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POLAR DECOMPOSITIONS IN FINITE DIMENSIONAL INDEFINITE SCALAR PRODUCT SPACES: SPECIAL CASES AND APPLICATIONS

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Polar decompositions $X = UA$ of real and complex matrices X with respect to the scalar product generated by a given indefinite nonsingular matrix H are studied in the following special cases: (1) X is an H-contraction, (2) X is an H-plus matrix, (3) H has only one positive eigenvalue, and (4) U belongs to the connected component of the identity in the group of H-unitary matrices. Applications to linear optics are presented.

1 Introduction

Let F be either the field of real numbers $\mathbf{R}$ or the field of complex numbers $\mathbf{C}$. Fix a real symmetric (if $F = \mathbf{R}$) or complex hermitian (if $F = \mathbf{C}$) invertible $n \times n$ matrix H. Consider the scalar product induced by H by the formula $[x, y] = \langle Hx, y\rangle$, $x, y \in F^n$. Here $\langle \cdot , \cdot \rangle$ stands for the standard scalar product in F^n defined by $\langle x, y\rangle = \sum_{j=1}^{n} x_j \bar{y}_j$, where $(x_1, \cdots, x_n)^T$ and $(y_1, \cdots, y_n)^T$ are column vectors in F^n. (Of course, $\bar{y}_j = y_j$ if $F = \mathbf{R}$.) The scalar product $[\cdot , \cdot]$ is nondegenerate ($[x, y] = 0$ for all $y \in F^n$ implies $x = 0$), but is indefinite in general. In other words, the real number $[x, x]$ can be positive, negative, or zero for various $x \in F^n$ (unless H is definite). The vector $x \in F^n$ is called *positive* if $[x, x] > 0$, *neutral* if $[x, x] = 0$, and *negative* if $[x, x] < 0$.

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Similarly, a subspace $\mathcal{M} \subset F^n$ is called *positive* (resp. *negative*) if all non-zero vectors $x \in \mathcal{M}$ are positive (resp. negative). We write H-positive or H-negative if we wish to emphasize the dependence of these definitions on H. If all vectors in a subspace are neutral, we say that the subspace is *isotropic* (or *H-isotropic*).

Well-known concepts related to the scalar product $[\,\cdot\,,\,\cdot\,]$ are defined in obvious ways. Thus, given an $n \times n$ matrix A over F, the adjoint $A^{[*]}$ is defined by $[Ax, y] = [x, A^{[*]}y]$ for all $x, y \in F^n$. The formula $A^{[*]} = H^{-1}A^*H$ is verified immediately (here and elsewhere we denote by A^* the conjugate transpose of A, so that $A^* = A^T$ if $F = \mathbf{R}$). A matrix A is called *H-selfadjoint* if $A^{[*]} = A$, or equivalently, if HA is hermitian. In particular, if HA is positive semidefinite hermitian, we say that the matrix A is *H-positive.* An $n \times n$ matrix U is called *H-unitary* if $[Ux, Uy] = [x, y]$ for all $x, y \in F^n$, or, equivalently, $U^*HU = H$. Observe that for every H-unitary matrix U we have $|\det U| = 1$; in particular, $\det U = \pm 1$ if $F = \mathbf{R}$.

In this article we continue to study decompositions of an $n \times n$ matrix X over F of the form

$$X = UA, \tag{1.1}$$

where U is H-unitary and A is H-selfadjoint (with or without additional restrictions). We call the decomposition (1.1) without additional restrictions on U and A an *H-polar decomposition* of X. Given non-negative integers p, q, (1.1) is called an *(H, p, q)-polar decomposition* if the number of positive (resp., negative) eigenvalues, when counted with multiplicities, of HA does not exceed p (resp., q). A general theory of H-polar decompositions has been developed in a preceding article [BMRRR1]; it is devoted to the problems of existence, uniqueness (up to equivalence) and basic properties of H-polar and (H, p, q)-polar decompositions, and to the existence of H-polar decompositions of H-normal matrices. Most of the concepts and notations used here are introduced in [BMRRR1]. In the present article we study H-polar decompositions of the type (1.1), where various constraints are imposed on the matrices X, U, A and H, and discuss its applications in linear optics.

We shall now briefly discuss these various subjects, some of their history, and the contents of the sections.

Motivated by the theory of the H-modulus for H-nonexpansive operators, i.e., operators X for which $H - X^*HX$ is positive semidefinite (see [P1,P2,AI]), and the theory of the H-modulus for H-plus operators, i.e., operators mapping positive vectors into positive or neutral vectors (see [AI,KS1,KS2]), we reprove and refine well-known results on the existence of an H-polar decomposition of H-contractions in Section 2 and of H-plus matrices in Section 3. In the case when H has precisely one positive eigenvalue, more specific H-polar decomposition results are obtained for H-plus matrices. Necessary and sufficient conditions are given for a matrix to be an H-plus matrix. In Section 4 we give a full description of all matrices X that allow an H-polar decomposition in the case when H has only one positive eigenvalue. The constraints that the structure of H imposes on the Jordan structure of A, make it possible to give a much more complete description than is given in [BMRRR1]. In Section 5 we seek H-polar decompositions where the H-unitary factor is required to belong to a prescribed connected component of the group of H-unitary matrices. For $F = \mathbf{R}$ and $n = 2, 3$, we give examples of matrices X having an H-polar decomposition, but where U cannot be chosen in just any prescribed connected component. For $F = \mathbf{R}$ and $n \geq 4$, such selections turn out to be always possible if H has only one positive eigenvalue and an H-polar decomposition exists. In Section 6 we apply our results on H-polar decomposition to linear optics, where we must study the case $F = \mathbf{R}$, $n = 4$, $H = \operatorname{diag}(1, -1, -1, -1)$, and U in the connected component of the identity. The polarization matrices involved are real H-plus matrices with respect to $H = \text{diag}(1, -1, -1, -1)$. Well-known results on two classes of polarization matrices, namely those satisfying the so-called Stokes criterion (see [K,MH,N,M]) and the so-called weighted sums of pure Mueller matrices (see [C,M]), are generalized. We indicate when matrices belonging to the larger one of these two classes (i.e., the class of matrices satisfying the Stokes criterion) have an H-polar decomposition. Necessary and sufficient conditions are given for a real 4×4 matrix to belong to either of these two classes, thus improving upon results given in [M].

The following notations will be used. The number of positive (negative, zero) eigenvalues of a hermitian matrix A is denoted by $\pi(A)$ $(\nu(A), \delta(A))$. F^n (where $F = \mathbf{R}$ or $F = \mathbf{C}$) stands for the vector space of n-dimensional columns over F. We denote by $F^{m\times n}$ the vector space of $m \times n$ matrices over F. The standard matrices are $J_k(\lambda)$ (the $k \times k$ upper triangular Jordan block with $\lambda \in \mathbf{C}$ on the main diagonal), I_m the $m \times m$ identity matrix, O_m the $m \times m$ zero matrix, and $Q_m = [\delta_{i+j,m+1}]_{i,j=1}^m$ the $m \times m$ matrix with 1's on the southwest–northeast diagonal and zeros elsewhere. The block diagonal matrix with matrices $Z_1, \dots, Z_k$ on the main diagonal is denoted by $Z_1 \oplus \cdots \oplus Z_k$ or $\text{diag}(Z_1, \dots, Z_k)$. The set of eigenvalues (including nonreal eigenvalues for real matrices) of a matrix X is denoted by $\sigma(X)$. $\text{Ker}\, A$ and $\text{Im}\, A$ stand for the null space and range of a matrix A. The symbol $\mathcal{M} \oplus \mathcal{N}$ denotes the direct sum of the subspaces $\mathcal{M}$ and $\mathcal{N}$.

Although we have sought to write the present paper in a self-contained way, occasionally we will draw on concepts and results from the previous paper [BMRRR1] and from the paper [BMRRR2] on H-unitary extensions and H-polar decompositions with HA positive semidefinite hermitian. The canonical form of an ordered pair $\{A, H\}$ where A is H-selfadjoint, is described in Section 2 of [BMRRR1] (as well as in Section I.3.2 of [GLR] and many other sources) and will not be redefined here. We will use freely the canonical form, in particular the sign characteristic, of the pair $\{A, H\}$.

For the reader's convenience, we quote here one result from [BMRRR1] (Theorem 4.4):

THEOREM 1.1. ($F = \mathbf{C}$ *or* $F = \mathbf{R}$) *An* $n \times n$ *matrix* X *admits* H*-polar decomposition if and only if all the conditions (i), (ii), and (iii) below are satisfied.*

(i) For each negative eigenvalue λ *of* $X^{[*]}X$ *the part of the canonical form of* $\{X^{[*]}X, H\}$ *corresponding to* λ *can be presented in the form*

$$\{\text{diag}\,(A_i)_{i=1}^m, \quad \text{diag}\,(H_i)_{i=1}^m\},$$

where, for $i = 1, \dots, m$,

$$A_i = J_{k_i}(\lambda) \oplus J_{k_i}(\lambda), \qquad H_i = Q_{k_i} \oplus -Q_{k_i}.$$

(ii) The part of the canonical form of $\{X^{[*]}X, H\}$ *corresponding to the zero eigenvalue can be presented in the form*

$$\{\text{diag}\,(B_i)_{i=0}^m, \quad \text{diag}\,(H_i)_{i=0}^m\},$$

where $B_0 = O_{k_0}$, $H_0 = I_{p_0} \oplus -I_{n_0}$, $p_0 + n_0 = k_0$ *and, for each* $i = 1, \dots, m$, *the pair* $\{B_i, H_i\}$ *is of one of the following two forms:*

$$B_i = J_{k_i}(0) \oplus J_{k_i}(0), \qquad H_i = Q_{k_i} \oplus -Q_{k_i}, \quad k_i > 1,$$

or

$$B_i = J_{k_i}(0) \oplus J_{k_i-1}, \qquad H_i = \varepsilon_i(Q_{k_i} \oplus Q_{k_i-1}),$$

with $\varepsilon = \pm 1$, *and* $k_i > 1$.

Assume that (ii) holds and denote the corresponding basis in $Ker(X^{[*]}X)^n$ *in which this is achieved by*

$$\{e_{i,j}\}_{i=0}^{m}{}_{j=1}^{l_i}$$

where $l_0 = k_0$ *and* $l_i = 2k_i$ *in case* B_i *is an even size matrix, and* $l_i = 2k_i - 1$ *in case* B_i *is an odd size matrix.*

(iii) There is a choice of basis $\{e_{i,j}\}_{i=0}^{m}{}_{j=1}^{l_i}$ *such that (ii) holds and*

$$\begin{aligned} KerX &= \operatorname{span}\{e_{i,1} + e_{i,\,k_i+1} |\ l_i = 2k_i,\ i = 1, \dots, m\} \oplus \\ &\oplus \operatorname{span}\{e_{i,1} |\ l_i = 2k_i - 1,\ i = 1, \dots, m\} \oplus \operatorname{span}\ \{e_{0,j}\}_{j=1}^{k_0}. \end{aligned}$$

2 H-Contractive Matrices

Let $F = \mathbf{R}$ or $F = \mathbf{C}$. We consider F^n together with the indefinite scalar product defined by the invertible hermitian matrix H over F.

An $n \times n$ matrix X (over F) is called *H-nonexpansive* if it does not increase the indefinite scalar product of two vectors, i.e., $[Xv, Xv] \le [v, v]$ for all $v \in F^n$, or, equivalently, the matrix $H - X^*HX = H(I - X^{[*]}X)$ is positive semidefinite hermitian. It was proved by Potapov [P2] (for the case $F = \mathbf{C}$) that every H-nonexpansive matrix X admits an H-polar decomposition. As a matter of fact, he showed that X can be factored as $X = UA$, where U is H-unitary and A is H-selfadjoint, with the additional conditions $\operatorname{Ker} A = \operatorname{Ker} A^2$ and $\sigma(A) \subset [0, \infty)$; such a matrix A was called an *H-modulus* of X. This result was later extended to the infinite dimensional case by Ju. P. Ginzburg ([Gi1,Gi2]), and by M. G. Krein and Ju. L. Shmul'jan ([KS1,KS2]). We will adopt the term *H-contraction* instead of H-nonexpansive matrix.

Given an arbitrary $n \times n$ matrix X, the question naturally arises when it is possible to find an H-polar decomposition $X = UA$ of X with the additional property that $\sigma(A) \subset \{\lambda |\ \lambda \ge 0\}$. The following theorem provides a complete answer to this question.

THEOREM 2.1. ($F = \mathbf{C}$ *or* $F = \mathbf{R}$) *An* $n \times n$ *matrix* X *admits an* H*-polar decomposition* $X = UA$ *with* $\sigma(A) \subset \{\lambda |\ \lambda \ge 0\}$ *if and only if the condition* (i) *below and the conditions* (ii), *and* (iii) *of Theorem 1.1 hold.*

(i) $\sigma(X^{[*]}X) \subset \{\lambda |\ \lambda \ge 0\}$.

Proof. First let $F = \mathbf{C}$. If X has an H-polar decomposition $X = UA$ with $\sigma(A) \subset \{\lambda | \lambda \ge 0\}$ then $X^{[*]}X = A^2$ has all its eigenvalues in $\{\lambda | \lambda \ge 0\}$. Thus (i) holds. Now apply Theorem 1.1. The converse follows as in the proof of Theorem 1.1 (see the proof of Theorem 4.4 of [BMRRR1]).

The proof for $F = \mathbf{R}$ is essentially the same, especially since Theorem 1.1 applies to both $F = \mathbf{C}$ and $F = \mathbf{R}$. □

Let us see how the theory of H-polar decompositions applies to the particular class of matrices studied by Potapov, i.e., the class of H-contractions. To do this we need some preliminary material on H-contractions.

