Inverse Problems **12** (1996) 1–25. Printed in the UK

Recovery of discontinuities in a non-homogeneous medium

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Received 27 April 1995

Abstract. The recovery of the coefficient $H(x)$ in the one-dimensional generalized Schrödinger equation $\frac{d^2\psi}{dx^2} + k^2 H(x)^2\psi = Q(x)\psi$, where $H(x)$ is a positive, piecewise continuous function with positive limits $H_\pm$ as $x \to \pm\infty$, is studied. This equation describes the wave propagation in a one-dimensional non-homogeneous medium in which the wavespeed $1/H(x)$ changes abruptly at a finite number of points and a restoring force $Q(x)$ is present. When there are no bound states, the uniqueness of $H(x)$ in the inversion is established for a proper choice of scattering data. When the transmission coefficient vanishes at $k = 0$, it is shown that the scattering data consisting of $Q(x)$ and a reduced reflection coefficient uniquely determine $H(x)$, and neither H_+ nor H_- need to be given as part of the scattering data. If the transmission coefficient does not vanish when $k = 0$, then one needs to include either H_+ or H_- in the scattering data to obtain $H(x)$ uniquely. A simple algorithm is described giving the travel times from $x = 0$ to any discontinuity of $H(x)$ and the relative changes in the wavespeed in terms of the large k-asymptotics of a (reduced) reflection coefficient. It is also shown that H_+ and the transmission coefficient alone do not determine the number of discontinuities of $H(x)$, let alone the travel times between them. Some examples are given to illustrate the algorithm.

1. Introduction

Consider the one-dimensional generalized Schrödinger equation

$$\psi''(k, x) + k^2 H(x)^2\psi(k, x) = Q(x)\psi(k, x) \qquad x \in \mathbb{R} \tag{1.1}$$

where the prime denotes the derivative with respect to the spatial coordinate and the coefficients are assumed to satisfy the following conditions:

(H1) $H(x)$ is strictly positive and piecewise continuous with jump discontinuities at x_n for $n = 1, \ldots, N$ such that $x_1 < \cdots < x_N$.

(H2) $H(x) \to H_\pm$ as $x \to \pm\infty$, where $H_\pm$ are positive constants.

(H3) $H - H_\pm \in L^1(\mathbb{R}^\pm)$, where $\mathbb{R}^- = (-\infty, 0)$ and $\mathbb{R}^+ = (0, +\infty)$.

(H4) H' is absolutely continuous on (x_n, x_{n+1}) and $2H''H - 3(H')^2 \in L^1_1(x_n, x_{n+1})$ for $n = 0, \ldots, N$, where $x_0 = -\infty$ and $x_{N+1} = +\infty$, and $L^1_\beta(I)$ denotes the space of measurable functions $f(x)$ on I such that $\int_I dx\,(1 + |x|)^\beta |f(x)| < \infty$.

(H5) $Q \in L^1_{1+\alpha}(\mathbb{R})$ for some $\alpha \in [0, 1]$.

Equation (1.1) describes, in the frequency domain, the propagation of waves in a one-dimensional non-homogeneous medium where k^2 is energy, $1/H(x)$ is the wavespeed, and $Q(x)$ is the restoring force per unit length. The discontinuities of $H(x)$ correspond to abrupt changes in the scattering properties of the medium in which the wave propagates, the relative changes $H(x_n - 0)/H(x_n + 0)$ correspond to the jumps in the wavespeed at the interfaces x_n, and $y_n = \int_0^{x_n} \mathrm{d}x\, H(x)$ correspond to the times required for the wave to propagate from the fixed location $x = 0$ to the interfaces y_n for $n = 1, \ldots, N$.

In [AKV95] we described a solution to the inverse problem of recovering $H(x)$ in terms of the scattering data consisting of $Q(x)$, a (reduced) reflection coefficient, H_+ or H_-, the bound state energies, and the bound state norming constants. In the present paper, for simplicity, we assume that there are no bound states and that $\alpha = 1$ in (H5). This will cover in particular the case $Q(x) = 0$. Under these assumptions the main steps in the procedure used in [AKV95] are the following. (1) Use a (reduced) reflection coefficient to formulate a singular integral equation, (2) solve this equation uniquely, (3) obtain $y(x)$ as the solution of an algebraic equation also containing as input the solution $f_l(0, x)$ of (1.1) for $k = 0$ satisfying $f_l(0, x) \to 1$ as $x \to +\infty$, (4) put $H(x) = y'(x)$. A similar procedure was given by Grinberg [Gr90, Gr91] when $Q(x) = 0$, in which case the unique solvability of the singular integral equation is immediate and the third step is trivial to implement.

