Bounds for the degree of polarization

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Relations between the degree of polarization of the incident beam and that of the outgoing beam are discussed for a number of optical devices and for scattering by particles and surfaces. Rigorous upper and lower bounds for the degree of polarization of the outgoing beam are given for cases in which the degree of polarization of the incident beam is known.

A quantity of considerable interest in studies of polarized light is the degree of polarization of a beam of radiation. Every transformation of Stokes parameters described by a Mueller matrix may change the degree of polarization. The nature of this change, however, is not so obvious (see, e.g., Ref. 1).

The main purpose of this Letter is to present simple upper and lower bounds for the change of the degree of polarization of a beam that occurs on interaction with matter, as described by a Mueller matrix. Such bounds can be used for testing optical devices and numerical techniques.

We consider a quasi-monochromatic beam of radiation with Stokes parameters I, Q, U, and V (see, e.g., Refs. 2 and 3). We write these Stokes parameters as elements of a column vector $\mathbf{I}$, called the Stokes vector, and define the degree of polarization as

$$p = (Q^2 + U^2 + V^2)^{1/2}/I\,. \tag{1}$$

We always have $0 \leq p \leq 1$. If a beam of radiation with Stokes vector $\mathbf{I}_1$ and degree of polarization p_1 creates through linear processes a beam of radiation with Stokes vector $\mathbf{I}_2$ and degree of polarization p_2, we can write

$$\mathbf{I}_2 = \mathbf{M}\mathbf{I}_1\,, \tag{2}$$

where the real 4×4 matrix $\mathbf{M}$ is called a Mueller matrix. Clearly, the corresponding change of the degree of polarization is, in general, a complicated function of the elements of $\mathbf{M}$ and $\mathbf{I}_1$. Instead of the elements M_{ij} of the Mueller matrix, we will sometimes use

$$m_{ij} = M_{ij}/M_{11}\,. \tag{3}$$

Since M_{11} is the largest element in absolute value,[4,5] we have

$$0 \leq |m_{ij}| \leq 1\,. \tag{4}$$

Similarly, we sometimes use the reduced Stokes parameters (i.e., the Stokes parameters divided by the first one) q_1, u_1, and v_1 for the incident beam.

A useful parameter turns out to be

$$s = (m_{21}^2 + m_{31}^2 + m_{41}^2)^{1/2}. \tag{5}$$

Apparently, s represents the degree of polarization of a beam of outgoing radiation if the incident beam is unpolarized. Consequently, s is the value of p_2 if $p_1 = 0$, and we have

$$0 \leq s \leq 1\,. \tag{6}$$

In the following discussion we restrict ourselves to pure Mueller matrices, i.e., matrices that can be derived from 2×2 Jones matrices. This holds for optical devices such as quarter-wave plates and retarders and for scattering by one particle. Pure Mueller matrices have many interesting properties.[5,6] For example, we can write

$$I_2^2(1 - p_2^2) = d^2 I_1^2(1 - p_1^2)\,, \tag{7}$$

where

$$d^2 = M_{11}^2(1 - s^2)\,. \tag{8}$$

Furthermore, we have

$$s^2 = m_{12}^2 + m_{13}^2 + m_{14}^2\,. \tag{9}$$

Combining Eqs. (7) and (8) and using Eq. (2) gives

$$1 - p_2^2 = \frac{M_{11}^2(1 - s^2)I_1^2(1 - p_1^2)}{(M_{11}I_1 + M_{12}Q_1 + M_{13}U_1 + M_{14}V_1)^2}\,, \tag{10}$$

which may be written in the form [cf. Eq. (3)]

$$\begin{aligned} 1 - p_2^2 &= \frac{(1 - s^2)(1 - p_1^2)}{(1 + m_{12}q_1 + m_{13}u_1 + m_{14}v_1)^2} \\ &= \frac{(1 - s^2)(1 - p_1^2)}{(1 + sp_1 \cos\gamma)^2}\,, \end{aligned} \tag{11}$$

where γ is the angle between the vectors $\{m_{12},\ m_{13},\ m_{14}\}$ and $\{q_1,\ u_1,\ v_1\}$.