THEOREM 2.2. *Let X be an $n \times n$ matrix which is an H-contraction. Then the following hold:*

(a) $\sigma(X^{[*]}X) \subset \{\lambda | \lambda \geq 0\}$;

(b) *let $\mathcal{M}_+$ be the spectral invariant subspace of $X^{[*]}X$ corresponding to eigenvalues in $[0,1)$, then $\mathcal{M}_+$ is H-positive; let $\mathcal{M}_-$ denote the spectral invariant subspace of $X^{[*]}X$ corresponding to eigenvalues in $(1,\infty)$ then $\mathcal{M}_-$ is H-negative; in other words, every Jordan block of $X^{[*]}X$ with eigenvalue $\lambda > 1$ (resp. $\lambda < 1$) is of order 1 and the corresponding sign in the sign characteristic of $\{X^{[*]}X, H\}$ is -1 (resp. $+1$).*

(c) *on $\operatorname{Ker}(X^{[*]}X - I)^n$ the matrix $X^{[*]}X$ has Jordan blocks of order at most two. The signs in the sign characteristic of $\{X^{[*]}X, H\}$ corresponding to blocks with eigenvalue one of order one may be both $+1$ and -1, the signs corresponding to blocks with eigenvalue one of order two are all -1.*

Conversely, if a matrix X is such that the canonical form of $\{X^{[*]}X, H\}$ satisfies (a)–(c), then X is an H-contraction.

Proof. Assume that X is an H-contraction. Part (a) was proved in [P2], where also the first part of (c) was observed.

We provide a full independent proof of the conditions (a), (b) and (c). If B is an H-selfadjoint matrix, the canonical form of the pair $\{B, H\}$ shows that, after reduction to the canonical form, for every non-real eigenvalue λ of B the matrix $H - HB$ has a 2×2 principal submatrix of the form $\begin{bmatrix} 0 & 1-\overline{\lambda} \\ 1-\lambda & * \end{bmatrix}$; this 2×2 matrix is never positive semidefinite. Applying this observation to $B = X^{[*]}X$, we obtain that all eigenvalues of $X^{[*]}X$ are real. Furthermore, let $X^{[*]}X = S^{-1}JS$ and $H = S^*H_0S$, where the pair $\{J, H_0\}$ is the canonical form of the pair $\{X^{[*]}X, H\}$. Then J is an H_0-contraction, and

$$H - X^*HX = H - HX^{[*]}X = S^*(H_0 - H_0J)S \geq 0;$$

so $H_0 - H_0J \geq 0$. Now $H_0 - H_0J$ is block diagonal. Suppose λ is an eigenvalue of $X^{[*]}X$ (hence also an eigenvalue of J), and let k be the order of one of the Jordan blocks in J with eigenvalue λ. Then $H_0 - H_0J$ contains a block of the form $\varepsilon(Q_k - Q_kJ_k(\lambda))$, where ε is the sign in the sign characteristic of $\{J, H_0\}$ corresponding to this block. Clearly, this block can only be positive semidefinite if (c) holds and every Jordan block of $X^{[*]}X$ with eigenvalue $\lambda > 1$ (resp. $\lambda < 1$) is of order 1 with the sign -1 (resp. $+1$).

It remains to prove that $X^{[*]}X$ has no negative eigenvalues. Let $\mathcal{M}_+$ be the spectral invariant subspace of $X^{[*]}X$ corresponding to the eigenvalues which are less than 1. By the already proved parts of (a), (b) and (c), $\mathcal{M}_+$ is H-positive. In other words, the scalar product induced by H on $\mathcal{M}_+$ is positive definite. On such a subspace $X^{[*]}X$ cannot have negative eigenvalues. This completes the proof of the properties (a), (b) and (c).

The converse statement follows easily from the canonical form of $\{X^{[*]}X, H\}$. □

The opposite concept is the concept of an H-expansive matrix. An $n \times n$ matrix X (over F) is called an *H-expansion* if $[Xv, Xv] \geq [v, v]$ for all $v \in F^n$. Using an obvious observation that a matrix is an H-expansion if and only if it is a $(-H)$-contraction, the result analogous to Theorem 2.2 holds for H-expansions. To obtain the statement of this result, replace in Theorem 2.2 "H-contraction" by "H-expansion," replace the signs in (b) and (c) by their opposites, and interchange "H-positive" and "H-negative" in (b).

We say that a matrix X is *H-monotone* if it is an H-expansion or an H-contraction. Another piece of information we need for H-monotone matrices, is the following.

LEMMA 2.3. *If X is H-monotone, then* $\mathrm{Ker} X^{[*]}X = \mathrm{Ker} X$.

Proof. Assume first that X is an H-contraction. It is proved in [BR] (Lemma 4.4) that

$$\mathrm{rank}(X^{[*]}X) \leq \mathrm{rank}(X) \leq \mathrm{rank}(X^{[*]}X) + d, \tag{2.1}$$

where

$$d = \min\{\pi(H) - \pi(HX^{[*]}X), \quad \nu(H) - \nu(HX^{[*]}X)\}.$$
$$\pi(H) - \pi(HX^{[*]}X) = p_1(0), \quad \nu(H) - \nu(HX^{[*]}X) = 0. \tag{2.2}$$

Here $p_1(0)$ is the number of 1×1 nilpotent blocks in the canonical form of $X^{[*]}X$ with the sign $+1$ in the sign characteristic of $\{X^{[*]}X, H\}$. Clearly, (2.1) and (2.2) yield $\mathrm{Ker} X^{[*]}X = \mathrm{Ker} X$. The case of X an H-expansion is considered analogously. □

Combining the results above easily yields the following theorem of Potapov [P2].

THEOREM 2.4. ($F = \mathbf{C}$ *or* $F = \mathbf{R}$) *Let X be H-monotone. Then X admits a unique H-polar decomposition $X = UA$, with the additional property that $\sigma(A) \subset \{\lambda | \lambda \geq 0\}$. For this H-polar decomposition we also have* $\mathrm{Ker}\, A = \mathrm{Ker}\, A^2$.

Proof. Let X be an H-contraction. Then $\sigma(X^{[*]}X) \subset [0, \infty)$ and there are no Jordan blocks of order ≥ 2 corresponding to any zero eigenvalue. Thus Theorem 2.1 implies that X has an H-polar decomposition with $\sigma(A) \subset [0, \infty)$. More precisely, if we write

$$X^{[*]}X = O_{k_0} \oplus \bigoplus_{i=1}^{m} Y_i$$

with $0 < \lambda_1 < \cdots < \lambda_m$ and $\sigma(Y_i) = \{\lambda_i\}$, then

$$A = O_{k_0} \oplus \bigoplus_{i=1}^{m} Z_i,$$

where $Z_i^2 = Y_i$ and $\sigma(Z_i) = \{\sqrt{\lambda_i}\}$, is an H-selfadjoint matrix such that $X^{[*]}X = A^2$ and $\mathrm{Ker}\, A = \mathrm{Ker}\, A^2 = \mathrm{Ker}\, X^{[*]}X$. Further, A is a real matrix if X is a real matrix.

Since any matrix with only positive eigenvalues has a unique square root that is a matrix with only positive eigenvalues, there exists a unique H-modulus A such that $X^{[*]}X = A^2$, where X is a given H-contraction. □

3 H-plus Matrices

Let $F = \mathbf{R}$ or $F = \mathbf{C}$. We consider F^n together with the indefinite scalar product $[\cdot,\cdot]$ defined by the invertible hermitian matrix H over F.

Krein and Shmul'jan [KS1,KS2] have developed a theory of plus operators, which are operators on an indefinite scalar product space that transform nonnegative vectors into nonnegative vectors. The results they obtained were formulated in an infinite dimensional setting and hence their term "plus *operator*" is appropriate. However, since we are working exclusively in a finite dimensional context, we will adopt the term "H-plus matrix" instead, where the matrix H generating the scalar product has been attached to our terminology.

An $n\times n$ matrix X (over F) will be called an *H-plus matrix* if $[X^{[*]}Xu,u] = [Xu,Xu] \geq 0$ whenever $[u,u] \geq 0$. Clearly, X is an H-plus matrix if $[X^{[*]}Xu,u] \geq 0$ whenever $[u,u] > 0$. Thus defining

$$\mu(X) = \inf_{[u,u]=1} [X^{[*]}Xu,u], \tag{3.1}$$

we see that X is an H-plus matrix if and only if $\mu(X) \geq 0$. Then

$$[X^{[*]}Xz,z] \geq \mu(X)[z,z], \qquad z \in F^n. \tag{3.2}$$

In the complex case the formula (3.2) is well-known (see [Bo], Theorem II.8.1); one can prove (3.2) in the complex case also using convexity of numerical ranges (see Theorem 1.6 in [A]). We relegate the proof of (3.2) for the case $F = \mathbf{R}$ to the appendix of this section. We call X a *strict H-plus matrix* if $\mu(X) > 0$. Finally, we call X a *doubly H-plus matrix* if both X and $X^{[*]}$ are H-plus matrices. As we will indicate below, every strict H-plus matrix is a doubly H-plus matrix. However, there exist H-plus matrices which are not doubly H-plus matrices.

Example 3.1. Let $H = \mathrm{diag}\,(1,-1)$ and $X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $X^{[*]} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}$. We now easily check that $[X^{[*]}Xu,u] = |u_2|^2$ and $[XX^{[*]}u,u] = -|u_1|^2$, where $u = (u_1,u_2)$. Thus X is an H-plus matrix, but $X^{[*]}$ is not. □

If H is positive definite, every $n\times n$ matrix is an H-plus matrix and $\mu(X)$ is the smallest (automatically nonnegative) eigenvalue of $X^{[*]}X$. Then X is a strict H-plus matrix if and only if it is invertible. On the other hand, if H is negative definite, every $n \times n$ matrix X is an H-plus matrix. Since the definition of $\mu(X)$ does not make sense in this case, we cannot even define strict H-plus matrices in the above way. In the rest of this section we will therefore assume that H is indefinite.

The next result allows us to derive the spectral properties of strict H-plus matrices from those of H-contractions.

LEMMA 3.1. *An $n \times n$ matrix X is a strict H-plus matrix if and only if, for some $c > 0$, cX is a $(-H)$-contraction. One may choose $c = \mu(X)^{-1/2}$. Moreover, every strict H-plus matrix is a doubly H-plus matrix.*

Proof. One easily checks the following string of implications: X is a strict H-plus matrix, if and only if there exists $\mu > 0$ such that $X^{[*]}X - \mu I$ is H-positive, if and only if there exists $c > 0$ such that $(cX)^{[*]}(cX) - I$ is H-positive, if and only if there exists $c > 0$

such that $I-(cX)^{[*]}(cX)$ is $(-H)$-positive, if and only if cX is a $(-H)$-contraction for some $c>0$. Obviously, $c=\mu^{-1/2}$.

The second part of the proposition follows from the fact that $Y^{[*]}$ is an H-contraction whenever Y is an H-contraction [P2]. □

We now discuss the spectral properties of $X^{[*]}X$ valid for H-plus matrices X. Although their infinite-dimensional versions (for $F=\mathbf{C}$) can be found in [KS1,KS2,Bo], for convenience we give full and concise proofs which apply to both the real and the complex cases.

PROPOSITION 3.2. *Let X be an H-plus matrix. Then the following hold:*

(a) $\sigma(X^{[*]}X)\subset\mathbf{R}$;

(b) *$X^{[*]}X$ does not have Jordan blocks of order ≥ 2 corresponding to eigenvalues different from $\mu(X)$, while there are no Jordan blocks of order ≥ 3 corresponding to the eigenvalue $\mu(X)$;*

(c) *The eigenvectors corresponding to the eigenvalues larger (resp. smaller) than $\mu(X)$ are positive (resp. negative);*

(d) *The Jordan blocks of size 2 corresponding to the eigenvalue $\mu(X)$ have the positive sign in the sign characteristic of $\{X^{[*]}X,H\}$;*

(e) *If X is a doubly H-plus matrix, then $\sigma(X^{[*]}X)\subset[0,\infty)$.*

Proof. If X be a strict H-plus matrix, parts (a), (b), (c) and (d) are immediate from Lemma 3.1 and the corresponding parts of Theorem 2.2. If X is an H-plus matrix, then $X^{[*]}X-\mu(X)I$ being H-positive implies parts (a), (b), (c) and (d). It remains to prove part (e).

Let X be an H-plus matrix such that $\mu(X)=0$. If $\lambda\in\sigma(X^{[*]}X)\cap(-\infty,0)$, then there exists u such that $X^{[*]}Xu=\lambda u$ and $[u,u]<0$. Using $X[\mathrm{Ker}\,(X^{[*]}X-\lambda I)]=\mathrm{Ker}\,(XX^{[*]}-\lambda I)$ and writing $v=Xu$, we find $[v,v]>0$ and $XX^{[*]}v=\lambda v$, which contradicts part (c) if $X^{[*]}$ is an H-plus matrix. Hence, if X is a doubly H-plus matrix, then $\sigma(X^{[*]}X)\subset[0,\infty)$. □

The following result has been proved in [M] in the case $F=\mathbf{R}$ and $n=4$. The main result in [M] can be simplified, since its author failed to observe that $\sigma(X^{[*]}X)\subset[0,\infty)$.

PROPOSITION 3.3. *Let X be an $n\times n$ matrix. Then X is a strict H-plus matrix if and only if $X^{[*]}X$ has the following properties:*

(a) $\sigma(X^{[*]}X)\subset[0,\infty)$;

(b) *There exists $\mu>0$ such that there are no Jordan blocks of order exceeding 1 corresponding to the eigenvalues of $X^{[*]}X$ different from μ. The eigenvectors corresponding to the eigenvalues smaller than μ are negative; those corresponding to the eigenvalues larger than μ are positive;*

(c) *There do not exist Jordan blocks of order exceeding 2 corresponding to the eigenvalue μ of $X^{[*]}X$; the blocks of size 2 have the positive sign in the sign characteristic of $\{X^{[*]}X,H\}$.*

X is a non-strict H-plus matrix if and only if $X^{[*]}X$ fails to satisfy at least one of (a), (b) *and* (c), *and has, in addition, the following properties*:

(d) $\sigma(X^{[*]}X) \subset \mathbf{R}$ *and* $0 \in \sigma(X^{[*]}X)$;

(e) *There are no Jordan blocks of order exceeding* 1 *corresponding to the eigenvalues of $X^{[*]}X$ different from* 0. *The eigenvectors corresponding to the negative eigenvalues are negative; those corresponding to the positive eigenvalues are positive*;

(f) *There do not exist Jordan blocks of order exceeding* 2 *corresponding to the zero eigenvalue of $X^{[*]}X$; the blocks of size* 2 *have the positive sign in the sign characteristic of* $\{X^{[*]}X, H\}$.

