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EXTENSION OF ISOMETRIES IN FINITE-DIMENSIONAL INDEFINITE SCALAR PRODUCT SPACES AND POLAR DECOMPOSITIONS*

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Abstract. Witt's theorem on the extension of H-isometries to H-unitary matrices with respect to the scalar product generated by a self-adjoint nonsingular matrix H is studied in detail. All possible extensions are given, and their structure as a real analytic manifold is described. Analogous problems with respect to skew-symmetric scalar products are studied as well.

The main motivation to study these problems, as well as the main applications of the results obtained, concerns polar decompositions in indefinite scalar product spaces. As another application, for given B all solutions of the matrix equation $XA = B$ with H-unitary X and upper triangular A are described. Equations of this type are of vital importance in hyperbolic QR decompositions.

Key words. indefinite scalar products, isometries, polar decompositions, hyperbolic QR decompositions

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1. Introduction.

Let F be either the field of real numbers $\mathbf{R}$ or the field of complex numbers $\mathbf{C}$. Fix a real symmetric (if $F = \mathbf{R}$) or complex Hermitian (if $F = \mathbf{C}$) invertible $n \times n$ matrix H. Consider the scalar product induced by H according to the formula $[x, y] = \langle Hx, y \rangle$, $x, y \in F^n$. Here $\langle \cdot\,, \cdot \rangle$ stands for the standard scalar product in F^n defined by $\langle x, y \rangle = \sum_{j=1}^{n} x_j \bar{y}_j$, where $(x_1, \ldots, x_n)^T$ and $(y_1, \ldots, y_n)^T$ are column vectors in F^n. (Of course, $\bar{y}_j = y_j$ if $F = \mathbf{R}$.) The scalar product $[\cdot\,, \cdot]$ is nondegenerate ($[x, y] = 0$ for all $y \in F^n$ implies $x = 0$) but is indefinite in general. In other words, the real number $[x, x]$ can be positive, negative, or zero for various $x \in F^n$ (unless H is definite). The vector $x \in F^n$ is called *positive* if $[x, x] > 0$, *neutral* if $[x, x] = 0$, and *negative* if $[x, x] < 0$.

Well-known concepts related to the scalar product $[\cdot\,, \cdot]$ are defined in obvious ways. Thus, given an $n \times n$ matrix A over F, the adjoint $A^{[*]}$ is defined by $[Ax, y] = [x, A^{[*]}y]$ for all $x, y \in F^n$. The formula $A^{[*]} = H^{-1}A^*H$ is verified immediately. (Here and elsewhere we denote by A^* the conjugate transpose of A, so that $A^* = A^T$ if $F = \mathbf{R}$.) A matrix A is called *H-self-adjoint* if $A^{[*]} = A$ or, equivalently, if HA is Hermitian. An $n \times n$ matrix U is called *H-unitary* if $[Ux, Uy] = [x, y]$ for all $x, y \in F^n$

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or, equivalently, $U^*HU = H$. Observe that for every H-unitary matrix U we have $|\det U| = 1$; in particular, $\det U = \pm 1$ if $F = \mathbf{R}$.

This article is the third of a series of four articles on decompositions of an $n \times n$ matrix X over F of the form

$$X = UA, \tag{1.1}$$

where U is H-unitary and A is H-self-adjoint (with or without additional restrictions). We call the decomposition (1.1) an *H-polar decomposition* of X. Our first article, henceforth called [BMRRR1], is devoted to the existence, uniqueness (up to equivalence), and basic properties of decompositions (1.1) and to the existence of H-polar decompositions of H-normal matrices. In our second article, subsequently referred to as [BMRRR2], we studied decompositions of the type (1.1), where various constraints are imposed on the matrices X, U, A, and H, and discussed their applications in linear optics.

In studying H-polar decompositions, we often face the problem of extending H-isometries between linear subspaces to H-isometries defined on the whole space. The theorem stating the existence of such extensions is a classical result in geometry called Witt's theorem (see, e.g., [A, Theorem III.3.9.], or [D]). However, the classical results are concerned with the existence of a Witt extension and do not address the problems of listing all possible Witt extensions and describing their topological and algebraic structure. In the present paper, we give a detailed proof of Witt's theorem in both the real and the complex cases, detailed enough to yield all Witt extensions that exist. This is the content of section 2. As a by-product, the connected components of the set of all Witt extensions are described in section 3. Section 4 is devoted to the analogous problem of finding real Witt extensions with respect to a real skew-symmetric scalar product.

Another aspect of the present paper concerns a particular class of H-polar decompositions (1.1) in which the matrix A is H-nonnegative (i.e., HA is positive definite Hermitian). Such decompositions will be called *semidefinite H-polar decompositions*. In section 5 the semidefinite H-polar decompositions are described and characterized in full detail using the general results of [BMRRR1] as a starting point and applying the results on Witt extensions of section 2.

In section 6 the description of all Witt extensions is applied to a class of matrix decompositions, namely, hyperbolic QR decompositions, which are crucial in certain algorithms based on the generalized Schur method (see, e.g., [B, OSB, V]).

We remark in passing that the results of this paper concerning the description of Witt's extensions carry over to certain fields other than $\mathbf{R}$ or $\mathbf{C}$. Indeed, our description involves H-unitary matrices; normalization of vectors needed to construct such matrices is only possible in number fields closed with respect to the square root operation on positive numbers, such as the field of real algebraic numbers.

The following notation will be used. The number of positive (negative, zero) eigenvalues of a Hermitian matrix A is denoted by $\pi(A)$ $(\nu(A), \delta(A))$. The symbol F^n (where $F = \mathbf{R}$ or $F = \mathbf{C}$) stands for the vector space of n-dimensional columns over F. We denote by $F^{m\times n}$ the vector space of $m \times n$ matrices over F. I_m is the $m \times m$ identity matrix. The block diagonal matrix with matrices $Z_1, \ldots, Z_k$ on the main diagonal is denoted by $Z_1 \oplus \cdots \oplus Z_k$ or $\operatorname{diag}(Z_1, \ldots, Z_k)$. The set of eigenvalues (including nonreal eigenvalues for real matrices) of a matrix X is denoted by $\sigma(X)$. A^T stands for the transpose of a matrix A. $\operatorname{Ker} A$ and $\operatorname{Im} A$ stand for the null space and range of a matrix A. The symbol $\mathcal{M} \oplus \mathcal{N}$ denotes the direct sum of the subspaces

$\mathcal{M}$ and $\mathcal{N}$. For a subspace $\mathcal{M} \subset F^n$ and an indefinite scalar product $[x, y] = \langle Hx, y\rangle$, we call the subspace

$$\mathcal{M}^{[\perp]} = \{x \in F^n | [x, y] = 0 \text{ for all } y \in \mathcal{M}\}$$

the *H-orthogonal companion* of $\mathcal{M}$.

2. Witt's theorem and its refinements. In this section we will derive a version of Witt's theorem which is suitable to our framework and describe all H-isometries to which a given partial H-isometry can be extended.

We start with Witt's theorem, which is a classical result (see, e.g., [A, D]). The proofs given in [A, D] are algebraic and do not easily yield the parametrization that we need. Although the proofs from [A, D] could be adapted, doing so would create a portion of the paper at odds in style with the linear algebra methods of the rest of the paper. On the other hand, results on extensions of isometries are well known in the theory of operators in infinite-dimensional spaces with indefinite scalar products; see, e.g., section 5.2 in [AI1] or section II.9 in [IKL].

THEOREM 2.1. *Let* $[\,\cdot\,,\,\cdot\,]_1$ *and* $[\,\cdot\,,\,\cdot\,]_2$ *be the two scalar products in* F^n *defined by the invertible Hermitian* $n \times n$ *matrices* H_1 *and* H_2, *respectively*:

$$[x, y]_1 = \langle H_1x, y\rangle, \quad [x, y]_2 = \langle H_2x, y\rangle, \qquad x, y \in F^n.$$

Assume $\pi(H_1) = \pi(H_2)$. *Let* $U_0 : V_1 \to V_2$, *where* V_1 *and* V_2 *are subspaces in* F^n, *be a nonsingular linear transformation that preserves the scalar products*

$$[U_0x, U_0y]_2 = [x, y]_1 \quad \textit{for every} \quad x, y \in V_1. \tag{2.1}$$

Then there exists a linear transformation $U : F^n \to F^n$ *such that*

$$[Ux, Uy]_2 = [x, y]_1 \quad \textit{for every} \quad x, y \in F^n \tag{2.2}$$

and

$$Ux = U_0x \quad \textit{for every} \quad x \in V_1. \tag{2.3}$$

It is easy to see that the condition $\pi(H_1) = \pi(H_2)$, the nonsingularity of U_0, and the equality (2.1) are necessary for the existence of U with the asserted properties. Note that any such U is necessarily invertible; however, a linear transformation U_0 that satisfies (2.1) need not be invertible. A linear transformation (or its matrix representation with respect to specified bases) U with the property (2.2) is called *H_1-H_2-unitary*.

Given U_0 as in Theorem 2.1, any linear transformation U satisfying (2.2) and (2.3) will be called a *Witt extension* of U_0.

The following well-known fact will be useful in the proof of Theorem 2.1.

PROPOSITION 2.2. *Let* $[x, y] = \langle Hx, y\rangle$ *be an indefinite scalar product on* F^n. *The following statements are equivalent for the subspace* $\mathcal{M} \subset F^n$:

(i) $\mathcal{M}$ *is H-nondegenerate; i.e.,* $x_0 \in \mathcal{M}$, $[x_0, y] = 0$ *for all* $y \in \mathcal{M}$ *implies* $x_0 = 0$.

(ii) *The H-orthogonal companion* $\mathcal{M}^{[\perp]}$ *is H-nondegenerate.*

(iii) $\mathcal{M}^{[\perp]}$ *is a direct complement to* $\mathcal{M}$ *in* F^n.

The proof is based on the simple observation that dim $\mathcal{M}$ + dim $\mathcal{M}^{[\perp]} = n$; see [GLR] or [Bo] for complete details.

