Integral equation methods for the inverse problem with discontinuous wave speed

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The recovery of the coefficient $H(x)$ in the one-dimensional generalized Schrödinger equation $d^2\psi/dx^2+k^2H(x)^2\psi=Q(x)\psi$, where $H(x)$ is a positive, piecewise continuous function with positive limits $H_\pm$ as $x\rightarrow\pm\infty$, is studied. The large-k asymptotics of the wave functions and the scattering coefficients are analyzed. A factorization formula is given expressing the total scattering matrix as a product of simpler scattering matrices. Using this factorization an algorithm is presented to obtain the discontinuities in $H(x)$ and $H'(x)/H(x)$ in terms of the large-k asymptotics of the reflection coefficient. When there are no bound states, it is shown that $H(x)$ is recovered from an appropriate set of scattering data by using the solution of a singular integral equation, and the unique solvability of this integral equation is established. An equivalent Marchenko integral equation is derived and is shown to be uniquely solvable; the unique recovery of $H(x)$ from the solution of this Marchenko equation is presented. Some explicit examples are given, illustrating the recovery of $H(x)$ from the solution of the singular integral equation and from that of the Marchenko equation. © *1996 American Institute of Physics.* [S0022-2488(96)02606-0]

I. INTRODUCTION

Consider the one-dimensional generalized Schrödinger equation,

$$\psi''(k,x)+k^2H(x)^2\psi(k,x)=Q(x)\psi(k,x), \quad x\in\mathbf{R}, \tag{1.1}$$

which describes the propagation of waves in a one-dimensional nonhomogeneous, nonabsorptive medium, where k^2 is energy, $1/H(x)$ is the wave speed, and $Q(x)$ is the restoring force density. The discontinuities of $H(x)$ correspond to abrupt changes in the properties of the medium in which the wave propagates. The prime denotes the derivative with respect to the spatial coordinate, and the coefficients $H(x)$ and $Q(x)$ are assumed to satisfy the following conditions:

(H1) $H(x)$ is strictly positive and piecewise continuous with jump discontinuities at x_n for $n=1,\dots,N$, such that $x_1<\cdots<x_N$.

(H2) $H(x)\rightarrow H_\pm$ as $x\rightarrow\pm\infty$, where $H_\pm$ are positive constants.

(H3) $H-H_\pm\in L^1(\mathbf{R}^\pm)$, where $\mathbf{R}^-=(-\infty,0)$ and $\mathbf{R}^+=(0,+\infty)$.

(H4) H' is absolutely continuous on (x_n,x_{n+1}) and $2H''H-3(H')^2\in L_1^1(x_n,x_{n+1})$, for $n=0,\dots,N$, where $x_0=-\infty$ and $x_{N+1}=+\infty$, and $L_\beta^1(I)$ denotes the space of measurable functions $f(x)$ on I, such that $\int_I dx\,(1+|x|)^\beta|f(x)|<+\infty$.

(H5) $Q(x)$ is real valued and belongs to $L_1^1(\mathbf{R})$.

The scattering solutions of (1.1) are those behaving like $e^{ikH_\pm x}$ or $e^{-ikH_\pm x}$ as $x\rightarrow\pm\infty$, and

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such solutions occur when $k^2>0$. Among the scattering solutions are the Jost solution from the left $f_l(k,x)$ and the Jost solution from the right $f_r(k,x)$ satisfying the boundary conditions

$$f_l(k,x)=\begin{cases} e^{ikH_+x}+o(1), & x\to+\infty, \\ \dfrac{1}{T_l(k)}\,e^{ikH_-x}+\dfrac{L(k)}{T_l(k)}\,e^{-ikH_-x}+o(1), & x\to-\infty, \end{cases}$$

$$f_r(k,x)=\begin{cases} \dfrac{1}{T_r(k)}\,e^{-ikH_+x}+\dfrac{R(k)}{T_r(k)}\,e^{ikH_+x}+o(1), & x\to+\infty, \\ e^{-ikH_-x}+o(1), & x\to-\infty, \end{cases}$$

where $T_l(k)$ and $T_r(k)$ are the transmission coefficients from the left and from the right, respectively, and $L(k)$ and $R(k)$ are the reflection coefficients from the left and from the right, respectively. For each fixed $x\in\mathbf{R}$, the Jost solutions have continuous extensions to the upper half complex plane $\mathbf{C}^+$, and they are analytic there.[1] The reduced transmission coefficient $\tau(k)$, the reduced reflection coefficients $\rho(k)$ from the right and $\ell(k)$ from the left, respectively, are defined as

$$\tau(k)=\sqrt{\frac{H_+}{H_-}}\,T_l(k)e^{ikA}=\sqrt{\frac{H_-}{H_+}}\,T_r(k)e^{ikA}, \tag{1.2}$$

$$\rho(k)=R(k)e^{2ikA_+}, \quad \ell(k)=L(k)e^{2ikA_-}, \tag{1.3}$$

where

$$A_\pm=\pm\int_0^{\pm\infty}ds[H_\pm-H(s)], \quad A=A_++A_- . \tag{1.4}$$

If $\tau(0)\neq0$, which is called the exceptional case, the Jost solutions $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent. If $\tau(0)=0$, which is called the generic case, $f_l(0,x)$ and $f_r(0,x)$ are linearly independent, and in this case $\tau(k)$ vanishes linearly as $k\to0$. Usually these two cases need to be analyzed separately, and the small-k analysis of the scattering problem in the exceptional case requires tedious estimates. However, the fact[2] that an exceptional case can always be decomposed into two generic cases is expected to simplify the analysis of the scattering problem in the exceptional case.

In general, (1.1) may have bound states, i.e. nontrivial solutions belonging to $L^2(\mathbf{R},H(x)^2dx)$. Since the treatment of bound states requires many separate arguments, we do not consider them in this paper. Bound states were already studied in Ref. 1 and further results may appear in the future. Thus, we assume that (1.1) does not have any bound states. The number of bound states for (1.1) is equal[3] to the number of bound states for the Schrödinger equation,

$$\Phi''(k,x)+k^2\Phi(k,x)=Q(x)\Phi(k,x), \quad x\in\mathbf{R}, \tag{1.5}$$

and hence our assumption can be restated by saying that $Q(x)$ does not have any bound states.

The inverse scattering problem in which we are interested consists of the recovery of $H(x)$ in (1.1) from an appropriate set of scattering data. The analysis of the scattering problem in a discontinuous medium is the first step to analyze the inverse scattering problem, and we mention the relevant work[4–7] of Sabatier and his collaborators on the scattering in a discontinuous medium in one and three dimensions governed by $\{\alpha(x)^{-2}\boldsymbol{\nabla}\cdot[\alpha(x)^2\boldsymbol{\nabla}]+k^2-V(x)\}\phi(k,x)=0$. In Ref. 4 Sabatier estimated the large-k asymptotics of the scattering data and also briefly discussed the inverse scattering problem in such a medium. Various authors have studied inverse scattering problems for differential equations with discontinuous coefficients, as exemplified by Krueger's work[8–10] and the bibliography of Ref. 1. Of more direct concern to us is the work by Sabatier[4] and Grinberg.[11,12] Grinberg, in the special (but still important) case $Q(x)=0$, developed a method to recover $H(x)$ using the solution of a singular integral equation; in this special case there are no bound states, the exceptional case occurs, and the norm of the associated singular integral operator is strictly less than unity so that the integral equation has a unique solution that can be obtained through iteration. The general case with nontrivial $Q(x)$ and with bound states was analyzed by a similar method in Ref. 1, and $H(x)$ was recovered from the solution of a singular integral equation under the assumption $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha\in(0,1]$. In Ref. 13 the scattering data leading to a unique solution of the inverse problem were specified.

In this paper, when there are no bound states, we develop a method to obtain $H(x)$ from the scattering data consisting of $Q(x)$, $\rho(k)$, and H_+. As already known,[13] H_+ must be omitted from the scattering data in the generic case, but in the exceptional case it needs to be specified in the scattering data in order to obtain $H(x)$ uniquely; this is also true in the method presented here. Note also that, in the scattering data, one can use $\ell(k)$ instead of $\rho(k)$ and one can also use H_- instead of H_+. The method given here and the method of Ref. 1 have some similarities and differences. The method used here holds whenever $Q\in L^1_1(\mathbf{R})$, whereas in Ref. 1, for technical reasons, we needed $Q\in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha\in(0,1]$. In both methods a singular integral equation is formulated and from its solution $H(x)$ is recovered; however, in the present paper we exploit the large-k behavior of the reduced scattering coefficients, thus avoiding complications encountered in Ref. 1 as $k\to 0$. A crucial result here is Proposition 2.1, which strengthens the result of Theorem 2.4 in Ref. 1. From the solution at $k=0$ of the singular integral equation one finds $H(x)$ as the x-derivative of the solution $y(x)$ of a separable differential equation under the initial condition $y(0)=0$. Furthermore, when the reduced reflection coefficient $\rho(k)$ is an almost periodic function, the singular integral equation of the present paper becomes trivial, and so does the computation of $H(x)$; in Ref. 1, even this relatively simple case required extensive calculations.

When $H(x)$ and $H'(x)$ have no discontinuities, the large-k asymptotics of the reduced scattering coefficients defined in (1.2)–(1.3) are known to be of the form $\tau(k)-1=O(1/k)$, $\rho(k)=O(1/k)$, and $\ell(k)=O(1/k)$. It is also known that each discontinuity of $H(x)$ contributes to the almost periodic part of the $O(1)$ terms in these asymptotics. We refer the reader to Refs. 1, 4, 11–13 for details. In this paper we show that the discontinuities in $H'(x)/H(x)$ are responsible for some of the $O(1/k)$ terms in these asymptotics; in fact, we develop an algorithm to recover the jumps in $H'(x)/H(x)$ from the large-k asymptotics of a reduced reflection coefficient.

This paper is organized as follows. In Sec. II we study the large-k asymptotics of the reduced scattering coefficients. In Sec. III we study the large-k asymptotics of certain wave functions defined in (3.1)–(3.2). In Sec. IV we present a factorization formula expressing the reduced scattering matrix as a matrix product of scattering matrices corresponding to potentials supported on a finite interval or on a half-line and those corresponding to discontinuities in $H(x)$ and $H'(x)/H(x)$. In Sec. V we present an algorithm to recover the discontinuities in $H(x)$ and $H'(x)/H(x)$ from the large-k asymptotics of the scattering data, thus generalizing the work of Ref. 13 regarding the discontinuities in $H(x)$. The results in Secs. II and III are used in Sec. VI in order to convert a key Riemann-Hilbert problem into a pair of uncoupled singular integral equations; in this section we also establish the unique solvability of these integral equations and show how to recover $H(x)$ from the solution of either singular integral equation. In Sec. VII we show that each singular integral equation can be converted into a Marchenko integral equation that is uniquely solvable, and we describe the recovery of $H(x)$ from the solution of a Marchenko equation. Hence, the inverse problem is solved by recovering $H(x)$ either by the method of Sec. VI or by that of Sec. VII. In Section VIII we present some examples illustrating the recovery of $H(x)$ using the solution of a singular integral equation and using the solution of a Marchenko equation; we also illustrate the algorithm of recovery of the discontinuities in $H'(x)/H(x)$.

II. SCATTERING COEFFICIENTS

In this section we analyze the large-k asymptotics of the reduced scattering coefficients defined in (1.2)–(1.3). Under the Liouville transformation

$$y=y(x)=\int_0^x ds\ H(s), \quad \psi(k,x)=\frac{1}{\sqrt{H(x)}}\ \phi(k,y), \tag{2.1}$$

the generalized Schrödinger equation (1.1) is transformed into

$$\frac{d^2\phi(k,y)}{dy^2}+k^2\phi(k,y)=V(y)\phi(k,y), \tag{2.2}$$

where

$$V(y(x))=\frac{H''(x)}{2H(x)^3}-\frac{3}{4}\,\frac{H'(x)^2}{H(x)^4}+\frac{Q(x)}{H(x)^2}. \tag{2.3}$$

Since $H(x)$ is assumed to have jump discontinuities at x_j for $j=1,\dots,N$, the quantity $V(y)$ is undefined at $y_j=y(x_j)$. However, $V(y)$ is well defined in each of the intervals (y_j,y_{j+1}) for $j=0,\dots,N$; thus, the Liouville transformation can be used on each interval (x_j,x_{j+1}) although it cannot be used on $\mathbf{R}$. Since $H(x)$ is strictly positive with positive limits as $x\to\pm\infty$, it follows that $y_0=y(x_0)=-\infty$ and $y_{N+1}=y(x_{N+1})=+\infty$. The constants q_j, defined by

$$q_j=\frac{H(x_j-0)}{H(x_j+0)}, \tag{2.4}$$

correspond to the relative jumps in the wave speed at the interfaces x_j, and y_j correspond to the times required for the wave to propagate from the fixed location $x=0$ to the interfaces x_j for $j=1,\dots,N$.

