdots&\ddots&\vdots&\vdots\\0&1&\dots&0&0\\1&0&\dots&0&0\end{bmatrix},\quad J:=\begin{bmatrix}J_1&0&\dots&0\\0&J_2&\dots&0\\0&0&\dots&0\\ \vdots&\vdots&\ddots&\vdots\\0&0&\dots&J_{m+n}\end{bmatrix},\tag{4.16}$$

where 1 appears along the trailing diagonal of J_j. Let us use the superscript T to denote the matrix transpose. Note that

$$J_j^{-1}=J_j,\quad J_j^T=J_j,\quad J^{-1}=J,\quad J^T=J.$$

It can be verified from (4.6) that $JAJ=A^T$. Using (4.7), (4.8), and (4.16), since $\hat C=CS^{-1}$ we get $\hat C=B^TJ$. Thus, we have

$$\Big(\text{col}_p(\hat C,A)\Big)^T=\text{row}_p(A^T,\hat C^T)=\text{row}_p(A^T,JB)=J\,\text{row}_p(A,B).$$

Since J is invertible, our proof is complete. ∎

5. EXPLICIT SOLUTIONS TO THE NLS EQUATION

In the previous section we have constructed A, B, C given in (4.6)-(4.8), respectively, from some rational scattering data of the Zakharov-Shabat system. In this section we solve the corresponding time-evolved Marchenko equation explicitly for $x\ge 0$ in terms of such A, B, C. Such solutions lead to explicit solutions to (3.1) via the formula given in (5.11). We then show that such solutions have analytic extensions to the whole xt-plane if the real parts of the eigenvalues of A are positive, which is equivalent to having $\lambda_j\in\mathbf{C}^+$ for $j=1,\dots,m+n$ in (4.3). We also analyze various properties of the key matrices $Q(x;t)$,

$N(x)$, and $\Gamma(x;t)$ that appear in (5.8) and (5.11) and that are used to construct our explicit solutions.

For $y \geq 0$, with the help (4.2), (4.5), and a contour integration along the boundary of $\mathbf{C}^+$, we evaluate the kernel $\Omega(y)$ defined in (2.5) as

$$\Omega(y) = Ce^{-Ay}B, \qquad y \geq 0. \tag{5.1}$$

Note that (5.1) yields a separable kernel for the Marchenko integral equation in (2.6) because from

$$\Omega(x+y) = Ce^{-Ax}e^{-Ay}B,$$

we see that $\Omega(x+y)$ is the Euclidean product of the row p-vector Ce^{-Ax} and the column p-vector $e^{-Ay}B$. As a result of this separability we are able to solve the Marchenko integral equation (2.6) exactly by algebraic means.

Using the time evolution of the scattering data as described in (3.2) and (3.3), from (2.5) and (4.4) it follows that the time-evolved Marchenko kernel is obtained from (5.1) as

$$\Omega(y;t) = Ce^{-Ay-4iA^2t}B, \qquad y \geq 0. \tag{5.2}$$

In other word, $\Omega(y;t)$ is obtained from $\Omega(y)$ by replacing C in (5.1) by Ce^{-4iA^2t}. Let us use a dagger to denote the matrix adjoint (complex conjugate and transpose). Since $\Omega(y;t)$ is a scalar, its complex conjugate is the same as its adjoint and we have

$$\Omega(y;t)^\dagger = B^\dagger e^{-A^\dagger y+4i(A^\dagger)^2t}C^\dagger. \tag{5.3}$$

Comparing with (2.6) we obtain the time-evolved Marchenko integral equation as

$$K(x,y;t)-\Omega(x+y;t)^\dagger+\int_x^\infty dz\int_x^\infty ds\, K(x,s;t)\,\Omega(s+z;t)\,\Omega(z+y;t)^\dagger = 0, \quad y > x. \tag{5.4}$$

Using (5.2) and (5.3) in (5.4), we see that we can look for a solution in the form

$$K(x,y;t) = H(x;t)\, e^{-A^\dagger y+4i(A^\dagger)^2t}C^\dagger, \tag{5.5}$$

where $H(x;t)$ is to be determined. Using (5.5) in (5.4), we obtain

$$H(x;t)\,\Gamma(x;t)=B^\dagger e^{-A^\dagger x}, \tag{5.6}$$

where we have defined

$$\Gamma(x;t):=I+Q(x;t)\,N(x), \tag{5.7}$$

with I denoting the $p\times p$ identity matrix and

$$Q(x;t):=\int_x^\infty ds\, e^{-A^\dagger s+4i(A^\dagger)^2 t}C^\dagger C e^{-As-4iA^2 t}, \tag{5.8}$$

$$N(x):=\int_x^\infty dz\, e^{-Az}BB^\dagger e^{-A^\dagger z}. \tag{5.9}$$

Using (5.6) in (5.5) we can write the solution to (5.4) as

$$K(x,y;t)=B^\dagger e^{-A^\dagger x}\Gamma(x;t)^{-1}e^{-A^\dagger y+4i(A^\dagger)^2 t}C^\dagger, \tag{5.10}$$

provided $\Gamma(x;t)$ is invertible. We will prove the invertibility of $\Gamma(x;t)$ in Theorem 5.2. In analogy to (2.7) we get the time-evolved potential as $u(x,t)=-2\,K(x,x;t)$, and hence the solution to (3.1) is obtained as

$$u(x,t)=-2B^\dagger e^{-A^\dagger x}\Gamma(x;t)^{-1}e^{-A^\dagger x+4i(A^\dagger)^2 t}C^\dagger. \tag{5.11}$$

It is possible [15] to write (5.11) as the ratio of two determinants as

$$u(x,t)=\frac{\det F(x;t)}{\det \Gamma(x;t)},$$

where the $(p+1)\times(p+1)$ matrix $F(x;t)$ is given by

$$F(x;t):=\begin{bmatrix} 0 & 2B^\dagger e^{-A^\dagger x} \\ e^{-A^\dagger x+4i(A^\dagger)^2 t}C^\dagger & \Gamma(x;t)\end{bmatrix}.$$

We end this section by listing some useful properties of the matrices $Q(x;t)$, $N(x)$, and $\Gamma(x;t)$.

