ON THE DIRECT AND INVERSE SCATTERING FOR THE MATRIX SCHRÖDINGER EQUATION ON THE LINE

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Abstract: The one-dimensional matrix Schrödinger equation is considered when the matrix potential is selfadjoint and satisfies certain general restrictions. The small-energy asymptotics of the scattering solutions and scattering coefficients are derived. The continuity of the scattering coefficients is established. The unique solvability of the corresponding matrix Marchenko integral equations is proved.

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I. INTRODUCTION

Consider the matrix Schrödinger equation

$$\psi''(k,x)+k^2\psi(k,x)=Q(x)\,\psi(k,x),\qquad x\in\mathbf{R}, \tag{1.1}$$

where $x\in\mathbf{R}$ denotes the spatial coordinate, the prime stands for the derivative with respect to x, k^2 is the energy, $Q(x)$ is an $n\times n$ selfadjoint matrix potential, i.e. $Q(x)^\dagger=Q(x)$ with the dagger standing for the matrix conjugate transpose, and $\psi(k,x)$ is either an $n\times 1$ or an $n\times n$ matrix function. Let $L^1_\alpha(I)$ with $\alpha\ge 0$ denote the Banach space of all measurable functions f for which $(1+|x|)^\alpha f(x)$ is integrable on the interval I. We will use $Q\in L^1_\alpha(\mathbf{R};\mathbf{C}^{n\times n})$ to mean that the entries of $Q(x)$ belong to $L^1_\alpha(\mathbf{R})$. In studying (1.1), we will generally use $Q\in L^1(\mathbf{R};\mathbf{C}^{n\times n})$ when $k\in\overline{\mathbf{C}^+}\setminus\{0\}$, $Q\in L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$ when $k=0$ in the generic case, and $Q\in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$ when $k=0$ in the exceptional case. The distinction between the generic and exceptional cases is indicated in Section IV. We use $\mathbf{C}^+$ to denote the upper-half complex plane and write $\overline{\mathbf{C}^+}$ for $\mathbf{C}^+\cup\mathbf{R}$.

The scattering solutions of (1.1) correspond to the solutions whose entries behave like e^{ikx} or e^{-ikx} as $x\to\pm\infty$. The bound state solutions are the solutions whose entries belong to $L^2(\mathbf{R})$, and it is known that such solutions can only occur when k takes certain values on the positive imaginary axis in $\mathbf{C}^+$. Among the $n\times n$ scattering solutions of (1.1) are the Jost solution from the left, $f_l(k,x)$, and the Jost solution from the right, $f_r(k,x)$, satisfying the boundary conditions

$$e^{-ikx}f_l(k,x)=I_n+o(1)\ \text{ and }\ e^{-ikx}f_l'(k,x)=ikI_n+o(1),\qquad x\to+\infty, \tag{1.2}$$

$$e^{ikx}f_r(k,x)=I_n+o(1)\ \text{ and }\ e^{ikx}f_l'(k,x)=-ikI_n+o(1),\qquad x\to-\infty, \tag{1.3}$$

where I_n denotes the identity matrix of order n. For each $k\in\mathbf{R}\setminus\{0\}$ we have

$$f_l(k,x)=a_l(k)e^{ikx}+b_l(k)e^{-ikx}+o(1),\qquad x\to-\infty, \tag{1.4}$$

$$(1.5) \qquad f_r(k,x) = a_r(k)e^{-ikx} + b_r(k)e^{ikx} + o(1), \qquad x \to +\infty,$$

where $a_l(k)$, $b_l(k)$, $a_r(k)$, and $b_r(k)$ are certain $n \times n$ matrix functions of k.

The direct scattering problem for (1.1) consists of the analysis of the scattering matrix given in (2.22) when $Q(x)$ is known; on the other hand, the inverse scattering problem deals with the determination of $Q(x)$ in terms of an appropriate set of scattering data involving a reflection coefficient and information on the bound states. The direct and inverse scattering problems for (1.1) were studied in Refs. 1-6. In our paper we will concentrate on what has not been done in the references cited, namely the analysis of the scattering solutions and coefficients of (1.1) at $k = 0$ and the unique solvability of the Marchenko equations arising in the analysis of the inverse scattering problem for (1.1). In Sections II and III we establish our notations and present a review of some basic known results; for the proofs and details we refer the reader to Refs. 3 and 5. We prove the continuity of the scattering matrix at $k = 0$ and obtain its asymptotics as $k \to 0$; the generic case is treated in Section IV and the exceptional case is treated in Section V. Finally, in Section VI we prove the unique solvability of the Marchenko integral equations.

The study of the inverse scattering problem for (1.1) with a selfadjoint $Q(x)$ is related to the analysis of the Cauchy problem for certain nonlinear evolution equations such as the n-component nonlinear Schrödinger equation[2,3] and the Calogero-Degasperis equations.[3,7,8] There are also other evolution equations related to (1.1) when $Q(x)$ is nonselfadjoint, but we will confine our analysis to the case with selfadjoint $Q(x)$.

The pioneering paper[1] for the study of the inverse scattering problem for (1.1) is that of Wadati and Kamijo, in which $Q(x)$ was assumed continuous, selfadjoint, and belonging to $L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$; however, some of the results stated and used in Ref. 1 such as the analytic extension of the reflection coefficients from $\mathbf{R}$ to $\mathbf{C}^+$ cannot hold under the stated conditions on $Q(x)$. In the review paper based on Ref. 1, Wadati[2] assumed that $Q(x)$ is selfadjoint and continuous and its entries "decrease sufficiently rapidly as $x \to \pm\infty$." Schuur[4] analyzed the inverse scattering problem for (1.1) when $Q(x)$ is nonselfadjoint, continuous, and belonging to $L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$. None of these three papers addressed the behavior of the scattering solutions and scattering coefficients as $k\to 0$. In order to avoid any problems that may arise at $k=0$, based on the results in Ref. 9 for the scalar case, $Q\in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$ was assumed in Refs. 3 and 5, where the inverse scattering problem for (1.1) was analyzed for a nonselfadjoint $Q(x)$ under some additional assumptions. Our work supplements the analysis in Refs. 3 and 5 for a selfadjoint $Q(x)$, as we provide the analysis of the scattering solutions and scattering coefficients as $k\to 0$ and prove the unique solvability of the Marchenko integral equations. Finally, when the reflection coefficient is a rational function of k, Alpay and Gohberg,[6] by using some results stated in Ref. 2, obtained the (selfadjoint) potential in terms of the minimal realization of the reflection coefficient by stating without a proof that the unique solvability of the Marchenko equation holds in the matrix case as it does in the scalar case.

II. SCATTERING SOLUTIONS AND SCATTERING COEFFICIENTS

In this section we review some basic results regarding (1.1). The proofs can be found in Refs. 3 and 5 or are simple generalizations of those for the scalar case.[9,10] Let $[F;G]=FG'-F'G$ denote the Wronskian of the square matrix functions F and G of x. Let us use an asterisk to denote complex conjugation and use $||\cdot||$ to denote the operator norm. The following standard result is based on the selfadjointness of Q.

Proposition 2.1 Let $\phi(k,x)$ be any $n\times p$ solution and $\psi(k,x)$ be any $n\times q$ solution of (1.1). Then, for $k\in\overline{\mathbf{C}^+}$, the $p\times q$ Wronskian matrix $[\phi(-k^*,x)^\dagger;\psi(k,x)]$ is independent of x. For real k, the Wronskian matrices $[\phi(\pm k,x)^\dagger;\psi(k,x)]$ are independent of x.