Let $V_{j,j+1}(y)$ be the potential defined by

$$V_{j,j+1}(y)=\begin{cases} V(y), & y\in(y_j,y_{j+1}),\\ 0, & \text{elsewhere,}\end{cases} \tag{2.5}$$

where $V(y)$ is the quantity in (2.3). From (H4) it follows that $V_{j,j+1}\in L_1^1(\mathbf{R})$ for $j=0,\dots,N$. Let $Y_{l;j,j+1}(k,y)$ and $Y_{r;j,j+1}(k,y)$ denote the Faddeev functions[1] from the left and from the right, respectively, associated with the potential $V_{j,j+1}(y)$. We have[1]

$$Y_{l;j,j+1}(k,y)=\begin{cases} \dfrac{1}{t_{j,j+1}(k)}\,[1+l_{j,j+1}(k)e^{-2iky}], & y\leqslant y_j, \quad j=1,\dots,N, \quad k\in\overline{C^+},\\ \dfrac{1}{t_{0,1}(k)}\,[1+l_{0,1}(k)e^{-2iky}]+o(1), & y\to-\infty, \quad j=0, \quad k\in\mathbf{R},\end{cases} \tag{2.6}$$

$$Y_{r;j,j+1}(k,y)=\begin{cases} \dfrac{1}{t_{j,j+1}(k)}\,[1+r_{j,j+1}(k)e^{2iky}], & y\geqslant y_{j+1}, \quad j=0,\dots,N-1, \quad k\in\overline{C^+},\\ \dfrac{1}{t_{N,N+1}(k)}\,[1+r_{N,N+1}(k)e^{2iky}]+o(1), & y\to+\infty, \quad j=N, \quad k\in\mathbf{R},\end{cases} \tag{2.7}$$

where $t_{j,j+1}(k)$, $r_{j,j+1}(k)$, and $l_{j,j+1}(k)$ denote the transmission coefficient and the reflection coefficients from the right and from the left, respectively, for the potential $V_{j,j+1}(y)$. Since $V_{j,j+1}\in L_1^1(\mathbf{R})$, it follows that for each fixed $y\in\mathbf{R}$ we have

$$Y_{l;j,j+1}(k,y)=1+O(1/k), \quad Y'_{l;j,j+1}(k,y)=o(1), \quad k\to\infty \text{ in } \overline{\mathbf{C}^+}, \tag{2.8}$$

$$Y_{r;j,j+1}(k,y)=1+O(1/k), \quad Y'_{r;j,j+1}(k,y)=o(1), \quad k\to\infty \text{ in } \overline{\mathbf{C}^+}. \tag{2.9}$$

Using (2.1) it can be shown that the functions defined by

$$\eta_{j,j+1}(k,x)=\frac{1}{\sqrt{H(x)}}\, e^{iky}Y_{l;j,j+1}(k,y), \quad \xi_{j,j+1}(k,x)=\frac{1}{\sqrt{H(x)}}\, e^{-iky}Y_{r;j,j+1}(k,y), \tag{2.10}$$

are solutions of (1.1). Let us introduce the matrices

$$\Gamma_{j,j+1}(k,x)=\begin{bmatrix} \eta_{j,j+1}(k,x) & \xi_{j,j+1}(k,x) \\ \eta'_{j,j+1}(k,x) & \xi'_{j,j+1}(k,x) \end{bmatrix}, \quad j=0,\dots,N, \tag{2.11}$$

$$\mathscr{G}(k)=\prod_{n=1}^{N} \Gamma_{n-1,n}(k,x_n-0)^{-1}\Gamma_{n,n+1}(k,x_n+0). \tag{2.12}$$

It was shown in Ref. 1 that

$$\frac{1}{\tau(k)}=\frac{1}{t_{0,1}(k)}\,[1 \quad 0]\mathscr{G}(k)\begin{bmatrix}1\\0\end{bmatrix}=\frac{1}{t_{N,N+1}(k)}\,[0 \quad 1]\mathscr{G}(k)^{-1}\begin{bmatrix}0\\1\end{bmatrix}, \tag{2.13}$$

$$\frac{\ell(k)}{\tau(k)}=\begin{bmatrix}\frac{l_{0,1}(k)}{t_{0,1}(k)} & 1\end{bmatrix}\mathscr{G}(k)\begin{bmatrix}1\\0\end{bmatrix}, \tag{2.14}$$

$$\frac{\rho(k)}{\tau(k)}=\begin{bmatrix}1 & \frac{r_{N,N+1}(k)}{t_{N,N+1}(k)}\end{bmatrix}\mathscr{G}(k)^{-1}\begin{bmatrix}0\\1\end{bmatrix}. \tag{2.15}$$

Moreover,

$$\det\,\Gamma_{n,n+1}(k,x)=-\frac{2ik}{t_{n,n+1}(k)}, \quad \det\,\mathscr{G}(k)=\frac{t_{0,1}(k)}{t_{N,N+1}(k)}.$$

Let

$$\alpha_n=\frac{1}{2}\left(\sqrt{q_n}+\frac{1}{\sqrt{q_n}}\right), \quad \beta_n=\frac{1}{2}\left(\sqrt{q_n}-\frac{1}{\sqrt{q_n}}\right), \tag{2.16}$$

$$E(k,x_n)=\begin{bmatrix}\alpha_n & \beta_n e^{-2iky_n}\\ \beta_n e^{2iky_n} & \alpha_n\end{bmatrix}, \tag{2.17}$$

with q_n as in (2.4); let us also define $a(k)$ and $b(k)$ by

$$\begin{bmatrix}a(k) & b(k)\\ b(-k) & a(-k)\end{bmatrix}=\prod_{n=1}^{N} E(k,x_n). \tag{2.18}$$

Let AP^W (almost periodic functions with Wiener norm) stand for the algebra of all complex-valued functions $f(k)$ on $\mathbf{R}$ that are of the form $f(k)=\sum_{j=-\infty}^{\infty} f_j e^{ik\lambda_j}$, where $f_j\in\mathbf{C}$ and $\lambda_j\in\mathbf{R}$ for all j and $\Sigma_j|f_j|<+\infty$. It is already known[1] that the functions $a(k)$, $b(k)$, $1/a(k)$, and $b(k)/a(k)$ belong to AP^W. In the next proposition we obtain the large-k asymptotics of the reduced scattering coefficients $\tau(k)$, $\rho(k)$, and $\ell(k)$.

Proposition 2.1: Under assumptions (H1)–(H5) we have

$$\tau(k)=\frac{1}{a(k)}+O\left(\frac{1}{k}\right),\quad k\to\infty \text{ in } \overline{\mathbf{C}^+}, \tag{2.19}$$

$$\rho(k)=-\frac{b(k)}{a(k)}+O\left(\frac{1}{k}\right),\quad k\to\pm\infty, \tag{2.20}$$

$$\ell(k)=\frac{b(-k)}{a(k)}+O\left(\frac{1}{k}\right),\quad k\to\pm\infty, \tag{2.21}$$

where $a(k)$ and $b(k)$ are the quantities defined in (2.18).

Proof: Using (2.8)–(2.10) we obtain

$$\begin{aligned}&\Gamma_{n,n+1}(k,x_{n+1}-0)^{-1}\Gamma_{n+1,n+2}(k,x_{n+1}+0)\\&=\begin{bmatrix}\alpha_{n+1}(1+O(1/k)) & \beta_{n+1}e^{-2iky_{n+1}}(1+O(1/k))\\ \beta_{n+1}e^{2iky_{n+1}}(1+O(1/k)) & \alpha_{n+1}(1+O(1/k))\end{bmatrix},\quad k\to\infty \text{ in } \overline{\mathbf{C}^+},\end{aligned} \tag{2.22}$$

where α_n and β_n are the constants defined in (2.16). Furthermore, using (2.13)–(2.15) and the fact[14] that

$$t_{j,j+1}(k)=1+O(1/k),\quad k\to\infty \text{ in } \overline{\mathbf{C}^+},$$

$$r_{j,j+1}(k)=O(1/k),\quad l_{j,j+1}(k)=O(1/k),\quad k\to\pm\infty,$$

we obtain (2.19)–(2.21). ■

Proposition 2.1 is an improvement over Theorem 2.4 in Ref. 13, where the error terms in (2.19)–(2.21) were only shown to be $o(1)$. We refer the reader to Refs. 1 and 13 for various other properties of the reduced scattering coefficients.

III. ESTIMATES ON WAVE FUNCTIONS

In this section we analyze the large-k behavior of the scattering solutions of (2.2). As in (5.1)–(5.2) of Ref. 1, let us define the Faddeev functions $Z_l(k,y)$ and $Z_r(k,y)$, from the left and from the right, respectively, associated with (2.2):

$$Z_l(k,y)=\sqrt{\frac{H(x)}{H_+}}\,e^{-iky-ikA_+}f_l(k,x), \tag{3.1}$$

$$Z_r(k,y)=\sqrt{\frac{H(x)}{H_-}}\,e^{iky-ikA_-}f_r(k,x), \tag{3.2}$$

where y is the quantity defined in (2.1) and $A_\pm$ are the constants in (1.4). Note that $e^{iky}Z_l(k,y)$ and $e^{-iky}Z_r(k,y)$ are the Jost solutions from the left and from the right, respectively, of (2.2). In this section we analyze the large-k asymptotics of $Z_l(k,y)$ and $Z_r(k,y)$.

The next proposition shows that, for each fixed $y\in\mathbf{R}\setminus\{y_1,\dots,y_N\}$, the Faddeev functions can be written as the sum of an almost periodic function and a continuous function, the latter vanishing as $k\to\infty$ in $\overline{\mathbf{C}^+}$.

Proposition 3.1: For each fixed $y\in\mathbf{R}\setminus\{y_1,\dots,y_N\}$, we have

$$Z_l(k,y)=J_l(k,y)+O(1/k),\quad Z_r(k,y)=J_r(k,y)+O(1/k),\quad k\to\infty \ \text{ in } \ \overline{\mathbf{C}^+}, \tag{3.3}$$

where

$$J_l(k,y)=[1 \quad e^{-2iky}]\left(\prod_{n=j+1}^{N} E(k,x_n)\right)\begin{bmatrix}1\\0\end{bmatrix},\quad y\in(y_j,y_{j+1}),\quad j=0,\dots,N-1, \tag{3.4}$$

$$J_l(k,y)=1,\quad y\in(y_N,+\infty), \tag{3.5}$$

$$J_r(k,y)=1,\quad y\in(-\infty,y_1), \tag{3.6}$$

$$J_r(k,y)=[e^{2iky} \quad -1]\left(\prod_{n=j}^{1} E(k,x_n)\right)\begin{bmatrix}0\\-1\end{bmatrix},\quad y\in(y_j,y_{j+1}),\quad j=1,\dots,N, \tag{3.7}$$

with $E(k,x_n)$ defined in (2.17). The product notation in (3.7) means that n decreases from j to 1.

Proof: When $y\in(y_N,+\infty)$, from (3.13), (3.15), (3.21) of Ref. 1 and (2.10) and (3.1), we have

$$Z_l(k,y)=Y_{l;N,N+1}(k,y),\quad y\in(y_N,+\infty), \tag{3.8}$$

and hence $Z_l(k,y)=1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Thus, we have (3.3) with $J_l(k,y)$ as in (3.5). Similarly, from (3.13), (3.15), (3.22) of Ref. 1 and (2.10) and (3.2), we get

$$Z_r(k,y)=Y_{r;0,1}(k,y),\quad y\in(-\infty,y_1), \tag{3.9}$$

and hence $Z_r(k,y)=1+O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Thus, we have (3.3) with $J_r(k,y)$ as in (3.6).