Proof of Theorem 2.1. We give a proof of Theorem 2.1 which will also serve as a basis for subsequent results concerning detailed descriptions of all Witt extensions. Put $m = m_+ + m_- + m_0$. Let $\{e_i\}_{i=1,2,\ldots,m}$ be a basis of V_1 such that $[e_j, e_j]_1 = 1$ for $j = m_0+1, m_0+2, \ldots, m_0+m_+$, $[e_k, e_k]_1 = -1$ for $k = m_0+m_++1,\ m_0+m_++2, \ldots, m$, and all the remaining indefinite scalar products of the basis vectors are zero (thus the Hermitian matrix defining the H_1-scalar product on V_1 has m_+ positive eigenvalues and m_- negative eigenvalues and the multiplicity of zero is m_0). Introduce the m linear functionals α_i on F^n as follows:

$$\alpha_i(x) = [x, e_i]_1, \qquad i = 1, 2, \ldots, m.$$

Since $\alpha_1, \ldots, \alpha_m$ are linearly independent, there exist vectors $\tilde{e}_i \in F^n$ such that $\alpha_i(\tilde{e}_j) = \delta_{ij}$, where $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$, i.e., such that $[e_i, \tilde{e}_j]_1 = \delta_{ij}$ for all $i, j = 1, 2, \ldots, m$. Let

$$W_k = \operatorname{span}\{e_k, \tilde{e}_k\}, \quad k = 1, 2, \ldots, m_0.$$

Since $[e_k, e_k]_1 = 0$ and $[e_k, \tilde{e}_k]_1 = 1$, each W_k is H_1-nondegenerate. Without loss of generality we can assume that, for $k = 1, 2, \ldots, m_0$, we have $[\tilde{e}_k, \tilde{e}_k]_1 = 0$. (Indeed, we can always replace the vector $\tilde{e}_k$ by the vector $\tilde{e}_k - \frac{1}{2}[\tilde{e}_k, \tilde{e}_k]_1 e_k$, which has the above property.) Let

$$e'_k = \frac{1}{\sqrt{2}}(e_k - \tilde{e}_k), \qquad e''_k = \frac{1}{\sqrt{2}}(e_k + \tilde{e}_k).$$

It is easy to see that

$$[e'_k, e'_k]_1 = -1, \quad [e''_k, e''_k]_1 = 1, \quad [e'_k, e''_k]_1 = 0.$$

The subspace $W = W_1 + \cdots + W_{m_0} + \operatorname{span}\{e_j\}_{i=m_0+1,\ldots,m}$ is H_1-nondegenerate; hence, $W^{[\perp]}$ is H_1-nondegenerate by Proposition 2.2. Therefore, we can append the vectors

$$e_s, \quad s = 2m_0 + m_+ + m_- + 1,\ 2m_0 + m_+ + m_- + 2, \ldots, n$$

to the set

$$\{e'_k, e''_k, e_{m_0+1}, e_{m_0+2}, \ldots, e_m\}_{k=1,2,\ldots,m_0}$$

of $2m_0 + m_+ + m_-$ vectors such that the resulting ordered set $\{g_1, \ldots, g_n\}$ will be a basis in F^n with the property that $[g_i, g_j]_1 = \epsilon_i \delta_{ij}$ for $i, j = 1, \ldots, n$, where $\epsilon_i = \pm 1$.

Now let $f_i = U_0 e_i$, $i = 1, 2, \ldots, m$. We introduce vectors f'_k and f''_k ($k = 1, 2, \ldots, m$) and vectors f_s ($s = 2m_0+m_++m_-+1,\ 2m_0+m_++m_-+2, \ldots, n$) in the same way we introduced the vectors e'_k, e''_k, and e_s but using $[\,\cdot\,,\,\cdot\,]_2$ instead of $[\,\cdot\,,\,\cdot\,]_1$, resulting in a basis $h_1, \ldots, h_n$ in F^n. The hypotheses on H_1 and H_2 ($\pi(H_1) = \pi(H_2)$) and on U_0 (U_0 being an isometry) guarantee that $[h_i, h_j]_2 = [g_i, g_j]_1$ ($i, j = 1, \ldots, n$).

Define the $n \times n$ matrix U by the equalities

$$\begin{aligned} &Ue'_k = f'_k,\ Ue''_k = f''_k, && k = 1, 2, \ldots, m_0, \\ &Ue_s = f_s, && s = 2m_0 + 1,\ 2m_0 + 2, \ldots, n. \end{aligned}$$

It is easy to see that the matrix U has all the properties required by the statement of the theorem. $\square$

We will use the bases

$$\mathcal{E} = \{e_1, e_2, \ldots, e_{m_0}, e_{m_0+1}, \ldots, e_m, \tilde{e}_1, \ldots, \tilde{e}_{m_0}, e_{2m_0+m_++m_-+1}, \ldots, e_n\} \tag{2.4}$$

and $\mathcal{F}$ (consisting of the vectors Ue, where $e \in \mathcal{E}$) of F^n constructed in the proof of Theorem 2.1. These bases will be more convenient than the ones we considered above because the subspaces V_1 and V_2 are spanned by the first m vectors of the corresponding bases. Recall that $U_0V_1 = V_2$, as U_0 is nonsingular. Thus, in particular, $\dim V_1 = \dim V_2$. With respect to $[\,\cdot\,,\,\cdot\,]_1$, the basis (2.4) has the Gramian matrix

$$\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & J_1 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & J_2 \end{bmatrix}, \tag{2.5}$$

where I is the $m_0 \times m_0$ identity matrix and J_1 is the diagonal $(m_+ + m_-) \times (m_+ + m_-)$ matrix such that its first m_+ diagonal elements are $+1$ and its remaining m_- diagonal elements are -1. Similarly, J_2 is the Gramian matrix of the basis $\{e_{m_0+m+1}, e_{m_0+m+2}, \ldots, e_n\}$ of the subspace spanned by these vectors; without loss of generality we can (and do) assume that J_2 is a diagonal matrix for which several diagonal entries are $+1$ and the remaining diagonal entries are -1.

The matrix (2.5) is also the Gramian matrix of the basis $\mathcal{F}$ with respect to $[\,\cdot\,,\,\cdot\,]_2$. The matrix U (constructed in the proof of Theorem 2.2), when understood as a linear transformation $F^n \to F^n$, is the $n \times n$ identity matrix with respect to the basis $\mathcal{E}$ (in F^n as the domain space of U) and the basis $\mathcal{F}$ (in F^n as the image space of U).

The Witt extensions of a given U_0 are described by the following theorem. (We represent the Witt extensions as linear transformations $F^n \to F^n$ with respect to the bases $\mathcal{E}$ and $\mathcal{F}$ constructed above.)

THEOREM 2.3 (extended Witt's theorem). *If a matrix $\tilde{U}$ is a Witt extension of the matrix U_0, then there exist a J_2-unitary matrix P_1 (of order $n - m - m_0$), an $(n - m - m_0) \times m_0$ matrix P_2, and a skew-self-adjoint $m_0 \times m_0$* matrix P_3 (*i.e.*, $P_3^* = -P_3$) *such that the matrix of $\tilde{U}$ has the form*

$$\tilde{U} = \begin{bmatrix} I_{m_0} & 0 & -\frac{1}{2}P_2^*J_2P_2 + P_3 & -P_2^*J_2P_1 \\ 0 & I_{m-m_0} & 0 & 0 \\ 0 & 0 & I_{m_0} & 0 \\ 0 & 0 & P_2 & P_1 \end{bmatrix}. \tag{2.6}$$

Here $m = \dim V_1$ and m_0 is the number of zero eigenvalues of the Gramian matrix of any basis in V_1 with respect to $[\,\cdot\,,\,\cdot\,]_1$.

Conversely, if P_1 is an arbitrary J_2-unitary matrix, P_2 is an arbitrary $(n - m - m_0) \times m_0$ matrix, and P_3 is an arbitrary skew-self-adjoint $m_0 \times m_0$ matrix, then the matrix $\tilde{U}$ defined by (2.6) *is a Witt extension of U_0.*

Proof. The proof is straightforward. Any extension $\tilde{U}$ of U_0 in the above bases has the matrix

$$\tilde{U} = \begin{bmatrix} I & 0 & A_1 & A_2 \\ 0 & I & A_3 & A_4 \\ 0 & 0 & A_5 & A_6 \\ 0 & 0 & A_7 & A_8 \end{bmatrix}. \tag{2.7}$$

The necessary and sufficient condition for the matrix (2.7) to be H_1-H_2-unitary is the identity $H_1^{-1}\tilde{U}^*H_2\tilde{U} = I$. Taking into account (2.5) and (2.7) we can rewrite the last relation in block form as

$$\begin{bmatrix} A_5^* & A_3^*J_1 & u_{13} & u_{14} \\ 0 & I & A_3 & A_4 \\ 0 & 0 & A_5 & A_6 \\ J_2A_6^* & J_2A_4^*J_1 & u_{43} & u_{44} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{2.8}$$

where

$$u_{13} = A_5^*A_1 + A_3^*J_1A_3 + A_1^*A_5 + A_7^*J_2A_7,$$

$$u_{14} = A_5^*A_2 + A_3^*J_1A_4 + A_1^*A_6 + A_7^*J_2A_8,$$

$$u_{43} = J_2A_6^*A_1 + J_2A_4^*J_1A_3 + J_2A_2^*A_5 + J_2A_8^*J_2A_7,$$

$$u_{44} = J_2A_6^*A_2 + J_2A_4^*J_1A_4 + J_2A_2^*A_6 + J_2A_8^*J_2A_8.$$

Equating the corresponding blocks in (2.8), we derive the theorem statement. ∎

If V_1 is H_1-nondegenerate (i.e., $m_0 = 0$), then necessarily V_2 is H_2-nondegenerate and the result of Theorem 2.3 is obvious.