Proposition 5.1 *The matrices $Q(x;t)$ and $N(x)$ defined in (5.8) and (5.9), respectively, satisfy*

$$Q(x;t) = e^{-A^\dagger x + 4i(A^\dagger)^2 t} Q(0;0)\, e^{-Ax - 4iA^2 t}, \quad N(x) = e^{-Ax} N(0)\, e^{-A^\dagger x}, \tag{5.12}$$

and the integrals in (5.8) and (5.9) converge for all $x, t \in \mathbf{R}$ as long as all the eigenvalues of A have positive real parts.

PROOF: By replacing s and z with $s - x$ and $z - x$ in (5.8) and (5.9), respectively, we obtain (5.12). From (5.8) and (5.9), we then get

$$Q(0;0) = \int_0^\infty ds\, [Ce^{-As}]^\dagger [Ce^{-As}], \quad N(0) = \int_0^\infty dz\, [e^{-Az}B][e^{-Az}B]^\dagger. \tag{5.13}$$

If $\epsilon > 0$ is chosen such that the real parts of the eigenvalues of A exceed ϵ, then in any matrix norm $||\cdot||$ we have $\left\| e^{-Az} \right\| = O(e^{-\epsilon z})$ and $\left\| e^{-A^\dagger z} \right\| = O(e^{-\epsilon z})$ as $z \to +\infty$. Hence, the integrals in (5.13) converge, and as a consequence of (5.12) the integrals in (5.8) and (5.9) converge for all $x, t \in \mathbf{R}$. ∎

The next theorem shows that the matrix $\Gamma(x;t)$ defined in (5.7) is invertible for all $x, t \in \mathbf{R}$ as long as the eigenvalues of A have positive real parts. In fact, in that case $\Gamma(x;t)$ has a positive determinant for all $x, t \in \mathbf{R}$.

Theorem 5.2 *Assume that the eigenvalues of A have positive real parts. Then, for every $x, t \in \mathbf{R}$ we have the following:*

(i) *The matrices $Q(x;t)$ and $N(x)$ defined in (5.8) and (5.9), respectively, are positive and selfadjoint. Consequently, there exist unique positive selfadjoint matrices $Q(x;t)^{1/2}$ and $N(x)^{1/2}$ such that $Q(x;t) = Q(x;t)^{1/2} Q(x;t)^{1/2}$ and $N(x) = N(x)^{1/2} N(x)^{1/2}$.*

(ii) *The matrix $\Gamma(x;t)$ defined in (5.7) is invertible.*

(iii) *The determinant of $\Gamma(x;t)$ is positive.*

PROOF: In our proof let us write Q and N for $Q(x;t)$ and $N(x)$, respectively. The positivity and selfadjointness of Q and N are a direct consequence of the fact that each of the integrands in (5.8) and (5.9) can be written as the product of a matrix and its adjoint; hence [19] we have proved (i). From the Sherman-Morrison-Woodbury formula [19] it follows that

$$[I+Q^{1/2}(Q^{1/2}N)]^{-1}=I-Q^{1/2}[I+(Q^{1/2}N)Q^{1/2}]^{-1}Q^{1/2}N,$$

and hence $(I+QN)$ is invertible if and only if $(I+Q^{1/2}NQ^{1/2})$ is invertible; on the other hand, the latter can be written as $[I+(Q^{1/2}N^{1/2})(Q^{1/2}N^{1/2})^\dagger]$ due to the selfadjointness of $Q^{1/2}$ and $N^{1/2}$, and hence it is invertible, establishing (ii). From the two matrix identities

$$\begin{bmatrix} I & 0 \\ Q^{1/2}N & I \end{bmatrix} \begin{bmatrix} I & Q^{1/2} \\ -Q^{1/2}N & I \end{bmatrix} \begin{bmatrix} I & -Q^{1/2} \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I+Q^{1/2}NQ^{1/2} \end{bmatrix},$$

$$\begin{bmatrix} I & -Q^{1/2} \\ 0 & I \end{bmatrix} \begin{bmatrix} I & Q^{1/2} \\ -Q^{1/2}N & I \end{bmatrix} \begin{bmatrix} I & 0 \\ Q^{1/2}N & I \end{bmatrix} = \begin{bmatrix} I+QN & 0 \\ 0 & I \end{bmatrix},$$

it follows that $I+QN$ and $(I+Q^{1/2}NQ^{1/2})$ have the same determinant. Thus, we have (iii) as a result of the fact that the determinant of $[I+(Q^{1/2}N^{1/2})(Q^{1/2}N^{1/2})^\dagger]$ is positive. ∎