When $y\in(y_j,y_{j+1})$ with $0\leqslant j\leqslant N-1$, from (3.25) of Ref. 1 and (3.1), we see that

$$\begin{aligned}Z_l(k,y)=&[1 \quad 0]\sqrt{H(x)}e^{-iky}\Gamma_{j,j+1}(k,x)\\&\times\left(\prod_{n=j}^{N-1}\Gamma_{n,n+1}(k,x_{n+1}-0)^{-1}\Gamma_{n+1,n+2}(k,x_{n+1}+0)\right)\begin{bmatrix}1\\0\end{bmatrix},\end{aligned} \tag{3.10}$$

where $\Gamma_{j,j+1}(k,x)$ is the matrix defined in (2.11). From (2.8)–(2.10) we have

$$[1 \quad 0]\sqrt{H(x)}e^{-iky}\Gamma_{j,j+1}(k,x)=[1+O(1/k) \quad e^{-2iky}(1+O(1/k))]. \tag{3.11}$$

Hence, using (2.22) and (3.11) in (3.10), we obtain

$$Z_l(k,y)=J_l(k,y)[1+O(1/k)],\quad k\to\infty \ \text{ in } \ \overline{\mathbf{C}^+}, \tag{3.12}$$

with $J_l(k,y)$ as in (3.4). Similarly, when $y\in(y_j,y_{j+1})$ with $1\leqslant j\leqslant N$, from (3.26) of Ref. 1 and (3.2), we see that

$$Z_r(k,y)=[1 \quad 0]\sqrt{H(x)}e^{iky}\Gamma_{j,j+1}(k,x)\left(\prod_{n=j}^{1}\Gamma_{n,n+1}(k,x_n+0)^{-1}\Gamma_{n+1,n+2}(k,x_n-0)\right)\begin{bmatrix}0\\1\end{bmatrix}. \tag{3.13}$$

From (2.8)–(2.10) we have

$$[1 \quad 0]\sqrt{H(x)}e^{iky}\Gamma_{j,j+1}(k,x)=[e^{2iky}(1+O(1/k)) \quad 1+O(1/k)]. \tag{3.14}$$

Using (2.22) and (3.14) in (3.13), we obtain

$$Z_r(k,y)=J_r(k,y)[1+O(1/k)], \quad k\to\infty \text{ in } \overline{\mathbf{C}^+}, \tag{3.15}$$

with $J_r(k,y)$ as in (3.7). Note that for each fixed $y\in\mathbf{R}\setminus\{y_1,\dots,y_N\}$ the functions $J_l(k,y)$ and $J_r(k,y)$ are uniformly bounded in $\overline{\mathbf{C}^+}$, and hence we see that (3.12) and (3.15) imply (3.3). ■

Recall that the Hardy spaces $\mathbf{H}^p_\pm(\mathbf{R})$ are defined as the spaces of all functions $f(k)$ that are analytic in $k\in\mathbf{C}^\pm$ and satisfy $\sup_{\epsilon>0}\int_{-\infty}^{\infty} dk \ |f(k\pm i\epsilon)|^p<+\infty$.

Theorem 3.2: For each fixed $y\in\mathbf{R}\setminus\{y_1,\dots,y_N\}$, the functions $Z_l(k,y)-J_l(k,y)$ and $Z_r(k,y)-J_r(k,y)$ belong to the Hardy space $\mathbf{H}^2_+(\mathbf{R})$.

Proof: It is proved in Theorem 2.1 of Ref. 1 that, for each fixed $x\in\mathbf{R}\setminus\{x_1,\dots,x_N\}$, $f_l(k,x)$ and $f_r(k,x)$ are continuous functions of k in $\overline{\mathbf{C}^+}$ and analytic in $\mathbf{C}^+$; therefore, for each fixed $y\in\mathbf{R}\setminus\{y_1,\dots,y_N\}$, the Faddeev functions $Z_l(k,y)$ and $Z_r(k,y)$ are continuous in $\overline{\mathbf{C}^+}$ and analytic in $\mathbf{C}^+$. From (3.4)–(3.7) we see that $J_l(k,y)$ and $J_r(k,y)$ are continuous in $\overline{\mathbf{C}^+}$ and analytic in $\mathbf{C}^+$. Hence, by Proposition 3.1 we can conclude that $Z_l(k,y)-J_l(k,y)$ and $Z_r(k,y)-J_r(k,y)$ belong to the Hardy space $\mathbf{H}^2_+(\mathbf{R})$. ■

Note that we can also conclude the analyticity in $\mathbf{C}^+$ and continuity in $\overline{\mathbf{C}^+}$ of $Z_l(k,y)$ and $Z_r(k,y)$ from (3.10) and (3.13), respectively, because the matrices there have these properties. At first the inverse matrices in (3.10) and (3.13) seem to have a $(1/k)$ singularity at $k=0$ in the exceptional case; however, if any $V_{n,n+1}(y)$ are exceptional potentials, we can divide each of those intervals (y_n,y_{n+1}) into two subintervals such that the fragments on the two subintervals are generic;[2] hence, even in the exceptional case, from (3.10) and (3.13), we can conclude that $Z_l(k,y)$ and $Z_r(k,y)$ are analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$.

Note that the matrix product $E(k,x_{j+1})\cdots E(k,x_N)$ in (3.4) can be explicitly evaluated in analogy to (2.28) of Ref. 13. Let us write

$$\prod_{n=j+1}^{N} E(k,x_n)=\begin{bmatrix} A_j(k) & B_j(k) \\ B_j(-k) & A_j(-k)\end{bmatrix},$$

where $A_j(k)$ and $B_j(k)$ will be explicitly evaluated. Thus, we can write (3.4)–(3.5) as

$$J_l(k,y)=[A_j(k)+e^{-2iky}B_j(-k)], \quad y\in(y_j,y_{j+1}), \tag{3.16}$$

with $A_N(k)=1$ and $B_N(k)=0$. Using induction, we can show that $A_j(k)$ and $e^{-2iky}B_j(-k)$ both are exponential polynomials having at most 2^{N-j} terms. All the coefficients in the exponential polynomials are real constants and all the exponentials are bounded by 1 in absolute value in $\overline{\mathbf{C}^+}$. For future reference, we list $A_j(k)$ and $B_j(k)$ for $j=N-1,\ N-2,\ N-3$.
If $j=N-1$,

$$A_{N-1}(k)=\alpha_N, \quad e^{2iky_N}B_{N-1}(k)=\beta_N.$$

If $j=N-2$,

$$A_{N-2}(k)=\alpha_{N-1}\alpha_N+\beta_{N-1}\beta_N e^{2ik(y_N-y_{N-1})},$$

$$e^{2iky_N}B_{N-2}(k)=\alpha_{N-1}\beta_N+\beta_{N-1}\alpha_N e^{2ik(y_N-y_{N-1})}.$$

If $j=N-3$,

$$A_{N-3}(k)=\alpha_{N-2}\alpha_{N-1}\alpha_N+\beta_{N-2}\beta_{N-1}\alpha_N e^{2ik(y_{N-1}-y_{N-2})}$$
$$+\alpha_{N-2}\beta_{N-1}\beta_N e^{2ik(y_N-y_{N-1})}+\beta_{N-2}\alpha_{N-1}\beta_N e^{2ik(y_N-y_{N-2})},$$

$$e^{2iky_N}B_{N-3}(k)=\alpha_{N-2}\alpha_{N-1}\beta_N+\beta_{N-2}\beta_{N-1}\beta_N e^{2ik(y_{N-1}-y_{N-2})}$$
$$+\alpha_{N-2}\beta_{N-1}\alpha_N e^{2ik(y_N-y_{N-1})}+\beta_{N-2}\alpha_{N-1}\alpha_N e^{2ik(y_N-y_{N-2})}.$$

We see that, for $j\leqslant N-1$, the term $e^{2iky_N}B_j(k)$ is obtained from $A_j(k)$ by interchanging β_N with α_N.

In a similar manner, using

$$E(k,x_j)^{-1}\cdots E(k,x_1)^{-1}=[E(k,x_1)\cdots E(k,x_j)]^{-1},$$

we can explicitly evaluate the matrix product $E(k,x_1)\cdots E(k,x_j)$ appearing in (3.7) in analogy to (2.28) of Ref. 13. Let us write

$$\prod_{n=1}^{j} E(k,x_n)=\begin{bmatrix} C_j(k) & D_j(k) \\ D_j(-k) & C_j(-k)\end{bmatrix},$$

where $C_j(k)$ and $D_j(k)$ will be explicitly evaluated. Thus, we can write (3.6)–(3.7) as

$$J_r(k,y)=[C_j(k)-e^{2iky}D_j(-k)],\quad y\in(y_j,y_{j+1}), \tag{3.17}$$

with $C_0(k)=1$ and $D_0(k)=0$. Using induction, we can show that $C_j(k)$ and $e^{2iky}D_j(-k)$ both are exponential polynomials having at most 2^j terms. All the coefficients in the exponential polynomials are real constants and all the exponentials are bounded by 1 in absolute value in $\overline{\mathbf{C}^+}$. For future reference, we list $C_j(k)$ and $D_j(k)$ for $j=1,\ 2,\ 3$.
If $j=1$,

$$C_1(k)=\alpha_1,\quad e^{2iky_1}D_1(k)=\beta_1.$$

If $j=2$,

$$C_2(k)=\alpha_1\alpha_2+\beta_1\beta_2 e^{2ik(y_2-y_1)},\quad e^{2iky_2}D_2(k)=\alpha_1\beta_2+\beta_1\alpha_2 e^{2ik(y_2-y_1)}.$$

If $j=3$,

$$C_3(k)=\alpha_1\alpha_2\alpha_3+\beta_1\beta_2\alpha_3 e^{2ik(y_2-y_1)}+\alpha_1\beta_2\beta_3 e^{2ik(y_3-y_2)}+\beta_1\alpha_2\beta_3 e^{2ik(y_3-y_1)},$$

$$e^{2iky_3}D_3(k)=\alpha_1\alpha_2\beta_3+\beta_1\beta_2\beta_3 e^{2ik(y_2-y_1)}+\alpha_1\beta_2\alpha_3 e^{2ik(y_3-y_2)}+\beta_1\alpha_2\alpha_3 e^{2ik(y_3-y_1)}.$$

We see that, for $j\geqslant 1$, the term $e^{2iky_j}D_j(k)$ is obtained from $C_j(k)$ by interchanging β_j with α_j.

IV. FACTORIZATION

In this section we generalize the factorization formula of Ref. 15 and show that the reduced scattering matrix corresponding to (1.1) can be expressed in terms of the scattering matrices corresponding to the potentials $V_{j,j+1}(y)$ defined in (2.5) and certain matrices associated with the discontinuities of $H(x)$ and $H'(x)/H(x)$. Using the scattering coefficients introduced in (2.6)–(2.7), let us define

$$\Lambda_{j,j+1}(k)=\begin{bmatrix} \frac{1}{t_{j,j+1}(k)} & -\frac{r_{j,j+1}(k)}{t_{j,j+1}(k)} \\ \frac{l_{j,j+1}(k)}{t_{j,j+1}(k)} & \frac{1}{t_{j,j+1}(-k)} \end{bmatrix}, \quad j=0,1,\dots,N, \tag{4.1}$$

$$\Lambda(k)=\begin{bmatrix} \frac{1}{\tau(k)} & -\frac{\rho(k)}{\tau(k)} \\ \frac{\ell(k)}{\tau(k)} & \frac{1}{\tau(-k)} \end{bmatrix}, \tag{4.2}$$

$$F_j(k)=\begin{bmatrix} \alpha_j+\frac{\nu_j}{2ik} & \left(\beta_j+\frac{\nu_j}{2ik}\right)e^{-2iky_j} \\ \left(\beta_j-\frac{\nu_j}{2ik}\right)e^{2iky_j} & \alpha_j-\frac{\nu_j}{2ik} \end{bmatrix}, \quad j=1,\dots,N, \tag{4.3}$$

where α_j and β_j are the constants defined in (2.16) and

$$\nu_j=\frac{1}{2\sqrt{H(x_j-0)H(x_j+0)}}\left[\frac{H'(x_j-0)}{H(x_j-0)}-\frac{H'(x_j+0)}{H(x_j+0)}\right]. \tag{4.4}$$

Note that $\nu_j=0$ if and only if $H'(x)/H(x)$ is continuous at x_j. Following Sabatier's terminology[4–7] we can refer to $F_j(k)$ as a "hard scatterer" and $\Lambda_{j,j+1}(k)$ as a "soft scatterer." The following theorem shows how the matrices defined in (4.1)–(4.3) are related to one another.

Theorem 4.1: We have

$$\Lambda=\Lambda_{0,1}F_1\Lambda_{1,2}F_2\Lambda_{2,3}\cdots F_N\Lambda_{N,N+1}, \tag{4.5}$$

where Λ, $\Lambda_{j,j+1}$, and F_j are the matrices defined in (4.2), (4.1), and (4.3), respectively.