Define the matrix Faddeev functions $m_l(k,x)$ and $m_r(k,x)$ as

$$m_l(k,x)=e^{-ikx}f_l(k,x),\qquad m_r(k,x)=e^{ikx}f_r(k,x). \tag{2.1}$$

Then, from (1.1)-(1.5) we obtain for $k \in \mathbf{R} \setminus \{0\}$

$$m_l(k,x) = I_n + \frac{1}{2ik} \int_x^\infty dy\, [e^{2ik(y-x)} - 1]\, Q(y)\, m_l(k,y), \tag{2.2}$$

$$m_r(k,x) = I_n + \frac{1}{2ik} \int_{-\infty}^x dy\, [e^{2ik(x-y)} - 1]\, Q(y)\, m_r(k,y). \tag{2.3}$$

The properties of the Faddeev functions are summarized in the next theorem. We use C to denote a generic constant that does not depend on x or k.

Theorem 2.2 Assume $Q \in L^1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then, for each $x \in \mathbf{R}$, the functions $m_l(k,x)$, $m_r(k,x)$, $m_l'(k,x)$, and $m_r'(k,x)$ are analytic in $k \in \mathbf{C}^+$ and continuous in $k \in \overline{\mathbf{C}^+} \setminus \{0\}$. For each $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, these four functions are continuous in $x \in \mathbf{R}$. Consequently, for each $x \in \mathbf{R}$, the Jost solutions and their x-derivatives are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+} \setminus \{0\}$; for each $k \in \overline{\mathbf{C}^+} \setminus \{0\}$ they are continuous in $x \in \mathbf{R}$. Moreover, for each $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, we have

$$\|m_l(k,x)\| \le Ce^{C/|k|}, \qquad \|m_r(k,x)\| \le Ce^{C/|k|},$$

$$m_l(k,x) = I_n + o(1), \qquad m_l'(k,x) = o(1), \quad x \to +\infty,$$

$$m_r(k,x) = I_n + o(1), \qquad m_r'(k,x) = o(1), \quad x \to -\infty.$$

Theorem 2.3 Assume $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then, the continuity of the functions in Theorem 2.2 is valid also as $k \to 0$ in $\overline{\mathbf{C}^+}$, and we have

$$\|m_l(k,x)\| \le C[1 + \max\{0,-x\}], \qquad \|m_r(k,x)\| \le C[1 + \max\{0,x\}], \tag{2.4}$$

$$m_l'(k,x) = o(1/x), \quad x \to +\infty; \qquad m_r'(k,x) = o(1/x), \quad x \to -\infty,$$

uniformly in $k \in \overline{\mathbf{C}^+}$.

Corollary 2.4 Assume Q belongs to $L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then, for each fixed $x \in \mathbf{R}$, the matrices $f_l(-k^*,x)^\dagger$, $f_r(-k^*,x)^\dagger$, $f_l'(-k^*,x)^\dagger$, and $f_r'(-k^*,x)^\dagger$ are analytic in $k \in \mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$.

Theorem 2.5 Assume $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then, for each fixed $x \in \mathbf{R}$, we have

$$(2.5) \qquad m_l(k,x) = I_n + \int_0^\infty dy\, B_l(x,y)\, e^{iky}, \qquad m_r(k,x) = I_n + \int_0^\infty dy\, B_r(x,y)\, e^{iky},$$

where every entry of $B_l(x,\cdot)$ and $B_r(x,\cdot)$ belongs to $L^1(\mathbf{R}^+) \cap L^2(\mathbf{R}^+)$.

The coefficient matrices $a_l(k)$, $b_l(k)$, $a_r(k)$, and $b_r(k)$ appearing in (1.4) and (1.5) can be expressed in terms of certain Wronskians of the Jost solutions, as follows:

$$(2.6) \qquad a_l(k) = \frac{1}{2ik}[f_r(-k^*,x)^\dagger; f_l(k,x)], \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\},$$

$$(2.7) \qquad a_r(k) = -\frac{1}{2ik}[f_l(-k^*,x)^\dagger; f_r(k,x)], \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\},$$

$$(2.8) \qquad b_l(k) = -\frac{1}{2ik}[f_r(k,x)^\dagger; f_l(k,x)], \qquad k \in \mathbf{R} \setminus \{0\},$$

$$(2.9) \qquad b_r(k) = \frac{1}{2ik}[f_l(k,x)^\dagger; f_r(k,x)], \qquad k \in \mathbf{R} \setminus \{0\}.$$

From (1.4), (1.5), and (2.1)-(2.3) it follows that

$$(2.10) \qquad a_l(k) = I_n - \frac{1}{2ik}\int_{-\infty}^\infty dx\, Q(x)\, m_l(k,x),$$

$$(2.11) \qquad b_l(k) = \frac{1}{2ik}\int_{-\infty}^\infty dx\, e^{2ikx}\, Q(x)\, m_l(k,x),$$

$$(2.12) \qquad a_r(k) = I_n - \frac{1}{2ik}\int_{-\infty}^\infty dx\, Q(x)\, m_r(k,x),$$

$$(2.13) \qquad b_r(k) = \frac{1}{2ik}\int_{-\infty}^\infty dx\, e^{-2ikx}\, Q(x)\, m_r(k,x).$$

Proposition 2.6 Assume $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then $ka_l(k)$ and $ka_r(k)$ are continuous in $\overline{\mathbf{C}^+}$ and analytic in $\mathbf{C}^+$, $\det\{a_l(k)\} = \det\{a_r(k)\}$ for $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, and

$$a_l(k) = I_n + O(1/k) \ \text{ and } \ a_r(k) = I_n + O(1/k), \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}.$$

Moreover, $kb_l(k)$ and $kb_r(k)$ are continuous in $\mathbf{R}$ and

$$b_l(k) = o(1/k) \quad \text{and} \quad b_r(k) = o(1/k), \qquad k \to \pm\infty.$$

In general $b_l(k)$ and $b_r(k)$ do not have extensions to $k \in \mathbf{C}^+$, and hence (1.4) and (1.5) cannot be valid for $k \in \mathbf{C}^+$; instead, one must use the following result for the spatial asymptotics of the Jost solutions for $k \in \mathbf{C}^+$, whose proof can be obtained as in the scalar case.

Proposition 2.7 If $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$, then for each fixed $k \in \mathbf{C}^+$ we have

$$m_l(k,x) = a_l(k) + o(1), \qquad x \to -\infty,$$

$$m_r(k,x) = a_r(k) + o(1), \qquad x \to +\infty.$$

Using (1.2)-(1.5) and some Wronskian relations for the Jost solutions, we obtain

$$a_r(-k^*)^\dagger = a_l(k), \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\}, \tag{2.14}$$

$$b_r(k) = -b_l(k)^\dagger, \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.15}$$

$$a_l(k)^\dagger a_l(k) = b_l(k)^\dagger b_l(k) + I_n, \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.16}$$

$$a_r(k)^\dagger a_r(k) = b_r(k)^\dagger b_r(k) + I_n, \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.17}$$

$$a_l(-k)^\dagger b_l(k) = b_l(-k)^\dagger a_l(k), \qquad k \in \mathbf{R} \setminus \{0\}, \tag{2.18}$$

$$a_r(-k)^\dagger b_r(k) = b_r(-k)^\dagger a_r(k), \qquad k \in \mathbf{R} \setminus \{0\}. \tag{2.19}$$

From (2.16) and (2.17) we have the following result.

Corollary 2.8 Assume $Q \in L^1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then $a_l(k)$ and $a_r(k)$ are nonsingular for $k \in \mathbf{R} \setminus \{0\}$.

Even though $a_l(k)$ and $a_r(k)$ extend to $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, $b_l(k)$ and $b_r(k)$ in general do not have extensions off the real axis. Such extensions can only be established under special restrictive conditions, such as in the following, where a proof can be obtained as in the scalar case.

Proposition 2.9 If $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$ vanishes when $x > 0$, then $b_r(k)$ has an analytic extension to $\mathbf{C}^+$ and that extension is continuous for $k \in \overline{\mathbf{C}^+}$. Similarly, if $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$ vanishes when $x < 0$, then $b_l(k)$ has an analytic extension to $\mathbf{C}^+$ and that extension is continuous for $k \in \overline{\mathbf{C}^+}$.