For given s and p_1, the maximum $f(p_1,\ s)$ of p_2 will be assumed for $\gamma = 0$, while the minimum $g(p_1,\ s)$ of p_2 will be assumed for $\gamma = \pi$. This gives

$$f(p_1,\ s) = \frac{p_1 + s}{1 + sp_1}\,, \tag{12}$$

$$g(p_1,\ s) = \frac{|p_1 - s|}{1 - sp_1}\,. \tag{13}$$

We have thus found very simple upper and lower bounds for p_2. Note that $f(p_1,\ s)$ and $g(p_1,\ s)$ are both symmetric in p_1 and s.

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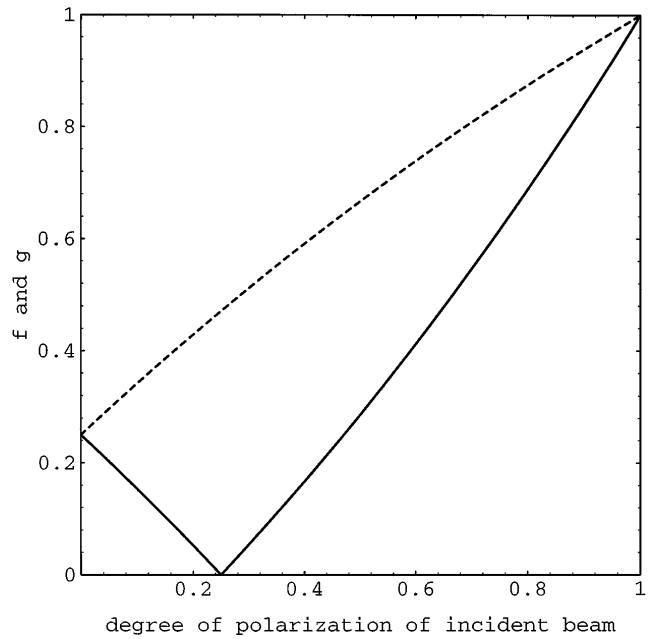

Fig. 1. Upper (dashed curve) and lower (solid curves) bounds for p_2 as functions of p_1. Here $s = 1/4$.

Figure 1 shows $f(p_1, s)$ and $g(p_1, s)$ as functions of p_1 for $s = 0.25$. If $0 < s < 1$ the bounds coincide for $p_1 = 0$ and $p_1 = 1$, yielding $p_2 = s$ and $p_2 = 1$, respectively [cf. Eqs. (12) and (13)]. If $0 < s < 1$ the bounds for p_2 are reached if $\{q_1, u_1, v_1\} = (\pm p_1/s)\{m_{12}, m_{13}, m_{14}\}$, where the plus refers to the upper bound and the minus refers to the lower bound. If $0 \le s < 1$, the maximal difference $f - g$ is $2s/(1 + s^2)$, which occurs if $p_1 = s$. If $s = 0$ the bounds coincide for all values of p_1, and $p_2 = p_1$. Similarly, the bounds coincide if $s = 1$, and we then have $p_2 = 1$ for all values of p_1. This means that if a primary beam of unpolarized light creates completely polarized light through a pure Mueller matrix, the light is completely polarized for every state of polarization of the primary beam.

For practical purposes several conclusions may be drawn from Fig. 1 or Eqs. (11)–(13). We give three examples:

(1) If one has a pure Mueller matrix with $s = 1/4$ it is not possible to obtain a beam with $p_2 \ge 2/3$ when beams are used with $p_1 < 0.5$.

(2) If one wishes an instrument, characterized by a pure Mueller matrix with $0 < s < 1$, to depolarize as much as possible one should use light with $p_1 = s$ and $\gamma = \pi$, i.e., $\{q_1, u_1, v_1\} = -\{m_{12}, m_{13}, m_{14}\}$, since then $p_2 = 0$.

(3) A pure Mueller matrix has the property that $p_2 = 1$ if $p_1 = 1$. For that reason a pure Mueller matrix is often called a totally polarizing Mueller matrix[7] or a nondepolarizing matrix.[8] It should be realized, however, that a pure Mueller matrix may give light with $p_2 < p_1$.

This concludes our discussion of pure Mueller matrices.