Proof: Note that we have $\Lambda_{j,j+1}=G_jD_j$, where we have defined

$$G_j=\begin{bmatrix} \frac{1}{t_{j,j+1}(k)} & 0 \\ \frac{l_{j,j+1}(k)}{t_{j,j+1}(k)} & 1 \end{bmatrix}, \quad D_j=\begin{bmatrix} 1 & -r_{j,j+1}(k) \\ 0 & t_{j,j+1}(k) \end{bmatrix}.$$

Using the displayed equation in Ref. 1 following (14.4), we can relate $\Lambda(k)$ and $\mathscr{G}(k)$ defined in (2.12) as $\Lambda=G_0\mathscr{G}D_N$. Inserting the identity matrices $G_jG_j^{-1}$ and $D_jD_j^{-1}$ in the appropriate places in (2.12), we obtain

$$\Lambda=G_0D_0\prod_{n=1}^{N}\,[D_{n-1}^{-1}\Gamma_{n-1,n}(k,x_n-0)^{-1}\Gamma_{n,n+1}(k,x_n+0)G_n^{-1}][G_nD_n]. \tag{4.6}$$

Using (2.11), it can be checked that

$$D_{n-1}^{-1}\Gamma_{n-1,n}(k,x_n-0)^{-1}\Gamma_{n,n+1}(k,x_n+0)G_n^{-1}=F_n, \tag{4.7}$$

where F_n are the matrices defined in (4.3). Thus, using (4.7) in (4.6), we get (4.5). ■

It is already known[13] that the function $H(x)$ given by

$$H(x)=\frac{h_{j,j+1}}{f_l(0,x)^2}, \quad x\in(x_j,x_{j+1}), \quad j=0,\dots,N,$$

$$h_{N,N+1}=H_+; \quad h_{j-1,j}=q_j h_{j,j+1}, \quad j=0,\dots,N,$$

corresponds to the scattering data $Q(x)$, $\rho(k)=-b(k)/a(k)$, and $\tau(k)=1/a(k)$; as seen from Proposition 2.1, the scattering coefficients in this case coincide with their asymptotic expressions as $k\to\pm\infty$. In this case, the matrix factorization given in (4.5) reduces to the factorization in (2.18). This is because in this case ν_j given in (4.4) vanishes, and hence the matrix $F_j(k)$ defined in (4.3) becomes equal to $E(k,x_j)$ defined in (2.17); in fact, $F_j(k)=E(k,x_j)$ if and only if $\nu_j=0$. Furthermore, in this case $V_{j,j+1}(y)=0$ and hence $\Lambda_{j,j+1}(k)=\mathbf{I}$; in fact, $\Lambda_{j,j+1}(k)=\mathbf{I}$ if and only if $V_{j,j+1}(y)=0$. In this case, we also have $Z_l(k,y)=J_l(k,y)$ and $Z_r(k,y)=J_r(k,y)$.

Now let us ask the following question. If we choose $V_{j,j+1}(y)=0$ for $j=0,1,\dots,N$, but still allow $\nu_j\neq 0$, what is the corresponding $H(x)$? From the factorization formula (4.5), by letting $\Lambda_{j,j+1}(k)=\mathbf{I}$, we can explicitly evaluate the corresponding scattering matrix. In this case, the corresponding $H(x)$ is given by

$$\sqrt{H(x)}=\frac{1}{a_j f_l(0,x)+b_j f_r(0,x)}, \quad x\in(x_j,x_{j+1}), \quad j=0,\dots,N, \tag{4.8}$$

$$a_N=\frac{1}{\sqrt{H_+}}, \quad b_N=0, \tag{4.9}$$

and a_j,b_j for $j=0,1,\dots,N-1$, will be determined recursively by using the jumps in $H(x)$ and $H'(x)/H(x)$ according to (2.4) and (4.4), respectively. Using (4.8) in (2.4), we obtain

$$\frac{a_j f_l(0,x_j)+b_j f_r(0,x_j)}{a_{j-1} f_l(0,x_j)+b_{j-1} f_r(0,x_j)}=\sqrt{q_j}, \quad j=1,\dots,N. \tag{4.10}$$

From (4.8) we have

$$\frac{H'(x)}{H(x)}=-2\,\frac{a_j f_l'(0,x)+b_j f_r'(0,x)}{a_j f_l(0,x)+b_j f_r(0,x)}, \tag{4.11}$$

and hence from (4.4) we get

$$\frac{a_{j-1} f_l'(0,x_j)+b_{j-1} f_r'(0,x_j)}{a_{j-1} f_l(0,x_j)+b_{j-1} f_r(0,x_j)}-\frac{a_j f_l'(0,x_j)+b_j f_r'(0,x_j)}{a_j f_l(0,x_j)+b_j f_r(0,x_j)}=-\nu_j\sqrt{H(x_j-0)H(x_j+0)}, \quad j=1,\dots,N. \tag{4.12}$$

Solving the linear system (4.10) and (4.12) with unknowns a_{j-1} and b_{j-1} in terms of a_j and b_j and known quantities, and using (4.9), we obtain

$$a_{j-1}=\frac{a_j}{\sqrt{q_j}}+\frac{\nu_j f_r(0,x_j)\sqrt{H(x_j+0)}}{[f_l(0,x);f_r(0,x)]}, \quad j=1,\dots,N; \quad a_N=\frac{1}{\sqrt{H_+}}, \tag{4.13}$$

$$b_{j-1}=\frac{b_j}{\sqrt{q_j}}-\frac{\nu_j f_l(0,x_j)\sqrt{H(x_j+0)}}{[f_l(0,x);f_r(0,x)]}, \quad j=1,\dots,N; \quad b_N=0, \tag{4.14}$$

where $[f_l(0,x);f_r(0,x)] = f_l(0,x)f_r'(0,x) - f_l'(0,x)f_r(0,x)$ is the Wronskian, which is a constant completely determined by $Q(x)$ alone. We can also obtain the Jost solutions for (1.1) explicitly. In this case, since $V_{j,j+1}(y)=0$, we have $Y_{l;j,j+1}(k,y)=1$ and $Y_{r;j,j+1}(k,y)=1$; thus, the matrix $\Gamma_{j,j+1}(k,x)$ defined in (2.11) is determined by using (2.10). Hence, using (3.8) and (3.10) the Faddeev function $Z_l(k,y)$ is determined, and using (3.9) and (3.13) the Faddeev function $Z_r(k,y)$ is determined. Then we obtain $f_l(k,x)$ and $f_r(k,x)$ as in (3.1)–(3.2).

Note that in the above procedure, in case $Q(x)$ is an exceptional potential, i.e., if $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent, in (4.8)–(4.14) we need to replace $f_r(0,x)$ by a zero-energy solution of (1.5) linearly independent of $f_l(0,x)$, such as $\psi(x)=f_l(0,x)\int_0^x dy/f_l(0,y)^2$; with this choice of $\psi(x)$, we have $[f_l(0,x);\psi(x)]=1$. In the exceptional case, it turns out that although different choices for $\psi(x)$ lead to different coefficients a_j and b_j, the resulting $H(x)$ is independent of the choice of $\psi(x)$. Also note that, if $N=1$, it is necessary that the generic case occurs; however, for $N\geqslant 2$ the exceptional case may occur.

V. AN ALGORITHM TO RECOVER JUMPS IN $H'(x)/H(x)$

In Ref. 13 we described an algorithm to recover N, y_j, and q_j associated with the discontinuities of $H(x)$ in terms of the leading asymptotic behavior of the scattering data as $k\to\pm\infty$. In this section we will analyze the $O(1/k)$ terms in the scattering data and will describe an algorithm to recover the constants ν_j associated with the discontinuities of $H'(x)/H(x)$ from the almost periodic part of the $O(1/k)$ terms in the scattering data. The algorithm of Ref. 13 must be applied first to recover N, y_j, and q_j before the algorithm to recover ν_j is used. In order to use the algorithm, one also needs to know the value of $w_{N,N+1}$, where we have defined

$$w_{j,j+1}=\int_{y_j}^{y_{j+1}} dz\; V_{j,j+1}(z),$$

with $V_{j,j+1}(y)$ being the quantity defined in (2.5). The constant $w_{N,N+1}$ can be obtained from a reduced reflection coefficient in various ways without solving the entire inverse problem. For example, as we will see in Sec. VII, we have $w_{N,N+1}=2h_l(0+,y_N)$, where $h_l(t,y)$ is the solution of the Marchenko equation (7.7) that is uniquely solvable; hence the solution of (7.7) at the fixed point y_N gives us $w_{N,N+1}$.

Since $V_{j,j+1}\in L_1^1(\mathbf{R})$, the scattering coefficients associated with $V_{j,j+1}(y)$ satisfy[14]

$$\frac{1}{t_{j,j+1}(k)}=1+\frac{w_{j,j+1}}{2ik}+o\left(\frac{1}{k}\right),\quad k\to\pm\infty,$$

$$\frac{r_{j,j+1}(k)}{t_{j,j+1}(k)}=o\left(\frac{1}{k}\right),\quad \frac{l_{j,j+1}(k)}{t_{j,j+1}(k)}=o\left(\frac{1}{k}\right),\quad k\to\pm\infty,$$

and hence from (4.1) we have

$$\Lambda_{j,j+1}(k)=\mathbf{I}+\frac{w_{j,j+1}}{2ik}\,\mathbf{J}+o\left(\frac{1}{k}\right),\quad k\to\pm\infty,$$

where we have defined $\mathbf{J}=\mathrm{diag}(1,-1)$. Let us write (4.3) in the form

$$F_j=E_j+\frac{\nu_j}{2ik}\,U_j,$$

where E_j is the matrix $E(k,x_j)$ defined in (2.17) and

$$U_j=\begin{bmatrix} 1 & e^{-2iky_j} \\ -e^{2iky_j} & -1 \end{bmatrix}.$$

Thus, as $k\to\pm\infty$, from (4.5) we obtain $\Lambda=E_1E_2\cdots E_N+O(1/k)$ and

$$2ik[\Lambda-E_1E_2\cdots E_N]=w_{0,1}\mathbf{J}E_1E_2\cdots E_N+w_{1,2}E_1\mathbf{J}E_2\cdots E_N+\cdots+w_{N,N+1}E_1E_2\cdots E_N\mathbf{J}$$
$$+\nu_1U_1E_2\cdots E_N+\nu_2E_1U_2E_3\cdots E_N+\cdots+\nu_NE_1E_2\cdots E_{N-1}U_N+o(1). \tag{5.1}$$

Thus, from (2.18) and (4.2) we see that (5.1) allows us to express

$$2ik\left[\frac{1}{\tau(k)}-a(k)\right]=\Delta(k)+o(1),\quad k\to\pm\infty, \tag{5.2}$$

$$-2ik\left[\frac{\rho(k)}{\tau(k)}+b(k)\right]=\Omega(k)+o(1),\quad k\to\pm\infty, \tag{5.3}$$

where $\Delta(k)$ and $\Omega(k)$ are linear combinations of $w_{0,1},\dots,w_{N,N+1}$ and $\nu_1,\dots,\nu_N$ with almost periodic polynomials as coefficients.

Let us now explain how to compute ν_N. When $N=1$ we have

$$\Delta(k)=(w_{0,1}+w_{1,2})\alpha_1+\nu_1\equiv\Delta_1, \tag{5.4}$$

$$e^{2iky_1}\Omega(k)=(w_{0,1}-w_{1,2})\beta_1+\nu_1\equiv\Omega_1. \tag{5.5}$$

Multiplying (5.4) by β_1 and (5.5) by α_1, and subtracting the resulting equations, we obtain

$$\nu_1=\frac{1}{\alpha_1-\beta_1}[2w_{1,2}\alpha_1\beta_1+\alpha_1\Omega_1-\beta_1\Delta_1]. \tag{5.6}$$

When $N=2$, we have

$$\Delta(k)=\Delta_1+e^{2ik(y_2-y_1)}\Delta_2, \tag{5.7}$$

$$e^{2iky_2}\Omega(k)=\Omega_1+e^{2ik(y_2-y_1)}\Omega_2, \tag{5.8}$$

where we have defined

$$\Delta_1=(w_{0,1}+w_{1,2}+w_{2,3})\alpha_1\alpha_2+\nu_1\alpha_2+\nu_2\alpha_1, \tag{5.9}$$

$$\Omega_1=(w_{0,1}+w_{1,2}-w_{2,3})\alpha_1\beta_2+\nu_1\beta_2+\nu_2\alpha_1, \tag{5.10}$$

$$\Delta_2=(w_{0,1}-w_{1,2}+w_{2,3})\beta_1\beta_2+\nu_1\beta_2-\nu_2\beta_1,$$

$$\Omega_2=(w_{0,1}-w_{1,2}-w_{2,3})\beta_1\alpha_2+\nu_1\alpha_2-\nu_2\beta_1.$$

Multiplying (5.9) by β_2 and (5.10) by α_2 and subtracting the resulting equations, we obtain

$$\alpha_1(\beta_2-\alpha_2)\nu_2=-2w_{2,3}\alpha_1\alpha_2\beta_2+\beta_2\Delta_1-\alpha_2\Omega_1,$$

and hence

$$\nu_2=\frac{1}{\alpha_2-\beta_2}\left[2w_{2,3}\alpha_2\beta_2+\frac{\alpha_2\Omega_1-\beta_2\Delta_1}{\alpha_1}\right].$$

As can be seen from (5.4), (5.5), (5.7), and (5.8), and in general be proved by induction, the quantity $e^{2iky_N}\Omega(k)$ is obtained from $\Delta(k)$ by interchanging β_N with α_N and by changing the sign of $w_{N,N+1}$. It can also be shown that $\Delta(k)$ and $e^{2iky_N}\Omega(k)$ both are exponential polynomials having at most 2^{N-1} nonzero terms. To compute ν_N for arbitrary N, we let Δ_1 and Ω_1 denote the constant terms in the almost periodic polynomials $\Delta(k)$ and $e^{2iky_N}\Omega(k)$, respectively. From (5.1) we have

$$\Delta_1=\left(\sum_{j=0}^{N} w_{j,j+1}+\sum_{j=1}^{N}\frac{\nu_j}{\alpha_j}\right)\prod_{n=1}^{N}\alpha_n,$$

$$\Omega_1=\left(-2w_{N,N+1}\beta_N+\beta_N\sum_{j=0}^{N} w_{j,j+1}+\beta_N\sum_{j=1}^{N-1}\frac{\nu_j}{\alpha_j}+\nu_N\right)\prod_{n=1}^{N-1}\alpha_n.$$