Observe also that the inverse of the matrix (2.6) is given by

$$\tilde{U}^{-1} = \begin{bmatrix} I & 0 & -\frac{1}{2}\hat{P}_2^*J_2\hat{P}_2 + \hat{P}_3 & -\hat{P}_2^*J_2\hat{P}_1 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & \hat{P}_2 & \hat{P}_1 \end{bmatrix}, \tag{2.9}$$

where

$$\hat{P}_1 = P_1^{-1} = J_2P_1^*J_2, \ \hat{P}_2 = -P_1^{-1}P_2 = -J_2P_1^*J_2P_2, \ \hat{P}_3 = -P_3.$$

Note that a Witt extension $\tilde{U}$ has the form (2.6) with respect to different bases in domain and image space, namely, with respect to the basis $\mathcal{E}$ given by (2.4) in the domain space and the basis $\mathcal{F}$ consisting of the vectors Ue in the image space. Keeping this in mind, we can easily reformulate Theorem 2.3 in the following way with respect to one basis $\mathcal{E}$ (the same for both the domain and the range of $\tilde{U}$). It is this form of the theorem that we shall apply later in Theorems 5.6 and 6.1.

Theorem 2.4 (extended Witt's theorem, second version). *Let U be a fixed Witt extension of U_0 as constructed in Theorem* 2.1. *Then any Witt extension $\tilde{U}$ of U_0 is given by $\tilde{U} = UM$, where M has the form of the right-hand side of* (2.6) *with respect to the basis $\mathcal{E}$ in* (2.4).

Proof. Observe that U maps the elements of the basis $\mathcal{E}$ into the corresponding elements of the basis $\mathcal{F}$ and that (2.6) is the matrix representation of U with respect to the basis $\mathcal{E}$ in the domain space and the basis $\mathcal{F}$ in the image space. ∎

It is of interest to compute the number of independent real parameters that describe all Witt extensions. Assume first $F = \mathbf{C}$. Then the formula (2.6), combined with the real analytic description of the group of J_2-unitary matrices (see, e.g., Theorem IV.3.1 in [GLR]), produces the following result.

Theorem 2.5 ($F = \mathbf{C}$). *The set $W(U_0)$ of all Witt extensions of a given isometry $U_0 : V_1 \to V_2$ is parametrized by $(n-m)^2$ independent real variables, where $m = \dim V_1$. More precisely, let*

$$p = \pi(H_1) - m_+ - m_0, \qquad q = \nu(H_1) - m_- - m_0, \tag{2.10}$$

where m_+, m_-, *and* m_0 *are the numbers of positive, negative, and zero eigenvalues, respectively, of the Gramian matrix of any basis in* V_1 *with respect to* $[\,\cdot\,,\,\cdot\,]_1$. *Then* $W(U_0)$ *is diffeomorphic* (*as a real analytic manifold*) *to*

$$SU(p) \times SU(q) \times T \times T \times \mathbf{R}^w, \qquad w = 2pq + 2(n-m-m_0)m_0 + m_0^2$$

if both p *and* q *are positive, and* $W(U_0)$ *is diffeomorphic to*

$$SU(p+q) \times T \times \mathbf{R}^w$$

if exactly one of p *and* q *is zero. Finally,* $W(U_0)$ *is diffeomorphic to* $\mathbf{R}^w$ *if* $p = q = 0$. *Here* T *is the unit circle and*

$$SU(k) = \{X \in \mathbf{C}^{k\times k} |\ X \text{ unitary},\ \det X = 1\}$$

is the $k \times k$ *special unitary group.*

Proof. We use the notation of Theorem 2.3. The matrix $\tilde{U}$ is parametrized by (P_1, P_2, P_3), where P_2 and P_3 are in turn parametrized by $2(n-m-m_0)m_0$ and m_0^2 independent real variables, respectively. Observe that $p = \pi(J_2)$ and $q = \nu(J_2)$. Thus, the group of all J_2-unitary matrices is diffeomorphic (as a real analytic manifold) either to $SU(p) \times SU(q) \times T \times T \times \mathbf{R}^{2pq}$ (if both p and q are positive) or to $SU(p+q) \times T$ (if exactly one of p and q is zero); see, e.g., Theorem IV.3.1 in [GLR]. In fact, explicit charts for the group of all J_2-unitary matrices can be constructed using the diffeomorphism mentioned above and the following two charts for $SU(p)$, namely, the sets

$$\left\{\exp K\ :\ K = -K^*,\ \text{trace}\ K = 0,\ \sigma(K) \subset \pm\left(-\frac{\pi}{2}i, \frac{3\pi}{2}i\right]\right\}.$$

The number of real parameters describing the group of J_2-unitary matrices is $(p^2-1)+(q^2-1)+1+1+2pq = (p+q)^2$ if $p, q > 0$. (Here we use the fact that $SU(k)$ has real dimension k^2-1, equal to the real dimension of the set of all skew-self-adjoint $k \times k$ matrices with trace 0, which is the Lie algebra of $SU(k)$.) The group of J_2-unitary matrices has real dimension $(p+q)^2$ also in the case where exactly one of p and q is zero. Thus, the total number of real parameters describing $\tilde{U}$ is

$$\begin{aligned}(p+q)^2 + 2(n-m-m_0)m_0 + m_0^2 &= (p+q+m_0)^2\\ &= (\pi(H_1) + \nu(H_1) - m_+ - m_- - m_0)^2\\ &= (n-m)^2.\end{aligned}$$

An analogous proof also works in the case $p = q = 0$. ◻

The real analogue of Theorem 2.5 runs as follows.

Theorem 2.6 ($F = \mathbf{R}$). *Let* $m = \dim V_1$, *and let* p *and* q *be defined by* (2.10). *Then the set* $W(U_0)$ *of all Witt extensions of an isometry* $U_0 : V_1 \to V_2$ *is connected if* $p = q = 0$, *has two connected components if exactly one of* p *and* q *is positive, and has four connected components if both* p *and* q *are positive. Every connected component of* $W(U_0)$ *is diffeomorphic* (*as a real analytic manifold*) *to*

$$SO(p) \times SO(q) \times \mathbf{R}^v, \qquad v = pq + (n-m-m_0)m_0 + \frac{1}{2}m_0(m_0-1),$$

where $SO(k)$ *is the group of real unitary* (*i.e., real orthogonal*) $k \times k$ *matrices with determinant* 1 *if both* p *and* q *are positive; every connected component of* $W(U_0)$ *is diffeomorphic to*

$$SO(p+q) \times \mathbf{R}^v$$

if exactly one of p and q is zero. Finally, $W(U_0)$ is diffeomorphic to $\mathbf{R}^v$ if $p = q = 0$. In all cases, every connected component of $W(U_0)$ can be parametrized by $\frac{1}{2}(n-m)(n-m-1)$ independent real variables.

The part of Theorem 2.6 concerning the number of connected components follows immediately from Theorems 2.3 and 3.1. (The latter is stated and proved in the next section.) The remainder of the proof of Theorem 2.6 is analogous to that of Theorem 2.5: one should use the real analogue of Theorem IV.3.1 in [GLR] and the fact that $SO(k)$ has (real) dimension $\frac{1}{2}k(k-1)$; this is the dimension of the Lie algebra of $SO(k)$ which consists of all real skew-symmetric $k \times k$ matrices.

It is a curious observation that the number of real parameters describing $W(U_0)$ depends only on n (the order of H_1) and on m (the dimension of V_1) and does not depend on m_0 (the degree of degeneracy of V_1 in the indefinite scalar product induced by H_1).

In particular, Theorems 2.5 and 2.6 allow one to identify the fundamental group of the set $W(U_0)$ using the well-known fact that $SU(k)$ and $\mathbf{R}^n$ are simply connected; the fundamental group of $SO(k)$ is of order 2 if $k \geq 3$, the infinite cyclic group $\mathbf{Z}$ if $k = 2$, and the trivial group if $k = 1$; and the fundamental group of the product of two arcwise connected topological spaces X and Y is the direct product of the fundamental groups of X and Y (see, e.g., sections II.VIII, II.X, and II.XI in [C]). Thus the fundamental group of $W(U_0)$ is $G_p \times G_q$ if both p and q are positive, G_{p+q} if one of p, q is positive and the other vanishes, and trivial if $p = q = 0$; here $G_p = \mathbf{Z}$ if $F = \mathbf{C}$, whereas $G_p = \mathbf{Z}_2$ if $p \geq 3$, $G_2 = \mathbf{Z}$, and G_1 is trivial if $F = \mathbf{R}$.

We conclude this section with two illustrative examples.

Example 2.1. Let $H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$; $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. Any linear transformation $U_0 : V \to V$ is an isometry. The linear transformation $U_0 : V \to V$ is defined by $U_0\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, where $\alpha \neq 0$ is a given complex number. We shall find the Witt extensions U of U_0. An elementary calculation shows that all such U have the form $\begin{bmatrix} \alpha & x \\ 0 & \bar{\alpha}^{-1} \end{bmatrix}$, where $x \in \mathbf{C}$ is any number such that $\bar{\alpha}x + \bar{x}\alpha = 0$. If we consider $F = \mathbf{R}$, then α is real and the unique Witt extension of U_0 is given by $\operatorname{diag}(\alpha, \alpha^{-1})$. □

Example 2.2. Let $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$; $V = \operatorname{span}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$. A linear transformation $U_0 : V \to V$ defined by $U_0\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix}$, $\alpha \neq 0$, is an H-isometry if and only if $|\alpha| = 1$. The Witt extensions U of U_0 are described by $U = \operatorname{diag}(\alpha, y)$, where $|y| = 1$. In the real case we have exactly *two* Witt extensions (corresponding to $y = \pm 1$). □

3. Connectivity of the H-unitary groups. Let H be an invertible Hermitian $n \times n$ matrix over F ($F = \mathbf{R}$ or $F = \mathbf{C}$). The set of H-unitary matrices (over F) is easily seen to be a group, denoted $\mathcal{U}(H; F)$. Its connected components are described as follows.

THEOREM 3.1.

(a) *The group $\mathcal{U}(H; \mathbf{C})$ is connected.*

(b) *If $F = \mathbf{R}$ and H is definite (positive or negative), then the group $\mathcal{U}(H; \mathbf{R})$ has two connected components. One of them contains all $X \in \mathcal{U}(H; \mathbf{R})$ with* $\det X = 1$; *the other contains all $X \in \mathcal{U}(H; \mathbf{R})$ with* $\det X = -1$.