Wherever $a_l(k)$ and $a_r(k)$ are invertible, we define the transmission coefficient from the left, $T_l(k)$, and the transmission coefficient from the right, $T_r(k)$, as

$$T_l(k) = a_l(k)^{-1}, \qquad T_r(k) = a_r(k)^{-1}, \tag{2.20}$$

and the reflection coefficient from the left, $L(k)$, and reflection coefficient from the right, $R(k)$, as

$$L(k) = b_l(k)\, a_l(k)^{-1}, \qquad R(k) = b_r(k)\, a_r(k)^{-1}. \tag{2.21}$$

Let $\mathbf{S}(k)$ denote the scattering matrix given by

$$\mathbf{S}(k) = \begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{bmatrix}. \tag{2.22}$$

From (2.15), (2.16), and (2.17), we respectively get

$$T_l(k)^\dagger R(k) + L(k)^\dagger T_r(k) = 0, \qquad k \in \mathbf{R} \setminus \{0\},$$

$$T_l(k)^\dagger T_l(k) + L(k)^\dagger L(k) = I_n, \qquad k \in \mathbf{R} \setminus \{0\},$$

$$T_r(k)^\dagger T_r(k) + R(k)^\dagger R(k) = I_n, \qquad k \in \mathbf{R} \setminus \{0\},$$

and hence $\mathbf{S}(k)$ is unitary for $k \in \mathbf{R} \setminus \{0\}$. Using (2.14), (2.18), and (2.19) we obtain

$$T_r(k) = T_l(-k^*)^\dagger, \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\}, \tag{2.23}$$

except at the poles of $T_r(k)$ in $\mathbf{C}^+$, and

$$L(-k)^\dagger = L(k), \qquad R(-k)^\dagger = R(k), \qquad k \in \mathbf{R} \setminus \{0\}.$$

We will study the behavior of $\mathbf{S}(k)$ as $k \to 0$ in Sections IV and V.

III. SINGULARITIES AND BOUND STATES

In terms of the Jost solutions of (1.1), let us define

$$F(k,x) = \begin{bmatrix} f_l(k,x) & f_r(k,x) \\ f_l'(k,x) & f_r'(k,x) \end{bmatrix}. \tag{3.1}$$

Proposition 3.1 Assume $Q \in L^1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then, the determinant of $F(k,x)$ is independent of x and we have

$$\det\{F(k,x)\} = (-2ik)^n \det\{a_l(k)\} = (-2ik)^n \det\{a_r(k)\}, \qquad k \in \overline{\mathbf{C}^+} \setminus \{0\}. \tag{3.2}$$

Theorem 3.2 Assume $Q \in L^1(\mathbf{R}; \mathbf{C}^{n\times n})$, and fix $k_0 \in \mathbf{C}^+ \setminus \{0\}$. Then, the following statements are equivalent:

(i) The $2n$ vectors formed by the columns of $f_l(k_0,x)$ and $f_r(k_0,x)$ are linearly dependent.

(ii) The Wronskian $[f_r(-k_0^*,x)^\dagger; f_l(k_0,x)]$ is singular.

(iii) The Wronskian $[f_l(-k_0^*,x)^\dagger; f_r(k_0,x)]$ is singular.

(iv) $a_l(k_0)$ and $a_r(k_0)$ are both singular.

(v) $a_l(k_0)$ is singular.

(vi) $a_r(k_0)$ is singular.

(vii) (1.1) has a nontrivial $n \times n$ matrix solution with entries in $L^2(\mathbf{R})$.

(viii) (1.1) has a nontrivial $n \times n$ matrix solution with entries that are continuous in x and that decay exponentially as $x \to \pm\infty$.

Theorem 3.3 Assume that $Q \in L^1(\mathbf{R}; \mathbf{C}^{n\times n})$. Except for possibly a countable set of points on the positive imaginary axis in $\mathbf{C}^+$, none of (i)-(viii) of Theorem 3.2 hold for $k \in \overline{\mathbf{C}^+} \setminus \{0\}$.

We will establish the continuity of $T_l(k)$ at $k = 0$ by assuming $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$ in the generic case (see Proposition 4.2) and $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$ in exceptional case (see Corollary 5.11). These two results then imply the finiteness of the number of zeros of $\det a_l(k)$. More precisely, we have the following result.

Proposition 3.4 Assume $Q \in L^1_1(\mathbf{R}; \mathbf{C}^{n\times n})$ in the generic case and $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$ in the exceptional case. Then, except at a finite set of points on the positive imaginary axis where $\det a_l(k) = 0$, $T_l(k)$ and $T_r(k)$ are continuous in $\overline{\mathbf{C}^+}$ and analytic in $\mathbf{C}^+$. Moreover, the poles of $T_l(k)$ and $T_r(k)$ on the positive imaginary axis are simple.

Because of (3.2) the poles of $T_l(k)$ and $T_r(k)$ coincide in $\mathbf{C}^+$. Let us denote these common poles as the distinct points $k = i\kappa_j$ with $j = 1, \dots, N$; they correspond to the bound states of (1.1).

Proposition 3.5 Assume $Q \in L^1(\mathbf{R}; \mathbf{C}^{n\times n})$ and let $k = i\kappa_j$ correspond to the bound states of (1.1). Then, there exist unique constant $n \times n$ matrices C_{lj} and C_{rj} such that

$$f_r(i\kappa_j, x)\,[\operatorname{Res} T_l(i\kappa_j)] = i\, f_l(i\kappa_j, x)\, C_{lj}, \qquad x \in \mathbf{R}, \tag{3.3}$$

$$f_l(i\kappa_j, x)\ [\operatorname{Res} T_r(i\kappa_j)] = i\, f_r(i\kappa_j, x)\, C_{rj}, \qquad x \in \mathbf{R}, \tag{3.4}$$

where $\operatorname{Res} T_l(i\kappa_j)$ and $\operatorname{Res} T_r(i\kappa_j)$ denote the residues of $T_l(k)$ and $T_r(k)$, respectively, at $k = i\kappa_j$. Moreover, C_{lj} and C_{rj} are positive selfadjoint matrices.

The proof of Proposition 3.5 can be found in Appendix A of Ref. 5. The matrices C_{lj} and C_{rj} are the analogs of the norming constants in the scalar case.

IV. SMALL k-BEHAVIOR IN THE GENERIC CASE

In this section we analyze the behavior of the Jost solutions and the scattering coefficients at $k = 0$.

When $k = 0$, from (2.1)-(2.3) we get

$$f_l(0, x) = m_l(0, x) = I_n + \int_x^{\infty} dy\,(y - x)\,Q(y)\,m_l(0, y),$$

$$f_r(0, x) = m_r(0, x) = I_n + \int_{-\infty}^{x} dy\,(x - y)\,Q(y)\,m_r(0, y),$$

from which the following result follows.

Proposition 4.1 Assume $Q \in L_1^1(\mathbf{R}; \mathbf{C}^{n\times n})$. Then $m_l(0, \cdot)$ is continuous on $\mathbf{R}$, $m_l(0, x) = I_n + o(1)$ as $x \to +\infty$, and $m_l(0, x) = O(x)$ as $x \to -\infty$. Similarly, $m_r(0, \cdot)$ is continuous, $m_r(0, x) = O(x)$ as $x \to +\infty$, and $m_r(0, x) = I_n + o(1)$ as $x \to -\infty$.