Using

$$\beta_N\Delta_1-\alpha_N\Omega_1=2w_{N,N+1}\beta_N\prod_{j=1}^{N}\alpha_j+\nu_N(\beta_N-\alpha_N)\prod_{j=1}^{N-1}\alpha_j,$$

we get

$$\nu_N=\frac{1}{\alpha_N-\beta_N}\left[2w_{N,N+1}\alpha_N\beta_N+\frac{\alpha_N\Omega_1-\beta_N\Delta_1}{\Pi_{j=1}^{N-1}\alpha_j}\right].$$

After obtaining ν_N, we can recover ν_{N-1} as follows. The solution of the Marchenko equation in the interval $(y_N,+\infty)$ yields $V_{N,N+1}(y)$ by (7.9); thus also we have the matrix $\Lambda_{N,N+1}(k)$ defined in (4.1) because it is determined by the scattering matrix of the potential $V_{N,N+1}(y)$. Note that from the unitarity of the scattering matrix corresponding to the potential $V_{j,j+1}(y)$, we have $\det \Lambda_{j,j+1}(k)=1$. Using (2.16) it can be shown that $\det F_j(k)=1$. Thus, we can easily form the matrix $\Lambda\Lambda_{N,N+1}^{-1}F_N^{-1}$ and recover ν_{N-1} from this matrix, as we have recovered ν_N from the matrix Λ. Note that the reduced reflection coefficient from the right associated with the matrix $\Lambda\Lambda_{N,N+1}^{-1}F_N^{-1}$ is given by

$$\rho^{[N-1]}(k)=-\frac{[1\quad 0]\Lambda\Lambda_{N,N+1}^{-1}F_N^{-1}\begin{bmatrix}0\\1\end{bmatrix}}{[1\quad 0]\Lambda\Lambda_{N,N+1}^{-1}F_N^{-1}\begin{bmatrix}1\\0\end{bmatrix}}.\tag{5.11}$$

Once ν_{N-1} is obtained, we recursively get the remaining $\nu_{N-2},...,\nu_1$.

VI. A SINGULAR INTEGRAL EQUATION

In this section, when there are no bound states, we formulate the singular integral equation (6.7) whose kernel and nonhomogeneous term are determined by the reduced reflection coefficient $\rho(k)$. We also show that (6.7) is uniquely solvable and its solution leads to the recovery of $H(x)$. In a similar manner, we formulate the singular integral equation (6.10) in terms of $\ell(k)$ and prove its unique solvability and show that its solution also leads to the recovery of $H(x)$.

For each fixed $y\in\mathbf{R}\backslash\{y_1,\ldots,y_N\}$, from (5.11) of Ref. 1, we have

$$\begin{bmatrix}Z_l(-k,y)\\Z_r(-k,y)\end{bmatrix}=\begin{bmatrix}\tau(k) & -\rho(k)e^{2iky}\\ -\ell(k)e^{-2iky} & \tau(k)\end{bmatrix}\begin{bmatrix}Z_r(k,y)\\Z_l(k,y)\end{bmatrix},\quad k\in\mathbf{R}.\tag{6.1}$$

Using (2.19)–(2.21) and (3.3), we obtain

$$\begin{bmatrix} J_l(-k,y) \\ J_r(-k,y) \end{bmatrix} = \begin{bmatrix} \dfrac{1}{a(k)} & \dfrac{b(k)}{a(k)}\, e^{2iky} \\ -\dfrac{b(-k)}{a(k)}\, e^{-2iky} & \dfrac{1}{a(k)} \end{bmatrix} \begin{bmatrix} J_r(k,y) \\ J_l(k,y) \end{bmatrix}, \quad k \in \mathbf{R}. \tag{6.2}$$

Subtracting (6.2) from (6.1), we get

$$Z_l(-k,y) - J_l(-k,y) = \left[\tau(k) - \frac{1}{a(k)} \right] Z_r(k,y) + \frac{1}{a(k)}\, [Z_r(k,y) - J_r(k,y)] \\ - \rho(k) e^{2iky} [Z_l(k,y) - J_l(k,y)] - \left[\rho(k) + \frac{b(k)}{a(k)} \right] e^{2iky} J_l(k,y), \tag{6.3}$$

$$Z_r(-k,y) - J_r(-k,y) = \left[\tau(k) - \frac{1}{a(k)} \right] Z_l(k,y) + \frac{1}{a(k)}\, [Z_l(k,y) - J_l(k,y)] \\ - \ell(k) e^{-2iky} [Z_r(k,y) - J_r(k,y)] - \left[\ell(k) - \frac{b(-k)}{a(k)} \right] e^{-2iky} J_r(k,y). \tag{6.4}$$

Let us analyze (6.3). Using Propositions 2.1 and 3.1 and Theorem 3.2, for each fixed y, in the absence of bound states, of the four terms on the right-hand side, we see that the first two belong to the Hardy space $\mathbf{H}^2_+(\mathbf{R})$ and the last two belong to $L^2(\mathbf{R})$; the term on the left-hand side belongs to $\mathbf{H}^2_-(\mathbf{R})$. Let $\Pi_\pm$ denote the orthogonal projection operators from $L^2(\mathbf{R})$ onto $\mathbf{H}^2_\pm(\mathbf{R})$, i.e.

$$(\Pi_\pm f)(k) = \frac{\pm 1}{2\pi i} \int_{-\infty}^{\infty} \frac{ds}{s - k \mp i0}\, f(s).$$

Let us define

$$X_l(k,y) = Z_l(-k,y) - J_l(-k,y), \quad X_r(k,y) = Z_r(-k,y) - J_r(-k,y). \tag{6.5}$$

Applying the projection Π_- on both sides of (6.3), we obtain

$$X_l(\cdot,y) + \Pi_-(\rho e^{2i(\cdot)y} \mathscr{J} X_l(\cdot,y)) = -\Pi_-\left(\left[\rho + \frac{b}{a} \right] e^{2i(\cdot)y} J_l(\cdot,y) \right), \tag{6.6}$$

where $(\mathscr{J} f)(k) = f(-k)$. Note that (6.6) is a singular integral equation and can be written as

$$X_l(k,y) + (\mathscr{O}_l X_l)(k,y) = P_l(k,y), \quad k \in \mathbf{R}, \tag{6.7}$$

where we have defined

$$(\mathscr{O}_l X)(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{ds}{s + k - i0}\, \rho(-s) e^{-2isy} X(s), \quad k \in \mathbf{R}, \tag{6.8}$$

$$P_l(k,y) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{ds}{s - k + i0} \left[\rho(s) + \frac{b(s)}{a(s)} \right] e^{2isy} J_l(s,y). \tag{6.9}$$

Notice that the integral operator $\mathscr{O}_l$ defined in (6.8) is the same as the operator defined in (5.23) of Ref. 1. Comparing (5.21) of Ref. 1 and (6.7), we see that the kernels in these two integral equations differ by a minus sign. We also recall that the solution of the singular integral equation of Ref. 1 is given by $X_l(k,y) = [Z_l(-k,y) - Z_l(0,y)]/[k\sqrt{H(x)}]$, where $Z_l(k,y)$ is the quantity defined in (3.1), whereas the solution of the integral equation of this paper is given by (6.5). The factor $1/k$ in the expression for $X_l(k,y)$ used in Ref. 1 was introduced to ensure that $X_l(k,y)$ belongs to an appropriate Hardy space, namely to $\mathbf{H}_-^p(\mathbf{R})$ if $p<1/(1-\alpha)$. However, this factor, while providing the desired behavior as $k\to\infty$, introduced some complications at $k=0$. With the present definition (6.5) it is easy to show that $X_l(k,y)$ is continuous as $k\to 0$ in $\overline{\mathbf{C}^+}$ and $X_l(k,y)=O(1/k)$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$, without imposing any stronger condition on $Q(x)$ than $Q\in L_1^1(\mathbf{R})$.

In a similar manner, in the absence of bound states, from (6.4) we obtain

$$X_r(\cdot,y)+\Pi_-(\ell e^{-2i(\cdot)y}\mathscr{J}X_r(\cdot,y))=-\Pi_-\left(\left[\ell-\frac{\mathscr{J}b}{a}\right]e^{-2i(\cdot)y}J_r(\cdot,y)\right),$$

which is equivalent to

$$X_r(k,y)+(\mathscr{O}_rX_r)(k,y)=P_r(k,y), \quad k\in\mathbf{R}, \tag{6.10}$$

where we have defined

$$(\mathscr{O}_rX)(k)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s+k-i0}\,\ell(-s)e^{2isy}X(s), \quad k\in\mathbf{R},$$

$$P_r(k,y)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s-k+i0}\left[\ell(s)-\frac{b(-s)}{a(s)}\right]e^{-2isy}J_r(s,y). \tag{6.11}$$

The solvability of (6.7) and (6.10) is analyzed in the next theorem.

Theorem 6.1: The singular integral equation (6.7) has a unique solution $X_l\in\mathbf{H}_-^2(\mathbf{R})$ for every nonhomogeneous term belonging to $\mathbf{H}_-^2(\mathbf{R})$, and the solution can be obtained through iteration. Similarly, (6.10) has a unique solution $X_r\in\mathbf{H}_-^2(\mathbf{R})$ for every nonhomogeneous term belonging to $\mathbf{H}_-^2(\mathbf{R})$ and the solution can be obtained through iteration.

Proof: The operator $\mathscr{O}_l$ defined in (6.8) is a strict contraction on $\mathbf{H}_-^2(\mathbf{R})$, which is proved in Theorem 7.1 of Ref. 1. Hence, (6.7) is uniquely solvable and its solution can be obtained through iteration. The proof for (6.10) is given in the same manner. ■

Next we will recover $H(x)$ from an appropriate set of scattering data. We will consider the generic and exceptional cases separately because the scattering data in these two cases are not the same.

Let us first consider the generic case; in this case an appropriate set of scattering data consists of $\{\rho(k),Q(x)\}$. We proceed as follows. Using the method of Ref. 16, from $\rho(k)$ we get $b(k)$ and $a(k)$; then from these we get N, $\{y_1,\ldots,y_N\}$, and $\{q_1,\ldots,q_N\}$ by using the method of Ref. 13. Hence, we have α_j and β_j for $j=1,\ldots,N$. Since $Q(x)$ is known, we also know the zero-energy Jost solutions of (1.5); these Jost solutions are identical to the zero-energy Jost solutions of (1.1). For example, we can get $f_l(0,x)$ by using (5.25) of Ref. 1. Next we obtain $J_l(k,y)$ using (3.4) and (3.5). Note that $J_l(k,y)$ is uniquely constructed from $\rho(k)$ because we already have y_j, α_j, and β_j for $j=1,\ldots,N$. From (3.1) and the fact that $H(x)=dy/dx$, we have

$$\frac{dy}{Z_l(0,y)^2}=H_+\,\frac{dx}{f_l(0,x)^2}. \tag{6.12}$$

Using $J_l(k,y)$ and $\rho(k)$ in (6.7), we obtain $X_l(k,y)$ uniquely. Then using (6.5), we write (6.12) in the form

$$\frac{dy}{[X_l(0,y)+J_l(0,y)]^2}=H_+\,\frac{dx}{f_l(0,x)^2}. \tag{6.13}$$

We get H_+ from (6.13) as

$$H_+=\frac{\int_{-\infty}^{0}dy/[X_l(0,y)+J_l(0,y)]^2}{\int_{-\infty}^{0}dx/f_l(0,x)^2}. \tag{6.14}$$

Note that both integrals in (6.14) converge because[17] in the generic case, $f_l(0,x)^2$ grows like x^2 as $x\to-\infty$ and $Z_l(0,y)^2$ grows like y^2 as $y\to-\infty$. Next, using a generalization of the method given in Theorem 5.1 of Ref. 13, we obtain $x_1,\ldots,x_N$. This is done as follows. If $N=1$ and $y_1=0$, then $x_1=0$. If $N=1$ and $y_1\neq 0$, then we can proceed as in the case $N\geqslant 2$. If $N\geqslant 2$, then at least $N-1$ of the points $y_1,\ldots,y_N$ must be nonzero. If at least one of these is positive, we can pick the smallest of them, say y_p. Then x_p is uniquely determined by

$$\int_0^{y_p}\frac{dy}{[X_l(0,y)+J_l(0,y)]^2}=H_+\int_0^{x_p}\frac{dx}{f_l(0,x)^2}, \tag{6.15}$$

and we recursively determine $x_{p+1},\ldots,x_N$ using

$$\int_{y_p}^{y_{p+1}}\frac{dy}{[X_l(0,y)+J_l(0,y)]^2}=H_+\int_{x_p}^{x_{p+1}}\frac{dx}{f_l(0,x)^2}.$$

Similarly, we can determine $x_{p-1},x_{p-2},\ldots,x_1$. If all y_j are nonpositive, then we pick the one with the smallest absolute value that is nonzero (either y_N or y_{N-1}) and find the corresponding x_j by using the appropriate integral of the form (6.15). Having found each x_j corresponding to y_j, we obtain $y(x)$ by solving the first-order separable ordinary differential equation (6.13) with the initial condition $y(x_i)=y_i$. Having $y(x)$ in each interval (x_j,x_{j+1}), we get $H(x)=dy/dx$.