(c) *If $F = \mathbf{R}$ and H is indefinite, then $\mathcal{U}(H, \mathbf{R})$ has four connected components which can be described as follows. We can assume $H = I_p \oplus -I_q$, where $p, q > 0$.*

Then, for every choice of signs $\delta_1 = \pm 1$, $\delta_2 = \pm 1$, *a connected component of* $\mathcal{U}(H, \mathbf{R})$ *is given by*

$$\mathcal{U}(H; \delta_1, \delta_2) = \left\{ V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix} \in \mathcal{U}(H; \mathbf{R}) | \ \delta_1 \det V_1 > 0, \ \delta_2 \det V_4 > 0 \right\},$$

where V_1 *is a* $p \times p$ *matrix and* V_4 *is a* $q \times q$ *matrix. In particular,*

$$\{X \in \mathcal{U}(H; \mathbf{R}) | \ \det X = 1\} = \mathcal{U}(H; 1, 1) \cup \mathcal{U}(H; -1, -1), \tag{3.1}$$

and this set consists of two connected components.

In all cases, each connected component of $\mathcal{U}(H; F)$ *is arcwise connected.*

Proof. This result is known; for the proof of (a) and (b) see Lemma I.3.8 and Theorem I.5.8, respectively, in [GLR].

For completeness, we provide a proof of (c). Let $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$ belong to $\mathcal{U}(H; \mathbf{R})$. Then the equation $V^T H V = H = I_p \oplus -I_q$ gives

$$V_1^T V_1 = I + V_3^T V_3, \quad V_4^T V_4 = I + V_2^T V_2, \quad V_2^T V_1 = V_4^T V_3. \tag{3.2}$$

It follows that $|\det V_1| \geq 1$, $|\det V_4| \geq 1$, and therefore the H-unitary matrices (over $\mathbf{R}$) $V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$ and $W = \begin{bmatrix} W_1 & W_2 \\ W_3 & W_4 \end{bmatrix}$ (here W_1 is $p \times p$ and W_4 is $q \times q$) belong to different connected components in $\mathcal{U}(H; \mathbf{R})$, provided at least one of the inequalities $\det V_1 \cdot \det W_1 < 0$, $\det V_4 \cdot \det W_4 < 0$ is valid. It remains to show that if $\det V_1 \cdot \det W_1 > 0$ and $\det V_4 \cdot \det W_4 > 0$, then V and W belong to the same connected component in $\mathcal{U}(H, \mathbf{R})$. It suffices to show that if $\det V_1 > 0$, $\det V_4 > 0$, then V can be continuously connected to I in $\mathcal{U}(H; \mathbf{R})$. As V is H-unitary, V^T is H-unitary as well (indeed, $V^T H V = H$ implies $V^{-1} = H^{-1} V^T H = H V^T H$, and therefore $V H V^T H = I$, or $V H V^T = H$). Thus, we also have

$$I + V_3 V_3^T = V_4 V_4^T. \tag{3.3}$$

Observe from (3.2) and (3.3) that $V_1 (I + V_3^T V_3)^{-\frac{1}{2}}$ and $V_4^T (I + V_3 V_3^T)^{-\frac{1}{2}}$ are real and unitary (with respect to I). Moreover, they both have determinant 1, as $\det V_1 > 0$ and $\det V_4 > 0$. So, by part (b), there is a continuous family of unitary matrices $U_1(t)$, $U_4(t)$ for $t \in [0, 1]$ such that

$$\begin{aligned} U_1(0) &= I, & U_4(0) &= I, \\ U_1(1) &= V_1 (I + V_3^T V_3)^{-\frac{1}{2}}, & U_4(1) &= V_4^T (I + V_3 V_3^T)^{-\frac{1}{2}}. \end{aligned}$$

Let

$$\begin{aligned} V_1(t) &= U_1(t) (I + t^2 V_3^T V_3)^{\frac{1}{2}}, & V_4(t) &= (I + t^2 V_3 V_3^T)^{\frac{1}{2}} U_4(t)^T, \\ V_3(t) &= t V_3, & V_2(t) &= t V_1(t)^{-T} V_3^T V_4(t), \end{aligned}$$

and

$$V(t) = \begin{bmatrix} V_1(t) & V_2(t) \\ V_3(t) & V_4(t) \end{bmatrix}.$$

Then $V(0) = I$ and $V(1) = V$, and one easily verifies that $V(t)$ is H-unitary for all $t \in [0, 1]$. $\square$

A basis independent description of the connected components of $\mathcal{U}(H; \mathbf{R})$, where H is indefinite, runs as follows. Let $\mathcal{M}_+$ and $\mathcal{M}_-$ be subspaces in $\mathbf{R}^n$ which are H-orthogonal complements of each other and such that $\mathcal{M}_+$ is H-positive and $\mathcal{M}_-$ is H-negative. Denote by P_+ (resp., P_-) the projector onto $\mathcal{M}_+$ (resp., $\mathcal{M}_-$) along $\mathcal{M}_-$ (resp., $\mathcal{M}_+$). Then $X \in \mathcal{U}(H; \delta_1, \delta_2)$ if and only if

$$\delta_1 \det(P_+X|M_+) > 0, \qquad \delta_2 \det(P_-X|\mathcal{M}_-) > 0. \tag{3.4}$$

The proof of this statement is analogous to the proof of Theorem 3.1 (part (c)) and therefore is omitted.

Observe that the inequalities (3.4) are independent of the choice of the pair of subspaces $\mathcal{M}_+$, $\mathcal{M}_-$ with the above properties.

THEOREM 3.2. *For any real invertible matrix S and any $X \in \mathcal{U}(H; \delta_1, \delta_2)$ the matrix $S^{-1}XS$ belongs to the connected component $\mathcal{U}(S^*HS; \delta_1, \delta_2)$ determined by the same δ_1, δ_2.*

Proof. The proof follows easily from the description of $\mathcal{U}(H; \delta_1, \delta_2)$ given by formula (3.4). Indeed, assume that $X \in \mathcal{U}(H; \delta_1, \delta_2)$. Choose a pair of subspaces $\mathcal{M}_+$ and $\mathcal{M}_-$ that are H-orthogonal complements to each other and such that $\mathcal{M}_+$ (resp., $\mathcal{M}_-$) is H-positive (resp., H-negative). Then $S^{-1}\mathcal{M}_+$ and $S^{-1}\mathcal{M}_-$ are S^*HS-orthogonal complements to each other and $S^{-1}\mathcal{M}_+$ (resp., $S^{-1}\mathcal{M}_-$) is S^*HS-positive (resp., S^*HS-negative). We conclude the proof by applying the formula (3.4) with X, P_+, P_- replaced by $S^{-1}XS$, $S^{-1}P_+S$, $S^{-1}P_-S$, respectively, and with $\mathcal{M}_\pm$ replaced by $S^{-1}\mathcal{M}_\pm$. $\square$

4. Witt's theorem for real skew-symmetric scalar products. Let $F = \mathbf{R}$ and let K be a real invertible skew-Hermitian $n \times n$ matrix (in particular, n is even). Define the skew-symmetric scalar product $\{\cdot, \cdot\}$ on $\mathbf{R}^n$ by

$$\{x, y\} = \langle Kx, y \rangle.$$

If A is an $n \times n$ matrix, its K-adjoint $A^{\{*\}}$ is defined by the identity $\{Ax, y\} = \{x, A^{\{*\}}y\}$, where $x, y \in \mathbf{R}^n$. It is easy to see that $A^{\{*\}} = K^{-1}A^*K$. A matrix A is called *K-self-adjoint* if $A^{\{*\}} = A$, and it is called *K-skew-self-adjoint* if $A^{\{*\}} = -A$. A *K-unitary* matrix A is defined by the property that it preserves the skew-symmetric scalar product, i.e., if, for any two vectors $x, y \in \mathbf{R}^n$, $\{Ax, Ay\} = \{x, y\}$. It is easy to verify that A is K-self-adjoint if and only if $KA = A^*K$, is K-skew-self-adjoint if and only if $KA = -A^*K$, and is K-unitary if and only if it is nonsingular and $K^{-1}A^*KA = I$.

Example 4.1. Consider the skew-Hermitian matrix

$$H = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We have $H^* = H^{-1} = -H$. Moreover, $X^{\{*\}} = H^{-1}X^*H$ is the cofactor matrix of X, so that $X^{\{*\}}X = (\det X)\, I$. Hence A is H-self-adjoint if and only if $A = cI$ for some $c \in \mathbf{R}$, and A is H-skew-self-adjoint (i.e., $HA = -A^*H$) if and only if $\operatorname{Tr} A = 0$. Furthermore, U is H-unitary (i.e., $U^*HU = H$) if and only if $\det U = +1$. $\square$

LEMMA 4.1. *Let $\{.,.\}$ be a skew-symmetric scalar product on $\mathbf{R}^n$ defined by the real invertible skew-symmetric $n \times n$ matrix K and let V be an m-dimensional subspace* *of* $\mathbf{R}^n$. *Let the defect of the restriction of* $\{.,.\}$ *to* V *be* m_0 (*so that the rank of the above restriction is* $m - m_0$). *Then*

(a) *There exists a basis*

$$\{e_1, \dots, e_{m_0}, e_{m_0+1}, \dots, e_{\frac{m+m_0}{2}}, f_{m_0+1}, \dots, f_{\frac{m+m_0}{2}}\} \tag{4.1}$$

of V *such that*

$$\{e_k, f_k\} = -\{f_k, e_k\} = 1, \qquad k = m_0 + 1, m_0 + 2, \dots, \frac{m+m_0}{2}, \tag{4.2}$$

while the scalar product of any other two vectors in (4.1) *is zero.*

(b) *There exist vectors*

$$\{f_1, f_2, \dots, f_{m_0}, e_{\frac{m+m_0}{2}+1}, e_{\frac{m+m_0}{2}+2}, \dots, e_{\frac{n}{2}}, f_{\frac{m+m_0}{2}+1}, f_{\frac{m+m_0}{2}+2}, \dots, f_{\frac{n}{2}}\} \tag{4.3}$$

such that the union of the sets (4.1) *and* (4.3) *is a canonical basis for* $\mathbf{R}^n$*; i.e.;*

$$\{e_k, f_k\} = -\{f_k, e_k\} = 1, \ \ k = 1, 2, \dots, \frac{n}{2}, \tag{4.4}$$

while the scalar product of any other two vectors from the union of (4.1) *and* (4.2) *is zero.*