In terms of the matrices $a_l(k)$ and $a_r(k)$ appearing in (1.4) and (1.5), let us define

$$\Delta_l = \lim_{k\to 0} 2ik\,a_l(k), \qquad \Delta_r = \lim_{k\to 0} 2ik\,a_r(k), \tag{4.1}$$

where the limits are taken from within $\overline{\mathbf{C}^+}$. From (2.6), (2.7), (2.10), and (2.12) we see that

$$\Delta_l = [f_r(0, x)^\dagger; f_l(0, x)] = -\int_{-\infty}^{\infty} dx\,Q(x)\,m_l(0, x), \tag{4.2}$$

$$\Delta_r = -[f_l(0, x)^\dagger; f_r(0, x)] = -\int_{-\infty}^{\infty} dx\,Q(x)\,m_r(0, x), \tag{4.3}$$

$$\Delta_l = \Delta_r^\dagger. \tag{4.4}$$

From (3.2) and (4.1) it follows that the $2n$ columns of $f_l(0,x)$ and $f_r(0,x)$ are linearly independent if and only if Δ_l is nonsingular. Let ρ denote the rank of Δ_l. From (4.4) it follows that Δ_l is singular if and only if Δ_r is singular, and that the ranks of Δ_l and Δ_r are the same. In analogy with the scalar Schrödinger equation, we will call $Q(x)$ a generic potential if $\rho = n$. We will call $Q(x)$ an exceptional potential of order $n-\rho$ when $\rho < n$. The case $\rho = 0$ will be called the purely exceptional case.

Proposition 4.2 Assume $Q(x)$ is a generic potential belonging to $L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$. Then, the scattering coefficients are continuous at $k=0$ and

$$R(0) = L(0) = -I_n, \qquad T_l(0) = T_r(0) = 0. \tag{4.5}$$

PROOF: Using (2.10), (2.12), (2.20), and (4.1) we get

$$T_l(k) = 2ik[\Delta_l^{-1} + o(1)], \quad T_r(k) = 2ik[\Delta_r^{-1} + o(1)], \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}. \tag{4.6}$$

From (2.8), (2.9), (2.11), (2.13), (4.2), and (4.3), we obtain

$$\lim_{k\to 0} 2ik\, b_l(k) = -\Delta_l, \qquad \lim_{k\to 0} 2ik\, b_r(k) = -\Delta_r. \tag{4.7}$$

Generically, Δ_l and Δ_r are nonsingular, and hence using (2.11), (2.13), (2.21), and (4.7), we get

$$L(k) = -I_n + o(1), \quad R(k) = -I_n + o(1), \qquad k \to 0 \text{ in } \mathbf{R}. \tag{4.8}$$

Thus, the proof is complete. ∎

Proposition 4.2, shows that the zeros of $\det a_l(k)$ and $\det a_r(k)$ cannot accumulate at $k=0$. Thus, we have the following result.

Corollary 4.3 Assume $Q(x)$ is a generic potential belonging to $L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$. Then, the zeros of $\det a_l(k)$ in $\overline{\mathbf{C}^+}$ are confined to the positive imaginary axis and their number is finite.

In the generic case, (4.5) holds and hence $||L(0)|| = ||R(0)|| = 1$. In the exceptional case of order $n-\rho$ for some positive rank ρ of Δ_l, we also have $||L(0)|| = ||R(0)|| = 1$, as we will see in Proposition 5.14. Thus, in general, we cannot expect $||L(0)|| < 1$ or $||R(0)|| < 1$ as stated by Alpay and Gohberg.[6] This is apparent in the scalar case or when $Q(x)$ is a diagonal matrix. Let $Q(x) = \text{diag}\,(v_1(x), v_2(x), \dots, v_n(x))$, and assume that exactly ρ of the scalar potentials are generic and the remaining $n-\rho$ are exceptional. Let us use $t_j(k)$, $l_j(k)$, and $r_j(k)$ to denote the transmission coefficient and the reflection coefficients from the left and right, respectively, for the scalar potential $v_j(k)$. We then get

$$T(k) = \text{diag}\,\big(t_1(k), t_2(k), \dots, t_n(k)\big),$$

$$R(k) = \text{diag}\,\big(r_1(k), r_2(k), \dots, r_n(k)\big),$$

$$L(k) = \text{diag}\,\big(l_1(k), l_2(k), \dots, l_n(k)\big).$$

Hence, unless $\rho = 0$, we have $||R(0)|| = ||L(0)|| = 1$. Note that if $\rho = 0$, then all the n scalar potentials are exceptional and we have

$$||L(0)|| = ||R(0)|| = \max\{|r_1(0)|, \dots, |r_n(0)|\} < 1.$$

Whenever $Q \in L^1(\mathbf{R}; \mathbf{C}^{n\times n})$, both in the generic and exceptional cases, from Proposition 4.1 it follows that the n columns of $f_l(0,x)$ are linearly independent, and similarly the n columns of $f_r(0,x)$ are linearly independent. As indicated earlier, in the exceptional case the $2n$ columns of $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent. Hence, there are $n\times n$ constant nonzero matrices M_l and M_r of maximal rank such that

$$f_l(0,x)\,M_l + f_r(0,x)\,M_r = 0. \tag{4.9}$$

Let

$$\phi(x) = f_l(0,x)\,M_l = -f_r(0,x)\,M_r. \tag{4.10}$$

Because of (1.2), (1.3), and Theorem 2.1, it follows that $\phi(x)$ is a continuous and bounded solution of (1.1), and in fact

$$\phi(x) = M_l + o(1), \quad x \to +\infty; \qquad \phi(x) = -M_r + o(1), \quad x \to -\infty. \tag{4.11}$$

From (2.15), (2.16), and (4.10) we see that

$$\phi(x) = M_l + \int_x^\infty dy\,(y-x)\,Q(y)\,\phi(y) = -M_r + \int_{-\infty}^x dy\,(x-y)\,Q(y)\,\phi(y). \tag{4.12}$$

In order for (4.11) to hold, from (4.12) we must have

$$\int_{-\infty}^\infty dy\,Q(y)\,\phi(y) = 0, \tag{4.13}$$

$$\int_{-\infty}^\infty dy\,y\,Q(y)\,\phi(y) = -(M_l + M_r). \tag{4.14}$$

Note that (4.13) is equivalent to Δ_l and Δ_r being singular.

Proposition 4.4 Assume $Q(x)$ is an exceptional potential belonging to $L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$, and suppose M_l and M_r are some constant $n\times n$ matrices satisfying (4.9). Then, the columns of M_l belong to the kernel of Δ_l and the maximum number of linearly independent such columns is equal to $n-\rho$, where ρ is the rank of Δ_l; the columns of $f_l(0,x)\,M_l$ are bounded vector solutions of (1.1) at $k=0$, and the total number of such linearly independent bounded vector solutions of (1.1) at $k=0$ is given by $n-\rho$. Similarly, the columns of M_r belong to the kernel of Δ_r and the maximum number of linearly independent such columns is equal to $n-\rho$; the columns of $f_r(0,x)\,M_r$ are bounded vector solutions of (1.1) at $k=0$, and the total number of such linearly independent bounded vector solutions of (1.1) at $k=0$ is given by $n-\rho$. Moreover, we have

$$\operatorname{Ker}\Delta_l \perp \operatorname{Im}\Delta_r, \qquad \operatorname{Ker}\Delta_r \perp \operatorname{Im}\Delta_l. \tag{4.15}$$

PROOF: If M_l satisfies (4.9) for some M_r, then $\phi(x)$ defined in (4.10) satisfies (4.13), which is equivalent to having $M_l \in \operatorname{Ker}\Delta_l$, because of (4.2). Conversely, if $M_l \in \operatorname{Ker}\Delta_l$, then $\phi(x)$ defined in (4.10) satisfies (4.11). The proof involving M_r and $f_r(0,x)$ is obtained in a similar way. Since the nullity and rank of Δ_l must add up to n, it follows that Ker Δ_l has dimension $n-\rho$. Finally, (4.15) follows from (4.4). ∎

Proposition 4.5 Assume $Q \in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$. Then, the rank of $F(0,x)$, where $F(k,x)$ is the matrix defined in (3.1), is equal to $n+\rho$, where ρ is the rank of Δ_l.