Now let us consider the exceptional case. In this case, we cannot use (6.14) to obtain H_+. In fact, for the unique recovery of $H(x)$ we need to include H_+ in the scattering data; otherwise, we get a one-parameter family of $H(x)$ corresponding to the set $\{\rho(k),Q(x)\}$. Thus, in the exceptional case, we recover $H(x)$ from the scattering data $\{\rho(k),H_+,Q(x)\}$ by the method outlined in the generic case.

Note that one can also recover $H(x)$ from the solution of the singular integral equation (6.10) using the scattering data $\{\ell(k),Q(x)\}$ in the generic case and using $\{\ell(k),Q(x),H_+\}$ in the exceptional case. One then needs to solve the analog of (6.12) given by

$$\frac{dy}{Z_r(0,y)^2}=H_-\,\frac{dx}{f_r(0,x)^2}, \tag{6.16}$$

with the condition $y(0)=0$. Note that from (6.5) we have $Z_r(0,y)=X_r(0,y)+J_r(0,y)$, and $f_r(0,x)$ is the zero-energy Jost solution from the right of (1.5) corresponding to $Q(x)$. The potential $Q(x)$ uniquely determines[14,18–20] $f_r(0,x)$, for example, by

$$f_r(0,x)=1+\int_{-\infty}^{x}dz\,(x-z)Q(z)f_r(0,z). \tag{6.17}$$

Once we obtain y as a function of x from (6.16), we recover $H(x)$ as

$$H(x)=H_- \frac{Z_r(0,y)^2}{f_r(0,x)^2}. \tag{6.18}$$

Note that, in the exceptional case, H_- can be expressed in terms of H_+ by using (5.29) of Ref. 1, namely,

$$H_-=H_+ \frac{1-\rho(0)}{1+\rho(0)} \left(\frac{1+R^{[0]}(0)}{T^{[0]}(0)}\right)^2, \tag{6.19}$$

where $R^{[0]}(k)$ and $T^{[0]}(k)$ are the reflection coefficient from the right and the transmission coefficient, respectively, associated with (1.5). Hence, in the exceptional case, one can use H_- in the scattering data instead of H_+ because of (6.19). Note also that in the exceptional case $f_l(0,x)$ and $f_r(0,x)$ are linearly dependent, and we have[17]

$$f_r(0,x)=\frac{1+R^{[0]}(0)}{T^{[0]}(0)} f_l(0,x). \tag{6.20}$$

Let $f_l^{[0]}(k,x)$ and $f_r^{[0]}(k,x)$ denote the Jost solutions of (1.5) from the left and from the right, respectively. In the generic case we have

$$f_r^{[0]}(k,x)=[f_l^{[0]}(k,x);f_r^{[0]}(k,x)]f_l^{[0]}(k,x)\int_{-\infty}^{x} \frac{dz}{f_l^{[0]}(k,z)^2}, \tag{6.21}$$

where the Wronskian $[f_l^{[0]}(k,x);f_r^{[0]}(k,x)]$ is equal to $-2ik/T^{[0]}(k)$. Hence, in the generic case from (6.21), after using the fact that $f_l^{[0]}(0,x)=f_l(0,x)$ and $f_r^{[0]}(0,x)=f_r(0,x)$, we have

$$f_r(0,x)=\left[\lim_{k\to 0} \frac{-2ik}{T^{[0]}(k)}\right] f_l(0,x)\int_{-\infty}^{x} \frac{dz}{f_l(0,z)^2}.$$

VII. MARCHENKO INTEGRAL EQUATION

In this section we show that the singular integral equation (6.7), with the use of the Fourier transform, can be transformed into the integral equation (7.7) generalizing the Marchenko integral equation[14,18–20] for the one-dimensional Schrödinger equation. We establish the unique solvability of (7.7) and describe how its solution leads to the recovery of $H(x)$.

Using (2.20) and the continuity of $\rho(k)$ and $b(k)/a(k)$, we see that $\rho+(b/a)\in L^p(\mathbf{R})$ for any $p\in(1,+\infty]$. We may then write

$$\rho(k)=-\frac{b(k)}{a(k)}+\int_{-\infty}^{\infty} dz\ e^{ikz}\varrho(z), \tag{7.1}$$

where $\varrho\in L^q(\mathbf{R})$ for $q\in[2,+\infty)$. The symmetry relation $F(-k)=\overline{F(k)}$ for $k\in\mathbf{R}$ valid for ρ, a, and b, implies that ϱ is real valued. Since b/a belongs to AP^W, we have $b(k)/a(k)=-\Sigma_s\gamma_s e^{ikb_s}$ where b_s are different real numbers and γ_s are real constants satisfying $\Sigma_s|\gamma_s|<+\infty$; thus we can write (7.1) in the form

$$\rho(k)=\sum_s \gamma_s e^{ikb_s}+\int_{-\infty}^{\infty} dz\ e^{ikz}\varrho(z). \tag{7.2}$$

Let us write (7.2) in the concise form

$$\rho(k)=\int_{-\infty}^{\infty} d\mu(t)\ e^{ikt}, \tag{7.3}$$

for a suitable real measure μ that is the sum of a discrete measure (with weights γ_s at the points b_s) and an absolutely continuous measure (with a Radon-Nikodym derivative ϱ). Let $\mathscr{F}$ denote the Fourier transform defined by

$$(\mathscr{F}g)(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ikt}g(k). \tag{7.4}$$

Since $X_l(\cdot,y)$ and $P_l(\cdot,y)$ appearing in (6.7) belong to $\mathbf{H}^2_-(\mathbf{R})$, their Fourier transforms are supported on the positive half-line; hence, we have

$$X_l(k,y)=\int_0^{\infty} dt\ e^{-ikt}h_l(t,y), \quad P_l(k,y)=\int_0^{\infty} dt\ e^{-ikt}h_{l0}(t,y), \tag{7.5}$$

where $h_l,h_{l0}\in L^q(\mathbf{R}^+)$ for any $q\in[2,+\infty)$. Furthermore, as seen from (3.16), $J_l(k,y)$ consists of a finite sum of exponential terms; hence we have $J_l(k,y)=\Sigma_s\omega_s(y)e^{ik\zeta_s(y)}$, where, in each interval $(y_j,y_{j+1}),\omega_s(y)$ is a constant and $\zeta_s(y)$ is either a constant or an affine function of y. Thus, from (6.9) we obtain

$$h_{l0}(t,y)=-\sum_s \omega_s(y)\varrho(-t-2y-\zeta_s(y)), \quad t\geqslant 0.$$

Now let us take the Fourier transform of both sides of (6.7). We have

$$h_l(\cdot,y)+(\mathscr{F}\mathscr{O}_l\mathscr{F}^{-1}h_l)(\cdot,y)=h_{l0}(\cdot,y). \tag{7.6}$$

Using (7.2) or (7.3) we can write (7.6) as the Marchenko-like integral equation

$$h_l(t,y)+\int_{-\infty}^{-(t+2y)} d\mu(z)\ h_l(-z-t-2y,y)=h_{l0}(t,y), \quad t\geqslant 0,$$

or equivalently

$$h_l(t,y)+\sum_{\{s:b_s<-t-2y\}} \gamma_s h_l(-t-2y-b_s,y)+\int_0^{\infty} ds\ \varrho(-s-t-2y)\ h_l(s,y)=h_{l0}(t,y), \quad t\geqslant 0. \tag{7.7}$$

We will call (7.7) a Marchenko equation. Note that when $N=0$, i.e. when $V(y)$ given in (2.3) is well defined for all $y\in\mathbf{R}$, the integral equation (7.7) reduces to

$$h_l(t,y)+\int_0^{\infty} ds\ \varrho(-s-t-2y)\ h_l(s,y)=-\varrho(-t-2y), \quad t\geqslant 0, \tag{7.8}$$

which is the Marchenko equation[14,18–20] for the ordinary Schrödinger equation. In a similar manner we can also obtain a Marchenko integral equation associated with the reflection coefficient ℓ(k), but we will not list it here. The next theorem shows that (7.7) is uniquely solvable.

Theorem 7.1: Equation (7.7) has a unique solution in $L^2(\mathbf{R}^+)$ for every nonhomogeneous term belonging to $L^2(\mathbf{R}^+)$, and the solution can be obtained through iteration.

Proof: The operator $\mathscr{O}_l$ in (7.6) is a strict contraction on $\mathbf{H}^2_-(\mathbf{R})$, as indicated in the proof of Theorem 6.1. Considering $L^2(\mathbf{R}^+)$ and $\mathbf{H}^2_-(\mathbf{R})$ as subspaces of $L^2(\mathbf{R})$, we see that $\sqrt{2\pi}\mathscr{F}$, where

$\mathscr{F}$ is the Fourier transformation defined in (7.4), is a unitary operator on $L^2(\mathbf{R})$ mapping $\mathbf{H}^2_-(\mathbf{R})$ onto $L^2(\mathbf{R}^+)$. Thus, the operator $\mathscr{F}\mathscr{O}_l\mathscr{F}^{-1}$ acting from $L^2(\mathbf{R}^+)$ into $L^2(\mathbf{R}^+)$ is a strict contraction. Hence, (7.7) is uniquely solvable and its solution can be obtained through iteration. ■

Let us now discuss the recovery of $H(x)$ from the solution of the Marchenko equation (7.7). Once $h_l(t,y)$ is obtained from (7.7), we can get $X_l(k,y)$ from (7.5) and recover $H(x)$ by repeating the procedure described in Sec. VI.

Let us also describe another way to recover $H(x)$. This is done in conjunction with the algorithm described in Sec. V, where N, y_j, q_j, are recovered first; recall that these are the parameters associated with the "hard scatterers." Next we recover the quantities associated with the "soft scatterers," namely we obtain $V_{j,j+1}(y)$. This is done recursively as follows. First we solve (7.7) only for $y>y_N$ and get $h_l(t,y)$ in the interval $(y_N,+\infty)$. Because of (3.8) we obtain[21]

$$V_{N,N+1}(y)=-2\,\frac{dh_l(0+,y)}{dy},\quad y_N<y<+\infty, \tag{7.9}$$

$$Z_l(0,y)=1+\int_y^\infty dz\,(z-y)V_{N,N+1}(z)\,Z_l(0,z),\quad y_N<y<+\infty. \tag{7.10}$$

Then, as described in Sec. V, we form the new reduced reflection coefficient $\rho^{[N-1]}(k)$ defined in (5.11) and obtain $V_{N-1,N}(y)$ from the solution of the Marchenko equation corresponding to $\rho^{[N-1]}(k)$ by using the analog of (7.9). Continuing in this manner, we then recover $V_{j,j+1}(y)$ for $j=0,1,\dots,N$. Then we obtain $Z_l(0,y)$ for $y\in\mathbf{R}\setminus\{y_1,\dots,y_N\}$ as follows. From (3.1) we have

$$Z_l(k,y_j-0)=\sqrt{q_j}\,Z_l(k,y_j+0),$$

$$Z_l'(k,y_j-0)=\frac{Z_l'(k,y_j+0)}{\sqrt{q_j}}-2ik\left(\beta_j-\frac{\nu_j}{2ik}\right)Z_l(k,y_j+0),$$

as well as $Z_l(k,+\infty)=1$ and $Z_l'(k,+\infty)=0$. Hence, $Z_l(0,y)$ and $Z_l'(0,y)$ satisfy the following internal boundary conditions:

$$Z_l(0,y_j-0)=\sqrt{q_j}\,Z_l(0,y_j+0), \tag{7.11}$$

$$Z_l'(0,y_j-0)=\frac{Z_l'(0,y_j+0)}{\sqrt{q_j}}+\nu_j\,Z_l(0,y_j-0). \tag{7.12}$$

Thus, in each interval (y_j,y_{j+1}), we can uniquely obtain $Z_l(0,y)$ from $V_{j,j+1}(y)$ by using

$$Z_l(0,y)=(y-y_{j+1})Z_l'(0,y_{j+1}-0)+Z_l(0,y_{j+1}-0)+\int_y^{y_{j+1}}dz\,(z-y)V(z)\,Z_l(0,z). \tag{7.13}$$

Thus, using (7.10), (7.11)–(7.13) we obtain $Z_l(0,y)$ for $y\in\mathbf{R}\setminus\{y_1,\dots,y_N\}$. Once we have $Z_l(0,y)$, we can recover $H(x)$ by using the procedure outlined starting with (6.12).