Proof. A strict H-plus matrix has the above properties (a)-(c) and a non-strict plus matrix has the above properties (d)-(f), as a consequence of Proposition 3.2. Conversely, suppose X is an $n \times n$ matrix having the above properties (a)-(c). Then with no loss of generality, we may assume that

$$H = \operatorname{diag}(\underbrace{+1,\cdots,+1,}_{p-\nu \text{ entries}}\ \underbrace{+1,-1,\cdots,+1,-1,}_{\nu \text{ pairs}}\ \underbrace{-1,\cdots,-1}_{k-\nu \text{ entries}}) \tag{3.3}$$

and

$$X^{[*]}X = \operatorname{diag}(\lambda_1,\cdots,\lambda_{p-\nu}, D(\varepsilon_1,\mu),\cdots,D(\varepsilon_\nu,\mu),\lambda_{p+\nu+1},\cdots,\lambda_n),$$

where $\lambda_j \geq \mu > 0$ if $j = 1,\cdots,p-\nu$; $0 \leq \lambda_j \leq \mu$ if $j = p+\nu+1,\cdots,n$; $\varepsilon_1,\cdots,\varepsilon_\nu > 0$; and

$$D(\varepsilon,\mu) = \begin{bmatrix} \mu+\varepsilon & \varepsilon \\ -\varepsilon & \mu-\varepsilon \end{bmatrix}.$$

Then one easily verifies that for any vector $z = (z_1,\cdots,z_n)$

$$\begin{aligned}
[X^{[*]}Xz,z] &= \sum_{j=1}^{p-\nu} \lambda_j|z_j|^2 - \sum_{j=p+\nu+1}^{n} \lambda_j|z_j|^2 \\
&+ \mu\sum_{j=1}^{\nu}\left(|z_{p-\nu+2j-1}|^2 - |z_{p-\nu+2j}|^2\right) \\
&+ \sum_{j=1}^{\nu} \varepsilon_j|z_{p-\nu+2j-1}+z_{p-\nu+2j}|^2.
\end{aligned}$$

As a result,

$$\begin{aligned}
[X^{[*]}Xz,z] - \mu[z,z] &= \sum_{j=1}^{p-\nu}(\lambda_j-\mu)|z_j|^2 + \sum_{j=p+\nu+1}^{n}(\mu-\lambda_j)|z_j|^2 \\
&+ \sum_{j=1}^{\nu} \varepsilon_j|z_{p-\nu+2j-1}+z_{p-\nu+2j}|^2 \geq 0,
\end{aligned}$$

where $\mu > 0$, which implies that X is a strict H-plus matrix.

Now let X be an $n \times n$ matrix having the properties (d)-(f). Then with no loss of generality, we may assume that H has the form (3.3) and

$$X^{[*]}X = \operatorname{diag}(\lambda_1, \cdots, \lambda_{p-\nu}, D(\varepsilon_1, 0), \cdots, D(\varepsilon_\nu, 0), \lambda_{p+\nu+1}, \cdots, \lambda_n),$$

where $\lambda_j \geq 0$ if $j = 1, \cdots, p-\nu$; $\lambda_j \leq 0$ if $j = p+\nu+1, \cdots, n$; and $\varepsilon_1, \cdots, \varepsilon_\nu > 0$. Then one easily verifies that for any vector $z = (z_1, \cdots, z_n)$

$$[X^{[*]}Xz, z] = \sum_{j=1}^{p-\nu} \lambda_j |z_j|^2 + \sum_{j=p+\nu+1}^{n} (-\lambda_j)|z_j|^2 + \sum_{j=1}^{\nu} \varepsilon_j |z_{p-\nu+2j-1} + z_{p-\nu+2j}|^2 \geq 0,$$

which implies that X is an H-plus matrix with $\mu(X) = 0$. □

Let us now characterize the H-plus matrices allowing an H-polar decomposition. The part pertaining to strict H-plus matrices has been proved before by Krein and Shmul'jan [KS2] in an infinite-dimensional setting under conditions that are satisfied in the finite-dimensional case. In fact, in [KS2] the existence of a unique H-modulus for strict H-plus matrices (and for a certain class of strict H-plus operators) is proved. Indeed, let X be a strict H-plus matrix. Then for $c = \mu(X)^{-1/2}$, the matrix cX is a $(-H)$-contraction and hence has a unique $(-H)$-modulus A_0. But then $A = \mu(X)^{1/2}A_0$ is an H-modulus for X. The uniqueness of A_0 is equivalent to the uniqueness of A. Of course, the proof using Proposition 3.1 breaks down in the infinite dimensional case, and therefore [KS2] needed a different proof for the existence of a unique H-modulus. Note that Condition (c) of Theorem 3.4 is redundant if X is a doubly H-plus matrix.

THEOREM 3.4. *Let X be an H-plus matrix. Then X has an H-polar decomposition if and only if the following conditions are satisfied:*

(a) *$X^{[*]}X$ is invertible, or $0 \in \sigma(X^{[*]}X)$ and there are at least as many linearly independent positive eigenvectors corresponding to the zero eigenvalue as there are Jordan blocks of order 2; in other words, the part of the canonical form of $\{X^{[*]}X, H\}$ corresponding to the zero eigenvalue of $X^{[*]}X$ can be presented in the form*

$$\{O_k \oplus B \oplus \cdots \oplus B,\ G \oplus K \oplus \cdots \oplus K\}, \tag{3.4}$$

where $G = I_p \oplus -I_q$ $(p+q=k)$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus (0)$, $K = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus (1)$, and the summands B and K are repeated m times each in (3.4).

(b) *In case $0 \in \sigma(X^{[*]}X)$, there is a basis $\{e_{ij}\}_{i=0\,j=1}^{m\,l_i}$ $(l_0 = k, l_i = 3$ for $i = 1, \ldots, m)$ in* $\operatorname{Ker}(X^{[*]}X)^n$ *with respect to the canonical form* (3.4) *is achieved and which has the additional property that*

$$\operatorname{Ker} X = \operatorname{span}\{e_{01}, \ldots, e_{0k}\} \oplus \operatorname{span}\{e_{11}, \ldots, e_{m1}\}.$$

(c) *$X^{[*]}X$ does not have negative eigenvalues.*

In particular, a strict H-plus matrix has an H-polar decomposition.

Proof. Lemma 3.1 and Theorem 2.4 imply that any strict H-plus matrix X has an H-polar decomposition $X = UA$ where A has only nonnegative eigenvalues and $\operatorname{Ker} A = \operatorname{Ker} A^2$.

To determine if a given H-plus matrix X has an H-polar decomposition, it suffices to examine the nilpotent part of $X^{[*]}X$ and the part corresponding to the negative eigenvalues. The theorem then is a straightforward application of Theorem 1.1, taking into account that (a) all Jordan blocks of order 2 corresponding to the zero eigenvalue have the positive sign in the sign characteristic of $\{X^{[*]}X, H\}$ and that there are no Jordan blocks of order exceeding 2, and (b) for each negative eigenvalue, all Jordan blocks are of order 1 and the corresponding eigenvectors are negative. □

If H has exactly one positive eigenvalue, then $X^{[*]}X$ and $XX^{[*]}$ are similar if X is an H-plus matrix. Indeed, if $X^{[*]}X$ and $XX^{[*]}$ were to have a different Jordan structure, then their nilpotent parts would not be similar (see [F] for a complete description of the relationships between the Jordan form of AB and that of BA). In view of Proposition 3.3 the only way in which this is possible is when one of the two matrices $X^{[*]}X$ and $XX^{[*]}$ has exactly one Jordan block of order 2 corresponding to the zero eigenvalue while the other matrix has all Jordan blocks of order 1 corresponding to the zero eigenvalue. But then in view of Theorem 3.4 one of X and $X^{[*]}$ would have an H-polar decomposition, whereas the other does not, which is impossible. Indeed, if $X = UA$ is an H-polar decomposition of X, then $X^{[*]} = U^{-1} \cdot UAU^{-1}$ is an H-polar decomposition of $X^{[*]}$.

The similarity of $X^{[*]}X$ and $XX^{[*]}$ for X an H-plus matrix can be used to refine Proposition 3.3. Namely, if H has exactly one positive eigenvalue and X is an $n \times n$ matrix, then X is a doubly H-plus matrix if and only if X has the following properties:

(g) $\sigma(X^{[*]}X) \subset [0, \infty)$;

(h) There exists $\mu \geq 0$ such that there are no Jordan blocks of order exceeding 1 corresponding to the eigenvalues of $X^{[*]}X$ different from μ. The eigenvectors corresponding to the eigenvalues smaller than μ are negative; those corresponding to the eigenvalues larger than μ are positive;

(i) There do not exist Jordan blocks of order exceeding 2 corresponding to the eigenvalue μ of $X^{[*]}X$; the blocks of size 2 have the positive sign in the sign characteristic of $\{X^{[*]}X, H\}$.

Indeed, suppose (g)-(i) hold with $\mu = 0$. (The case $\mu > 0$ implies that X is a strict H-plus matrix and hence a doubly H-plus matrix). Since (g)-(i) imply that X is an H-plus matrix (see (d)-(f) in Proposition 3.3), we have $X^{[*]}X$ and $XX^{[*]}$ similar. But then $XX^{[*]}$ satisfies (g)-(i), and therefore $X^{[*]}$ is an H-plus matrix.

In connection with the remark made two paragraphs ago, observe that in general the matrices $XX^{[*]}$ and $X^{[*]}X$ need not be similar:

Example 3.2 (based on the formula (7.1) in [BR].) Let

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \oplus (1) \oplus (-1),$$

and let

$$X = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

A calculation shows that

$$X^{[*]}X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus (0) \oplus (0),$$

whereas the matrix $XX^{[*]}$ has 1 only in the positions (1,5), (3,4) and (5,2), all other positions in $XX^{[*]}$ being zero. Clearly, $X^{[*]}X$ and $XX^{[*]}$ are not similar (they have different ranks). □

Let $F = \mathbf{R}$ and $n \geq 2$, and let H have exactly one positive eigenvalue. With no loss of generality, we take $H = \operatorname{diag}(1, -1, \cdots, -1)$. Then the set

$$\mathcal{C} = \{x = (x_1, \cdots, x_n) \in \mathbf{R}^n : [x, x] \geq 0,\ x_1 \geq 0\}$$

is a positive cone in $\mathbf{R}^n$, i.e., $u + v \in \mathcal{C}$ and $\lambda u \in \mathcal{C}$ whenever $u, v \in \mathcal{C}$ and $\lambda \in [0, \infty)$. In fact, if $u = (u_1, \cdots, u_n)$ and $v = (v_1, \cdots, v_n)$ belong to $\mathcal{C}$, then Schwarz' inequality implies that

$$\begin{aligned} [u+v, u+v] &= \{u_1^2 - (u_2^2 + \cdots + u_n^2)\} + \{v_1^2 - (v_2^2 + \cdots + v_n^2)\} \\ &+ 2\{u_1 v_1 - (u_2 v_2 + \cdots + u_n v_n)\} \geq 0. \end{aligned}$$

The cone $\mathcal{C}$ has the following properties [BP,Kr]:

1. The interior of $\mathcal{C}$, relative to the usual topology of $\mathbf{R}^n$, coincides with the set of positive vectors in $\mathbf{R}^n$ with positive first component, and it is obviously nonempty.

2. $e = (1, 0, \cdots, 0)$ is an order unit relative to the partial order of $\mathbf{R}^n$ generated by the cone $\mathcal{C}$ (i.e., $x \geq y$ if and only if $x - y \in \mathcal{C}$). Indeed, if $x = (x_1, \cdots, x_n) \in \mathcal{C}$, then $\lambda_+ e - x \in \mathcal{C}$ and $x - \lambda_- e \in \mathcal{C}$ where $\lambda_\pm = x_1 \pm \sqrt{x_2^2 + \cdots + x_n^2}$. As a result, a real $n \times n$ matrix X maps the interior of $\mathcal{C}$ into itself if and only if $(Xe)_1 > 0$ and Xe is a positive vector.

3. The dual cone, i.e., the set of all vectors $x \in \mathbf{R}^n$ satisfying $[x, y] \geq 0$ for every $y \in \mathcal{C}$ (where we note that we have defined duality with respect to the indefinite scalar product rather than with respect to the usual scalar product of $\mathbf{R}^n$), coincides with $\mathcal{C}$. Indeed, if $[x, y] \geq 0$ for every $y \in \mathcal{C}$, then (using $y = (\sqrt{x_2^2 + \cdots + x_n^2}, x_2, \cdots, x_n)$ if $(x_2, \cdots, x_n)$ is nontrivial, and using $y = e$ if $x_2 = \cdots = x_n = 0$) we have $x_1 \geq \sqrt{x_2^2 + \cdots + x_n^2}$. Thus $x \in \mathcal{C}$. Conversely, if $x, y \in \mathcal{C}$, then $[x, y] = x_1 y_1 - (x_2 + \cdots + x_n y_n) = x_1 y_1 - [x_2^2 + \cdots + x_n^2]^{1/2}[y_2^2 + \cdots + y_n^2]^{1/2} \geq 0$. As a result, if a real $n \times n$ matrix X satisfies $X[\mathcal{C}] \subset \mathcal{C}$, this is also true for $X^{[*]}$.

PROPOSITION 3.5. *Let $F = \mathbf{R}$ and $H = \operatorname{diag}(1, -1, \cdots, -1)$. Then a real $n \times n$ matrix $X = [X_{ij}]_{i,j=1}^n$ satisfies $X[\mathcal{C}] \subset \mathcal{C}$ if and only if it is a doubly H-plus matrix with $X_{11} \geq 0$.*

Proof. Let X be a real $n \times n$ matrices leaving invariant the cone $\mathcal{C}$. Suppose $u \in \mathbf{R}^n$ and $[u, u] \geq 0$. If $u_1 \geq 0$, then $u \in \mathcal{C}$ and hence $Xu \in \mathcal{C}$, so that $[Xu, Xu] \geq 0$. On the other hand, if $u_1 \leq 0$, then $(-u) \in \mathcal{C}$ and hence $(-Xu) \in \mathcal{C}$, so that $[Xu, Xu] = [-Xu, -Xu] \geq 0$. Thus X is an H-plus matrix. Further, $X^{[*]}[\mathcal{C}] \subset \mathcal{C}$ (see item 3 above), and therefore X is a doubly H-plus matrix. Finally, since X_{11} is the first component of Xe and $Xe \in \mathcal{C}$, we get $X_{11} \geq 0$.