Proposition 5.3 *Assume that the eigenvalues of A have positive real parts. Then, for all $x,t\in\mathbf{R}$ the matrices $Q(x;t)$, $N(x)$, $\Gamma(x;t)$ defined in (5.8)-(5.10) satisfy:*

$$Q_x=-A^\dagger Q-QA,\quad N_x=-AN-NA^\dagger,\quad Q_t=4i[(A^\dagger)^2Q-QA^2], \tag{5.14}$$

$$\Gamma^\dagger=I+NQ,\quad \Gamma^{-1}Q=Q(\Gamma^\dagger)^{-1},\quad (\Gamma^\dagger)^{-1}N=N\Gamma^{-1}. \tag{5.15}$$

PROOF: We obtain (5.14) from (5.8) and (5.9) through differentiation. Using the selfadjointness of Q and N proved in Theorem 5.2, from (5.7) we obtain (5.15). ∎

Theorem 5.4 *For every $x,t\in\mathbf{R}$, the matrices $Q(x;t)$ and $N(x)$ defined in (5.8) and (5.9), respectively, are simultaneously invertible for all $x,t\in\mathbf{R}$ if and only if the realization in (5.1) of $\Omega(y)$ with the triplet A, B, C is minimal and the eigenvalues of A have positive real parts.*

PROOF: From (5.12) we see that it is enough to prove that $Q(0;0)$ and $N(0)$ defined in (5.13) are invertible. The integrals in (5.13) are convergent as a result of the positivity of the real parts of the eigenvalues of A. If $Q(0;0)g=0$ for some vector $g\in\mathbf{C}^p$, then from

(5.13) we see that $Ce^{-As}g = 0$ for all $s \geq 0$. By analytic continuation this implies that $Ce^{-As}g = 0$ for all $s \in \mathbf{C}$ and hence

$$CA^k g = 0, \qquad k = 0, 1, \ldots. \tag{5.16}$$

Similarly, if $N(0)h = 0$ for some vector $h \in \mathbf{C}^P$, using (5.13) we conclude that

$$B^\dagger (A^\dagger)^k h = 0, \qquad k = 0, 1, \ldots. \tag{5.17}$$

It is known [8] that the realization in (4.9) or (5.1) for the triplet A, B, C is minimal if and only if the two matrices given in (4.10) both have rank p, where we recall that the size of A is $p \times p$, that of B is $p \times 1$, and that of C is $1 \times p$. On the other hand, the ranks of the two matrices in (4.10) are both p if and only if (5.16) and (5.17) have only the trivial solutions $g = 0$ and $h = 0$, respectively. ∎

For any fixed $x_0 \in \mathbf{R}$, by shifting the dummy integration variable in (5.9) we get

$$N(x) = e^{-A(x-x_0)} N(x_0) e^{-A^\dagger (x-x_0)},$$

and similarly from (5.8) for any $x_0, t_0 \in \mathbf{R}$ we get

$$Q(x;t) = e^{-A^\dagger (x-x_0) + 4i(A^\dagger)^2 (t-t_0)} Q(x_0; t_0) e^{-A(x-x_0) - 4iA^2 (t-t_0)}.$$

Thus, we have the following observations.

Corollary 5.5 *Assume that the eigenvalues of A have positive real parts. Then, the matrix $N(x)$ defined in (5.9) is invertible for all $x \in \mathbf{R}$ if and only if it is invertible at any one particular value of x. Similarly, $Q(x;t)$ defined in (5.8) is invertible for all $x, t \in \mathbf{R}$ if and only if it is invertible at any one particular point on the xt-plane.*

Proposition 5.6 *If the eigenvalues of A have positive real parts, then the matrix $\Gamma(x;t)$ defined in (5.7) satisfies $\Gamma(x;t) \to I$ as $x \to +\infty$. Additionally, if $Q(0;0)$ and $N(0)$ defined in (5.13) are invertible, then $\Gamma(x;t)^{-1} \to 0$ exponentially as $x \to -\infty$, where I and 0 are the $p \times p$ unit and zero matrices, respectively. Equivalently stated, $\Gamma(x;t)^{-1} \to 0$*

exponentially as $x \to -\infty$ if and only if the triplet A, B, C is minimal and the eigenvalues of A have positive real parts.

PROOF: As stated in Proposition 5.1, since the integrals in (5.8) and (5.9) converge, $\Gamma(x;t) \to I$ as $x \to +\infty$ follows from (5.7). To obtain the limit for $\Gamma(x;t)^{-1}$ as $x \to -\infty$, let us first define

$$Y(x;t) := e^{A^\dagger x}\Gamma(x;t)\, e^{A^\dagger x}. \tag{5.18}$$

Using (5.12) in (5.18) we get

$$Y(x;t) = Q(0;t)\, e^{-2Ax}N(0)\left[I + N(0)^{-1}e^{2Ax}Q(0;t)^{-1}e^{2A^\dagger x}\right]. \tag{5.19}$$

Note that, from Theorem 5.2 it follows that $N(0)^{-1}$ and $e^{2Ax}Q(0;t)^{-1}e^{2A^\dagger x}$ are positive selfadjoint matrices. Using the Sherman-Morrison-Woodbury formula [19] as in the proof of Theorem 5.2, we see that the inverse of the matrix in the brackets in (5.19) exists, and for all $x \in \mathbf{R}$ we have

$$Y(x;t)^{-1} = \left[I + N(0)^{-1}e^{2Ax}Q(0;t)^{-1}e^{2A^\dagger x}\right]^{-1} N(0)^{-1}e^{2Ax}Q(0;t)^{-1}. \tag{5.20}$$