Note that although we assume that there are no bound states associated with (1.1), some of the $V_{j,j+1}(y)$ may have bound states. In terms of the factorization formula (4.5), this happens when the hard scatterers $F_j(k)$ in (4.5) overcome the bound states from the soft scatterers $\Lambda_{j,j+1}(k)$, resulting in no bound states for (1.1); in other words, the poles of $t_{j,j+1}(k)$ in $\mathbf{C}^+$ are canceled by the terms in $F_j(k)$, resulting in no poles in $\mathbf{C}^+$ for $\tau(k)$. The recovery of $V_{j,j+1}(y)$, even in the presence of bound states, is well understood;[22] since each $V_{j,j+1}(y)$ has support contained in a half-line, the reflection coefficient $r_{j,j+1}(k)$ uniquely determines $V_{j,j+1}(y)$ without needing the bound state energies and the bound state norming constants; in fact, both the bound state energies and the norming constants are uniquely determined by $r_{j,j+1}(k)$ alone.

We can also obtain $H(x)$ by modifying the procedures described earlier. For example, using the reduced reflection coefficient from the left $\ell(k)$, the analog of (7.7) associated with $\ell(k)$ can be used to obtain $V_{j,j+1}(y)$ starting with the interval (y_0,y_1) and moving to the interval (y_1,y_2) and continuing in this manner. One can also solve the Marchenko equations associated with $\ell(k)$ and $\rho(k)$, respectively, simultaneously starting with the intervals (y_0,y_1) and (y_N,y_{N+1}), respectively, and moving to the intervals (y_1,y_2) and (y_{N-1},y_N), respectively, and continuing in this manner until all $V_{j,j+1}(y)$ are obtained. Then, using (7.11)–(7.13) one gets $Z_l(0,y)$ or $Z_r(0,y)$, from which $H(x)$ is obtained using (6.12) or (6.16).

VIII. EXAMPLES

In this section we illustrate the methods described in Secs. V–VII through explicitly solved examples. In Examples 8.1–8.3 we illustrate the recovery of $H(x)$ using the solution of the Marchenko integral equation (7.7). In Example 8.4 we illustrate the method of Sec. V to recover the discontinuities in $H'(x)/H(x)$. In Example 8.5 we illustrate the alternative procedure described in Sec. VII using (5.11). Finally, In Example 8.6 we illustrate the recovery of $H(x)$ in terms of the solutions of the singular integral equations (6.7) and (6.10).

Example 8.1: Let us demonstrate the Marchenko method of Sec. VII. As our scattering data, for a given $Q(x)$ with no bound states and a given H_+, let us use

$$\rho(k)=\epsilon\,\frac{k+i\alpha}{k+i\gamma}, \tag{8.1}$$

where ϵ, α, and γ are real constants satisfying $-1<\epsilon<1$, $\gamma>0$, and $\gamma^2>\alpha^2\epsilon^2$. It is straightforward but tedious to show that for $y\leqslant 0$ the denominator in (8.11) and (8.12) is nonzero if and only if $(\alpha+\beta)\epsilon\neq 0$. Thus, in this example, we assume $(\alpha+\beta)\epsilon\neq 0$ and postpone the case $(\alpha+\beta)\epsilon=0$ to Example 8.2. Using the method of Ref. 16 we construct $\tau(k)$ by solving the Wiener-Hopf factorization problem $\tau(k)\tau(-k)=1-|\rho(k)|^2$ for $k\in\mathbf{R}$, and we obtain

$$\tau(k)=\sqrt{1-\epsilon^2}\,\frac{k+i\beta}{k+i\gamma}, \tag{8.2}$$

where we have defined the positive constant

$$\beta=\sqrt{\frac{\gamma^2-\alpha^2\epsilon^2}{1-\epsilon^2}}. \tag{8.3}$$

It can be verified that $|\tau(k)|^2+|\rho(k)|^2=1$ and that $\tau(k)$ has no poles or zeros in $\mathbf{C}^+$. Since $\tau(0)\neq 0$, we are in the exceptional case. Using the method of Ref. 13, we obtain

$$N=1,\quad q_1=\frac{1-\epsilon}{1+\epsilon},\quad y_1=0,\quad a(k)=\frac{1}{\sqrt{1-\epsilon^2}},\quad b(k)=-\frac{\epsilon}{\sqrt{1-\epsilon^2}}. \tag{8.4}$$

From (3.16) we get

$$J_l(k,y)=\begin{cases}\dfrac{1-\epsilon e^{-2iky}}{\sqrt{1-\epsilon^2}}, & y<0,\\ 1, \quad y>0.\end{cases} \tag{8.5}$$

Thus, from (6.9) we obtain

$$P_l(k,y)=\begin{cases} \dfrac{i\epsilon}{k+i\gamma}\,(\alpha-\gamma)\sqrt{1-e^2}\,[e^{2\gamma y}-e^{2iky}], & y<0,\\ 0, & y>0. \end{cases} \tag{8.6}$$

Using (8.6) in (7.5) we have

$$h_{l0}(t,y)=\begin{cases} -\dfrac{\epsilon(\alpha-\gamma)}{\sqrt{1-\epsilon^2}}\,e^{\gamma(t+2y)}, & t>0, \quad t+2y<0,\\ 0, \quad t>0, & t+2y>0. \end{cases}$$

From (8.1) we see that we can write (7.2) as

$$\rho(k)=\epsilon+\int_{-\infty}^{\infty}dt\ e^{ikt}\varrho(t),$$

with

$$\varrho(t)=\begin{cases} 0, & t<0\\ \epsilon(\alpha-\gamma)e^{-\gamma t}, & t>0, \end{cases} \tag{8.7}$$

and hence $\varrho(t)$ is supported only on $t\geqslant 0$. The Marchenko equation (7.7) has the following form:

$$h_l(t,y)=0, \quad t>0, \quad t+2y>0, \tag{8.8}$$

$$h_l(t,y)+\epsilon h_l(-t-2y,y)+\epsilon(\alpha-\gamma)e^{\gamma(t+2y)}\int_0^{-(t+2y)}ds\ e^{\gamma s}h_l(s,y)=-\frac{\epsilon(\alpha-\gamma)}{\sqrt{1-\epsilon^2}}\,e^{\gamma(t+2y)},$$

$$t>0, \quad t+2y<0. \tag{8.9}$$

Notice that from (8.8) we obtain $X_l(k,y)=0$ for $y>0$, and hence using (8.1) and (8.5), from (6.13) we conclude that

$$H(x)=\frac{H_+}{f_l(0,x)^2}, \quad y=H_+\int_0^x \frac{dz}{f_l(0,z)^2}, \quad x>0, \tag{8.10}$$

where $f_l(0,x)$ is the zero-energy Jost solution from the left associated with $Q(x)$. We can solve (8.9) exactly and obtain

$$h_l(t,y)=\frac{(\beta^2-\gamma^2)e^{\beta t}+\epsilon(\gamma-\beta)(\alpha+\beta)e^{-\beta(t+2y)}}{\sqrt{1-e^2}[(\alpha+\beta)\epsilon e^{-2\beta y}+\beta-\gamma]}, \quad t>0, \quad t+2y<0, \tag{8.11}$$

where β is the constant in (8.3) and the denominator does not vanish. Using (8.11) in (7.5), for $y<0$, we get

$$X_l(k,y)=\frac{(\beta+ik)(\beta^2-\gamma^2)[e^{2y(ik-\beta)}-1]+(\beta-ik)\epsilon(\gamma-\beta)(\alpha+\beta)e^{-2\beta y}[1-e^{2y(\beta+ik)}]}{(k^2+\beta^2)\sqrt{1-\epsilon^2}[(\alpha+\beta)\epsilon e^{-2\beta y}+\beta-\gamma]}. \tag{8.12}$$

Hence, using (8.5) and (8.12), we find

$$Z_l(0,y)=\sqrt{\frac{\gamma-\epsilon\alpha}{\gamma+\epsilon\alpha}}\,\frac{\epsilon(\alpha+\beta)e^{-2\beta y}+\gamma-\beta}{\epsilon(\alpha+\beta)e^{-2\beta y}-y+\beta},\quad y<0. \tag{8.13}$$

Using (8.13) in (6.12), we obtain

$$\frac{\gamma+\epsilon\alpha}{\gamma-\epsilon\alpha}\left[y+\frac{2(\gamma-\beta)/\beta}{\epsilon(\alpha+\beta)+\gamma-\beta}-\frac{2(\gamma-\beta)/\beta}{\epsilon(\alpha+\beta)e^{-2\beta y}+\gamma-\beta}\right]=H_+\int_0^x\frac{dz}{f_l(0,z)^2},\quad y<0, \tag{8.14}$$

$$H(x)=\frac{H_+}{f_l(0,x)^2}\,\frac{\gamma-\epsilon\alpha}{\gamma+\epsilon\alpha}\left[\frac{\epsilon(\alpha+\beta)e^{-2\beta y}+\gamma-\beta}{\epsilon(\alpha+\beta)e^{-2\beta y}-\gamma+\beta}\right]^2,\quad y<0, \tag{8.15}$$

where y in (8.15) is obtained in terms of x from (8.14).

Example 8.2: In this example we consider the same scattering data as in Example 8.1 but with the additional condition $(\alpha+\beta)\epsilon=0$, where β is the constant in (8.3). If $\epsilon=0$ then $\rho(k)=0$ and $\tau(k)=1$, and the Marchenko equation (7.7) gives us $h_l(t,y)=0$ for $t>0$ and $y\in\mathbf{R}$; thus, there are no discontinuities in $H(x)$ or $H'(x)/H(x)$, and we have

$$H(x)=\frac{H_+}{f_l(0,x)^2},\quad x\in\mathbf{R}.$$

If $\beta=-\alpha$ but $\epsilon\neq0$, then $\gamma=\beta$; in this case we have

$$\rho(k)=\epsilon\,\frac{k-i\gamma}{k+i\gamma},\quad \tau(k)=\sqrt{1-\epsilon^2}.$$

In this case, for $x>0$, (8.10) is still valid. When $x<0$, we proceed as follows. In the Marchenko equation (8.9), putting $\alpha=-\gamma$, we obtain

$$h_l(t,y)+\epsilon h_l(-t-2y,y)-2\gamma\epsilon e^{\gamma(t+2y)}\int_0^{-(t+2y)}ds\ e^{\gamma s}h_l(s,y)=\frac{2\epsilon\gamma}{\sqrt{1-\epsilon^2}}\,e^{\gamma(t+2y)},$$
$$t>0,\quad t+2y<0. \tag{8.16}$$

The solution of (8.16) is given by

$$h_l(t,y)=\frac{2\epsilon\gamma}{\sqrt{1-\epsilon^2}}\,\frac{e^{\gamma(t+2y)}}{1+\epsilon e^{2\gamma y}},\quad t>0,\quad t+2y<0. \tag{8.17}$$

Using (6.5), (7.5), (8.5), (8.8), and (8.17), we obtain

$$Z_l(0,y)=\sqrt{\frac{1+\epsilon}{1-\epsilon}}\,\frac{1-\epsilon e^{2\gamma y}}{1+\epsilon e^{2\gamma y}},\quad y<0. \tag{8.18}$$

Using (8.18) in (6.12), we obtain

$$\frac{1-\epsilon}{1+\epsilon}\left[y+\frac{2/\gamma}{1-\epsilon e^{2\gamma y}}-\frac{2/\gamma}{1-\epsilon}\right]=H_+\int_0^x\frac{dz}{f_l(0,z)^2},\quad x<0, \tag{8.19}$$

$$H(x)=\frac{H_+}{f_l(0,x)^2}\,\frac{1+\epsilon}{1-\epsilon}\left(\frac{1-\epsilon e^{2\gamma y}}{1+\epsilon e^{2\gamma y}}\right)^2, \quad y<0, \tag{8.20}$$

where y in (8.20) is obtained in terms of x from (8.19).