Conversely, let X be a doubly H-plus matrix with $X_{11} \geq 0$. First note that every $x = (x_1, \cdots, x_n) \in \mathcal{C}$ can be written as the sum of three vectors from the boundary of $\mathcal{C}$:

$$x = \frac{x_1 - \sigma}{2} \begin{pmatrix} 1 \\ q_2 \\ \vdots \\ q_n \end{pmatrix} + \frac{x_1 - \sigma}{2} \begin{pmatrix} 1 \\ -q_2 \\ \vdots \\ -q_n \end{pmatrix} + \begin{pmatrix} \sigma \\ x_2 \\ \vdots \\ x_n \end{pmatrix},$$

where $\sigma = \sqrt{x_2^2 + \cdots + x_n^2}$ and $(q_2, \cdots, q_n) \in \mathbf{R}^{n-1}$ has unit length. Thus in order to prove $X[\mathcal{C}] \subset \mathcal{C}$ it suffices to prove that $Xu \in \mathcal{C}$ for every vector u of the form $u = \operatorname{col}(1, q)$ where $q \in \mathbf{R}^{n-1}$ has unit length. First of all, $X^{[*]}$ being an H-plus matrix implies $[X^{[*]}e, X^{[*]}e] \geq 0$. With $X_{11} \geq 0$ this yields $X_{11} \geq [X_{12}^2 + \cdots + X_{1n}^2]^{1/2}$. Now applying X to $u = \operatorname{col}(1, q)$ with $q = (q_2, \cdots, q_n) \in \mathbf{R}^{n-1}$ having unit length, we obtain $[Xu, Xu] \geq 0$ because X is an H-plus matrix, as well as

$$[Xu]_1 = X_{11} + X_{12}q_2 + \cdots + X_{1n}q_n \geq X_{11} - \sqrt{X_{12}^2 + \cdots + X_{1n}^2} \geq 0,$$

which implies $Xu \in \mathcal{C}$. Hence $X[\mathcal{C}] \subset \mathcal{C}$. □

The next result is immediate from the Perron-Fröbenius theory (e.g., [BP]).

PROPOSITION 3.6. *Suppose $H = \operatorname{diag}(1, -1, \cdots, -1)$, X is a nontrivial doubly H-plus matrix, and σ_0 is the sign of X_{11}. Then the following statements hold:*

(a) *The spectral radius ρ of X is an eigenvalue of $\sigma_0 X$ and there exists a corresponding eigenvector in $\mathcal{C}$.*

(b) *Let m be the order of the largest Jordan block corresponding to the eigenvalue $\sigma_0\rho$. Then the Jordan blocks corresponding to any eigenvalue λ of X of absolute value ρ have orders not exceeding m.*

(c) *Let either $X_{11}^2 > X_{21}^2 + \cdots + X_{n1}^2$ or $X_{11}^2 > X_{12}^2 + \cdots + X_{1n}^2$. Then*

  (c_1) $\rho > 0$;

  (c_2) *$\sigma_0\rho$ is an algebraically simple eigenvalue of both X and $X^{[*]}$ to which correspond positive eigenvectors;*

(c_3) *X and $X^{[*]}$ have no other eigenvalues on the spectral circle $|z| = \rho$.*

Proof. Parts (a) and (b) follow from Theorem 1.3.2 of [BP]. Part (c) follows from Theorem 1.3.26 of [BP], because under the additional assumption either Xe or $X^{[*]}e$ belongs to the interior of $\mathcal{C}$. □

If $F = \mathbf{R}$ and $H = \operatorname{diag}(1, -1, \cdots, -1)$, then any real $n \times n$ matrix X leaving invariant the cone $\mathcal{C}$ has an H-polar decomposition $X = UA$ where U is H-unitary, A is H-selfadjoint, and both U and A leave invariant $\mathcal{C}$, unless $X^{[*]}X$ is nilpotent and different from the zero matrix. This follows almost immediately from Theorem 3.4 and Proposition 3.5, provided we can prove, in the cases where X allows an H-polar decomposition, that one may choose U such that U (and hence also $U^{-1} = U^{[*]}$) leaves invariant $\mathcal{C}$. However, any H-unitary matrix U has the property that either $U_{11} \geq 1$ or $U_{11} \leq -1$; thus an H-unitary matrix U leaves invariant $\mathcal{C}$ if and only if $U_{11} \geq 1$. If $U_{11} \leq 1$ for the H-unitary matrix U in the H-polar decomposition $X = UA$, we can simply replace $X = UA$ by $X = (-U)(-A)$. Since in that case both X and $(-U)^{-1}(= (-U)^{[*]})$ map $\mathcal{C}$ into itself, this is also the case for $(-A)$.

Appendix: Proof of (3.2) in the real case.

Throughout the appendix (with the exception of Corollary 3.7) we assume that $F = \mathbf{R}$, and that X is a real H-plus matrix of order n. Without loss of generality we can (and do) assume that the pair $\{X^{[*]}X, H\}$ is in the canonical form (see, e.g., Theorem I.5.3 in [GLR], or Section 2 of [BMRRR1]).

We denote by $J_k(\lambda \pm i\mu)$ the $k \times k$ real Jordan block with complex conjugate eigenvalues $\lambda \pm i\mu$ (the integer k is necessarily even). More explicitly, $J_k(\lambda \pm i\mu)$ is a block $k/2 \times k/2$ matrix with 2×2 blocks, where the block diagonal consists of the blocks $\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$, the first block superdiagonal consists of the blocks I_2, and all other blocks are zeros.

Observe that all eigenvalues of $X^{[*]}X$ are real. Indeed, if $\lambda \pm i\mu$ is a pair of non-real complex conjugate eigenvalues of $X^{[*]}X$, then the pair $\{X^{[*]}X, H\}$ contains a pair of blocks $\{J_k(\lambda \pm i\mu), Q_k\}$. Denote by e_p the standard unit vector having 1 in the p-th position and zeros elsewhere. If $m = k/2$ is odd, then, denoting $Z = J_k(\lambda \pm i\mu)^{[*]}J_k(\lambda \pm i\mu)$, we have

$$[Ze_m, e_m] = -\mu, \quad [Ze_{m+1}, e_{m+1}] = \mu, \quad [e_m, e_m] = [e_{m+1}, e_{m+1}] = 0;$$

if m is even, then

$$[Z(e_{m-1} + e_{m+1}), e_{m-1} + e_{m+1}] = -2\mu, \quad [Z(e_m + e_{m+2}), e_m + e_{m+2}] = 2\mu,$$

$$[e_{m-1} + e_{m+1}, e_{m-1} + e_{m+1}] = [e_m + e_{m+2}, e_m + e_{m+2}] = 0;$$

and in both cases a contradiction to X being an H-plus matrix is obtained.

Assume that $\{X^{[*]}X, H\}$ contains a pair of blocks $\{J_{2m}(\lambda), \varepsilon Q_{2m}\}$. The set of vectors $x = \sum_{j=1}^{m} x_j e_j \in \mathbf{R}^{2m}$ satisfying the equation $\langle \varepsilon Q_{2m} x, x \rangle = 1$ is described by the formula

$$2\varepsilon(x_1 x_{2m} + x_2 x_{2m-1} + \cdots + x_m x_{m+1}) = 1. \tag{3.5}$$

A calculation shows that if (3.5) holds, then

$$\langle \varepsilon Q_m J_{2m}(\lambda)^{[*]} J_{2m}(\lambda) x, x \rangle = \lambda + 2\varepsilon(x_2 x_{2m} + \cdots + x_m x_{m+2}) + \varepsilon x_{m+1}^2.$$

Since X is H-plus, the equality

$$\lambda + 2\varepsilon(x_2 x_{2m} + \cdots + x_m x_{m+2}) + \varepsilon x_{m+1}^2 \geq 0 \tag{3.6}$$

holds for every x satisfying (3.5). If $m > 1$, then letting $x_3 = \cdots = x_{2m-1} = 0$, $x_{2m} = 1$, x_2 arbitrary (and x_1 determined from (3.5)), it follows that $\lambda + 2\varepsilon x_2 \geq 0$ for all $x_2 \in \mathbf{R}$, which is impossible. Thus, $m = 1$. Now (3.6) takes the form $\lambda + \varepsilon x_2^2 \geq 0$, and since this inequality must be satisfied for all non-zero x_2 (because for any such x_2 the value of x_1 can be determined from (3.5)), we obtain $\varepsilon = +1$ and $\lambda \geq 0$. Conclusion: the even size Jordan blocks of $X^{[*]}X$ must have size 2, their signs in the sign characteristic are all $+1$, and their eigenvalues are all nonnegative.

Analogously we verify that $X^{[*]}X$ cannot have odd size Jordan blocks of size larger than 1. Thus:

$$X^{[*]}X = J_2(\lambda_1) \oplus \cdots \oplus J_2(\lambda_k) \oplus (\mu_1) \oplus \cdots \oplus (\mu_s),$$
$$H = Q_2 \oplus \cdots \oplus Q_2 \oplus (\varepsilon_1) \oplus \cdots \oplus (\varepsilon_s),$$

where $\lambda_j \geq 0$; $\mu_1, \cdots, \mu_s$ are real, and $\varepsilon = \pm 1$. Assume first $k \neq 0$ (i.e., $X^{[*]}X$ is not diagonalizable). Let $u = (x_1, y_1, \cdots, x_k, y_k, z_1, \cdots, z_s)^T$, where x_j, y_j, z_j are real numbers. We have

$$[u,u] = \sum_{i=1}^{k} 2x_j y_j + \sum_{j=1}^{s} \varepsilon_j z_j^2, \tag{3.7}$$

$$[X^{[*]}Xu, u] = \sum_{i=1}^{k} (2\lambda_i x_i y_i + y_i^2) + \sum_{j=1}^{s} \varepsilon_j \mu_j z_j^2. \tag{3.8}$$

Therefore, if $[u,u] = 1$, then

$$[X^{[*]}Xu, u] = \lambda_k + \sum_{i=1}^{k-1} 2(\lambda_i - \lambda_k) x_i y_i + \sum_{i=1}^{k} y_i^2 + \sum_{j=1}^{s} (\mu_j - \lambda_k)\varepsilon_j z_j^2 \geq 0 \tag{3.9}$$

by the H-plus property of X. Since the parameters $x_1, \cdots, x_{k-1}, y_1 \cdots, y_k, z_1, \cdots, z_s$ can be chosen arbitrarily in (3.7) (as long as $y_k \neq 0$, to ensure existence of $x_k \in \mathbf{R}$ such that $[u,u] = 1$), we conclude that $\lambda_i - \lambda_k = 0$ for $i = 1, \cdots, k-1$ and $(\mu_j - \lambda_k)\varepsilon_j \geq 0$ for $j = 1, \cdots, s$. In other words, all λ_i's are equal to the same number, call it λ, and $\varepsilon_j = +1$ (resp. $\varepsilon_j = -1$) for every eigenvalue $\mu_j > \lambda$ (resp. $\mu_j < \lambda$). The formula (3.9) shows also that $\mu(X) = \lambda$, where $\mu(X)$ is defined by (3.1); indeed, it suffices to take $z_j = 0$ and y_i as close to zero as we wish in (3.9). Now

$$[X^{[*]}Xu, u] - \mu(X)[u,u] = \sum_{i=1}^{k} y_i^2 + \sum_{j=1}^{s} (\mu_j - \lambda)\varepsilon_j z_j^2 \geq 0$$

for all u, which proves (3.2) in the case when $k \neq 0$.

Finally, assume $k = 0$, i.e., $X^{[*]}X$ is diagonalizable. Write

$$X^{[*]}X = (\mu_1) \oplus \cdots \oplus (\mu_s), \quad H = (\varepsilon_1) \oplus \cdots \oplus (\varepsilon_s),$$

where $\varepsilon_j = 1$ for $j = 1, \cdots, p$; $\varepsilon_j = -1$ for $j = p+1, \cdots, s$ $(1 \leq p < s)$. For $u = (x_1, \cdots, x_s)^T$ we have

$$[u,u] = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_s^2, \tag{3.10}$$

$$[X^{[*]}Xu, u] = \mu_1 x_1^2 + \cdots + \mu_p x_p^2 - \mu_{p+1} x_{p+1}^2 - \cdots - \mu_s x_s^2. \tag{3.11}$$

If $[u, u] = 1$, then for a fixed index j, $(1 \le j \le p)$, we have

$$[X^{[*]}Xu, u] = \mu_j + (\mu_k - \mu_j)x_2^2 + \cdots + (\mu_p - \mu_j)x_p^2 + (\mu_j - \mu_{p+1})x_{p+1}^2 + \cdots + (\mu_j - \mu_s)x_s^2, \tag{3.12}$$

which must be nonnegative. Letting here $x_2 = \cdots = x_p = 0$, and observing that $x_{p+1}, \cdots, x_s$ can attain arbitrary real values independently, it follows that $\mu_j \ge 0$ and $\mu_j \ge \mu_k$ for $j = 1, \cdots, p$ and $k = p+1, \cdots, s$. Applying (3.12) with $\mu_j = \min(\mu_1, \cdots, \mu_p)$ yields the value of $\mu(X)$: $\mu(X) = \min\{\mu_j \in \sigma(X^{[*]}X) |$ there is a nonnegative eigenvector of $X^{[*]}X$ corresponding to $\mu_j\}$. Using this formula for $\mu(X)$, the equalities (3.10) and (3.11) easily yield the inequality (3.2) for every $u \in \mathbf{R}^n$. This concludes the proof of formula (3.2) in the real case.

We remark that the above proof can be adapted to the complex case as well.

As a byproduct of the above proof, a characterization of $\mu(X)$ is obtained:

COROLLARY 3.7. ($F = \mathbf{R}$ or $F = \mathbf{C}$.) *Let X be an H-plus matrix. Then $\mu(X)$ coincides with the minimal eigenvalue of $X^{[*]}X$ for which there exists an eigenvector v satisfying $[v, v] \ge 0$.*

4 Indefinite Scalar Products with Only One Positive Square

In this section $H = H^*$ is an invertible $n \times n$ matrix with only one positive eigenvalue and $n-1$ negative eigenvalues. We are interested in H-polar decompositions.