Further, since the eigenvalues of A and $A^\dagger$ have strictly positive real parts, for each fixed $t \in \mathbf{R}$ we conclude, as in the proof of Proposition 5.1, that there exists $\epsilon > 0$ such that $\left\|e^{Ax}\right\| = O(e^{\epsilon x})$ and $\left\|e^{A^\dagger x}\right\| = O(e^{\epsilon x})$ as $x \to -\infty$ in any matrix norm $||\cdot||$. Hence, from (5.20) we see that $Y(x;t)^{-1} \to 0$ exponentially as $x \to -\infty$, and writing (5.18) in the form

$$\Gamma(x;t)^{-1} = e^{A^\dagger x}Y(x;t)^{-1}\, e^{A^\dagger x},$$

we also see that $\Gamma(x;t)^{-1} \to 0$ exponentially as $x \to -\infty$. The equivalent statement is a consequence of Theorem 5.4. ∎

6. FURTHER PROPERTIES OF OUR EXPLICIT SOLUTIONS

We have obtained certain explicit solutions to (3.1) in the form of (5.11) by starting with some rational scattering data for (2.1) and by constructing the corresponding matrices A, B, and C given in (4.6)-(4.8), respectively. In this section we will show that (5.11) is a solution to (3.1) no matter how the triplet A, B, C is chosen, as long as the matrix $\Gamma(x;t)$ defined in (5.7) is invertible. For example, from Theorem 5.2 it follows that $\Gamma(x;t)^{-1}$ exists on the whole xt-plane and thus (5.11) is a solution to (3.1) when the eigenvalues of A have positive real parts.

The purpose of this section is threefold. We will first obtain some useful representations for $|u(x,t)|^2$ corresponding to $u(x,t)$ given in (5.11). Next, we will prove that $u(x,t)$ given in (5.11) is a solution to (3.1) as long as $\Gamma(x;t)^{-1}$ exists. Then, we will consider further properties of such solutions.

We can evaluate $|u(x,t)|^2$ from (5.11) directly. Alternatively, we can recover it by using the time-evolved analog of (2.10), namely

$$\int_x^\infty dz\,|u(z,t)|^2 = -2\,G(x,x;t), \quad |u(x,t)|^2 = 2\,\frac{\partial G(x,x;t)}{\partial x}, \tag{6.1}$$

where, in comparison with (2.8), we see that

$$G(x,y;t) := -\int_x^\infty dz\,\Omega(y+z;t)^\dagger K(x,z;t)^\dagger. \tag{6.2}$$

From (5.3), (5.8), (5.10), and (6.2), we get

$$G(x,y;t) = -B^\dagger e^{-A^\dagger y}\Gamma(x;t)^{-1}Q(x;t)e^{-Ax}B. \tag{6.3}$$

Using (6.3) in (6.1), with the help of (5.14), (5.15), and

$$(\Gamma^{-1})_x = -\Gamma^{-1}\Gamma_x\Gamma^{-1}, \quad (\Gamma^{-1})_t = -\Gamma^{-1}\Gamma_t\Gamma^{-1}, \tag{6.4}$$

we obtain

$$|u(x,t)|^2 = 4B^\dagger e^{-A^\dagger x}\Gamma(x;t)^{-1}[A^\dagger Q(x;t) + Q(x;t)A][\Gamma(x;t)^\dagger]^{-1}e^{-Ax}B. \tag{6.5}$$

Next we show that $|u(x,t)|^2$ can be expressed in a simple form in terms of the matrix $\Gamma(x;t)$ defined in (5.7). As indicated in Theorem 5.2, recall that $\Gamma(x;t)$ has a positive determinant for all $x,t \in \mathbf{R}$ when the real parts of the eigenvalues of A are positive.

Theorem 6.1 *The absolute square* $|u(x,t)|^2$ *of the solution to the NLS equation can be written directly in terms of the determinant of the matrix* $\Gamma(x;t)$ *defined in (5.7) so that*

$$|u(x,t)|^2 = \frac{\partial}{\partial x}\left[\frac{\partial \det\Gamma(x;t)/\partial x}{\det\Gamma(x;t)}\right] = \frac{\partial^2}{\partial x^2}\left[\log\left(\det\Gamma(x;t)\right)\right]. \tag{6.6}$$

PROOF: In terms of a matrix trace, from (6.1) and (6.3) we get

$$|u(x,t)|^2 = -2\left[B^\dagger e^{-A^\dagger x}\Gamma^{-1}Q\, e^{-Ax}B\right]_x = 2\text{tr}\left[\Gamma^{-1}QN_x\right]_x, \tag{6.7}$$

where we have used (5.9) and the fact that in evaluating the trace of a product of two matrices the order in the product can be changed. With the help of (5.7), (5.14), (5.15), and the trace properties we obtain

$$\text{tr}\left[\Gamma^{-1}QN_x\right] = \text{tr}\left[-A - A^\dagger + (\Gamma^\dagger)^{-1}A + \Gamma^{-1}A^\dagger\right], \tag{6.8}$$

$$\text{tr}\left[\Gamma^{-1}Q_xN\right] = \text{tr}\left[-A - A^\dagger + (\Gamma^\dagger)^{-1}A + \Gamma^{-1}A^\dagger\right]. \tag{6.9}$$

Thus, from (6.7)-(6.9) with the help of (5.7) we get

$$2\text{tr}\left[\Gamma^{-1}QN_x\right] = \text{tr}\left[\Gamma^{-1}Q_xN + \Gamma^{-1}QN_x\right] = \text{tr}\left[\Gamma^{-1}\Gamma_x\right],$$

and hence

$$|u(x,t)|^2 = \text{tr}\left[\Gamma^{-1}\Gamma_x\right]_x,$$

which can also be written as (6.6), as indicated in Theorem 7.3 on p. 38 of [14]. ∎

We remark that (6.6) is a generalization of the formula given at the end of Section 3 of [32], where the formula was obtained for the n-soliton solution with simple poles. Thus, our method handles the bound states with nonsimple poles easily even though nonsimple poles have always caused complications in other methods and have mostly been avoided in the literature.