Example 8.3: In this example, we consider the scattering data of Example 8.1 with $\gamma^2=\alpha^2\epsilon^2$. When $\gamma=\pm\alpha\epsilon$, we have $\beta=0$, and hence $\rho(k)=(\epsilon k\pm i\gamma)/(k+i\gamma)$. Since $\rho(0)=+1$ is not allowed (cf. Theorem 4.2 of Ref. 1), we cannot have $\gamma=+\alpha\epsilon$. Thus, the inverse scattering problem to be solved corresponds to the scattering data

$$\rho(k)=\frac{\epsilon k-i\gamma}{k+i\gamma}, \quad Q(x),$$

when there are no bound states. We have $\tau(k)=\sqrt{1-\epsilon^2}k/(k+i\gamma)$, and hence this corresponds to the generic case; thus H_+ cannot be specified arbitrarily in the scattering data, and it is determined as in (6.14). In this case, (8.8) still holds. Putting $\alpha=-\gamma/\epsilon$ in (8.9), we obtain

$$h_l(t,y)+\epsilon h_l(-t-2y,y)-\gamma(1+\epsilon)e^{\gamma(t+2y)}\int_0^{-(t+2y)}ds\; e^{\gamma s}h_l(s,y)=\frac{\gamma(1+\epsilon)}{\sqrt{1-\epsilon^2}}\,e^{\gamma(t+2y)},$$

$$t>0, \quad t+2y<0. \tag{8.21}$$

The solution of (8.21) is given by

$$h_l(t,y)=\frac{\gamma}{\sqrt{1-\epsilon^2}}, \quad t>0, \quad t+2y<0. \tag{8.22}$$

Using (7.5), (8.5), (8.8), and (8.22), we obtain

$$Z_l(0,y)=\frac{1-\epsilon-2\gamma y}{\sqrt{1-\epsilon^2}}, \quad y<0. \tag{8.23}$$

Using (8.23) in (6.12), we have

$$\frac{(1+\epsilon)y}{1-\epsilon-2\gamma y}=H_+\int_0^x\frac{dz}{f_l(0,z)^2}, \quad x<0. \tag{8.24}$$

Letting $x,y\to-\infty$ in (8.24), as in (6.14), we get

$$H_+=\frac{1+\epsilon}{2\gamma\int_{-\infty}^0 dz/f_l(0,z)^2}. \tag{8.25}$$

Thus, from (8.24) and (8.25) we find

$$y=\frac{1-\epsilon}{2\gamma}\,\frac{\int_0^x dz/f_l(0,z)^2}{\int_{-\infty}^x dz/f_l(0,z)^2}, \quad x<0,$$

$$H(x)=\frac{1-\epsilon}{2\gamma f_l(0,x)^2}\,\frac{\int_{-\infty}^0 dz/f_l(0,z)^2}{[\int_{-\infty}^x dz/f_l(0,z)^2]^2}, \quad x<0. \tag{8.26}$$

Alternatively, by using (6.21) we can write (8.26) as

$$H(x)=\frac{(1-\epsilon)[f_l^{[0]}(0,x);f_r^{[0]}(0,x)]^2}{2\gamma}\,\frac{\int_{-\infty}^{0}dz/f_l(0,z)^2}{f_r(0,x)^2},\quad x<0.$$

This expression agrees with that obtained in (6.51) of Example 6.2 in Ref. 1, but the method used here is simpler.

Example 8.4: In this example we describe how to obtain ν_j defined in (4.4) related to discontinuities in $H'(x)/H(x)$ using the method outlined in Sec. V. Let us use the scattering data of Example 8.1, and hence $\rho(k)$ is given by (8.1) and $\tau(k)$ is given by (8.2). We proceed as in Example 8.1 until (8.7); we then set up the Marchenko equation only for $y>0$, which, by (8.8), yields $h_l(t,y)=0$. At this point we can conclude that $V_{1,2}(y)=0$ and hence $w_{1,2}=0$. Using (5.2)–(5.5), we obtain

$$\Delta_1=\frac{2(\beta-\gamma)}{\sqrt{1-\epsilon^2}},\quad \Omega_1=\frac{2\epsilon(\alpha-\beta)}{\sqrt{1-\epsilon^2}}.$$

Thus, from (5.6) we get

$$\nu_1=\frac{2\epsilon(\alpha-\gamma)}{(1+\epsilon)\sqrt{1-\epsilon^2}}. \tag{8.27}$$

Hence, $H'(x)/H(x)$ is continuous at $x=0$ if and only if $\epsilon(\alpha-\gamma)=0$, i.e. if and only if $\rho(k)$ in (8.1) is a constant.

Example 8.5: In this example we illustrate the iterative method outlined in Sec. VII to recover $H(x)$, based on the matrix factorization in (4.5). Let us again use the scattering data of Example 8.1. We proceed as in Example 8.4 and get $H(x)$ given in (8.10) for $x>0$, $V_{1,2}(y)=0$, and ν_1 given in (8.27). Thus, we have $\Lambda_{1,2}=\mathbf{I}$ and

$$F_1(k)=\frac{1}{\sqrt{1-\epsilon^2}}\begin{bmatrix} 1+\dfrac{\epsilon(\alpha-\gamma)}{ik(1+\epsilon)} & -\epsilon+\dfrac{\epsilon(\alpha-\gamma)}{ik(1+\epsilon)} \\ -\epsilon-\dfrac{\epsilon(\alpha-\gamma)}{ik(1+\epsilon)} & 1-\dfrac{\epsilon(\alpha-\gamma)}{ik(1+\epsilon)} \end{bmatrix},$$

where $\Lambda_{j,j+1}(k)$ and $F_j(k)$ are the matrices defined in (4.1) and (4.3), respectively. From (4.1) and (4.5) we obtain $\Lambda_{0,1}(k)$. Note that, in this case, $\rho^{[0]}(k)$ defined in (5.11) and $r_{0,1}(k)$ corresponding to $V_{0,1}(y)$ coincide. We have

$$r_{0,1}(k)=\frac{-k_+k_-}{(k-k_+)(k-k_-)},\quad t_{0,1}(k)=\frac{k(k+i\beta)}{(k-k_+)(k-k_-)}, \tag{8.28}$$

where k_+ and k_- are the constants defined as

$$k_\pm=-\frac{i}{2}\,\frac{\gamma+\epsilon\alpha}{1+\epsilon}\,[1\pm\sqrt{1+E}],\quad E=\frac{4\epsilon(\gamma-\alpha)}{(1-\epsilon)(\gamma+\epsilon\alpha)}. \tag{8.29}$$

Next, we will solve the Marchenko equation (7.7) for $y<0$ with the input of (8.28) and (8.29). In fact, since there are no discontinuities associated with the reflection coefficient in (8.28), the Marchenko equation (7.7) reduces to (7.8). Note that the sign of E in (8.29) is the same as the sign of $\epsilon(\gamma-\alpha)$. There are three cases to consider, namely $E=0$, $E<0$, and $E>0$. When $E=0$, i.e. when $\epsilon=0$ or $\alpha=\gamma$, we have $r_{0,1}(k)=0$, and hence $h_l(t,y)=0$. Thus $V_{0,1}(y)=0$, because in analogy to (7.9) we have

$$V_{0,1}(y)=-2\,\frac{dh_l(0+,y)}{dy},\quad -\infty<y<0. \tag{8.30}$$

Thus $Y_{l;0,1}(k,y)=1$, and so $H(x)$ is given by (8.10) for all $x\in\mathbf{R}$. Next, we consider the case $E<0$. In this case both k_+ and k_- lie in $\mathbf{C}^-$, and hence using (8.28) in (7.1) we obtain

$$\varrho(t)=\begin{cases}0, & t<0,\\ \dfrac{ik_+k_-}{k_+-k_-}\,[e^{-ik_+t}-e^{-ik_-t}], & t>0.\end{cases} \tag{8.31}$$

The solution of the Marchenko equation (7.8) with the integral kernel in (8.31) is given by

$$h_l(t,y)=\begin{cases}0, & t>-2y,\\ \dfrac{k_+k_-}{\beta}\,\dfrac{(\beta+\gamma)[e^{\beta t}-e^{-2\beta y}]+\epsilon(\beta-\alpha)[1-e^{-\beta(t+2y)}]}{\epsilon(\beta-\alpha)+(\beta+\gamma)e^{-2\beta y}}, & t<-2y,\end{cases}$$

where β is the constant in (8.3). Again, using (8.30), we obtain

$$V_{0,1}(y)=\begin{cases}0, & y>0,\\ -\dfrac{8\beta^2\epsilon(\beta-\alpha)(\beta+\gamma)e^{-2\beta y}}{[\epsilon(\beta-\alpha)+(\beta+\gamma)e^{-2\beta y}]^2}, & y<0.\end{cases} \tag{8.32}$$

Corresponding to $V_{0,1}(y)$ in (8.32), we have the zero-energy Jost solution from the right given by

$$Y_{r;0,1}(0,y)=\frac{-\epsilon(\beta-\alpha)+(\beta+\gamma)e^{-2\beta y}}{\epsilon(\beta-\alpha)+(\beta+\gamma)e^{-2\beta y}},\quad y<0. \tag{8.33}$$

Using (3.9) we see that for $y<0$, $Z_r(0,y)$ is given by (8.33). Using (6.16)–(6.18) and (8.33) we obtain

$$y-\frac{2\epsilon(\beta-\alpha)/\beta}{\epsilon(\beta-\alpha)+\beta+\gamma}+\frac{2\epsilon(\beta-\alpha)/\beta}{\epsilon(\beta-\alpha)+(\beta+\gamma)e^{-2\beta y}}=H_-\int_0^x\frac{dz}{f_r(0,z)^2},\quad x<0, \tag{8.34}$$

$$H(x)=\frac{H_-}{f_r(0,x)^2}\left[\frac{-\epsilon(\beta-\alpha)+(\beta+\gamma)e^{-2\beta y}}{\epsilon(\beta-\alpha)+(\beta+\gamma)e^{-2\beta y}}\right]^2,\quad x<0, \tag{8.35}$$

where y in (8.35) is obtained in terms of x from (8.34), and $f_r(0,x)$ is the zero-energy Jost solution from the right associated with $Q(x)$. Using (6.19) and (6.20), one can show that (8.34) and (8.35) are identical to (8.14) and (8.15), respectively. Finally, let us briefly consider the case where the constant E defined in (8.29) is positive. In this case, k_+ is in $\mathbf{C}^-$ and k_- is in $\mathbf{C}^+$. Thus, $V_{0,1}(y)$ has one bound state. However, since $V_{0,1}(y)$ is supported on a half-line, its bound state norming constant cannot be chosen arbitrarily and is determined by $r_{0,1}(k)$ alone.[22] Routine computations[21] lead us again to $H(x)$ as given in (8.14).

Example 8.6: In this example, we demonstrate the recovery of $H(x)$ by the method outlined in Sec. VI, namely by solving the singular integral equations (6.7) or (6.10). As our scattering data, let us use the same scattering data as in Example 8.1, with the same restrictions on the parameters ϵ, α, and γ. First, using the method of Ref. 13 we get the quantities given in (8.4). When $y>0$, we will solve (6.7); for this, using (3.16), we get $J_l(k,y)=1$ and from (6.9) we have

$P_l(k,y)=0$. Thus, the solution of (6.7) for $y>0$ is given by $X_l(k,y)=0$; hence from (6.13) we obtain $H(x)$ for $x>0$ as given in (8.10). Now let us consider the situation when $y<0$; in this case, it is easier to obtain $\ell(\mathrm{k})$ and solve (6.10). Using the method of Ref. 16 we construct $\tau(k)$ given in (8.2) and using $\ell(\mathrm{k})=-\rho(-\mathrm{k})\tau(\mathrm{k})/\tau(-\mathrm{k})$, we get

$$\ell(k)=-\epsilon\,\frac{k-i\alpha}{k+i\gamma}\,\frac{k+i\beta}{k-i\beta}. \tag{8.36}$$

Using (8.36) in (6.11), we obtain

$$P_r(k,y)=\frac{2i\epsilon\beta}{k-i\beta}\,\frac{\beta-\alpha}{\beta+\gamma}\,e^{2\beta y},\quad y<0. \tag{8.37}$$

Since $X_r(k,y)$ is analytic in $\mathbf{C}^-$, a contour integration along the boundary of $\mathbf{C}^-$ converts (6.10) into the algebraic equation,

$$X_r(k,y)-\frac{2i\epsilon\beta}{k-i\beta}\,\frac{\beta-\alpha}{\beta+\gamma}\,e^{2\beta y}X_r(-i\beta,y)=P_r(k,y),\quad y<0.$$

Using (8.37) and the analyticity requirement on $X_r(k,y)$ to evaluate $X_r(-i\beta,y)$, we get

$$X_r(k,y)=\frac{2i\epsilon\beta}{k-i\beta}\,\frac{\beta-\alpha}{\beta+\gamma}\,\frac{(\beta+\gamma)e^{2\beta y}}{\beta+\gamma+\epsilon(\beta-\alpha)e^{2\beta y}},\quad y<0. \tag{8.38}$$

From (3.17) we have $J_r(k,y)=1$ for $y<0$. Thus, using (6.5) and (8.38), we get

$$Z_r(0,y)=\frac{(\beta+\gamma)e^{-2\beta y}-\epsilon(\beta-\alpha)}{(\beta+\gamma)e^{-2\beta y}+\epsilon(\beta-\alpha)},\quad y<0.$$

Thus using (6.16) and (6.18)–(6.20), we obtain $H(x)$ given in (8.15).

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