Proof. This is an elementary exercise in linear algebra. Namely, if $m = m_0$ then V is isotropic and any basis of V does the job. If $m > m_0$ there exist vectors e_{m_0+1} and f_{m_0+1} such that $\{e_{m_0+1}, f_{m_0+1}\} = 1$. If $m-m_0 = 2$ then the orthogonal companion V_1 of the subspace span$\{e_{m_0+1}, f_{m_0+1}\}$ in V is isotropic and any basis of V_1 appended to vectors e_{m_0+1}, f_{m_0+1} produces a desired basis. If $m - m_0 > 2$ then V_1 is not isotropic and we can find vectors $e_{m_0+2}, f_{m_0+2} \in V_1$ such that $\{e_{m_0+2}, f_{m_0+2}\} = 1$. Continuing this process we will find a desired basis of V. This proves (a). To prove (b) we first introduce the $(n - m + m_0)$-dimensional subspace W of $\mathbf{R}^n$, which is K-orthogonal to the subspace

$$\text{span}\{e_{m_0+1}, f_{m_0+1}, e_{m_0+2}, f_{m_0+2}, \dots, e_{\frac{m+m_0}{2}}, f_{\frac{m+m_0}{2}}\}.$$

Obviously, W is nondegenerate and $e_1, e_2, \dots, e_{m_0} \in W$. Since W is nondegenerate, there exists a vector $f_1 \in W$ such that $\{e_1, f_1\} = 1$ and $\{e_k, f_1\} = 0$ for $k = 2, 3, \dots, m_0$. Let W_1 be the K-orthogonal complement of span$\{e_1, f_1\}$ in W. If $m_0 = 1$ then any basis of W_1 appended to vectors e_1 and f_1 already found will produce a desired basis. If $m_0 > 1$ then $e_2 \in W_1$ and we can find a vector $f_2 \in W_1$ such that $\{e_2, f_2\} = 1$ and $\{e_k, f_2\} = 0$ for $k = 1, 3, 4, \dots, m_0$. Continuing this process we will finally find a basis of $\mathbf{R}^n$ that satisfies all the requirements of (b). ∎

THEOREM 4.2. *Let* $\{\cdot,\cdot\}_1$ *and* $\{\cdot,\cdot\}_2$ *be two skew-symmetric scalar products on* $\mathbf{R}^n$ *defined by the skew-symmetric* $n \times n$ *matrices* K_1 *and* K_2*, respectively:*

$$\{x, y\}_1 = \langle K_1 x, y\rangle, \ \ \{x, y\}_2 = \langle K_2 x, y\rangle, \ \ x, y \in \mathbf{R}^n.$$

Let $U_0 : V_1 \to V_2$*, where* V_1 *and* V_2 *are subspaces in* $\mathbf{R}^n$*, be a nonsingular linear transformation that preserves the scalar products; namely,*

$$\{U_0 x, U_0 y\}_2 = \{x, y\}_1$$

for every $x, y \in V_1$. *Then there exists a linear transformation* $U : \mathbf{R}^n \to \mathbf{R}^n$ *such that*

$$\{Ux, Uy\}_2 = \{x, y\}_1$$

for every $x, y \in V_1$ *and*

$$Ux = U_0 x$$

for every $x \in V_1$.

Proof. Let the vectors

$$\{e_1, \dots, e_{m_0}, e_{m_0+1}, \dots, e_{\frac{m+m_0}{2}}, f_{m_0+1}, \dots, f_{\frac{m+m_0}{2}}\} \tag{4.5}$$

be as in (a) of Lemma 4.1 and let

$$g_t = U_0 e_t, \quad h_s = U_0 f_s, \quad t = 1, 2, \dots, \frac{m+m_0}{2}, \; s = m_0+1, m_0+2, \dots, \frac{m+m_0}{2}. \tag{4.6}$$

Next, let the vectors

$$f_1, f_2, \dots, f_{m_0}, e_{\frac{m+m_0}{2}+1}, e_{\frac{m+m_0}{2}+2}, \dots, e_{\frac{n}{2}}, f_{\frac{m+m_0}{2}+1}, f_{\frac{m+m_0}{2}+2}, \dots, f_{\frac{n}{2}} \tag{4.7}$$

be as in (b) of Lemma 4.1; i.e., combined with the vectors (4.1) they produce a canonical basis

$$\begin{gathered}\{e_1, e_2, \dots, e_{m_0}, e_{m_0+1}, e_{m_0+2}, \dots, e_{\frac{m+m_0}{2}}, f_{m_0+1}, f_{m_0+2}, \dots, f_{\frac{m+m_0}{2}}, \\ f_1, f_2, \dots, f_{m_0}, e_{\frac{m+m_0}{2}+1}, e_{\frac{m+m_0}{2}+2}, \dots, e_{\frac{n}{2}}, f_{\frac{m+m_0}{2}+1}, f_{\frac{m+m_0}{2}+2}, \dots, f_{\frac{n}{2}}\}\end{gathered} \tag{4.8}$$

of $\mathbf{R}^n$,

$$\{e_s, f_s\}_1 = -\{f_s, e_s\}_1 = 1, \qquad s = 1, 2, \dots, \frac{n}{2}. \tag{4.9}$$

The remaining scalar products of the basis are zero. Similarly, let the vectors

$$h_1, h_2, \dots, h_{m_0}, g_{m+1}, g_{m+2}, \dots, g_{\frac{n}{2}}, h_{m+1,}, h_{m+2}, \dots, h_{\frac{n}{2}} \tag{4.10}$$

be as in (b) of Lemma 4.1; i.e., combined with the vectors (4.6) they produce a canonical basis

$$\begin{gathered}\{g_1, g_2, \dots, g_{m_0}, g_{m_0+1}, g_{m_0+2}, \dots, g_{\frac{m+m_0}{2}}, h_{m_0+1}, h_{m_0+2}, \dots, h_{\frac{m+m_0}{2}}, \\ h_1, h_2, \dots, h_{m_0}, g_{\frac{m+m_0}{2}+1}, g_{\frac{m+m_0}{2}+2}, \dots, g_{\frac{n}{2}}, h_{\frac{m+m_0}{2}+1}, h_{\frac{m+m_0}{2}+2}, \dots, h_{\frac{n}{2}}\}\end{gathered} \tag{4.11}$$

of $\mathbf{R}^n$,

$$\{g_s, h_s\}_2 = -\{h_s, g_s\}_2 = 1, \qquad s = 1, 2, \dots, \frac{n}{2}. \tag{4.12}$$

The remaining scalar products of the basis are zero. Define the linear transformation U as follows:

$$Ue_s = g_s, \quad Uf_s = h_s, \quad s = 1, 2, \dots, \frac{n}{2}. \tag{4.13}$$

It is easy to see that the matrix defined by (4.13) satisfies all the conditions of the theorem. $\square$

We will use the bases (4.8) and (4.11) constructed in the proof of Theorem 4.2. With respect to $\{.,.\}_1$, the basis (4.8) has the skew-symmetric Gramian matrix

$$K = \begin{bmatrix} 0 & 0 & -I & 0 \\ 0 & J_1 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & J_2 \end{bmatrix}. \tag{4.14}$$

Here I is the $m_0 \times m_0$ identity matrix, J_1 is an $(m - m_0) \times (m - m_0)$ matrix of the form $J_1 = \left[\begin{smallmatrix} 0 & -I \\ I & 0 \end{smallmatrix}\right]$, and J_2 is an $(n - m - m_0) \times (n - m - m_0)$ matrix of the same form as J_1.

As in previous sections, any linear transformation (or its matrix representation with respect to fixed bases) U from Theorem 4.2 will be called a *Witt extension* of U_0. All the Witt extensions of a given U_0 are described by the following theorem (we represent the Witt extensions as linear transformations $\mathbf{R}^n \to \mathbf{R}^n$ with respect to the bases (4.8) and (4.11) above).

THEOREM 4.3 (extended Witt's theorem for a skew-symmetric scalar product). *If a matrix $\tilde{U}$ is a Witt extension of the matrix U_0, then there exist a J_2-unitary matrix P_1 (of order $n - m - m_0$), a real $(n - m - m_0) \times m_0$ matrix P_2, and a real symmetric $m_0 \times m_0$ matrix P_3 (i.e., $P_3^* = P_3$) such that the matrix of $\tilde{U}$ has the form*

$$\tilde{U} = \begin{bmatrix} I & 0 & -\frac{1}{2}P_2^* J_2 P_2 + P_3 & -P_2^* J_2 P_1 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & P_2 & P_1 \end{bmatrix}. \tag{4.15}$$

Conversely, if P_1 is an arbitrary J_2-unitary matrix, P_2 is an arbitrary real $(n - m - m_0) \times m_0$ matrix, and P_3 is an arbitrary real symmetric $m_0 \times m_0$ matrix, then the matrix $\tilde{U}$ defined by (4.15) is a Witt extension of U_0.

Proof. The proof is similar to that of Theorem 2.3. Namely, any extension $\tilde{U}$ of U_0 in the bases (4.8), (4.11) has the matrix

$$\tilde{U} = \begin{bmatrix} I & 0 & A_1 & A_2 \\ 0 & I & A_3 & A_4 \\ 0 & 0 & A_5 & A_6 \\ 0 & 0 & A_7 & A_8 \end{bmatrix}. \tag{4.16}$$

The necessary and sufficient condition for the matrix $\tilde{U}$ to be K_1-K_2-unitary is the identity $K_1^{-1}\tilde{U}^* K_2 \tilde{U} = I$. Taking into account (4.14), (4.16), and the facts that $K_1 = K_2 = K$ and that $K^{-1} = -K$, we can rewrite the last relation in block form as

$$\begin{bmatrix} A_5^* & A_3^* J_1 & u_{13} & u_{14} \\ 0 & I & A_3 & A_4 \\ 0 & 0 & A_5 & A_6 \\ -J_2 A_6^* & -J_2 A_4^* J_1 & u_{43} & u_{44} \end{bmatrix} = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \tag{4.17}$$

where

$$u_{13} = -A_1^* A_5 + A_3^* J_1 A_3 + A_5^* A_1 + A_7^* J_2 A_7,$$

$$u_{14} = -A_1^* A_6 + A_3^* J_1 A_4 + A_5^* A_2 + A_7^* J_2 A_8,$$

$$u_{43} = J_2 A_2^* A_5 - J_2 A_4^* J_1 A_3 - J_2 A_6^* A_1 - J_2 A_8^* J_2 A_7,$$

$$u_{44} = J_2 A_2^* A_6 - J_2 A_4^* J_1 A_4 - J_2 A_6^* A_2 - J_2 A_8^* J_2 A_8.$$

Equating the corresponding blocks in (4.17), we easily derive the statement of the theorem. The only appropriate clarification to make is the following. After we establish that $A_3 = A_4 = A_6 = 0$ and that $A_5 = I$ we can rewrite the equation $u_{13} = 0$ as

$$A_1 - A_1^* + A_7^* J_2 A_7 = 0. \tag{4.18}$$

Having represented the matrix A_1 as a sum of symmetric and skew-symmetric matrices, we get $A_1 = A_+ + A_-$, where $A_+^* = A_+$ and $A_-^* = -A_-$. Substituting $A_+ + A_-$ for A_1 and $A_+ - A_-$ for A_1^* into (4.18) we conclude that

$$A_- = -\frac{1}{2} A_7^* J_2 A_7$$

and that, for an arbitrary self-adjoint matrix P_3, the matrix $A_1 = P_3 - \frac{1}{2} A_7^* J_2 A_7$ satisfies the equation (4.18). □

The formula (2.9) for the inverse of $\tilde{U}$ is valid here as well.