Let us also remark that (3.1) has infinitely many conserved quantities expressed as trace formulas. One such trace formula is given in the following.

Proposition 6.2 *When the eigenvalues of the matrix A have positive real parts, the function $u(x,t)$ given in (5.11) satisfies the trace formula*

$$\int_{-\infty}^{\infty} dx\, |u(x,t)|^2 = \operatorname{tr}\left[A + A^\dagger\right] = 2\sum_{j=1}^{m+n} n_j \operatorname{Im}[\lambda_j], \tag{6.10}$$

where λ_j and n_j are the poles in $\mathbf{C}^+$ and the corresponding multiplicities, respectively, as in (4.3).

PROOF: From (6.7) and (6.8) we see that

$$\int_{-\infty}^{\infty} dx\, |u(x,t)|^2 = \operatorname{tr}\left[-A - A^\dagger + (\Gamma^\dagger)^{-1}A + \Gamma^{-1}A^\dagger\right]\Big|_{-\infty}^{\infty}.$$

As indicated in Proposition 5.6, we have $\Gamma(x;t) \to I$ as $x \to +\infty$ and $\Gamma(x;t)^{-1} \to 0$ as $x \to -\infty$. Thus, we get the first equality in (6.10). Using (4.3) and (4.6), we can write the trace of $(A + A^\dagger)$ in terms of the multiplicities and imaginary parts of λ_j as indicated in the second equality in (6.10). ∎

Theorem 6.3 *The function $u(x,t)$ given in (5.11) satisfies (3.1) with any $p \times p$ matrix A, column p-vector B, and row p-vector C as long as the matrix $\Gamma(x;t)$ defined in (5.7) is invertible. In particular, if all eigenvalues of A have positive real parts, then $u(x,t)$ given in (5.11) satisfies (3.1) on the whole xt-plane.*

PROOF: With the help of (5.14), (5.15), and (6.4), through straightforward differentiation and after some simplifications, from (5.11) we get

$$iu_t = 8B^\dagger e^{-A^\dagger x}\Gamma^{-1}[(A^\dagger)^2 + QA^2N]\Gamma^{-1}e^{-A^\dagger x + 4i(A^\dagger)^2 t}C^\dagger, \tag{6.11}$$

$$\begin{aligned} u_x &= 4B^\dagger e^{-A^\dagger x}\Gamma^{-1}[A^\dagger - QAN]\Gamma^{-1}e^{-A^\dagger x + 4i(A^\dagger)^2 t}C^\dagger, \\ u_{xx} &= 8B^\dagger e^{-A^\dagger x}\Gamma^{-1}[(A^\dagger)^2 - 2QAN\Gamma^{-1}QAN + 2A^\dagger\Gamma^{-1}QAN - 2A^\dagger\Gamma^{-1}A^\dagger \\ &\qquad + 2QAN\Gamma^{-1}A^\dagger + QA^2N]\Gamma^{-1}e^{-A^\dagger x + 4i(A^\dagger)^2 t}C^\dagger, \end{aligned} \tag{6.12}$$

$$2uu^\dagger u = -16B^\dagger e^{-A^\dagger x}\Gamma^{-1}[(A^\dagger Q + QA)(\Gamma^\dagger)^{-1}(AN + NA^\dagger)]\Gamma^{-1}e^{-A^\dagger x + 4i(A^\dagger)^2 t}C^\dagger. \tag{6.13}$$

Using (5.15) and (6.11)-(6.13), and noting that $u^\dagger = \overline{u}$, we verify that (3.1) is satisfied. Let us note that (6.13) could also be obtained directly by multiplying (5.11) and (6.5). ∎

Theorem 6.4 *Assume that the eigenvalues of A have positive real parts and that the matrices $Q(0;0)$ and $N(0)$ defined in (5.13) are invertible, or equivalently, assume that the representation in (4.9) with the triplet A, B, C is minimal and the eigenvalues of A have positive real parts. Then, for each fixed $t \in \mathbf{R}$ the solution $u(x,t)$ given in (5.11) vanishes exponentially as $x \to \pm\infty$.*

PROOF: From (5.11) and the fact that $\Gamma(x;t) \to I$ as $x \to +\infty$, it follows that $u(x,t) \to 0$ exponentially as $x \to +\infty$ for each fixed $t \in \mathbf{R}$. Let us write (5.11) as

$$u(x,t) = -2B^\dagger Y(x;t)^{-1} e^{4i(A^\dagger)^2 t} C^\dagger, \tag{6.14}$$

where $Y(x;t)$ is the matrix defined in (5.18). In the proof of Proposition 5.6, we have shown that $Y(x;t)^{-1} \to 0$ exponentially as $x \to -\infty$. Hence, from (6.14) we can conclude that for each fixed $t \in \mathbf{R}$ we have $u(x,t) \to 0$ exponentially as $x \to -\infty$. ∎

Let us remark that, if the eigenvalues of A have positive real parts, when extended to the entire x-axis the solutions given in (5.11) become multisoliton solutions, where the number of solitons, multiplicity of the corresponding poles, and norming constants can be chosen at will. This can also be seen by analytically continuing the time-evolved Jost solution $\psi(\lambda, x; t)$ to the entire x-axis, by using (2.3), (2.9), and

$$\frac{L(\lambda;t)}{T(\lambda;t)} = \lim_{x \to -\infty} \int_x^\infty dy\, K(x,y;t)\, e^{i\lambda(y-x)}, \tag{6.15}$$

by evaluating the integral with help of (5.10), and by observing that the limit in (6.15) vanishes.