PROOF: In the generic case, the columns of $f_l(0,x)$ and $f_r(0,x)$ form $2n$ linearly independent vectors solutions of (1.1) at $k=0$. In the exceptional case, by Proposition 4.4, exactly $2n-(n-\rho)$ among the $2n$ columns of $f_l(0,x)$ and $f_r(0,x)$ are linearly independent. Hence the rank of $F(0,x)$ is equal to $n+\rho$. In fact, with the help of (1.3), (1.4), (4.1), and (4.7), we can show that $F(0,+\infty)$ is row equivalent to $\begin{bmatrix} I_n & 0 \\ 0 & \Delta_r \end{bmatrix}$ and $F(0,-\infty)$ is row equivalent to $\begin{bmatrix} I_n & 0 \\ 0 & \Delta_l \end{bmatrix}$. Since the rank of $F(0,x)$ is independent of x, it is equal to the rank of $\begin{bmatrix} I_n & 0 \\ 0 & \Delta_l \end{bmatrix}$, which is equal to $n+\rho$. ∎

V. SMALL k-BEHAVIOR IN THE EXCEPTIONAL CASE

Now let us turn to the exceptional case, i.e. when Δ_l and Δ_r are singular. In this case we will analyze the asymptotics of the scattering coefficients as $k\to 0$ under the stronger condition $Q \in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. This condition also allows us to improve the estimates in (4.6) and (4.8) in the generic case. Let an overdot denote the derivative with respect to k.

Proposition 5.1 Assume $Q \in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. Then, for each $x\in\mathbf{R}$, $\dot{m}_l(\cdot,x)$ and $\dot{m}_r(\cdot,x)$ are analytic in $\mathbf{C}^+$, are continuous in $\overline{\mathbf{C}^+}$, and satisfy

$$||\dot{m}_l(\cdot,x)|| \le C(1+x^2), \quad ||\dot{m}_r(\cdot,x)|| \le C(1+x^2), \qquad x\in\mathbf{R}.$$

PROOF: Similar to the proof given in the scalar case on pp. 134–136 of Ref. 9. ∎

Proposition 5.2 Assume $Q \in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. Then we have

$$a_l(k) = \frac{1}{2ik}\Delta_l + I_n + \frac{i}{2}G_l + o(1), \qquad k\to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{5.1}$$

$$a_r(k) = \frac{1}{2ik}\Delta_r + I_n + \frac{i}{2}G_r + o(1), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}, \tag{5.2}$$

$$b_l(k) = -\frac{1}{2ik}\Delta_l + E_l - \frac{i}{2}G_l + o(1), \qquad k \to 0 \text{ in } \mathbf{R}, \tag{5.3}$$

$$b_r(k) = -\frac{1}{2ik}\Delta_r - E_r - \frac{i}{2}G_r + o(1), \qquad k \to 0 \text{ in } \mathbf{R}, \tag{5.4}$$

where Δ_l and Δ_r are the matrices defined in (4.2) and (4.3), respectively, and

$$E_l = \int_{-\infty}^{\infty} dx\, x\, Q(x)\, m_l(0,x), \qquad E_r = \int_{-\infty}^{\infty} dx\, x\, Q(x)\, m_r(0,x), \tag{5.5}$$

$$G_l = \int_{-\infty}^{\infty} dx\, Q(x)\, \dot{m}_l(0,x), \qquad G_r = \int_{-\infty}^{\infty} dx\, Q(x)\, \dot{m}_r(0,x).$$

PROOF: These follow by expanding the integrals in (2.10)-(2.13) and by using Proposition 5.1 and the mean value theorem. ▮

In general, as seen from (5.1)-(5.4), the quantities $a_l(0)$, $a_r(0)$, $b_l(0)$, and $b_r(0)$ do not exist. However, under the assumption $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$, both $a_l(k) + b_l(k)$ and $a_r(k) + b_r(k)$ have well-defined limits as $k \to 0$, which we will denote by $a_l(0) + b_l(0)$ and $a_r(0) + b_r(0)$, respectively.

Proposition 5.3 Assume $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$. Then, $a_l(0) + b_l(0)$ and $a_r(0) + b_r(0)$ are both nonsingular and are inverses of each other.

PROOF: Because of (2.4) and (5.1)-(5.5), both $a_l(0) + b_l(0)$ and $a_r(0) + b_r(0)$ are well defined when $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$. Using (2.14)-(2.19) in the limit as $k \to 0$ in $\mathbf{R}$, we get

$$[a_l(0) + b_l(0)][a_r(0) + b_r(0)] = [a_r(0) + b_r(0)][a_l(0) + b_l(0)] = I_n. \tag{5.6}$$

Thus, the proof is complete. ▮

Using (2.10), (2.11), (4.2), and (4.10) we can write (4.13) and (4.14) as

$$\Delta_l M_l = 0, \qquad M_r = -[a_l(0) + b_l(0)]\, M_l.$$

From (5.1)-(5.4), we see that

$$a_l(0)+b_l(0)=I_n+E_l, \qquad a_r(0)+b_r(0)=I_n-E_r, \tag{5.7}$$

and hence, with the help of (5.6) we conclude that

$$G_r=-G_l^\dagger, \qquad E_r=E_l^\dagger+iG_l^\dagger, \qquad E_l-E_r=E_lE_r=E_rE_l.$$

Proposition 5.4 Assume $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$. Then, in the generic case we have

$$T_l(k)=2ik\,\Delta_l^{-1}+k^2\,\Delta_l^{-1}[4-G_l]\Delta_l^{-1}+o(k^2), \qquad k\to 0 \text{ in } \overline{\mathbf{C}^+},$$

$$T_r(k)=2ik\,\Delta_r^{-1}+k^2\,\Delta_r^{-1}[4-G_r]\Delta_r^{-1}+o(k^2), \qquad k\to 0 \text{ in } \overline{\mathbf{C}^+},$$

$$L(k)=-I_n+2ik\,[I_n+E_l]\,\Delta_l^{-1}+o(k), \qquad k\to 0 \text{ in } \mathbf{R},$$

$$R(k)=-I_n+2ik\,[I_n-E_r]\,\Delta_r^{-1}+o(k), \qquad k\to 0 \text{ in } \mathbf{R}.$$

PROOF: These follow from (2.20) and (2.21) with the help of Proposition 5.3. ∎

Proposition 5.5 Assume $Q\in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$ and let $1\le q\le n$. Then, for any $n\times q$ matrix M_l that satisfies (4.9), there is a unique $n\times q$ matrix M_r satisfying (4.9). Conversely, for each $n\times q$ matrix M_r satisfying (4.9), there corresponds a unique $n\times q$ matrix M_l that satisfies (4.9).

PROOF: In (4.9), by letting $x\to\pm\infty$, we get

$$M_r=-[a_l(0)+b_l(0)]\,M_l, \tag{5.8}$$

$$M_l=-[a_r(0)+b_r(0)]\,M_r. \tag{5.9}$$

Because of (5.6), (5.8) and (5.9) establish a unique link between M_l and M_r. ∎

Proposition 5.6 Assume $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$ and let $1 \le q \le n$. Then, for any $n \times q$ matrices M_l and M_r satisfying (5.8) or (5.9), we have

$$\lim_{k\to 0} M_r^\dagger\, a_l(k)\, M_l = -\frac{1}{2}\left[M_r^\dagger M_r + M_l^\dagger M_l\right] + o(1), \qquad k \in \overline{\mathbf{C}^+}. \tag{5.10}$$