Note that a Witt extension $\tilde{U}$ has the form (4.17) with respect to different bases in domain and image space, namely, with respect to the basis (4.8) in the domain space and the basis (4.11) consisting of the vectors Ue, in the image space. Keeping this in mind, we can easily reformulate Theorem 4.2 in the following way with respect to one basis (4.8) (the same for both the domain and the range of $\tilde{U}$) and obtain a statement similar to Theorem 2.4.

THEOREM 4.4 (extended Witt's theorem, second version). *Let U be a fixed Witt extension of U_0 as constructed in Theorem* 4.1. *Then any Witt extension $\tilde{U}$ of U_0 is given by $\tilde{U} = UM$, where M has the form of the right-hand side of* (4.16) *with respect to the basis* (4.8).

Proof. Observe that U maps the elements of the basis (4.8) into the corresponding elements of the basis (4.11) and that (4.15) is the matrix representation of U with respect to the basis (4.8) in the domain space and the basis (4.11) in the image space. □

The set of all Witt extensions of an isometry between two real skew-symmetric scalar product spaces is described as follows.

THEOREM 4.5. *Let H_1, H_2, V_1, and U_0 be as in Theorem* 4.2. *Then the set $W(U_0)$ of all Witt extensions of U_0 is connected and can be parametrized by $\frac{1}{2}(n - m)(n - m + 1)$ real variables. More precisely, let*

$$m_0 = \delta \left[z_j^T (iH_1) z_k \right]_{j,k=1}^{m}$$

for some (*every*) *basis $\{z_1, \ldots, z_m\}$ in V_1; in other words, m_0 is the defect of the restriction of H_1 to V_1. Then $W(U_0)$ is diffeomorphic* (*as a real analytic manifold*) *to*

$$SU\left(\frac{n - m - m_0}{2}\right) \times T \times \mathbf{R}^u,$$

where

$$u = \frac{n - m - m_0}{2} \left(\frac{n - m + 3m_0}{2} + 1 \right) + \frac{m_0(m_0 + 1)}{2}.$$

The proof is obtained by combining Theorem 4.3 and the parametrization of the group of all real matrices that are orthogonal with respect to a skew-symmetric scalar product (see Theorem II.1.7 in [GLR]). Observe that this group is connected (see the same theorem in [GLR]). Also observe that the set of (real) J_2-unitary matrices is diffeomorphic (as a real analytic manifold) to $\mathbf{R}^{k(k+1)} \times SU(k) \times T$, where $k = \frac{n-m-m_0}{2}$, and hence can be described by $2k^2 + k$ real parameters; a detailed proof is found in section II.1.5 of [GLR].

As in Theorems 2.5 and 2.6, the number of independent parameters that describe the set of Witt extensions in Theorem 4.5 depends only on n and m and does not depend on m_0.

5. Polar decompositions. Let $F = \mathbf{C}$ or $F = \mathbf{R}$, and let H be an invertible Hermitian $n \times n$ matrix over F. A factorization $X = UA$ will be called a *semidefinite H-polar decomposition* if U is H-unitary, A is H-nonnegative, and both U and A are over F. Recall that an $n \times n$ matrix A is said to be H-nonnegative if HA is positive semidefinite Hermitian.

More general classes and concepts of polar decompositions in indefinite scalar product spaces are studied in [BMRRR1]. If H is positive definite, then the concept of semidefinite H-polar decomposition reduces to the well-known and widely used notion of polar decompositions for real and complex matrices. For an indefinite H, polar decompositions have been studied in [P1, P2, AI1, AI2, BMRRR2] in connection with Potapov's theory of H-nonexpansive operators, in [KS1, KS2] in connection with plus operators, and in [BR] in connection with H-unitary equivalence. Such polar decompositions play an important role in certain applications in linear optics [M, MH, BMRRR2]. A general approach to polar decompositions is developed in [K]. Other variants of factorizations of matrices of the polar decomposition type have also been studied extensively in the literature; see, e.g., [HM1, HM2, CH].

In this section we characterize the matrices X which admit semidefinite H-polar decompositions (note that in contrast to the standard polar decompositions not every real or complex matrix admits H-polar decompositions if H is indefinite; see [BMRRR1] for examples). Furthermore, in the case when semidefinite H-polar decompositions exist, we provide a full description of the H-nonnegative and H-unitary factors.

We start by recalling the canonical forms of H-self-adjoint matrices (more precisely, of the pairs $\{A, H\}$, where A is H-self-adjoint). We denote by $J_k(\lambda)$ the $k \times k$ upper triangular Jordan block with $\lambda \in \mathbf{C}$ on the main diagonal and by $J_k(\lambda \pm i\mu)$ the $k \times k$ almost upper triangular real Jordan block with eigenvalues $\lambda \pm i\mu$ (here λ, μ are real and $\mu > 0$; k is necessarily even). We also use the notation $Q_m = [\delta_{i+j,m+1}]_{i,j=1}^m$ for the $m \times m$ matrix with ones on the southwest–northeast diagonal and zeros elsewhere.

THEOREM 5.1. *Let H be an $n \times n$ invertible Hermitian matrix* (*over F*), *and let $A \in F^{n\times n}$ be H-self-adjoint. Then there exists an invertible S over F such that $S^{-1}AS$ and S^*HS have the form*

$$S^{-1}AS = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_\alpha}(\lambda_\alpha) \oplus \left[J_{k_{\alpha+1}}(\lambda_{\alpha+1}) \oplus J_{k_{\alpha+1}}(\bar{\lambda}_{\alpha+1})\right] \oplus \cdots \oplus \left[J_{k_\beta}(\lambda_\beta) \oplus J_{k_\beta}(\bar{\lambda}_\beta)\right] \tag{5.1}$$

if $F = \mathbf{C}$, where $\lambda_1, \dots, \lambda_\alpha$ are real and $\lambda_{\alpha+1}, \dots, \lambda_\beta$ are nonreal with positive imaginary parts;

$$S^{-1}AS = J_{k_1}(\lambda_1) \oplus \cdots \oplus J_{k_\alpha}(\lambda_\alpha) \oplus J_{2k_{\alpha+1}}(\lambda_{\alpha+1} \pm i\mu_{\alpha+1}) \oplus \cdots \oplus J_{2k_\beta}(\lambda_\beta \pm i\mu_\beta) \tag{5.2}$$

if $F = \mathbf{R}$*, where* $\lambda_1, \dots, \lambda_\beta$ *are real and* $\mu_{\alpha+1}, \dots, \mu_\beta$ *are positive;*

$$S^* HS = \epsilon_1 Q_{k_1} \oplus \cdots \oplus \epsilon_\alpha Q_{k_\alpha} \oplus Q_{2k_{\alpha+1}} \oplus \cdots \oplus Q_{2k_\beta} \tag{5.3}$$

for both cases ($F = \mathbf{R}$ *or* $F = \mathbf{C}$)*, where* $\epsilon_1, \dots, \epsilon_\alpha$ *are* ± 1*. For a given pair* $\{A, H\}$*, where* A *is* H*-self-adjoint, the canonical form* (5.1), (5.2), (5.3) *is unique up to permutation of orthogonal components in* (5.3), *and the same simultaneous permutation of the corresponding blocks in* (5.1) *or* (5.2), *as the case may be.*

Theorem 5.1 is well known and goes back to Weierstrass and Kronecker. A complete proof of this theorem can be found in many sources; see, e.g., [GLR, T].

The signs ϵ_j in (5.3) form the *sign characteristic* of the pair $\{A, H\}$. Thus, the sign characteristic consists of signs $+1$ or -1 attached to every partial multiplicity (= size of a Jordan block in the Jordan form) of A corresponding to a real eigenvalue.

An existence result concerning general classes of polar decompositions with respect to indefinite scalar products was proved in [BMRRR1, Theorem 4.1]. In particular, this theorem contains the following statement.

Proposition 5.2. *An* $n \times n$ *matrix* X (*over* F) *admits a semidefinite* H*-polar decomposition if and only if* $X^{[*]}X = A^2$ *for some* H*-nonnegative matrix* A *such that* Ker $A =$ Ker X*; moreover, for any such* A *there is an* H*-unitary* U *such that* $X = UA$.

This existence result can be given a much more tractable formulation.