7. GENERALIZATION

In some parts of Sections 4-6 we have assumed that λ_j values appearing in (4.3) and in the matrix A given in (4.6) are all located in $\mathbf{C}^+$. In this section we relax that restriction and allow some or all λ_j to be located in the lower half complex plane $\mathbf{C}^-$. Our only restriction will be that no λ_j will be real and no two distinct λ_j will be symmetrically located with respect to the real axis in the complex plane. This restriction is mathematically equivalent to the disjointness of the sets $\{\lambda_j\}_{j=1}^{m+n}$ and $\{\overline{\lambda}_j\}_{j=1}^{m+n}$. Under this restriction we will show that $u(x,t)$ given in (5.11) is a solution to (3.1) in any region on the xt-plane in which the matrix $\Gamma(x;t)$ defined in (5.7) is invertible. The only change we need is that $Q(x;t)$ and $N(x)$ will no longer be defined as in (5.8), but instead they will be given as in (5.12), where we now let

$$Q(0;0) = \frac{1}{2\pi}\int_\gamma d\lambda\,(\lambda+iA^\dagger)^{-1}C^\dagger C(\lambda-iA)^{-1}, \tag{7.1}$$

$$N(0) = \frac{1}{2\pi}\int_\gamma d\lambda\,(\lambda-iA)^{-1}BB^\dagger(\lambda+iA^\dagger)^{-1}, \tag{7.2}$$

with γ being any positively oriented closed contour enclosing all λ_j in such a way that all $\overline{\lambda}_j$ lie outside γ.

As the following proposition shows, the quantities given in (7.1) and (7.2) are the unique (selfadjoint) solutions to the respective Lyapunov equations

$$Q(0;0)\,A + A^\dagger Q(0;0) = C^\dagger C, \tag{7.3}$$

$$A\,N(0) + N(0)\,A^\dagger = BB^\dagger. \tag{7.4}$$

We note that, using (5.12), we could also write (7.3) and (7.4) in the equivalent form

$$Q(x;t)\,A + A^\dagger Q(x;t) = e^{-A^\dagger x+4i(A^\dagger)^2 t}C^\dagger C e^{-Ax-4iA^2 t}, \tag{7.5}$$

$$A\,N(x) + N(x)\,A^\dagger = e^{-Ax}BB^\dagger e^{-A^\dagger x}. \tag{7.6}$$

Proposition 7.1 *Assume that none of the eigenvalues of A are purely imaginary and that no two eigenvalues of A are symmetrically located with respect to the imaginary axis. Equivalently, assume that $\{\lambda_j\}_{j=1}^{m+n}$ and $\{\overline{\lambda}_j\}_{j=1}^{m+n}$ are disjoint, where λ_j are the complex constants appearing in (4.3) and (4.6). We then have the following:*

(i) *The matrix equations given in (7.3) and (7.4) are each uniquely solvable.*

(ii) *The unique solutions $Q(0;0)$ and $N(0)$ are selfadjoint matrices.*

(iii) *The unique solutions are given by (7.1) and (7.2), respectively.*

PROOF: Note that (i) and (iii) directly follow from Theorem 4.1 in Section I.4 of [17]. It is straightforward to show that the adjoint of any solution to (7.3) or (7.4) is also a solution to the same equation, and hence the unique solutions $Q(0;0)$ and $N(0)$ must be selfadjoint. ∎

Next, without requiring that all λ_j appearing in (4.6) be located in $\mathbf{C}^+$, we will prove that the matrix $u(x,t)$ given in (5.11) is a solution to (3.1) as long as $\Gamma(x;t)$ defined in (5.7) is invertible. First, we will write (5.11) in a slightly different but equivalent form. Define

$$\Lambda(x;t) := I + P(x;t)^\dagger\, Q(0;0)\, P(x;t)\, N(0), \quad P(x;t) := e^{-2Ax-4iA^2t}. \tag{7.7}$$

Note that $\Gamma(x;t)$ is invertible if and only if $\Lambda(x;t)$ is invertible because, by using (5.7), (5.12), and (7.7), we see that

$$\Gamma(x;t) = e^{A^\dagger x}\Lambda(x;t)\, e^{-A^\dagger x}. \tag{7.8}$$

With the help of (7.8) we can write (5.11) in the equivalent form

$$u(x,t) = -2B^\dagger \Lambda(x;t)^{-1} P(x;t)^\dagger C^\dagger. \tag{7.9}$$

Theorem 7.2 *Assume that none of the eigenvalues of the matrix A in (4.6) are purely imaginary and that no two eigenvalues of A are symmetrically located with respect to the imaginary axis. Equivalently, assume that $\{\lambda_j\}_{j=1}^{m+n}$ and $\{\overline{\lambda}_j\}_{j=1}^{m+n}$ are disjoint, where λ_j are the complex constants appearing in (4.3) and (4.6). Then, the quantity $u(x,t)$ given in (5.11) or equivalently in (7.9) is a solution to (3.1) in any region of the xt-plane where the matrix $\Lambda(x;t)$ defined in (7.7) or equivalently the matrix $\Gamma(x;t)$ given in (5.7) is invertible.*