PROOF: Note that M_l belongs to $\operatorname{Ker}\Delta_l$ as shown in Proposition 4.4, and M_r is determined by M_l as indicated Proposition 5.5. Hence, from (5.1) we see that the limit in (5.10) exists and unique and is equal to $M_r^\dagger[I_n + iG_l/2]M_l$ as $k \to 0$ in $\overline{\mathbf{C}^+}$. Thus, in order to obtain (5.10) it is sufficient to evaluate the limit as $k \to 0$ in $\mathbf{R}$. Note that

$$\begin{aligned} a_l(k) &= \frac{1}{2}\left[a_l(k) + b_l(k)\right] + \frac{1}{2}\left[a_l(k) - b_l(k)\right] \\ &= \frac{1}{2}\left[a_l(k) + b_l(k)\right] + \frac{1}{2}\left[a_r(-k)^\dagger + b_r(k)^\dagger\right], \end{aligned} \tag{5.11}$$

where we have used (2.14) and (2.15). From (5.1)-(5.4) we obtain

$$a_r(-k) + b_r(k) = -\frac{1}{ik}\Delta_r + a_r(k) + b_r(k) + o(1), \qquad k \to 0 \text{ in } \mathbf{R}. \tag{5.12}$$

Using (5.12) in (5.11) we get

$$M_r^\dagger\, a_l(k)\, M_l = \frac{1}{2} M_r^\dagger \left[a_l(k) + b_l(k)\right] M_l + \frac{1}{2}\left[\left(-\frac{1}{ik}\Delta_r + a_r(k) + b_r(k)\right) M_r\right]^\dagger M_l + o(1), \tag{5.13}$$

By Proposition 4.4 we have $\Delta_r M_r = 0$, and by using (5.8) and (5.9) in (5.13) we obtain (5.10). ∎

Proposition 5.7 Assume $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$. If $M_l \in \operatorname{Ker}\Delta_l$ and M_r is as in (5.8), then (5.10) holds. Similarly, if $M_r \in \operatorname{Ker}\Delta_r$ and M_l is as in (5.9), then (5.10) holds.

PROOF: Because of (5.6), (5.8) and (5.9) are equivalent. Using (5.8), (5.9), and the fact that $M_l \in \operatorname{Ker}\Delta_l$ on the right hand side of (5.13), we get (5.10). The proof of the second assertion is obtained in a similar manner. ∎

Proposition 5.8 Assume $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$. Then

$$\Delta_l[a_r(0) + b_r(0)] = [a_r(0) + b_r(0)]^\dagger \Delta_r, \tag{5.14}$$

$$\Delta_r[a_l(0)+b_l(0)]=[a_l(0)+b_l(0)]^\dagger\Delta_l. \tag{5.15}$$

PROOF: In terms of the Jost solutions of (1.1) we get

$$[f_l(k,x)^\dagger;f_l(k,x)]=2ikI_n=2ik\left[a_l(k)^\dagger a_l(k)-b_l(k)^\dagger b_l(k)\right],\qquad k\in\mathbf{R}, \tag{5.16}$$

$$[f_r(k,x)^\dagger;f_r(k,x)]=-2ikI_n=-2ik\left[a_r(k)^\dagger a_r(k)-b_r(k)^\dagger b_r(k)\right],\qquad k\in\mathbf{R}. \tag{5.17}$$

Using (5.1)-(5.4) and (5.7), we see that the terms proportional to k on both sides of (5.16) and (5.17) lead to (5.14) and (5.15). ∎

Having obtained the small-k asymptotics of the scattering coefficients in the generic case, we now turn to finding such asymptotics in the exceptional case. Since Δ_l defined in (4.1) is not invertible in the exceptional case, we cannot directly use (2.20) and (2.21) to determine such asymptotics.

Let us use $W(k)$ to denote $2ika_l(k)$. By (5.1) we see that $W(0)=\Delta_l$. In general we do not expect Δ_l to be diagonalizable, but we can always put it in a Jordan normal form by using a special Jordan basis. Let $\tilde W(k)$ denote the matrix $W(k)$ in the special Jordan basis of $W(0)$, i.e.

$$\tilde W(k)=\bigoplus_{\alpha=1}^{\gamma}\tilde W_\alpha(k),\qquad \tilde W(0)=\bigoplus_{\alpha=1}^{\gamma}J_{n_\alpha}(\lambda_\alpha), \tag{5.18}$$

with $J_{n_\alpha}(\lambda_\alpha)$ is the $n_\alpha\times n_\alpha$ given by

$$J_{n_\alpha}(\lambda_\alpha)=\begin{bmatrix}\lambda_\alpha&1&0&\dots&0&0\\0&\lambda_\alpha&1&\dots&0&0\\0&0&\lambda_\alpha&\dots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\dots&\lambda_\alpha&1\\0&0&0&\dots&0&\lambda_\alpha\end{bmatrix}, \tag{5.19}$$

where λ_α denotes the corresponding eigenvalue of Δ_l. Let us assume that $\lambda_1=\cdots=\lambda_\mu=0$ and the remaining $\lambda_\alpha\neq0$ for $\alpha=\mu+1,\dots,\gamma$. Since $\operatorname{Ker}\Delta_l$ has dimension $n-\rho$, we have

$\sum_{\alpha=1}^{\mu} n_\alpha = n-\rho$ and $\sum_{\alpha=\mu+1}^{\gamma} n_\alpha = \rho$. Let $\{\xi_{\alpha j}\}$ denote our special Jordan basis with $\alpha=1,\dots,\gamma$ and $j=1,\dots,n_\alpha$ so that

$$\xi_{\alpha j} = (\Delta_l - \lambda_\alpha)^{n_\alpha - j}\xi_{\alpha n_\alpha}, \qquad (\Delta_l-\lambda_\alpha)^{n_\alpha}\xi_{\alpha n_\alpha} = 0. \tag{5.20}$$

We can construct another basis $\{\chi_{\alpha j}\}$ with $\alpha=1,\dots,\gamma$ and $j=1,\dots,n_\alpha$ so that

$$\chi_{\alpha j}^\dagger \xi_{\rho t} = \delta_{\alpha\rho}\delta_{jt}, \tag{5.21}$$

where δ_{jt} denotes the Kronecker delta. Since

$$[(\Delta_r - \lambda_\alpha^*)\chi_{\alpha(j-1)} - \chi_{\alpha j}]^\dagger \xi_{\alpha t} = \chi_{\alpha(j-1)}^\dagger \xi_{\alpha(t-1)} - \chi_{\alpha j}^\dagger \xi_{\alpha t},$$

with the help of (5.20) and (5.21), it follows that

$$\chi_{\alpha j} = (\Delta_r - \lambda_\alpha^*)^{j-1}\chi_{\alpha 1}, \qquad (\Delta_r - \lambda_\alpha^*)^{n_\alpha}\chi_{\alpha 1} = 0. \tag{5.22}$$

Proposition 5.9 Assume $Q \in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$, and let $[\tilde{W}_\alpha(k)]_{n_\alpha,1}$ denote the $(n_\alpha,1)$ entry of the matrix $\tilde{W}_\alpha(k)$ defined in (5.18). Then, as $k\to 0$ in $\mathbf{R}$ we have

$$[\tilde{W}_\alpha(k)]_{n_\alpha,1} = c_\alpha k + o(k), \qquad \alpha = 1,\dots,\mu, \tag{5.23}$$

for some nonzero constant c_α.