As H has only one positive eigenvalue the possibilities for H-selfadjoint matrices A are rather restricted. We shall list them below in terms of the canonical forms (as in Theorem 2.1 of [BMRRR1], or Section I.3.2 of [GLR], for example) of $\{A, H\}$. Let $F = \mathbf{C}$. Given an H-selfadjoint matrix A, there exists an invertible matrix S such that either one of the following six alternatives occurs:

(a)
$$S^{-1}AS = \operatorname{diag}(\lambda, \bar\lambda) \oplus \operatorname{diag}(\lambda_i^2)_{i=1}^{n-2},$$

where $\lambda \notin \mathbf{R}$, $\lambda_i \in \mathbf{R}$ (not necessarily distinct), and

$$S^*HS = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus -I_{n-2}.$$

In this case $(S^{-1}AS)^2$ is given by

$$S^{-1}A^2S = \operatorname{diag}(\lambda^2, \bar\lambda^2) \oplus \operatorname{diag}(\lambda_i^2)_{i=1}^{n-2}.$$

It is important to distinguish between $\lambda \in i\mathbf{R}$ and $\lambda \notin i\mathbf{R}$. In the first case $\lambda^2 = \bar\lambda^2 < 0$.

(b)
$$S^{-1}AS = (\lambda) \oplus \operatorname{diag}(\lambda_i)_{i=1}^{n-1},$$
with $\lambda \in \mathbf{R}$, $\lambda_i \in \mathbf{R}$ not necessarily distinct, and
$$S^*HS = (1) \oplus -I_{n-1}.$$
In this case
$$S^{-1}A^2S = (\lambda^2) \oplus \operatorname{diag}(\lambda_i^2)_{i=1}^{n-1}.$$

(c)
$$S^{-1}AS = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \oplus \operatorname{diag}(\lambda_i)_{i=1}^{n-2},$$
where $\lambda \in \mathbf{R} \setminus \{0\}$, $\lambda_i \in \mathbf{R}$, not necessarily distinct, and
$$S^*HS = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus -I_{n-2}.$$
In this case,
$$S^{-1}A^2S = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix} \oplus \operatorname{diag}(\lambda_i^2)_{i=1}^{n-2}.$$

(d)
$$S^{-1}AS = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \oplus \operatorname{diag}(\lambda_i)_{i=1}^{n-3},$$
where $\lambda \in \mathbf{R} \setminus \{0\}$, $\lambda_i \in \mathbf{R}$, not necessarily distinct, and
$$S^*HS = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \oplus -I_{n-3}.$$
In this case,
$$S^{-1}A^2S = \begin{bmatrix} \lambda^2 & 2\lambda & 1 \\ 0 & \lambda^2 & 2\lambda \\ 0 & 0 & \lambda^2 \end{bmatrix} \oplus \operatorname{diag}(\lambda_i^2)_{i=1}^{n-3}.$$

(e)
$$S^{-1}AS = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \oplus \operatorname{diag}(\lambda_i)_{i=1}^{n-2},$$
where $\lambda_i \in \mathbf{R}$, not necessarily distinct, and
$$S^*HS = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus -I_{n-2}.$$
In this case,
$$S^{-1}A^2S = O_2 \oplus \operatorname{diag}(\lambda_i^2)_{i=1}^{n-2}.$$

(f)

$$S^{-1}AS = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \oplus \operatorname{diag}(\lambda_i)_{i=1}^{n-3},$$

where $\lambda \in \mathbf{R} \setminus \{0\}$, $\lambda_i \in \mathbf{R}$, not necessarily distinct, and

$$S^*HS = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \oplus -I_{n-3}.$$

In this case,

$$S^{-1}A^2S = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \operatorname{diag}(\lambda_i^2)_{i=1}^{n-3}.$$

In the case $F = \mathbf{R}$, the same classification (a)-(f) is valid, with the only exception that in (a), $\operatorname{diag}(\lambda, \bar{\lambda})$ is replaced by $\begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}$, $\lambda, \mu \in \mathbf{R}$, $\mu \neq 0$.

We now state the main result of this section.

THEOREM 4.1. ($F = \mathbf{C}$ *or* $F = \mathbf{R}$) *Let* $H = H^*$ *be an invertible* $n \times n$ *matrix with one positive eigenvalue. An* $n \times n$ *matrix* X *allows* H*-polar decomposition if and only if* X *has precisely one of the following mutually exclusive properties*:

(i) $X^{[*]}X$ *has a non-real eigenvalue*,

(ii) $X^{[*]}X$ *has a negative eigenvalue* λ *of algebraic and geometric multiplicity two, and* H *is indefinite on* $\operatorname{Ker}(X^{[*]}X - \lambda)$,

(iii) $X^{[*]}X$ *has all its eigenvalues in* $\{\lambda \in \mathbf{R} |\ \lambda \geq 0\}$, *and there is a positive* λ *such that* $\operatorname{Ker}(X^{[*]}X - \lambda)^n$ *is* H*-indefinite*,

(iv) $X^{[*]}X$ *has all its eigenvalues in* $\{\lambda \in \mathbf{R} |\ \lambda \geq 0\}$, *is diagonalizable and* $\operatorname{Ker} X$ *contains a* k*-dimensional* H*-nonpositive subspace and a* p*-dimensional* H*-nonnegative subspace, where* k (*respectively*, p) *is the number of negative* (*respectively, positive*) *signs in the sign characteristic of* $\{X^{[*]}X, H\}$ *corresponding to the zero eigenvalue of* $X^{[*]}X$ (*observe that* $p \leq 1$),

(v) $X^{[*]}X$ *has all its eigenvalues in* $\{\lambda \in \mathbf{R} |\ \lambda \geq 0\}$,

$$\operatorname{rank}(X^{[*]}X)|_{\operatorname{Ker}(X^{[*]}X)^n} = 1, \quad \dim \operatorname{Ker} X^{[*]}X \geq 2,$$

in the canonical form of $\{X^{[*]}X, H\}$ *there is a block of the form*

$$\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \right),$$

and $\operatorname{Ker} X$ *is the direct sum of an* $(r-3)$*-dimensional strictly* H*-negative subspace and the subspace* $[H(\operatorname{Ker}(X^{[*]}X))]^{\perp}$, *where* $r = \dim \operatorname{Ker}[(X^{[*]}X)^n]$.

Proof. We give the proof here only in the complex case. Suppose $X = UA$ where U is H-unitary and A is H-selfadjoint. Then A is as in one of the cases (a)-(f) described above. In case (a) precisely one of (i) and (ii) in the statement of the theorem holds. In case (b), (iv) holds, in cases (c) and (d), (iii) holds. In case (e), (iv) holds, and finally, in case (f), (v) holds.

Conversely, suppose precisely one of (i)-(v) holds for X. Because of Lemma 4.3 of [BM-RRR1], we may assume either $\sigma(X^{[*]}X) = \{\lambda\}$ with $\lambda \in \mathbf{R}$ or $\sigma(X^{[*]}X) = \{\lambda, \bar{\lambda}\}$ with $\lambda \notin \mathbf{R}$, and H has at most one positive eigenvalue.

Assume $\sigma(X^{[*]}X) = \{\lambda, \bar{\lambda}\}$ with $\lambda \notin \mathbf{R}$. As H has only one positive eigenvalue, in this case there is an S such that

$$S^{-1}X^{[*]}XS = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix}, \quad S^*HS = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let μ be such that $\mu^2 = \lambda$, and put $A = S \text{ diag } (\mu, \bar{\mu})\ S^{-1}$. Then A is H-selfadjoint and $A^2 = X^{[*]}X$. So X admits H-polar decomposition by Theorem 4.1(e) of [BMRRR1].

Assume $\sigma(X^{[*]}X) = \{\lambda\}$, $\lambda < 0$, then (ii) holds. So there is an S such that

$$S^{-1}X^{[*]}XS = \text{diag}\,(\lambda, \lambda), \quad S^*HS = \text{diag}\,(1, -1).$$

Then there is also a V such that

$$V^{-1}X^{[*]}XV = X^{[*]}X = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}, \quad V^*HV = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Take A as defined by

$$A = V \text{ diag } \left(i\sqrt{-\lambda}, -i\sqrt{-\lambda}\right) V^{-1}.$$

Then A is H-selfadjoint and $A^2 = \lambda I = X^{[*]}X$.

Now assume $\sigma(X^{[*]}X) = \{\lambda\}$, $\lambda > 0$, and $\text{Ker}\,(X^{[*]}X - \lambda)^n$ is H-indefinite, i.e., (iii) holds. Then for some invertible S there are three possibilities:

$$S^{-1}X^{[*]}XS = \lambda I_n, \quad S^*HS = (1) \oplus -I_{n-1},$$

or

$$S^{-1}X^{[*]}XS = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \oplus \lambda I_{n-2}, \quad S^*HS = \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \oplus -I_{n-2},$$

or

$$S^{-1}X^{[*]}XS = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} \oplus \lambda I_{n-3}, \quad S^*HS = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \oplus -I_{n-3}.$$

In the first case, put $A = \sqrt{\lambda}\, I_n$. Then $A^2 = X^{[*]}X$ and A is H-selfadjoint. In the second case, put

$$A = S \left(\begin{bmatrix} \sqrt{\lambda} & \dfrac{1}{2\sqrt{\lambda}} \\ 0 & \sqrt{\lambda} \end{bmatrix} \oplus \sqrt{\lambda}\, I_{n-2} \right) S^{-1}.$$

Then A is H-selfadjoint and $A^2 = X^{[*]}X$. In the third case, put

$$A = S\left(\begin{bmatrix} \sqrt{\lambda} & \frac{1}{2\sqrt{\lambda}} & \frac{-1}{8\sqrt{\lambda^3}} \\ 0 & \sqrt{\lambda} & \frac{1}{2\sqrt{\lambda}} \\ 0 & 0 & \sqrt{\lambda} \end{bmatrix} \oplus \sqrt{\lambda}\, I_{n-3}\right) S^{-1}.$$

Then A is H-selfadjoint and $A^2 = X^{[*]}X$.

Next, assume $\sigma(X^{[*]}X) = \{\lambda\}$, $\lambda > 0$, and $\operatorname{Ker}(X^{[*]}X - \lambda)^n$ is H-definite. Then

$$S^{-1}X^{[*]}XS = \lambda I_n, \quad S^*HS = -I_n$$

for some S (recall that in this part of the proof we assume that H has at most one positive eigenvalue; in particular, the case of a negative definite H is not excluded). Taking $A = \sqrt{\lambda}\, I_n$ we find that A is H-selfadjoint and $A^2 = X^{[*]}X$.

Finally, assume $\sigma(X^{[*]}X) = \{0\}$. Then either (iv) or (v) holds. In case (iv) holds we can apply Theorem 5.3 of [BMRRR2] to show that X admits an $(H, 0, n)$-polar decomposition. So it remains to consider case (v).

In case (v) holds and $\sigma(X^{[*]}X) = \{0\}$, observe that $\operatorname{rank} X^{[*]}X = 1$, $\dim \operatorname{Ker} X^{[*]}X \geq 2$ and $X^{[*]}X$ has one Jordan block of order 2, and we have $r = n \geq 3$. Put $M = \operatorname{Ker} X$ and denote $\operatorname{Ker} X^{[*]}X$ by N. Then $N \cap (HN)^{\perp} = (HN)^{\perp}$, as both are one dimensional, and $(HN)^{\perp} \subset N$ (these facts can be easily verified using the canonical form of $\{X^{[*]}X, H\}$). Because of the hypothesis on $\operatorname{Ker} X$, we also have $(HN)^{\perp} \subset M$. Let e_0 be a vector such that $\operatorname{span}\{e_0\} = (HN)^{\perp}$ and choose a basis $e_0, e_1, \ldots, e_{n-3}$ for M such that

$$\begin{aligned} \langle He_i, e_j\rangle &= 0 \quad \text{for } i \neq j \quad (\text{and for } i = j = 0) \\ \langle He_i, e_i\rangle &= -1 \text{ for } i = 1, \ldots, n-3. \end{aligned}$$

As one sees from the canonical form of $\{X^{[*]}X, H\}$ (or proves quite easily directly), $\operatorname{Im} X^{[*]}X = (HN)^{\perp}$. Choose any f_0 such that $X^{[*]}Xf_0 = e_0$, $\langle Hf_0, e_0\rangle = -1$, and $\langle Hf_0, f_0\rangle = 0$. (Note that this choice is possible by the hypothesis (v) and by the canonical form of $\{X^{[*]}X, H\}$.)

Next, we choose a vector $g \in (HM)^{\perp} \cap N$ such that $g \notin M$. To see that such a choice is possible, argue as follows. Since

$$\operatorname{Ker} X \subset \operatorname{Ker} X^{[*]}X,$$

and $(\dim \operatorname{Ker} X^{[*]}X) - (\dim \operatorname{Ker} X) = 1$, there is $g_0 \in (\operatorname{Ker} X^{[*]}X) \setminus (\operatorname{Ker} X)$. Put

$$g = g_0 + \sum_{j=1}^{n-3} \langle Hg_0, e_j\rangle e_j.$$

Clearly, $g \in \operatorname{Ker} X^{[*]}X$ and $g \notin \operatorname{Ker} X$. Also, for $i = 1, \ldots, n-3$,

$$\langle Hg, e_i\rangle = \langle Hg_0, e_i\rangle - \langle Hg_0, e_i\rangle = 0,$$

and

$$\langle Hg, e_0\rangle = 0,$$

because $e_0 \in (H \operatorname{Ker} X^*X)^\perp$. It follows that $g \in (HM)^\perp$.

We note that $\langle Hg, g\rangle < 0$, otherwise span $\{e_0, g\}$ would be a two-dimensional H-nonnegative subspace, and an H-nonnegative subspace can have dimension at most one as H has only one positive eigenvalue. Scaling g we may assume that $\langle Hg, g\rangle = -1$.