PROOF: In our proof let us write u, Λ, P, Q, N for $u(x,t)$, $\Lambda(x;t)$, $P(x;t)$, $Q(0;0)$, $N(0)$, respectively. Without explicitly mentioning it, we will use the selfadjointness $Q^\dagger = Q$ and $N^\dagger = N$ established in Proposition 7.1 as well as the fact that P is invertible. Proceeding as in the proof of Theorem 6.3, using straightforward differentiation on (7.9) and after some simplification we obtain

$$iu_t = 8B^\dagger \Lambda^{-1}[(A^\dagger)^2 + P^\dagger Q A^2 P N]\Lambda^{-1} P^\dagger C^\dagger, \tag{7.10}$$

where we have used the fact that

$$\Lambda = I + P^\dagger QPN, \quad (\Lambda^{-1})_t = -\Lambda^{-1}\Lambda_t\Lambda^{-1}, \quad P_t = -4iA^2P, \quad AP = PA. \tag{7.11}$$

Similarly, by using (7.11) and

$$P_x = -2AP, \quad (\Lambda^{-1})_x = -\Lambda^{-1}\Lambda_x\Lambda^{-1},$$

after some simplifications we obtain

$$u_x = 4B^\dagger\Lambda^{-1}[A^\dagger - P^\dagger QAPN]\Lambda^{-1}P^\dagger C^\dagger,$$

$$\begin{aligned} u_{xx} = 8B^\dagger\Lambda^{-1}\big[&(A^\dagger)^2 - 2A^\dagger\Lambda^{-1}A^\dagger + P^\dagger QA^2PN + 2A^\dagger\Lambda^{-1}P^\dagger QAPN \\ &+2P^\dagger QAPN\Lambda^{-1}A^\dagger - 2P^\dagger QAPN\Lambda^{-1}P^\dagger QAPN\big]\Lambda^{-1}P^\dagger C^\dagger. \end{aligned} \tag{7.12}$$

Next, with the help of (7.3) and (7.4) and using $|u|^2u = uu^\dagger u$, we obtain

$$\begin{aligned} 2|u|^2u = -16B^\dagger\Lambda^{-1}\big[&P^\dagger QAP(\Lambda^\dagger)^{-1}AN + P^\dagger QAP(\Lambda^\dagger)^{-1}NA^\dagger \\ &+P^\dagger A^\dagger QP(\Lambda^\dagger)^{-1}AN + P^\dagger A^\dagger QP(\Lambda^\dagger)^{-1}NA^\dagger\big]\Lambda^{-1}P^\dagger C^\dagger. \end{aligned} \tag{7.13}$$

We see that (3.1) is satisfied, which is verified by adding (7.10), (7.12), and (7.13) side by side and by using

$$QPN = (P^\dagger)^{-1}(\Lambda - I), \quad (\Lambda^\dagger)^{-1}N = N\Lambda^{-1}, \quad NP^\dagger Q = (\Lambda^\dagger - I)P^{-1},$$

which directly follows from (7.7) and the selfadjointness of Q and N. ∎

As the next theorem shows, if we remove the restriction $\lambda_j \in \mathbf{C}^+$ then the result in Theorem 6.1 still remains valid in any region in the xt-plane where $\Gamma(x;t)$ or equivalently $\Lambda(x;t)$ is invertible.

Theorem 7.3 *Assume that none of the eigenvalues of the matrix A in (4.6) are purely imaginary and that no two eigenvalues of A are symmetrically located with respect to the imaginary axis. Equivalently, assume that $\{\lambda_j\}_{j=1}^{m+n}$ and $\{\overline{\lambda}_j\}_{j=1}^{m+n}$ are disjoint, where λ_j are the complex constants appearing in (4.3) and (4.6). Then, in any region of the xt-plane where the matrix $\Lambda(x;t)$ defined in (7.7) or equivalently the matrix $\Gamma(x;t)$ given in (5.7)* *is invertible, the solution $u(x,t)$ given in (5.11) or equivalently in (7.9) satisfies (6.6) or equivalently*

$$|u(x,t)|^2 = \text{tr}\left[\frac{\partial}{\partial x}\left(\Gamma(x,t)^{-1}\frac{\partial\Gamma(x;t)}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\partial \det\Gamma(x;t)/\partial x}{\det\Gamma(x;t)}\right], \tag{7.14}$$

$$|u(x,t)|^2 = \text{tr}\left[\frac{\partial}{\partial x}\left(\Lambda(x,t)^{-1}\frac{\partial\Lambda(x;t)}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\frac{\partial \det\Lambda(x;t)/\partial x}{\det\Lambda(x;t)}\right]. \tag{7.15}$$

PROOF: Let us write u, Λ, P, Q, N for $u(x,t)$, $\Lambda(x;t)$, $P(x;t)$, $Q(x;t)$, $N(x)$, respectively. Using the fact that, in evaluating the trace of a product of two matrices we can change the order in the matrix product, from (7.8) we obtain

$$\text{tr}\left[\Gamma^{-1}\Gamma_x\right] = \text{tr}\left[\Lambda^{-1}\Lambda_x\right],$$

and hence it is sufficient to prove only (7.14). From (5.12) it follows that (7.5) and (7.6) are equivalent to the first two equations, respectively, in (5.14). Note that (5.15) is still valid and is a direct consequence of (5.7) and the selfadjointness of Q and N. Proceeding as in the proof of Theorem 6.1, with the help of (5.14), (5.15), and (6.4) we obtain

$$\begin{aligned} \text{tr}\left[\Gamma^{-1}\Gamma_x\right] &= 2\text{tr}\left[-A - A^\dagger + (\Gamma^\dagger)^{-1}A + \Gamma^{-1}A^\dagger\right], \\ \text{tr}\left[\Gamma^{-1}\Gamma_x\right]_x &= 4\text{tr}\left[\Gamma^{-1}(A^\dagger)^2 + (\Gamma^\dagger)^{-1}A^2 - \Gamma^{-1}A^\dagger\Gamma^{-1}A^\dagger \right. \\ &\quad \left. -(\Gamma^\dagger)^{-1}A(\Gamma^\dagger)^{-1}A + 2\Gamma^{-1}QAN\Gamma^{-1}A^\dagger\right]. \end{aligned} \tag{7.16}$$