Theorem 5.3 ($F = \mathbf{C}$ *or* $F = \mathbf{R}$). *An* $n \times n$ *matrix* X *admits a semidefinite* H*-polar decomposition if and only if* $X^{[*]}X$ *has eigenvalues only in* $\{\lambda \in \mathbf{R} | \ \lambda \geq 0\}$ *and is diagonalizable and moreover, if* $\operatorname{Ker} X$ *contains a* k*-dimensional* H*-nonpositive subspace, where* k *is the number of negative signs in the sign characteristic of* $\{X^{[*]}X, H\}$ *corresponding to the zero eigenvalue, and* $\operatorname{Ker} X$ *contains a* p*-dimensional* H*-nonnegative subspace, where* p *is the number of positive signs of* H *corresponding to the zero eigenvalue of* $X^{[*]}X$*. Moreover,* A *can be chosen as to satisfy* $\operatorname{Ker}(A^2) = \operatorname{Ker} A$ *if and only if the subspace* $\operatorname{Ker} X^{[*]}X = \operatorname{Ker} X$ *is* H*-nondegenerate.*

Proof. Suppose X admits a semidefinite H-polar decomposition $X = UA$. Then

$$X^{[*]}X = A^2. \tag{5.4}$$

Since A is H-nonnegative, the canonical form (Theorem 5.1) for $\{A, H\}$ implies that there is an invertible matrix S (over F) such that

$$S^{-1}AS = \operatorname{diag}(\lambda_i)_{i=1}^{\nu_1} \oplus 0_{\nu_2} \oplus \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)_{i=1}^{\nu_3} \oplus \operatorname{diag}(\mu_i)_{i=1}^{\nu_4}, \tag{5.5}$$

where λ_i are negative, μ_i are positive, and

$$S^* HS = -I_{\nu_1} \oplus \operatorname{diag}(\epsilon_i)_{i=1}^{\nu_2} \oplus \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)_{i=1}^{\nu_3} \oplus I_{\nu_4}, \tag{5.6}$$

where $\epsilon_i = \pm 1$. Then

$$S^{-1}A^2S = \operatorname{diag}(\lambda_i^2)_{i=1}^{\nu_1} \oplus 0_{(\nu_2 + 2\nu_3)} \oplus \operatorname{diag}(\mu_i^2)_{i=1}^{\nu_4},$$

and thus A^2 is diagonalizable with nonnegative eigenvalues. In view of (5.4) the same thing is true of $X^{[*]}X$. Now we show that $\operatorname{Ker} X = \operatorname{Ker} A$ contains a k-dimensional

H-nonpositive subspace and a p-dimensional H-nonnegative subspace. This follows easily from (5.5), (5.6), as in the notation introduced there:

$$p = \nu_3 + \#\{\epsilon_i | \ \epsilon_i = +1, \ i = 1, \dots, \nu_2\},$$
$$k = \nu_3 + \#\{\epsilon_i | \ \epsilon_i = -1, \ i = 1, \dots, \nu_2\}.$$

To prove the converse part we will need the following lemma (its proof can be found in [BMRRR1]).

LEMMA 5.4. *Let $H = H^*$ be an invertible $n \times n$ matrix, and let X be an $n \times n$ matrix. Let S be an invertible $n \times n$ matrix such that*

$$S^{-1} X^{[*]} X S = \operatorname{diag}(Z_i)_{i=1}^{\nu}, \quad S^* H S = \operatorname{diag}(H_i)_{i=1}^{\nu},$$

with $\sigma(Z_i) \cap \sigma(Z_j) = \emptyset$ for $i \neq j$. Then there exists an H-self-adjoint, respectively, H-nonnegative, matrix A such that $X^{[*]} X = A^2$ if and only if for each i there exists an H_i-self-adjoint, respectively, H_i-nonnegative, matrix A_i such that $Z_i = A_i^2$.

To prove the "if" part of Theorem 5.3, we now only have to consider the case where $X^{[*]} X$ has a single eigenvalue, $\sigma(X^{[*]} X) = \{\lambda\}$. The cases $\lambda > 0$ and $\lambda = 0$ will be considered separately.

Suppose $X^{[*]} X$ is diagonalizable and $\sigma(X^{[*]} X) = \{\lambda\}$, $\lambda > 0$. Let S be an invertible matrix such that

$$S^{-1} X^{[*]} X S = \begin{bmatrix} \lambda I_{n_1} & 0 \\ 0 & \lambda I_{n_2} \end{bmatrix}, \qquad S^* H S = \begin{bmatrix} I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix}.$$

The existence of S is guaranteed; in fact, one brings the pair $\{X^{[*]} X, H\}$ to the canonical form in this way (Theorem 5.1). Let

$$A = S \begin{bmatrix} \sqrt{\lambda}\, I_{n_1} & 0 \\ 0 & -\sqrt{\lambda}\, I_{n_2} \end{bmatrix} S^{-1}.$$

Then A is H-nonnegative and $A^2 = X^{[*]} X$.

Finally, assume $X^{[*]} X$ is diagonalizable, $\sigma(X^{[*]} X) = \{0\}$ (then $X^{[*]} X = 0$), and $\operatorname{Ker} X$ contains a k-dimensional H-nonpositive subspace and a p-dimensional H-nonnegative subspace. It is easy to see that $k + p = n$ in this case, so $\operatorname{Ker} X$ contains a maximal H-nonpositive subspace and a maximal H-nonnegative subspace. For the sake of convenience write $M = \operatorname{Ker} X$. Put $N = M \cap (HM)^{\perp}$, and let M_1 be such that $M = N \oplus M_1$, where this direct sum is orthogonal. This direct sum is also H-orthogonal. Select a basis $f_1, \dots, f_{\nu^0}$ in N and a basis $e_1, \dots, e_{\nu^+}, e_{\nu^+ + 1}, \dots, e_{\nu^+ + \nu^-}$ in M_1 such that

$$\langle He_i, e_j \rangle = 0 \quad \text{for} \quad i \neq j,$$

$$\langle He_i, e_i \rangle = 1 \quad \text{if} \quad i \leq \nu^+, \qquad \langle He_i, e_i \rangle = -1 \quad \text{if} \quad i > \nu^+.$$

We shall construct a subspace K such that $M \oplus K = F^n$ and $(HK)^{\perp} = K \oplus M_1$.

We shall construct an H-nonnegative matrix A such that $A^2 = 0$ and $\operatorname{Ker} A = \operatorname{Ker} X$. The matrix A will be constructed so that N coincides with the linear span of eigenvectors of A corresponding to Jordan blocks of length 2, while $N \oplus K$ is spanned by the eigenvectors, as well as by the generalized eigenvectors of A.

As M contains a maximal H-nonnegative and a maximal H-nonpositive subspace we have

$$\nu^+ + \nu^0 = k,\ \nu^- + \nu^0 = p,$$

and therefore $\dim M = \nu^0 + \nu^+ + \nu^- = k + p - \nu^0 = n - \nu^0$. Consider $(HM_1)^\perp$. The dimension of this subspace is $n - \nu^+ - \nu^- = k + p - \nu^+ - \nu^- = 2\nu^0$; moreover, $(HM_1)^\perp$ contains N. Take any subspace K' such that $(HM_1)^\perp = N \oplus K'$. Then K' is a direct complement of M. Indeed, as $\nu^0 = \dim N = \operatorname{codim} M$ we have $(HM)^\perp = N, (HN)^\perp = M$. Therefore,

$$N = (HM)^\perp = (H(N \oplus M_1))^\perp = (HN)^\perp \cap (HM_1)^\perp = M \cap (HM_1)^\perp.$$

So $K' \cap M = (0)$. Also, $\dim K' = \nu^0$. Take vectors $g'_1, \ldots, g'_{\nu^0}$ in K' such that $\langle Hf_i, g'_j\rangle = \delta_{ij}$ for $i, j = 1, \ldots, \nu^0$. Construct

$$g_i = g'_i - \frac{1}{2}\sum_{\nu=1}^{\nu^0} \langle Hg'_i, g'_\nu\rangle f_\nu, \quad i = 1, \ldots, \nu^0,$$

and let $K = \operatorname{span}\{g_1, \ldots, g_{\nu^0}\}$. Then

$$\langle Hg_i, g_j\rangle = 0 \quad \text{for all } i, j,$$

$$\langle Hf_i, g_j\rangle = \delta_{ij} \text{ for all } i, j,$$

and $K \subset N \oplus K' = (HM_1)^\perp$. By construction, K is H-neutral, so $(HK)^\perp = K \oplus M_1$. Consider the vectors

$$e_1, \ldots, e_{\nu^+}, e_{\nu^+ + 1}, \ldots, e_{\nu^+ + \nu^-}, f_1, g_1, f_2, g_2, \ldots, f_{\nu^0}, g_{\nu^0}$$

as a basis for F^n, and let S be the matrix with these basis vectors as its columns in the order in which they appear here. Then

$$S^* H S = I_{\nu^+} \oplus -I_{\nu^-} \oplus \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)_{i=1}^{\nu^0}.$$

Construct A as follows:

$$S^{-1} A S = 0_{(\nu^+ + \nu^-)} \oplus \operatorname{diag}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)_{i=1}^{\nu^0}.$$

Then, A is H-nonnegative, $A^2 = 0$, and

$$\operatorname{Ker} A = \operatorname{span}\{e_1, \ldots, e_{\nu^+}, e_{\nu^+ + 1}, \ldots, e_{\nu^+ + \nu^-}, f_1, \ldots, f_{\nu^0}\} = \operatorname{Ker} X.$$

By Proposition 5.2 X admits a semidefinite H-polar decomposition.

The statement on choosing A to satisfy $\operatorname{Ker}(A^2) = \operatorname{Ker} A$ is clear because it is equivalent to the nondegeneracy of $\operatorname{Ker} A$. Further, when constructing such A as above, one has $\nu^0 = 0$, which implies $\operatorname{Ker}(A^2) = \operatorname{Ker} A$. ∎

We now give a description of all semidefinite H-polar decompositions (when they exist). The description of all possible H-nonnegative factors A is as follows.