Consider

$$f = f_0 + \sum_{j=1}^{n-3} \langle Hf_0, e_j\rangle e_j + \langle Hf_0, g\rangle g.$$

Then $X^{[*]}Xf = X^{[*]}Xf_0 = e_0$, and for $i \geq 1$,

$$\begin{aligned} \langle Hf, e_i\rangle &= \langle Hf_0, e_i\rangle + \sum_{j=1}^{n-3} \langle Hf_0, e_j\rangle \langle He_j, e_i\rangle \\ &+ \langle Hf_0, g\rangle \langle Hg, e_i\rangle = 0 \end{aligned}$$

as $\langle He_j, e_j\rangle = -\delta_{ij}$ and $\langle Hg, e_i\rangle = 0$, since $g \in (HM)^\perp$. Likewise $\langle Hf, g\rangle = 0$. Furthermore, we have $\langle Hf, e_0\rangle \neq 0$. Indeed, suppose $\langle Hf, e_0\rangle = 0$. Then (since $N = M + \operatorname{span}\{g\}$) $\langle Hf, x\rangle = 0$ for all $x \in \operatorname{Ker} X^{[*]}X$, so $f \in (H \operatorname{Ker} X^{[*]}X)^\perp = \operatorname{span}\{e_0\}$. But then, $X^{[*]}Xf = e_0 = 0$ and $N = \operatorname{Ker} X^{[*]}X$. Contradiction. As $\langle Hf, e_0\rangle = \langle Hf, X^{[*]}Xf\rangle = \langle HX^{[*]}Xf, f\rangle$ and $HX^{[*]}X \leq 0$ (considering the canonical form of $\{X^{[*]}X, H\}$ the latter fact is easily seen), we have $\langle Hf, e_0\rangle < 0$. Observe also $\langle Hf, e_0\rangle = \langle Hf_0, e_0\rangle = -1$. Next,

$$\langle Hf, f\rangle = \langle Hf, f_0\rangle = \langle Hf_0, f_0\rangle = 0.$$

Take as a basis in F^n the vectors $e_0, g, f, e_1, e_1, \ldots, e_{n-3}$, and let S be the $n \times n$ matrix with these vectors as its columns in the order in which they appear here. Then

$$S^{-1}X^{[*]}XS = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus 0,$$

and

$$S^*HS = \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \oplus -I_{n-3}.$$

Take A as follows:

$$A = S \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \oplus 0 \right) S^{-1}.$$

Then $A^2 = X^{[*]}X$, and

$$\operatorname{Ker} A = \operatorname{span}\{e_0, e_1, \ldots, e_{n-3}\} = \operatorname{Ker} X.$$

Clearly, also A is H-selfadjoint. By Theorem 4.1(e) of [BMRRR1], X allows an H-polar decomposition. □

5 Polar Decompositions with Special Unitary Factors

The H-polar decomposition (1.1) will be called *special* if $\det U = 1$ and *connected* if U belongs to the connected component of I in the group of H-unitary $n \times n$ matrices (over F). If $F = \mathbf{C}$, then every H-polar decomposition is connected; if H is definite and $F = \mathbf{R}$, then the classes of connected and of special H-polar decompositions coincide. In this section we study special and connected H-polar decompositions.

First, we find the possible values of $\det U$ in H-polar decompositions $X = UA$ of a given $n \times n$ matrix X. Two H-polar decompositions $X = UA$ and $X = \tilde{U}\tilde{A}$ of X are called *equivalent* if the matrices A and $\tilde{A}$ are H-unitarily similar, i. e., $\tilde{A} = W^{-1}AW$ for some H-unitary matrix W. A complete description of the equivalence classes is given in [BMRRR1]. Clearly, if two H-polar decompositions are equivalent, then they are (H, p, q)-polar decompositions for the same (p, q), but the converse is false in general: Two (H, p, q)-polar decompositions with the same (p, q) need not be equivalent. Fix an H-polar decomposition $X = \tilde{U}\tilde{A}$. By Proposition 7.1 of [BMRRR1], all values of $\det U$ in equivalent H-polar decompositions $X = UA$ are given by the formula

$$\{\det(VW) \cdot \det \tilde{U} |\ V \text{ and } W \text{ are } H\text{-unitary such that } X = VXW\}. \tag{5.1}$$

It is convenient to study the formula (5.1) in two steps, by considering separately premultiplication and postmultiplication by H-unitary matrices.

We fix an invertible hermitian $n \times n$ matrix H (over F). For any $X \in F^{n \times n}$, denote by $\mathcal{U}_\ell(X)$ (resp., $\mathcal{U}_r(X)$) the group of H-unitary matrices U such that $UX = X$ (resp., $XU = X$). Denote also by $\mathcal{DU}_\ell(X)$ (resp., $\mathcal{DU}_r(X)$) the set $\{\det U |\ U \in \mathcal{U}_\ell(X)\}$ (resp., $\{\det U |\ U \in \mathcal{U}_r(X)\}$) of values of the determinant function on $\mathcal{U}_\ell(X)$ (resp., $\mathcal{U}_r(X)$). As usual, we distinguish the two cases $F = \mathbf{R}$ and $F = \mathbf{C}$. We denote by $d(V)$ the *defect* of a subspace V with respect to $[\,\cdot\,,\,\cdot\,]$ (the indefinite scalar product induced by H), i.e., the number of zero eigenvalues of the Gram matrix (relative to $[\,\cdot\,,\,\cdot\,]$) of any basis in V. The defect is zero precisely when the subspace is *H-nondegenerate*. It is well known that a subspace $\mathcal{M}$ is H-nondegenerate if and only if its orthogonal companion

$$\mathcal{M}^{[\perp]} = \{x \in F^n |\ [x, y] = 0 \text{ for all } y \in \mathcal{M}\}$$

is actually a direct complement of $\mathcal{M}$ in F^n.

The invertibility of H easily implies

$$\dim V + d(V) \le n$$

for every subspace $V \subset F^n$.

THEOREM 5.1.

(i) $(F = \mathbf{C})$. *$\mathcal{DU}_\ell(X)$ coincides with the unit circle if and only if*

$$\dim(\operatorname{Im} X) + d(\operatorname{Im} X) < n; \tag{5.2}$$

otherwise, $\mathcal{DU}_\ell(X) = \{1\}$.

(ii) $(F = \mathbf{R})$. $\mathcal{DU}_\ell(X) = \{1, -1\}$ *if and only if* (5.2) *holds; otherwise,* $\mathcal{DU}_\ell(X) = \{1\}$.

Proof. $U \in \mathcal{U}_\ell(X)$ if and only if U is H-unitary and $Ux = x$ for every $x \in \operatorname{Im} X$. In other words, U is a Witt extension (in the terminology of [BMRRR2]) of the identity linear transformation on $\operatorname{Im} X$. The formula for the Witt extensions (given in Theorem 2.3 of [BMRRR2]) shows that all such Witt extensions either have a constant determinant (if $\dim(\operatorname{Im} X) + d(\operatorname{Im} X) = n$) or can have an arbitrary value of the determinant on the unit circle (if $F = \mathbf{C}$) or in the set $\{1, -1\}$. □

THEOREM 5.2. *$\mathcal{DU}_r(X)$ coincides with the unit circle (if $F = \mathbf{C}$) or with $\{1, -1\}$ (if $F = \mathbf{R}$) if and only if $\operatorname{Ker} X$ is not H-isotropic. Otherwise, i.e., if $\operatorname{Ker} X$ is H-isotropic, then $\mathcal{DU}_r(X) = \{1\}$ in both of the cases $F = \mathbf{C}$ and $F = \mathbf{R}$.*

Proof. Using the obvious equality

$$\mathcal{DU}_r(X) = \mathcal{DU}_\ell(X^{[*]}),$$

and applying Theorem 5.1 (with X replaced by $X^{[*]}$) we see that $\mathcal{DU}_r(X) \neq \{1\}$ if and only if

$$\dim(\operatorname{Im} X^{[*]}) + d(\operatorname{Im} X^{[*]}) < n. \tag{5.3}$$

It is well known (and easy to verify) that

$$\operatorname{Im} X^{[*]} = (\operatorname{Ker} X)^{[\perp]},$$

and that $\dim(\operatorname{Ker} X)^{[\perp]} = n - \dim \operatorname{Ker} X$. Also,

$$d(\mathcal{M}) = d(\mathcal{M}^{[\perp]}) \tag{5.4}$$

for every subspace $\mathcal{M} \subset F^n$ (to verify (5.4), simply observe that $d(\mathcal{M}) = \dim(\mathcal{M} \cap \mathcal{M}^{[\perp]})$). Using all these observations, we see that (5.3) is equivalent to $d(\operatorname{Ker} X) < \dim(\operatorname{Ker} X)$, i.e., $\operatorname{Ker} X$ is not H-isotropic. □

Combining Theorems 5.1 and 5.2 with formula (5.1), the following result is obtained.

THEOREM 5.3. *Let $X = \tilde{U}\tilde{A}$ be an H-polar decomposition. If*

$$d(\operatorname{Ker} X) = d(\operatorname{Im} X) = \dim(\operatorname{Ker} X), \tag{5.5}$$

then $\det U = \det \tilde{U}$ for every H-polar decomposition of X which is equivalent to $X = \tilde{U}\tilde{A}$. If at least one of the equalities (5.5) fails, then for every $\alpha \in F$, $|\alpha| = 1$, there exists an H-polar decomposition $X = UA$ with $\det U = \alpha$ and which is equivalent to $X = \tilde{U}\tilde{A}$.

Observe that (5.5) holds for every invertible X.

We emphasize that Theorem 5.3 holds only for equivalent H-polar decompositions (in particular, the inertia of HA and that of $H\tilde{A}$ must be the same). If one considers H-polar decompositions $X = UA$ irrespective of the inertia of HA, then there is considerably more freedom in the values of $\det U$. For example, in the case when $F = \mathbf{R}$ and n is odd, the trivial equality $X = UA = (-U)(-A)$ shows that both 1 and -1 appear as values of $\det U$.

In the remainder of this section we assume $F = \mathbf{R}$ and focus on the more subtle problem of having U in a prescribed connected component of the group $\mathcal{U}(H; \mathbf{R})$ of H-unitary matrices (since the group $\mathcal{U}(H;\mathbf{C})$ is connected, this question is trivial for $F=\mathbf{C}$). We start with two examples.

Example 5.1. Let

$$X=\begin{bmatrix}0&0\\0&1\end{bmatrix},\quad H=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

We have an H-polar decomposition

$$X=H\begin{bmatrix}0&1\\0&0\end{bmatrix}=U\cdot A$$

with $U=H$. Then if $X=U_1A_1$ is another H-polar decomposition there is a real H-unitary V such that $A_1=VA$ and $U_1=UV^{-1}$. Write $V=(v_{ij})_{i=1}^{2}{}_{j=1}^{2}$. Then $VA=\begin{bmatrix}0&v_{11}\\0&v_{21}\end{bmatrix}$ is H-selfadjoint if and only if $v_{21}=0$. Hence $V=\begin{bmatrix}v_{11}&v_{12}\\0&v_{22}\end{bmatrix}$. From the H-unitarity of V (and V being real), it follows that also $v_{12}=0$ and $v_{11}^{-1}=v_{22}$. So $V=\operatorname{diag}(v_{11},v_{11}^{-1})$, for some nonzero real number v_{11}. We see that for any H-polar decomposition of X the H-unitary factor is of the form

$$\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}v_{11}^{-1}&0\\0&v_{11}\end{bmatrix}=\begin{bmatrix}0&v_{11}\\v_{11}^{-1}&0\end{bmatrix};$$

in particular, any H-unitary factor has determinant -1, and hence cannot be in the connected component of I in $\mathcal{U}(H;\mathbf{R})$. □

Example 5.2. As a second example, let

$$X=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}=UA$$

with $H=U$. Let V be a real H-unitary matrix such that VA is H-selfadjoint. Put

$$V=\begin{bmatrix}v_{11}&v_{12}&v_{13}\\v_{21}&v_{22}&v_{23}\\v_{31}&v_{32}&v_{33}\end{bmatrix}.$$

Then the H-selfadjointness of VA implies $v_{31}=v_{32}=0$ and $v_{11}=v_{22}$, as one checks easily, while the H-unitarity of V implies that V must be of the form

$$V=\begin{bmatrix}v_{11}&v_{12}&-\frac{1}{2}v_{11}v_{12}^2\\0&v_{11}&-v_{12}\\0&0&v_{11}\end{bmatrix}\quad\text{with}\quad v_{11}^2=1.$$

Any H-unitary matrix U_1 that is the H-unitary component in an H-polar decomposition of X is then necessarily of the form HV^{-1}:

$$U_1 = \begin{bmatrix} 0 & 0 & v_{11} \\ 0 & v_{11} & v_{12} \\ v_{11} & -v_{12} & -\frac{1}{2}v_{11}v_{12}^2 \end{bmatrix}.$$

Put

$$S = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Then $S^*HS = \operatorname{diag}(1,1,-1)$, and S^*U_1S is of the form

$$\begin{bmatrix} \frac{1}{4}v_{11}(4-v_{12}^2) & -\frac{v_{12}}{\sqrt{2}} & \frac{1}{4}v_{11}v_{12}^2 \\ \frac{v_{12}}{\sqrt{2}} & v_{11} & -\frac{v_{12}}{\sqrt{2}} \\ \frac{1}{4}v_{11}v_{12}^2 & \frac{v_{12}}{\sqrt{2}} & -\frac{1}{4}v_{11}(4+v_{12}^2) \end{bmatrix}.$$

As $\det \begin{bmatrix} \frac{1}{4}v_{11}(4-v_{12}^2) & -\frac{1}{2}v_{12}\sqrt{2} \\ \frac{1}{2}v_{12}\sqrt{2} & v_{11} \end{bmatrix} = 1 + \frac{1}{4}v_{12}^2$ and the sign of $-\frac{1}{4}v_{11}(4+v_{12}^2)$ depends on v_{11}, it is clear that U_1 can only be in two of the four connected components of $\mathcal{U}(H;\mathbf{R})$; see Theorem 5.2. In particular, U_1 can be in the connected component of the identity for this choice of X. However, for $X = S\operatorname{diag}(1,-1,1)\,S^*\cdot A$, the same argument shows that for this choice of X, there is no H-polar decomposition with the unitary factor in the connected component of I. □

The following results give sufficient conditions for existence of H-polar decompositions with the H-unitary factor in an arbitrary connected component of $\mathcal{U}(H;\mathbf{R})$.

THEOREM 5.4. *Suppose $n \geq 4$, and let H be a real invertible selfadjoint $n \times n$ matrix with exactly one positive eigenvalue. Assume X admits an H-polar decomposition. Then X has an H-polar decomposition with the H-unitary factor in any preselected connected component of the group of real H-unitary matrices.*

Observe that Theorem 5.4 is sharp in the sense that it does not hold for $n = 2$ or $n = 3$, as shown by Examples 5.1 and 5.2.

THEOREM 5.5. *Let H be a real symmetric indefinite invertible $n \times n$ matrix, and suppose X is an $n \times n$ matrix which allows an $(H,n,0)$-polar decomposition. Suppose either*

(a) *n is odd, or*

(b) *$X^{[*]}X \neq 0$, or*

(c) *n is even and* $\dim[\operatorname{Ker} X \cap (H \operatorname{Ker} X)^{\perp}] \neq \frac{n}{2}$.