PROOF: Because of (5.21) we have

$$\chi_{\alpha t}^\dagger W(k)\,\xi_{\rho j} = \xi_{\alpha t}^\dagger W(k)\,\xi_{\rho j}.$$

Recall that $\lambda_\alpha = 0$ for $\alpha = 1,\dots,\mu$, and hence $\Delta_r \xi_{\alpha n_\alpha} = 0$; thus, with the help of (5.13) we obtain

$$[\tilde{W}_\alpha(k)]_{n_\alpha,1} = ik\,\chi_{\alpha n_\alpha}^\dagger[a_l(0)+b_l(0)]\xi_{\alpha 1} + ik\left([a_r(0)+b_r(0)]\chi_{\alpha n_\alpha}\right)^\dagger \xi_{\alpha 1} + o(k). \tag{5.24}$$

By (5.20), $\xi_{\alpha 1} \in \operatorname{Ker}\Delta_l$ because $\lambda_\alpha = 0$; then (5.15) implies that $[a_l(0)+b_l(0)]\xi_{\alpha 1} \in \operatorname{Ker}\Delta_r$. Therefore, with the help of (5.21) we conclude that $[a_l(0)+b_l(0)]\xi_{\alpha 1} = \omega_\alpha \chi_{\alpha n_\alpha}$, for some nonzero ω_α, because $\chi_{\alpha j}$ with $1 \le j \le n_\alpha - 1$ do not belong to $\operatorname{Ker}\Delta_r$ for this particular α. In a similar way, using (5.14), (5.15), (5.21), and (5.22), we conclude that $[a_r(0)+b_r(0)]\chi_{\alpha n_\alpha} = \dfrac{1}{\omega_\alpha}\xi_{\alpha 1}$. Thus, (5.24) implies that

$$[\tilde W_\alpha(k)]_{n_\alpha,1} = ik\,\frac{1}{\omega_\alpha^*}[|\omega_\alpha|^2\chi_{\alpha n_\alpha}^\dagger\chi_{\alpha n_\alpha} + \xi_{\alpha 1}^\dagger\xi_{\alpha 1}] + o(k),$$

which gives us (5.23). ∎

Proposition 5.10 Assume $Q \in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$, and let c_α be the constant appearing in (5.23). Then, as $k \to 0$ in $\mathbf{R}$ we have

$$2ik\tilde W_\alpha(k)^{-1} = \begin{cases} 2ikD_\alpha[I_{n_\alpha}+o(1)], & \alpha = 1,\dots,\mu, \\ 2ikJ_{n_\alpha}(\lambda_\alpha)^{-1}[I_{n_\alpha}+o(1)], & \alpha = \mu+1,\dots,\gamma, \end{cases} \tag{5.25}$$

where $J_{n_\alpha}(\lambda_\alpha)$ is the Jordan matrix defined in (5.19) and

$$D_\alpha(k) = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1/(ikc_\alpha) \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

PROOF: The proof for $\alpha = \mu+1,\dots,\gamma$ follows from (5.18). For $\alpha = 1,\dots,\mu$, we proceed as follows. With the help of Proposition 5.9 we obtain

$$\tilde W_\alpha(k) = \begin{bmatrix} O(k) & 1+O(k) & O(k) & \dots & O(k) & O(k) \\ O(k) & O(k) & 1+O(k) & \dots & O(k) & O(k) \\ O(k) & O(k) & O(k) & \dots & O(k) & O(k) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O(k) & O(k) & O(k) & \dots & O(k) & 1+O(k) \\ ikc_\alpha + o(k) & O(k) & O(k) & \dots & O(k) & O(k) \end{bmatrix}.$$

Writing

$$\tilde W_\alpha(k) = A_\alpha(k)\,[H_\alpha + o(1)], \qquad \alpha = 1,\dots,\mu,$$

where

$$A_\alpha(k) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & ik \end{bmatrix}, \qquad H_\alpha = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ c_\alpha & 0 & 0 & \dots & 0 & 0 \end{bmatrix},$$

we see that

$$\tilde{W}_\alpha(k)^{-1} = [H_\alpha^{-1} + o(1)]A_\alpha(k)^{-1},$$

from which (5.25) follows for $\alpha = 1, \dots, \mu$. ▮.

Since $T_l(k) = 2ik\, W(k)$, from Proposition 5.10 we obtain the following result.

Corollary 5.11 Assume $Q \in L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$, and let c_α be the constant appearing in (5.23). Then, $T_l(k)$ is continuous and differentiable at $k = 0$; moreover, as $k \to 0$, in the special basis in (5.20) the transmission coefficient $T_l(k)$ is given by

$$\tilde{T}_l(k) = \tilde{T}_l(0) + k\,\dot{\tilde{T}}_l(0) + o(k), \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+},$$

where

$$\tilde{T}_l(0) = \bigoplus_{\alpha=1}^{\gamma} \frac{2}{c_\alpha} P_\alpha, \qquad \dot{\tilde{T}}_l(0) = -\frac{i}{2} \bigoplus_{\alpha=1}^{\gamma} U_\alpha,$$

with

$$P_\alpha = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad U_\alpha = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \qquad \alpha = 1, \dots, \mu,$$

P_α is the $n_\alpha \times n_\alpha$ zero matrix and $U_\alpha = J_{n_\alpha}(\lambda_\alpha)^{-1}$ for $\alpha = \mu + 1, \dots, \gamma$.

As an analog of Corollary 4.3, using Corollary 5.11, we conclude that the zeros of $\det a_l(k)$ and $\det a_r(k)$ cannot accumulate at $k = 0$. Thus, we have the following result.

Corollary 5.12 Assume $Q(x)$ is an exceptional potential belonging to $L^1_2(\mathbf{R}; \mathbf{C}^{n\times n})$. Then, the zeros of $\det a_l(k)$ in $\overline{\mathbf{C}^+}$ are confined to the positive imaginary axis and their number is finite.

Having found the small-k asymptotics of $\tilde{T}_l(k)$, we can evaluate the small-k asymptotics of $T_l(k)$ by using

$$T_l(k) = \mathcal{M}^{-1}\, \tilde{T}_l(k)\, \mathcal{M},$$

where $\mathcal{M}$ is the transition matrix from the basis $\{\xi_{\alpha j}\}$ to the standard basis.

The small-k asymptotics of $T_r(k)$, $L(k)$, and $R(k)$ can be obtained in terms of the small-k asymptotics of $T_l(k)$ as follows.

Proposition 5.13 Assume $Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$. Then, the scattering coefficients are continuous and differentiable at $k=0$, and we have

$$T_r(k) = T_l(0)^\dagger - k^*\, \dot{T}_l(0)^\dagger + o(k), \qquad k \in \overline{\mathbf{C}^+}, \tag{5.26}$$

$$L(k) = -I_n + [I_n + E_l]\, T_l(0) + k\, [I_n + E_l]\, \dot{T}_l(0) + o(k), \qquad k \in \mathbf{R}, \tag{5.27}$$

$$R(k) = -I_n + [I_n - E_r]\, T_l(0)^\dagger - k\, [I_n - E_r]\, \dot{T}_l(0)^\dagger + o(k), \qquad k \in \mathbf{R}, \tag{5.28}$$

where E_l and E_r are the matrices defined in (5.5).

PROOF: We get (5.26) from (2.23), and we obtain (5.27) and (5.28) from (2.20), (2.21), and (5.7). ∎

Proposition 5.14 Assume $Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$. Then, for all $h_l \in \operatorname{Im}\Delta_l$ and $h_r \in \operatorname{Im}\Delta_r$, we have

$$\lim_{k\to 0} L(k)\, h_l = -h_l, \qquad \lim_{k\to 0} R(k)\, h_r = -h_r, \qquad k \to 0 \text{ in } \mathbf{R}, \tag{5.29}$$

$$\lim_{k\to 0} T_l(k)\, h_l = 0, \qquad \lim_{k\to 0} T_r(k)\, h_r = 0, \qquad k \to 0 \text{ in } \overline{\mathbf{C}^+}. \tag{5.30}$$

As a result, unless Δ_l has zero rank, we have $||L(0)|| = ||R(0)|| = 1$.