On the other hand, using the fact that $|u|^2 = uu^\dagger$, from (5.11) we obtain

$$|u|^2 = 4\text{tr}\left[(AN + NA^\dagger)\Gamma^{-1}(QA + A^\dagger Q)(\Gamma^\dagger)^{-1}\right], \tag{7.17}$$

where we have also used (7.5) and (7.6). Using (5.15) and the aforementioned property of the matrix trace, we can simplify the right hand side of (7.17) and show that it is equal to the right hand side of (7.16). Finally, as indicated in the proof of Theorem 6.1, the second equalities in (7.14) and (7.15) follow from Theorem 7.3 on p. 38 of [14]. ∎

8. EXAMPLES

Our explicit solutions can be obtained from (5.11) by specifying A, B, and C, where $\Gamma(x;t)$ is the matrix defined in (5.7). We have made available various Mathematica notebooks [34] in which the user can easily perform the following steps:

(i) Specify $m+n$, λ_j, and n_j; form the matrix A and the column vector B defined in (4.6) and (4.7), respectively.

(ii) Specify c_{js} for $j=1,\dots,m+n$ and $s=0,1,\dots,n_j-1$; form the row vector C defined in (4.8).

(iii) Evaluate the matrix $\Gamma(x;t)$ as in (5.7), where $Q(x;t)$ and $N(x)$ are the matrices appearing in (5.12). In case all λ_j lie in $\mathbf{C}^+$, evaluate $Q(0;0)$ and $N(0)$ explicitly as in the integrals in (5.13) with the help of `MatrixExp`, which is used to evaluate matrix exponentials in Mathematica. In case some or all λ_j lie in $\mathbf{C}^-$, use (7.1) and (7.2) instead in order to evaluate explicitly $Q(0;0)$ and $N(0)$, respectively.

(iv) Having obtained $\Gamma(x;t)$, use (5.11) to display $u(x,t)$ explicitly.

(v) Using (6.6), $|u(x,t)|^2$ can be obtained explicitly and $|u(x,t)|$ can be animated.

(vi) As an option, evaluate the quantities iu_t, u_{xx}, and $2|u|^2u$, and verify directly that (3.1) is satisfied.

The well-known "n-soliton" to (3.1) is obtained when $R(\lambda)\equiv 0$ and $T(\lambda)$ has n simple bound-state poles in $\mathbf{C}^+$. In this case, from (4.6)-(4.8) we see that A, B, and C are given by

$$A=\begin{bmatrix} -i\lambda_1 & 0 & \dots & 0 \\ 0 & -i\lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -i\lambda_n \end{bmatrix},\quad B=\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix},\quad C=\begin{bmatrix}c_1 & c_2 & \dots & c_n\end{bmatrix}, \tag{8.1}$$

where λ_j are distinct and all lie in $\mathbf{C}^+$. Using (5.7)-(5.9), the (α,β)-entries of the matrices $Q(x;t)$, $N(x)$, and $\Gamma(x;t)$ are easily evaluated as

$$N_{\alpha\beta}=\frac{ie^{i(\lambda_\alpha-\overline{\lambda}_\beta)x}}{\lambda_\alpha-\overline{\lambda}_\beta},\quad Q_{\alpha\beta}=\frac{i\overline{c}_\alpha c_\beta\, e^{i(\lambda_\beta-\overline{\lambda}_\alpha)x+4i(\lambda_\beta^2-\overline{\lambda}_\alpha^2)t}}{\lambda_\beta-\overline{\lambda}_\alpha},$$

$$\Gamma_{\alpha\beta} = \delta_{\alpha\beta} - \sum_{\gamma=1}^{n} \frac{\overline{c}_\alpha c_\gamma \, e^{i(2\lambda_\gamma - \overline{\lambda}_\alpha - \overline{\lambda}_\beta)x + 4i(\lambda_\gamma^2 - \overline{\lambda}_\alpha^2)t}}{(\lambda_\gamma - \overline{\lambda}_\alpha)(\lambda_\gamma - \overline{\lambda}_\beta)}, \tag{8.2}$$

where $\delta_{\alpha\beta}$ is the Kronecker delta. A Mathematica notebook [34] is available, where the user can choose n and $\{\lambda_j, c_j\}$ for $j = 1, \dots, n$ and obtain the corresponding $u(x,t)$ explicitly and animate $|u(x,t)|$. In other Mathematica notebooks available [34], the user can display explicit solutions with single solitons having nonsimple poles by using a single block in (4.6)-(4.8) as well as multisolitons having nonsimple poles by using more blocks.

We have other Mathematica notebooks available [34] displaying and animating various exact solutions to (3.1) when some or all λ_j lie in $\mathbf{C}^-$. In fact, some simple such solutions can readily be obtained by choosing in (8.1) and (8.2) some or all λ_j in $\mathbf{C}^-$.

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