Then, for any preselected connected component of the group of real H-unitary matrices X admits an $(H, n, 0)$-polar decomposition with the unitary factor in this connected component.

In a way Theorem 5.5 is sharp, as testified by Example 5.2.

Both Theorems 5.4 and 5.5 will be deduced from Theorem 5.6 below, which is of independent interest.

THEOREM 5.6. *Let H be a real symmetric invertible $n \times n$ matrix, and suppose X is a real $n \times n$ matrix which allows an H-polar decomposition $X = UA$. Suppose the canonical form of $\{A, H\}$ is of the form*

$$H = \varepsilon_1 P_1 \oplus (\varepsilon_2) \oplus H_3, \quad A = J_1 \oplus (\lambda_2) \oplus A_3, \tag{5.6}$$

where J_1 is a real Jordan block of order n_1, $\varepsilon_1 = \pm 1$ if $\sigma(J_1)$ is real, and $\varepsilon_1 = 1$ if $\sigma(J_1)$ is nonreal (in which case n_1 is necessarily even), $\varepsilon_2 = \pm 1$, $\lambda_2 \in \mathbf{R}$, and A_3 is some real $n_3 \times n_3$ matrix. Furthermore, assume that one of the following three properties are satisfied:

(i) $\sigma(J_1) = \{\lambda_1\} \subset \mathbf{R}$, *$n_1$ is even but not divisible by* 4;

(ii) $\sigma(J_1) = \{\lambda_1\} \subset \mathbf{R}$, *$n_1$ is odd, and either $\varepsilon_2 = 1$ and $\frac{n_1 + \varepsilon_1}{2}$ is even, or $\varepsilon_2 = -1$ and $\frac{n_1 + \varepsilon_1}{2}$ is odd*;

(iii) $\sigma(J_1) = \{\alpha \pm i\beta\}$, $\beta \neq 0$, *in which case $\varepsilon_1 = 1$ and n_1 is necessarily even, but we assume that $\frac{n_1}{2}$ is odd.*

Then, for any preselected connected component of the group of real H-unitary matrices, there is an H-polar decomposition of X such that its unitary factor is in this preselected connected component.

Proof. Put

$$V = aI_{n_1} \oplus (b) \oplus I_{n_3},$$

where $a^2 = b^2 = 1$. Then V is H-unitary and VA is H-selfadjoint. Hence $X = (UV^{-1})(VA)$ is an alternative H-polar decomposition. We may assume without loss of generality that $H_3 = I_{p_3} \oplus -I_{m_3}$, where $p_3 + m_3 = n_3$. Let S_1 be an arbitrary invertible matrix such that $S_1^*(\varepsilon_1 P_1)S_1 = I_{p_1} \oplus -I_{m_1}$, where $p_1 + m_1 = n_1$. Then

$$\begin{bmatrix} S_1^* & 0 \\ 0 & I_{n_3+1} \end{bmatrix} H \begin{bmatrix} S_1 & 0 \\ 0 & I_{n_3+1} \end{bmatrix} = I_{p_1} \oplus -I_{m_1} \oplus (\varepsilon_2) \oplus I_{p_3} \oplus -I_{m_3}.$$

Observe: $p_1 = m_1 = n_1/2$ if n_1 is even; $p_1 = (n_1 + \varepsilon_1)/2$, $m_1 = (n_1 - \varepsilon_1)/2$ if n_1 is odd.

Consider $S^{-1}UV^{-1}S$ and $S^{-1}US$, where $S = S_1 \oplus I_{n_3+1}$, and partition

$$S^{-1}US = (U_{ij})_{i,j=1}^{5}, \quad S^{-1}UV^{-1}S = (W_{ij})_{i,j=1}^{5},$$

where U_{11}, W_{11} are $p_1 \times p_1$; U_{22}, W_{22} are $m_1 \times m_1$; U_{33}, W_{33} are 1×1; U_{44}, W_{44} are $p_3 \times p_3$; and U_{55}, W_{55} are $m_3 \times m_3$.

First consider the case n_1 is even, $n_1/2$ odd, and $\varepsilon_2 = +1$. Then to see in which connected component UV^{-1} is we have to look at

$$\det \begin{bmatrix} W_{11} & W_{13} & W_{14} \\ W_{31} & W_{33} & W_{34} \\ W_{41} & W_{43} & W_{44} \end{bmatrix} = \det \begin{bmatrix} aU_{11} & bU_{13} & U_{14} \\ aU_{31} & bU_{33} & U_{34} \\ aU_{41} & bU_{43} & U_{44} \end{bmatrix} = a^{n_1/2} b \det \begin{bmatrix} U_{11} & U_{13} & U_{14} \\ U_{31} & U_{33} & U_{34} \\ U_{41} & U_{43} & U_{44} \end{bmatrix},$$

and at

$$\det \begin{bmatrix} W_{22} & W_{25} \\ W_{52} & W_{55} \end{bmatrix} = \det \begin{bmatrix} aU_{22} & U_{25} \\ aU_{52} & U_{55} \end{bmatrix} = a^{n_1/2} \det \begin{bmatrix} U_{22} & U_{25} \\ U_{52} & U_{55} \end{bmatrix}.$$

Taking $a, b = \pm 1$ we can choose the signs of these determinants arbitrarily, independently of each other. This proves the theorem in this case. The case when n_1 even, $n_1/2$ odd, and $\varepsilon_2 = -1$ is handled the same way.

Next, consider case (ii), n_1 is odd. Assume first $\varepsilon_2 = 1$, $(n_1 + \varepsilon_1)/2$ is even. Then to see in which connected component UV^{-1} is we have to consider again

$$\det \begin{bmatrix} W_{11} & W_{13} & W_{14} \\ W_{31} & W_{33} & W_{34} \\ W_{41} & W_{43} & W_{44} \end{bmatrix} = a^{(n_1+\varepsilon_1)/2} b \det \begin{bmatrix} U_{11} & U_{13} & U_{14} \\ U_{31} & U_{33} & U_{34} \\ U_{41} & U_{43} & U_{44} \end{bmatrix}, \tag{5.7}$$

and

$$\det \begin{bmatrix} W_{22} & W_{25} \\ W_{52} & W_{55} \end{bmatrix} = a^{(n_1-\varepsilon_1)/2} \det \begin{bmatrix} U_{22} & U_{25} \\ U_{52} & U_{55} \end{bmatrix}. \tag{5.8}$$

Again, choosing $a, b = \pm 1$ we can obtain any signs for these determinants, independently of each other, because $a^{(n_1+\varepsilon_1)/2} b = b$ and $a^{(n_1-\varepsilon_1)/2} = a$. This proves (ii) in case $\varepsilon_2 = 1$. The case when $\varepsilon_2 = -1$ and $(n_1 + \varepsilon_1)/2$ is odd is done likewise.

Finally, consider case (iii), and again assume first $\varepsilon_2 = 1$. The argument now is the same as the one for case (i). The case $\varepsilon_2 = -1$ is treated in the same manner. □

Proof of Theorem 5.4. In the case when $n \geq 4$ and H has only one positive eigenvalue there is a limited number of possibilities for H-selfadjoint matrices A. Each of them is listed below, and it will turn out that all possibilities fall under one of the three cases (i), (ii), (iii) of Theorem 5.6.

Case 1. A is diagonalizable and $\sigma(A) \subset \mathbf{R}$. In this case we are in case (ii) of Theorem 5.6 for $n_1 = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $H_3 = -I_{n-2}$.

Case 2. A has precisely one Jordan block of order 2 with a real eigenvalue, all eigenvalues are real. We are in case (i) of Theorem 5.6 with $n_1 = 2$, $\varepsilon_2 = -1$, $H_3 = -I_{n-3}$.

Case 3. A has precisely one Jordan block of order 3 with a real eigenvalue, the corresponding sign in the sign characteristic is -1, and $\sigma(A) \subset \mathbf{R}$. We are in case (ii) of Theorem 5.6 with $n_1 = 3$, $\varepsilon_1 = -1$, $\varepsilon_2 = -1$. Note that $(n_1 + \varepsilon_1)/2 = 1$ is odd indeed.

Case 4. A has precisely one pair of complex conjugate eigenvalues $\alpha \pm i\beta$ with multiplicity one, all other eigenvalues are real. We are in case (iii) of Theorem 5.6 with $n_1 = 2$.

As in all cases Theorem 5.6 is applicable, Theorem 5.4 is proved. □

Proof of Theorem 5.5. Let $X = UA$ be an $(H, n, 0)$-polar decomposition. Then A is H-nonnegative, and by assumptions (a)-(c) the canonical form of $\{A, H\}$ is as in (5.6), and by the indefiniteness of H either case (i) of Theorem 5.6 holds (with $\lambda_1 = 0$ and $n_1 = 2$) or case (ii) holds with $n_1 = 1$ (here the indefiniteness of H plays a crucial role). The result now follows from Theorem 5.6. □

6 Applications: Linear Optics

In this section $F = \mathbf{R}$ and all vectors and matrices are real.

In linear optics, a beam of light may be described by a vector $I = (i, q, u, v)^T$ where i ($i > 0$) denotes intensity, q/i, u/i, and v/i describe the state of polarization, and the degree of polarization $p = (q^2 + u^2 + v^2)^{1/2}/i$ belongs to $[0, 1]$. In linear optics, transformations of one beam of light into another are described by 4×4 matrices that transform vectors $I_0 = (i_0, q_0, u_0, v_0)^T$ satisfying

$$i_0 \geq \sqrt{q_0^2 + u_0^2 + v_0^2} \tag{6.1}$$

into vectors $I = (i, q, u, v)^T$ satisfying the same inequality, a property of 4×4 matrices which we call the *Stokes criterion.* It is therefore of interest to give necessary and sufficient conditions on a 4×4 matrix to satisfy the Stokes criterion. This problem has been studied extensively, see, e.g., [K,MH,N,M]. In [K,MH,N] the Stokes criterion is studied by minimizing a quadratic form under a quadratic constraint. In [M,N] the eigenvalue structure of the matrix is exploited.

Clearly, the matrices satisfying the Stokes criterion are precisely the real 4×4 matrices that leave invariant the positive cone of vectors $I_0 = (i_0, q_0, u_0, v_0)^T$ satisfying (6.1). Using this point of view, in [K] necessary and sufficient conditions were obtained for a (special) direct sum of two 2×2 matrices to satisfy the Stokes criterion; a different proof of this result was given in [MH]. To generalize this result to general real 4×4 matrices, the indefinite scalar product generated by $H = \operatorname{diag}(1, -1, -1, -1)$ had to be employed. This has led to necessary and sufficient conditions for general real 4×4 matrices to satisfy the Stokes criterion (see [M]). These results have been sharpened and generalized in Section 3 of the present paper.

The Stokes criterion is obviously satisfied for the matrices U belonging to the orthochronous Lorentz group, i.e., those matrices U orthogonal with respect to $H = \operatorname{diag}(1, -1, -1, -1)$ such that $U_{11} > 0$. Given a 4×4 matrix M and two elements U_1 and U_2 of the orthochronous Lorentz group, M satisfies the Stokes criterion if and only if $U_1 M U_2$ does. It is therefore useful to know which 4×4 matrices allow an H-polar decomposition where the H-unitary factor belongs to the orthochronous Lorentz group. The H-selfadjoint factor can then easily be analyzed through its eigenvalue structure [M]. This problem may also be solved using the eigenvalue structure of $M^{[*]}M$ (cf., [M]). The idea to diagonalize either M or $M^{[*]}M$ appears also in [N] and [X]. In [N], the Minkowski space of special relativity (in mathematical terms, $\mathbf{R}^4$ equipped with the indefinite scalar product generated by $H = \operatorname{diag}(1, -1, -1, -1)$) is employed.

In the most important problems of linear optics, polarization matrices are obtained as weighted sums of so-called pure Mueller matrices. Pure Mueller matrices are derived from the complex 2×2 transformation matrix for the associated electric field vectors and hence transform fully polarized beams represented by real vectors $I_0 = (i_0, q_0, u_0, v_0)^T$ satisfying $i_0 = \sqrt{(q_0^2 + u_0^2 + v_0^2)}$ into fully polarized beams (represented by real vectors $I = (i, q, u, v)^T$

satisfying $i = \sqrt{(q^2+u^2+v^2)}$). In other words, pure Mueller matrices are exactly the matrices of the form cU where $c \geq 0$ and U belongs to the proper Lorentz group of all real H-unitary matrices U for which $U_{11} > 0$ and $\det U = +1$ (Note that the proper Lorentz group coincides with the connected component of I in the group of H-unitary matrices). Thus every pure Mueller matrix satisfies the Stokes criterion. The weighted sums of pure Mueller matrices are exactly the matrices belonging to the set

$$\mathcal{W} = \left\{ \sum_{j=1}^{n} c_j U_j : n \in \mathbf{N},\ c_1, \cdots, c_n \geq 0,\ U_1, \cdots, U_n \in \mathcal{G} \right\},$$

where $\mathcal{G}$ is the proper Lorentz group. Thus $\mathcal{W}$ is a subset of the set of matrices satisfying the Stokes criterion. In particular, the elements of the Lorentz group that belong to $\mathcal{W}$ are precisely the elements of the proper Lorentz group. Moreover, given a 4×4 matrix M and two elements U_1 and U_2 of the proper Lorentz group, $M \in \mathcal{W}$ if and only if $U_1 M U_2 \in \mathcal{W}$.

Necessary and sufficient conditions for a 4×4 matrix to belong to $\mathcal{W}$ have been given in [C], where a bijective linear transformation $T : \mathbf{C}^{4\times 4} \to \mathbf{C}^{4\times 4}$ was constructed mapping the real matrices bijectively onto the complex hermitian matrices. Then $M \in \mathcal{W}$ if and only if the so-called coherency matrix $T[M]$ is positive semidefinite (see [C,M] for two different proofs). On the other hand, there exists a criterion in terms of the eigenvalue structure of the given matrix M if it is H-selfadjoint (cf. [M]). In order to generalize this criterion to arbitrary 4×4 matrices M, it is therefore useful to have an H-polar decomposition of M where the H-unitary factor belongs to the connected component of I. Contrary to what is sometimes suggested in the literature [X], there exist matrices in $\mathcal{W}$ that do not allow an H-polar decomposition (see [M] for an example); this circumstance will slightly complicate the proof of Theorem 6.4 below.