Let us now consider a Mueller matrix of the type

$$\mathbf{M} = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_1 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_3 \end{bmatrix}, \tag{14}$$

which holds, e.g., for the scattering matrix of an assembly of optically inactive homogeneous spheres, also called Mie scattering. In this case [cf. Eq. (2)]

$$I_2^2(1 - p_2^2) = (a_1^2 - b_1^2)(I_1^2 - Q_1^2) - (a_3^2 + b_2^2)(U_1^2 + V_1^2). \tag{15}$$

When we use Eq. (81) of Ref. 9,

$$a_3^2 + b_2^2 \le a_1^2 - b_1^2, \tag{16}$$

Eq. (15) yields

$$1 - p_2^2 \ge \frac{a_1^2(1 - s^2)I_1^2(1 - p_1^2)}{(a_1 I_1 + b_1 Q_1)^2}, \tag{17}$$

which can be written as

$$p_2 \le \frac{p_1 + s}{1 + p_1 s} = f(p_1, s). \tag{18}$$

Thus we find exactly the same upper bound as for a pure Mueller matrix.

Equation (15) also gives

$$I_2^2(1 - p_2^2) \le (a_1^2 - b_1^2)(I_1^2 - Q_1^2), \tag{19}$$

yielding

$$1 - p_2^2 \le \frac{(1 - s^2)(1 - q_1^2)}{(1 - s|q_1|)^2}, \tag{20}$$

so that

$$p_2 \ge \left| \frac{s - |q_1|}{1 - s|q_1|} \right|. \tag{21}$$

Suppose we now have a Mueller matrix of the form

$$\mathbf{M} = \mathrm{diag}(a_1, a_2, a_3, a_4). \tag{22}$$

Here

$$p_2 = \frac{(a_2^2 q_1^2 + a_3^2 u_1^2 + a_4^2 v_1^2)^{1/2}}{a_1}, \tag{23}$$

which satisfies the inequalities

$$\left[\frac{\min(|a_2|, |a_3|, |a_4|)}{a_1}\right] p_1 \le p_2 \le \left[\frac{\max(|a_2|, |a_3|, |a_4|)}{a_1}\right] p_1. \tag{24}$$

In particular, since $\max(|a_2|, |a_3|, |a_4|) \le a_1$,[9] we have

$$p_2 \le p_1. \tag{25}$$

Consequently, a diagonal Mueller matrix can never enhance the degree of polarization.

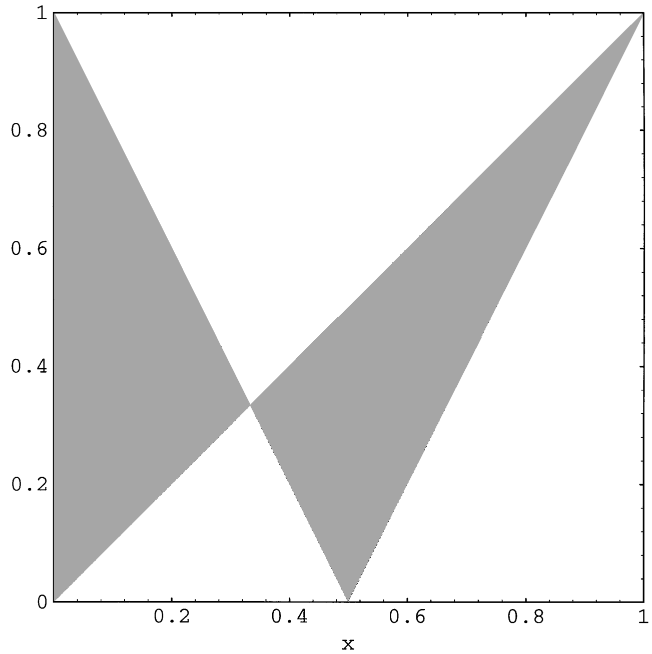

Fig. 2. Domains for p_2/p_1 (the shaded areas) for exact backscattering by the assemblies of particles described in the text.

An interesting special case is provided by exact backscattering by a small volume element comprising (i) randomly oriented particles having a plane of symmetry, such as ellipsoids, and/or (ii) particles and their mirror particles in equal numbers and in random orientation. The scattering matrix then is

$$\mathbf{M} = \mathrm{diag}(a_1, a_2, -a_2, a_1 - 2a_2), \qquad (26)$$

where $a_1 \geq a_2 \geq 0$.[10] When we write $x = a_2/a_1$, inequalities (24) provide in this case

$$x p_1 \leq p_2 \leq (1 - 2x) p_1 \qquad \text{if } x < 1/3,$$

$$|1 - 2x| p_1 \leq p_2 \leq x p_1 \qquad \text{if } x > 1/3,$$

$$p_2 = p_1/3 \qquad \text{if } x = 1/3.$$

This is illustrated in Fig. 2.

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