PROOF: If $h_l \in \operatorname{Im}\Delta_l$, there exists a vector g_l such that $h_l = \Delta_l g_l$. Using (2.21), (4.1), (4.7), and (5.2) we have

$$\begin{aligned} L(0)\, h_l = L(0)\, \Delta_l\, g_l &= \lim_{k\to 0} [2ik\, b_l(k)][2ik\, a_l(k)]^{-1}[2ik\, a_l(k)]\, g_l \\ &= \lim_{k\to 0} [2ikb_l(k)]\, g_l = -\Delta_l\, g_l = -h_l. \end{aligned}$$

The result for h_r in (5.29) is obtained in a similar way. Similarly, using (4.1) we get

$$T_l(0)\, h_l = T_l(0)\, \Delta_l\, g_l = \lim_{k\to 0} 2ik\, [2ik\, a_l(k)]^{-1}[2ik\, a_l(k)]\, g_l = \lim_{k\to 0} 2ik\, g_l = 0.$$

The result for h_r in (5.30) is obtained in a similar way. ∎

By Proposition 3.4, the $2n$ columns of the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ are linearly independent at each $k \in \mathbf{R}\setminus\{0\}$. Since k appears as k^2 in (1.1), the functions $f_l(-k,x)$ and $f_r(-k,x)$ are also solutions of (1.1). From (1.2) and (1.3) it is clear that the $2n$ columns of $f_l(-k,x)$ and $f_l(k,x)$ are linearly independent for $k \in \mathbf{R}\setminus\{0\}$; similarly, the $2n$ columns of $f_r(-k,x)$ and $f_r(k,x)$ are linearly independent. With the help of (1.4) and (1.5) we get

$$f_r(k,x) = f_l(k,x)\, b_r(k) + f_l(-k,x)\, a_r(k), \qquad k \in \mathbf{R}\setminus\{0\}, \tag{5.31}$$

$$f_l(k,x) = f_r(k,x)\, b_l(k) + f_r(-k,x)\, a_l(k), \qquad k \in \mathbf{R}\setminus\{0\}. \tag{5.32}$$

Using (2.20) and (2.21) we can write (5.31) and (5.32) in the matrix form as

$$\begin{bmatrix} f_l(-k,x) & f_r(-k,x) \end{bmatrix} = \begin{bmatrix} f_r(k,x) & f_l(k,x) \end{bmatrix} \begin{bmatrix} T_r(k) & -L(k) \\ -R(k) & T_l(k) \end{bmatrix}, \qquad k \in \mathbf{R}\setminus\{0\}. \tag{5.33}$$

Proposition 5.15 Assume $Q \in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$ and let Δ_l and Δ_r be the matrices defined in (4.1). Then, the columns of $T_l(0)$ and of $I_n + R(0)$ belong to $\operatorname{Ker}\Delta_l$; the columns of $T_r(0)$ and of $I_n + L(0)$ belong to $\operatorname{Ker}\Delta_r$.

PROOF: Letting $k \to 0$ in (5.33) we get

$$f_l(0,x)\, [I_n + R(0)] - f_r(0,x)\, T_r(0) = 0, \tag{5.34}$$

$$f_r(0,x)\, [I_n + L(0)] - f_l(0,x)\, T_l(0) = 0. \tag{5.35}$$

Comparing (5.34) with (4.9) and using Proposition 4.4, we see that in the exceptional case the columns of $I_n + R(0)$ belong to $\operatorname{Ker}\Delta_l$ and the maximum number of linearly independent columns of $I_n+R(0)$ is equal to $n-\rho$. Similarly, comparing (5.35) with (4.9) and using Proposition 4.4, we conclude that the columns of $T_l(0)$ belong to $\operatorname{Ker}\Delta_l$ and the maximum number of linearly independent columns is $n-\rho$. ∎

VI. MARCHENKO EQUATIONS

In this section we present the Marchenko integral equations and prove their unique solvability. We assume that $Q\in L^1_1(\mathbf{R};\mathbf{C}^{n\times n})$ in the generic case and $Q\in L^1_2(\mathbf{R};\mathbf{C}^{n\times n})$ in the exceptional case.

Using (2.1), we can rewrite (5.33) as

$$\text{(6.1)}\quad [\,m_l(-k,x)\quad m_r(-k,x)\,]=[\,m_r(k,x)\quad m_l(k,x)\,]\begin{bmatrix} T_r(k) & -L(k)\,e^{-2ikx}\\ -R(k)\,e^{2ikx} & T_l(k)\end{bmatrix},$$

which holds for $k\in\mathbf{R}$ due to the continuity of the scattering coefficients. Let us write (6.1) as the two matrix equations

$$\text{(6.2)}\qquad m_l(-k,x)=m_r(k,x)\,T_r(k)-m_l(k,x)\,R(k)\,e^{2ikx},\qquad k\in\mathbf{R},$$

$$\text{(6.3)}\qquad m_r(-k,x)=m_l(k,x)\,T_l(k)-m_r(k,x)\,L(k)\,e^{-2ikx},\qquad k\in\mathbf{R}.$$

By the results in Section III, the bound states correspond to the simple poles of $T_r(k)$ occurring on the positive imaginary axis at $k=i\kappa_j$ for $j=1,\dots,N$. At $k=i\kappa_j$, as indicated in Proposition 3.5 the $2n$ columns of the Jost solutions $f_l(i\kappa_j,x)$ and $f_r(i\kappa_j,x)$ are linearly dependent and are related to each other as in (3.3) and (3.4). Recalling that $B_l(x,y)$ and $B_r(x,y)$ are defined by (2.5), we introduce

$$\text{(6.4)}\qquad \Omega_l(y)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,R(k)\,e^{iky}-\sum_{j=1}^{N}C_{lj}\,e^{-\kappa_j y},$$

$$\Omega_r(y)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}dk\,L(k)\,e^{iky}-\sum_{j=1}^{N}C_{rj}\,e^{-\kappa_j y},$$

where C_{lj} and C_{rj} are the norming constants given in Proposition 3.5. As in the scalar case by taking the Fourier transforms of (6.2) and (6.3) and by using (3.3) and (3.4), we are led to the $n \times n$ matrix Marchenko equations

$$B_l(x,y) = \Omega_l(2x+y) + \int_0^\infty dz\, B_l(x,z)\,\Omega_l(2x+y+z), \qquad y>0, \tag{6.5}$$

$$B_r(x,y) = \Omega_r(-2x+y) + \int_0^\infty dz\, B_r(x,z)\,\Omega_r(-2x+y+z), \qquad y>0. \tag{6.6}$$

The potential $Q(x)$ can be obtained from the solution of either one of the Marchenko equations as

$$Q(x) = -2\frac{dB_l(x,0^+)}{dx} = 2\frac{dB_r(x,0^+)}{dx}.$$

Theorem 6.1 Assume $Q \in L_1^1(\mathbf{R};\mathbf{C}^{n\times n})$ in the generic case and $Q \in L_2^1(\mathbf{R};\mathbf{C}^{n\times n})$ in the exceptional case. Then, for each fixed $x \in \mathbf{R}$, each of the Marchenko equations (6.5) and (6.6) has a unique solution belonging to $L^2(\mathbf{R}^+;\mathbf{C}^{n\times n})$.

PROOF: We will adapt the proof[11] given in the scalar case to the matrix case. The proof for (6.6) is similar the proof of (6.5), and hence we will only present the latter. From (6.4) and (6.5) we see the Marchenko integral operator is a selfadjoint perturbation of the identity and has the form $I+\mathcal{O}+\mathcal{C}$, where I is the identity, the kernel of $\mathcal{O}$ is related to the integral term in (6.4), and the kernel of $\mathcal{C}$ is related to the finite sum in (6.4). In the purely exceptional case we have $||R(k)|| < 1$ for all $k \in \mathbf{R}$, and hence the norm of $\mathcal{O}$ is strictly less than one; otherwise for every real x we can find $\varepsilon > 0$ such that $\sup_{k\in\mathbf{R}} ||R(k)\, e^{2ikx} + \varepsilon I_n|| < 1$. Using Nehari's theorem as in the scalar case,[11] we conclude that the operator $\mathcal{O}$ on $L^2(\mathbf{R}^+;\mathbf{C}^{n\times n})$ is a strict contraction and hence the corresponding Marchenko integral equation is uniquely solvable when there are no bound states. On the other hand, when there are bound states, due to the positive selfadjointness of the matrices C_{lj} as indicated in Proposition 3.5, the operator $\mathcal{C}$ is positive selfadjoint, and $I+\mathcal{O}+\mathcal{C}$ is boundedly invertible. ∎

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