In [M] criteria have been given for a 4×4 matrix to satisfy the Stokes criterion and, for an H-selfadjoint matrix, to belong to $\mathcal{W}$.

THEOREM 6.1. *Let M be an H-selfadjoint matrix. Then M satisfies the Stokes criterion (resp. $M \in \mathcal{W}$) if and only if one of the following two situations occurs:*

(1) *M has the one nonnegative eigenvalue λ_0 corresponding to a positive eigenvector and three real eigenvalues λ_1, λ_2 and λ_3 corresponding to negative eigenvectors, and $\lambda_0 \geq \max(|\lambda_1|, |\lambda_2|, |\lambda_3|)$ (resp. both of $\lambda_0 \pm \lambda_1 \geq |\lambda_2 \pm \lambda_3|$);*

(2) *M has the real eigenvalues λ, μ and ν but is not diagonalizable. The eigenvectors corresponding to μ and ν are negative, whereas to the double eigenvalue λ there corresponds one Jordan block of order 2 with the positive sign in the sign characteristic of $\{M^{[*]}M, H\}$. Moreover, $\lambda \geq \max(|\mu|, |\nu|)$ (resp. both $\mu = \nu$ and $\lambda \geq |\mu|$).*

For general 4×4 matrices the situation is more complicated. Since a real 4×4 matrix M satisfies the Stokes criterion if and only if M is a doubly H-plus matrix with $M_{11} \geq 0$, necessary and sufficient conditions for M to satisfy the Stokes criterion follow from Proposition 3.3 (with $F = \mathbf{R}$ and $H = \mathrm{diag}(1, -1, -1, -1)$), the third paragraph following the proof of Theorem 3.4, and Proposition 3.5 (with $n = 4$). Necessary and sufficient conditions for such a matrix to have an H-polar decomposition follow from Theorem 3.4 (with $F = \mathbf{R}$

and $H = \mathrm{diag}(1,-1,-1,-1)$). We will formulate these results below in the context of this section and then go on to characterize the matrices belonging to $\mathcal{W}$.

THEOREM 6.2. *Let M be a 4×4-matrix satisfying $M_{11} \geq 0$. Then M satisfies the Stokes criterion if and only if one of the following two situations occurs:*

(1) *$M^{[*]}M$ has the one nonnegative eigenvalue λ_0 corresponding to a positive eigenvector and three nonnegative eigenvalues λ_1, λ_2 and λ_3 corresponding to negative eigenvectors, and $\lambda_0 \geq \max(\lambda_1,\lambda_2,\lambda_3)$;*

(2) *$M^{[*]}M$ has the nonnegative eigenvalues λ, μ and ν but is not diagonalizable. The eigenvectors corresponding to μ and ν are negative, whereas to the double eigenvalue λ there corresponds one Jordan block of order 2 with the positive sign in the sign characteristic of $\{M^{[*]}M, H\}$. Moreover, $\lambda \geq \max(\mu,\nu)$.*

To derive conditions for a 4×4 matrix to belong to $\mathcal{W}$, we need a result on H-polar decomposition of matrices satisfying the Stokes criterion. The result is immediate from Theorem 3.4.

PROPOSITION 6.3. *A matrix M satisfying the Stokes criterion allows an H-polar decomposition, unless all of the eigenvalues of $M^{[*]}M$ vanish and $M^{[*]}M$ has one Jordan block of order 2 with the positive sign in the sign characteristic of $\{M^{[*]}M, H\}$.*

Suppose M satisfies the Stokes criterion and is invertible, and let S be an H-unitary matrix S such that either

$$S^{-1}M^{[*]}MS = \mathrm{diag}(\lambda_0,\lambda_1,\lambda_2,\lambda_3), \tag{6.2}$$

or

$$S^{-1}M^{[*]}MS = \begin{bmatrix} \lambda+\varepsilon & -\varepsilon \\ \varepsilon & \lambda-\varepsilon \end{bmatrix} \oplus (\mu) \oplus (\nu), \tag{6.3}$$

where $\varepsilon > 0$. Then one may choose S such that $S_{11} > 0$ and $\det S = +1$. Further, there exists an H-selfadjoint matrix A such that $M^{[*]}M = A^2$ and $M = UA$ for some H-unitary matrix U. Writing $A = S^{-1}DS$ and $N = MS^{-1}$, we obtain

$$US^{-1} = ND^{-1},$$

where N satisfies the Stokes criterion. In the case of (6.2), D^{-1} is a diagonal matrix with nonnegative entries and hence $U_{11} > 0$. In the case of (6.3), we find for the $(1,1)$-component of US^{-1}

$$[US^{-1}]_{11} = \frac{N_{11}}{\sqrt{\lambda}} + \frac{\varepsilon(N_{11}-N_{12})}{2\lambda\sqrt{\lambda}} > 0,$$

because $\varepsilon > 0$ and $N_{11} \geq |N_{12}|$. Hence in either case the H-unitary factor U satisfies $U_{11} > 0$, because $U_{11} \geq [US^{-1}]_{11}S_{11} - \sqrt{\sum_{j=2}^{4}[US^{-1}]_{jj}^2}\sqrt{\sum_{j=2}^{4}S_{jj}^2} > 0$. Moreover, the identities $M^{[*]}M = A^2$, $M = UA$ and $\det A > 0$ imply that $\det U = +1$ if $\det M > 0$ and $\det U = -1$ if $\det M < 0$.

We have the following result.

THEOREM 6.4. *Let M be a 4×4-matrix satisfying $M_{11} \geq 0$. Let σ be the sign ± 1 of the product of the nonzero eigenvalues of M. Then $M \in \mathcal{W}$ if and only if one of the following four situations occurs:*

(1) *$M^{[*]}M$ has the positive eigenvalue λ_0 corresponding to a positive eigenvector and a positive and two nonnegative eigenvalues λ_1, λ_2 and λ_3 (with $\lambda_1 \geq \lambda_2 \geq \lambda_3$) corresponding to negative eigenvectors, and $\sqrt{\lambda_0} \pm \sqrt{\lambda_1} \geq |\sqrt{\lambda_2} \pm \sigma\sqrt{\lambda_3}|$;*

(2) *$M^{[*]}M$ is diagonalizable with one positive and three zero eigenvalues, and $\sigma = +1$;*

(3) *$M^{[*]}M$ has the positive eigenvalue λ and the nonnegative eigenvalues μ and ν but is not diagonalizable. The eigenvectors corresponding to μ and ν are negative, whereas to the double eigenvalue λ there corresponds one Jordan block of order 2 with the positive sign in the sign characteristic of $\{M^{[*]}M, H\}$. Moreover, $\sigma = +1$, $\mu = \nu$* and *$\lambda \geq \mu$;*

(4) *$M^{[*]}M$ has only zero eigenvalues. When $M^{[*]}M$ is not diagonalizable, it has one Jordan block of order 2 with the positive sign in the sign characteristic of $\{M^{[*]}M, H\}$.*

Proof. Suppose M is an invertible matrix satisfying the Stokes criterion. Then either of the two cases of Theorem 6.2 applies. Further, according to Proposition 6.3 and Theorem 5.4, M has an H-polar decomposition of the form $M = UA$ where U belongs to the connected component of I in the group of H-unitary matrices and

$$A = S^{-1}\mathrm{diag}\left(\sqrt{\lambda_0}, \sqrt{\lambda_1}, \sqrt{\lambda_2}, \sigma\sqrt{\lambda_3}\right) S$$

in the first case and

$$A = S^{-1} \begin{bmatrix} \sqrt{\lambda} + \frac{1}{2}\varepsilon\lambda^{-1/2} & \frac{1}{2}\varepsilon\lambda^{-1/2} & 0 & 0 \\ -\frac{1}{2}\varepsilon\lambda^{-1/2} & \sqrt{\lambda} - \frac{1}{2}\varepsilon\lambda^{-1/2} & 0 & 0 \\ 0 & 0 & \sqrt{\mu} & 0 \\ 0 & 0 & 0 & \sigma\sqrt{\nu} \end{bmatrix} S \tag{6.4}$$

in the second case, where S is an H-unitary matrix in the connected component of I. The theorem follows by applying Theorem 6.1 to the matrix SAS^{-1}.

Now let M satisfy the Stokes criterion, be singular, and have the H-polar decomposition $H = UA$ constructed in the paragraph following the proof of Proposition 6.3. Then $\mathrm{Ker}\, A = \mathrm{Ker}\, M^{[*]}M = \mathrm{Ker}\, M$. Further, there exists a nondegenerate A-invariant subspace L complementary to $\mathrm{Ker}\, A$ on which A is invertible. Then U is a Witt extension of the restriction U_0 of U to L acting as a matrix from L onto $U[L]$. One easily sees from Theorem 2.6 of [BMRRR2] (in the case $n = 4$, $m = 4 - d$, $m_+ = 1$, $m_0 = 0$, $m_- = 3 - d$, $p = 0$ and $q = d$, where $d = \dim \mathrm{Ker}\, A$) that there are two connected components of Witt extensions of U_0 and that there exist in fact U with $\det U = +1$ and U with $\det U = -1$. So let us choose U with $\det U = +1$, so that U belongs to the connected component of the identity in the group of H-unitary matrices. Now let σ be the sign of the nonzero eigenvalues of M. If $M^{[*]}M$ is diagonal, we find SAS^{-1} equal to $\mathrm{diag}(\sqrt{\lambda_0}, \sqrt{\lambda_1}, \sigma\sqrt{\lambda_2}, 0)$ if $\lambda_0 \geq \max(\lambda_1, \lambda_2) > \lambda_3 = 0$, to $\mathrm{diag}(\sqrt{\lambda_0}, \sigma\sqrt{\lambda_1}, 0, 0)$ if $\lambda_0 \geq \lambda_1 > \lambda_2 = \lambda_3 = 0$, and to diag $(\sigma\sqrt{\lambda_0}, 0, 0, 0)$ if $\lambda_0 > \lambda_1 = \lambda_2 = \lambda_3 = 0$, where S is an H-unitary matrix in the connected component of I. Thus, applying Theorem 6.1 to these A, we conclude that $M \in \mathcal{W}$ if and only if $\sqrt{\lambda_0} \pm \sqrt{\lambda_1} \geq \sqrt{\lambda_2}$, $\sqrt{\lambda_0} \pm \sigma\sqrt{\lambda_1} \geq 0$, and $\sigma\sqrt{\lambda_0} \geq 0$, respectively. On the other hand, if $M^{[*]}M$ has a Jordan block of order 2 corresponding to $\lambda > 0$ with the positive sign in the sign characteristic of $\{M^{[*]}M, H\}$, then, writing the 2×2 block in the left upper corner of the matrix in the right-hand side of (6.4) as B, SAS^{-1} is equal to diag $(B, \sigma\sqrt{\mu}, 0)$ if $\lambda \geq \mu > \nu = 0$, and to diag $(\sigma B, 0, 0)$ if $\lambda > \mu = \nu = 0$, where S is an H-unitary matrix in the connected component of I. Thus, applying Theorem 6.1 to these A, we conclude that $M \in \mathcal{W}$ if and only if $\lambda > \mu = \nu = 0$ and $\sigma = 1$.

When $M^{[*]}M$ has only zero eigenvalues and one Jordan block of order 2 with the positive sign in the sign characteristic of $\{M^{[*]}M, H\}$, there exist an H-unitary S in the connected component of I and $\varepsilon > 0$ such that

$$M^{[*]}M = S\left(\begin{bmatrix} \varepsilon & -\varepsilon \\ \varepsilon & -\varepsilon \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right) S^{-1}.$$

Further, $MM^{[*]}$ has the same Jordan structure as $M^{[*]}M$ (see the paragraph following the proof of Theorem 3.4). Now let D be an H-unitary matrix in the connected component of I mapping $\operatorname{Ker} M$ onto $\operatorname{Ker} M^{[*]}$; such a matrix exists. Then $D^{[*]}$ maps $\operatorname{Im} M$ onto $\operatorname{Im} M^{[*]}$. Put $M_\delta = M + \delta D$ for $\delta > 0$. Then M_δ satisfies the Stokes criterion whenever M does. Further,

$$M_\delta^{[*]}M_\delta = M^{[*]}M + \delta^2 I + \delta\left(M^{[*]}D + D^{[*]}M\right),$$

so that $M_\delta^{[*]}M_\delta x = \delta^2 x$ for every $x \in \operatorname{Ker} M$. If $\dim \operatorname{Ker} M = 3$ and hence $\operatorname{Ker} M = \operatorname{Ker} M^{[*]}M$, then $M_\delta^{[*]}M_\delta$ must have the same Jordan structure as $M^{[*]}M$, but then at the eigenvalue δ^2. Further, if ξ and η are vectors such that $M^{[*]}M\eta = \xi$ and $M^{[*]}M\xi = 0$, we have $(M^{[*]}D + D^{[*]}M)\eta$ proportional to ξ but nonzero. Thus the vectors ξ, η form a Jordan chain for $M_\delta^{[*]}M_\delta$ at the eigenvalue δ^2. Hence the Jordan block of $M_\delta^{[*]}M_\delta$ of order 2 has the positive sign in the sign characteristic of $\{M_\delta^{[*]}M_\delta, H\}$. Thus $M_\delta \in \mathcal{W}$. On the other hand, if $\dim \operatorname{Ker} M = 2$ and hence $\operatorname{Ker} M = S[(0) \oplus \mathbf{R}^2]$, then besides δ^2 the eigenvalues of $M_\delta^{[*]}M_\delta$ consist of two numbers $\geq \delta^2$, because M_δ satisfies the Stokes criterion whenever M does. Then $M_\delta^{[*]}M_\delta$ is either diagonalizable or has a Jordan block of order 2 with the positive sign in the sign characteristic of $\{M_\delta^{[*]}M_\delta, H\}$. As a result, $M_\delta \in \mathcal{W}$.

Note that $\mathcal{W}$ is closed, since there exists a linear invertible transformation $T: \mathbf{C}^{4\times 4} \to \mathbf{C}^{4\times 4}$ mapping $\mathcal{W}$ onto the set of hermitian matrices with nonnegative eigenvalues, which is closed in $\mathbf{C}^{4\times 4}$ (actually, T maps polarization matrices onto their corresponding coherency matrices; cf. [C,M]). Since M can be perturbed by an arbitrarily close element of $\mathcal{W}$ and $\mathcal{W}$ is a closed set, $M \in \mathcal{W}$. □

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