THEOREM 5.5. *Let X be an $n \times n$ matrix that admits a semidefinite H-polar decomposition. Let S be an invertible matrix (over F) such that*

$$S^{-1}(X^{[*]}X)S = \operatorname{diag}(\lambda_i)_{i=1}^{\tau_1} \oplus 0_p \oplus \operatorname{diag}(\mu_i)_{i=1}^{\tau_2}, \tag{5.7}$$

$$S^*HS = -I_{\tau_1} \oplus H_0 \oplus I_{\tau_2}, \tag{5.8}$$

where $\lambda_i > 0,\ \mu_i > 0$, *and*

$$H_0 = \begin{bmatrix} 0 & 0 & I \\ 0 & H_2 & 0 \\ I & 0 & 0 \end{bmatrix}, \tag{5.9}$$

with respect to the decomposition

$$F^p = (\operatorname{Ker} X \cap (H \operatorname{Ker} X)^{\perp}) \oplus M_1 \oplus K, \tag{5.10}$$

where $\operatorname{Ker} X = (\operatorname{Ker} X \cap (H\operatorname{Ker} X)^{\perp}) \oplus M_1$. (*Such S exists by the proof of Theorem* 5.3.) *Then $X = UA$ for some H-unitary U and H-nonnegative A if and only if A has the form*

$$A = S\left(\operatorname{diag}\left(-\sqrt{\lambda_i}\right)_{i=1}^{\tau_1} \oplus A_0 \oplus \operatorname{diag}(\sqrt{\mu_i})_{i=1}^{\tau_2}\right) S^{-1},$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & Y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in F^{p \times p}, \tag{5.11}$$

with respect to the decomposition (5.10), *and where Y is positive definite.*

Observe that by Theorem 5.3 the existence of S such that $S^{-1}(X^{[*]}X)S$ and S^*HS have the forms (5.7) and (5.8), respectively, is necessary for X to have a semidefinite H-polar decomposition.

Proof. By Proposition 5.2, $X = UA$ for some H-unitary U if and only if the H-nonnegative matrix A is such that $X^{[*]}X = A^2$ and $\operatorname{Ker} X = \operatorname{Ker} A$. These conditions are easily translated (using the invertibility of H_0 and H_2) into the statement of Theorem 5.5. ∎

For a fixed A, all possible H-unitary matrices U in the semidefinite H-polar decompositions $X = UA$ are given by an application of Theorem 2.4. This works as follows. Consider the decomposition of F^n,

$$F^n = N_1 \oplus N_2 \oplus N_3 \oplus N_4 \oplus N_5, \tag{5.12}$$

into five components as indicated in Theorem 5.5. With respect to this decomposition, let us write $S^{-1}US = [U_{ij}]_{i,j=1}^5$, $S^{-1}XS = [X_{ij}]_{i,j=1}^5$, $S^{-1}AS = A_1 \oplus A_0 \oplus A_5$. Assume that $X = UA$ and $X = \tilde{U}A$ are semidefinite H-polar decompositions of X. Also write $S^{-1}\tilde{U}S = [\tilde{U}_{ij}]_{i,j=1}^5$. Observing that A_1, A_5, and Y are invertible, we obtain from $X = UA = \tilde{U}A$ that

$$U_{j1} = \tilde{U}_{j1} = X_{j1}A_1^{-1}, \quad U_{j2} = \tilde{U}_{j2} = X_{j2}Y^{-1}, \quad U_{j5} = \tilde{U}_{j5} = X_{j5}A_5^{-1}.$$

Let $\hat{U} = \mathrm{col}\,[U_{j1}\ U_{j2}\ U_{j5}]_{j=1}^{5}$. Take $V_1 = N_1 \oplus N_2 \oplus N_5$ and $V_2 = UV = \tilde{U}V = \hat{U}V$. Then for all $x, y \in V_1$ we have

$$\langle H\hat{U}x, \hat{U}y\rangle = \langle Hx, y\rangle.$$

We see that both U and $\tilde{U}$ are Witt extensions of $\hat{U} : V_1 \to V_2$. Conversely, for any Witt extension V of $\hat{U}$ we have $X = VA$. Applying Theorem 2.4 to this situation gives the following.

THEOREM 5.6. *Suppose $X = UA$ is a semidefinite H-polar decomposition of X, and let A have the form as described in Theorem* 5.5, *with respect to the decomposition* (5.12) *of F^n. Then any H-unitary $\tilde{U}$ such that $X = \tilde{U}A$ is given by $\tilde{U} = UM$, where*

$$M = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & -P_2^* H_2 P_1 & P_3 - \frac{1}{2}P_2^* H_2 P_2 & 0 \\ 0 & 0 & P_1 & P_2 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

with respect to the decomposition (5.12). *Here P_2 is arbitrary, $P_3 = -P_3^*$, and P_1 is an arbitrary H_2-unitary matrix.*

In the real case, Theorem 2.6, together with Theorem 5.6, describes the number of connected components of $\mathcal{U}(H; \mathbf{R})$ from which the H-unitary factor in the semidefinite H-polar decompositions of X may be chosen.

COROLLARY 5.7 ($F = \mathbf{R}$). *Let X be an $n \times n$ matrix that admits a semidefinite H-polar decomposition. If $(H\ \mathrm{Ker} X)^{\perp} \supset \mathrm{Ker}\ X$, then all possible H-unitary factors in the semidefinite H-polar decompositions of X belong to the same connected component of $\mathcal{U}(U; \mathbf{R})$. Otherwise, let M_1 be the H-orthogonal complement of $\mathrm{Ker}\ X \cap (H\mathrm{Ker}\ X)^{\perp}$ in $\mathrm{Ker}\ X$. Then the H-unitary factors belong to two connected components of $\mathcal{U}(H; \mathbf{R})$ having determinants of opposite signs if $H|M_1$ is definite and to all four connected components of $\mathcal{U}(H; \mathbf{R})$ if $H|M_1$ is indefinite.*

The descriptions of the H-nonnegative and H-unitary factors in the polar decompositions of X obtained in Theorems 5.5 and 5.6, together with the real analytic structure of all Witt extensions (Theorems 2.5 and 2.6), allow one to describe the set of all possible H-polar decompositions of a given X in terms of a diffeomorphism (as a real analytic manifold). Using the results mentioned above, such a description is routine and is left to the interested readers.

6. Applications: Hyperbolic QR decompositions. The results of sections 2 and 4 have obvious applications to matrix equations of the form

$$A = UX, \tag{6.1}$$

where A is a given matrix, and X and U are matrices to be found such that U is H-unitary (usually additional requirements are imposed on X and/or U as well). Here A and X are $m \times n$ matrices over F (as usual, we assume that either $F = \mathbf{C}$ or $F = \mathbf{R}$), and H is an invertible $m \times m$ matrix over F which is either Hermitian or skew-symmetric (in the latter case we assume $F = \mathbf{R}$). Indeed, if U and V are solutions of (6.1) with the same A and X, then obviously $Ux = Vx$ for all x in the range of X. Thus, all H-unitary solutions of (6.1) can be treated as Witt extensions of $U|\mathrm{Range}\ X$, where U is one fixed H-unitary solution of (6.1). We will not explicitly present the straightforward statements that are obtained in this way. We focus instead on an important special case of equations (6.1) which is fundamental for a certain class of algorithms for computing the eigenvalues of a matrix using the generalized Schur method, namely, hyperbolic QR decompositions (see, e.g., [B, OSB, V] and references therein).

In a typical version of hyperbolic QR decompositions, one seeks factorizations of the form (6.1), where $m \geq n$ and X is an upper triangular matrix

$$X = \begin{bmatrix} X_1 \\ 0 \end{bmatrix}$$

with invertible $n \times n$ matrix X_1. The factor U is H-unitary, where H is a fixed invertible Hermitian $m \times m$ matrix. Factorizations (6.1) of a given matrix A with the above properties will be called *hyperbolic QR decompositions* in this paper.

In what follows, we will use a basis $\{f_1, \dots, f_m\}$ in $\mathbf{F}^m$ such that $\{f_1, \dots, f_n\}$ forms a basis in $\begin{bmatrix} \mathbf{F}^n \\ 0 \end{bmatrix}$ and with respect to which the indefinite scalar product $[x, y] = \langle Hx, y \rangle$, $x, y \in \mathbf{F}^m$, induced by H has the Gramian matrix

$$\begin{bmatrix} 0 & 0 & I & 0 \\ 0 & J_1 & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & J_2 \end{bmatrix}, \tag{6.2}$$

where I is the $n_0 \times n_0$ identity matrix and J_1 is the diagonal $(n_+ + n_-) \times (n_+ + n_-)$ matrix having the first n_+ diagonal elements equal to $+1$ and the remaining n_- diagonal elements equal to -1; here $n_0 + n_+ + n_- = n$. Similarly, J_2 is a diagonal matrix with entries $+1$ and -1 on the main diagonal. (Compare with (2.5).) A basis $\{f_1, \dots, f_m\}$ with the above properties will be called *admissible*.

THEOREM 6.1. *Let $A = U_0 X_0$ be a hyperbolic QR decomposition of a given $m \times n$ matrix A. Then every hyperbolic QR decomposition $A = \tilde{U} X_0$ of A with the same factor X_0 is given by the following formula, written as a block 4×4 matrix* (*compatible with* (6.2)) *with respect to an admissible basis: $\tilde{U} = U_0 M$, where*

$$M = \begin{bmatrix} I & 0 & -\frac{1}{2} P_2^* J_2 P_2 + P_3 & -P_2^* J_2 P_1 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & P_2 & P_1 \end{bmatrix}.$$

Here P_1 is J_2-unitary, P_3 is a skew-self-adjoint $n_0 \times n_0$ matrix, and P_2 is an arbitrary $(m - n - n_0) \times n_0$ matrix.

The proof is a straightforward application of Theorem 2.4.

Applying Theorem 2.6, we have the following corollary in the real case.

COROLLARY 6.2. *The set of all hyperbolic QR decompositions $A = \tilde{U} X_0$ of a given $m \times n$ matrix A with a given $m \times n$ factor X_0 is connected if*

$$\pi(H) = n_+ + n_0, \qquad \nu(H) = n_- + n_0$$

has two connected components if exactly one of the numbers $p = \pi(H) - n_+ - n_0$ and $q = \nu(H) - n_- - n_0$ is positive and has four connected components if both p and q are positive.

We do not discuss here the problem of existence of hyperbolic QR decompositions for a given $m \times n$ matrix A and a given invertible Hermitian $m \times m$ matrix H and only mention that the obvious necessary condition for A to have full column rank is not sufficient. A characterization of all square matrices A that admit a decomposition $A = UX$, where U is H_1-H_2-unitary and X is upper triangular and nonsingular, is given in Theorem 2.3 of [B]. (The paper [B] considers only diagonal matrices H_1 and H_2, which is the most important case for the development of algorithms based on the generalized Schur method.) An extension to the case of rectangular matrices is presented in [V].

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