After introducing the scattering and reduced scattering matrices and reviewing their small and large k-asymptotics in section 2, we will study two problems associated with the recovery of $H(x)$ in (1.1). The first problem deals with the inclusion or exclusion of $H_\pm$ in the scattering data. The second problem deals with the recovery of the discontinuities of $H(x)$ in terms of the large k-asymptotics of the scattering data. As for the first problem, an example was given in [AKV95] where a unique $H(x)$ was recovered although neither H_+ nor H_- was included in the scattering data. We now understand the general theory concerning that surprising result, and in section 3 we investigate the proper choice of the scattering data that lead to a unique $H(x)$. In the exceptional case, i.e. when the transmission coefficient associated with (1.1) does not vanish at $k = 0$, we show that one needs to include either H_+ or H_- in the scattering data; otherwise, as the example in (3.1) indicates, a one-parameter family of $H(x)$ with different H_+ leads to the same scattering data. In the generic case, i.e. when the transmission coefficient vanishes at $k = 0$, if one uses a reduced reflection coefficient in the scattering data, then neither H_+ nor H_- need to be included in the scattering data, and in fact $H_\pm$ are determined by using the condition (3.40) without including either H_+ or H_- in the scattering data. On the other hand, in the generic case, if one uses a reflection coefficient instead of a reduced reflection coefficient, in order to determine $H(x)$ uniquely, one can omit $H_\pm$ from the scattering data if and only if (3.41) is satisfied. All the details are given in section 3, and some examples are provided to illustrate the proper choice of the scattering data.

The inversion method described in [AKV95] is based on a singular integral equation whose solution eventually leads to $H(x)$. From this method it is not clear how simple properties of the medium, such as the number of and the travel times between discontinuities of $H(x)$, can be found in an elementary way without solving an integral equation. In section 4 of the present paper, we describe an algorithm that allows one to find the number N of discontinuities of $H(x)$, the travel times $y_1, \ldots, y_N$ to these discontinuities from the fixed location $x = 0$, and the jumps $H(x_n - 0)/H(x_n + 0)$ in the wavespeed at the interfaces by using the large k-asymptotics of a (reduced) reflection coefficient. This algorithm does not involve any integral equations and, as some illustrative examples show, can be implemented by hand. An algorithm to recover the travel times $y_{n+1} - y_n$ and the jumps $H(x_n - 0)/H(x_n + 0)$ in terms of the large k-asymptotics of the modulus of the transmission coefficient was described by Grinberg [Gr90, Gr91] under certain technical restrictions. Our algorithm given in section 4 does not have these restrictions. As example 4.2 indicates, H_+ and the transmission coefficient alone do not in general determine even the number of discontinuities of $H(x)$, let alone the travel times between the successive discontinuities of

$H(x)$. When the function $H(x) f_l(0,x)^2$ is known to be piecewise constant, the algorithm described in section 4 allows us to recover $H(x)$ exactly. When the conditions (H1)–(H5) are satisfied, the large k-asymptotics of a (reduced) reflection coefficient are given by an almost-periodic function of k. In section 5, we characterize those functions $H(x)$ that satisfy (H1)–(H4) and for which the corresponding (reduced) scattering coefficients are almost periodic functions of k.

Concerning scattering and inverse scattering problems with discontinuous coefficients, we remark that Sabatier and his co-workers [SD88, Sa89, DS92, MS94] studied the scattering for the impedance-potential equation and that Krueger [Kr76, Kr78] studied the inverse scattering problem for $u_{xx} - u_{tt} + c_1(x)u_x + c_2(x)u_t + c_3(x)u = 0$, where $x, t \in \mathbb{R}$ and the coefficients c_1, c_2, c_3 are piecewise continuous functions with support in a finite interval. Krueger [Kr82] also considered $u_{xx} - \varepsilon(x)u_{tt} = 0$ when $\varepsilon(x)$ is constant for $x < 0$ and piecewise continuous for $x > 0$, and he developed an iterative method to recover $\varepsilon(x)$ when the incoming and reflected waves are given.

2. Preliminaries

In this section we review the small and large k-asymptotics of the scattering matrix associated with (1.1). The reader is referred to [AKV95] for the details and proofs. The scattering coefficients associated with (1.1) are defined in terms of the Jost solution from the left $f_l(k,x)$ and the Jost solution from the right $f_r(k,x)$, which satisfy the boundary conditions

$$
\begin{aligned}
f_l(k,x) &= \begin{cases} e^{ikH_+x} + o(1) & x \to +\infty \\ \dfrac{1}{T_l(k)} e^{ikH_-x} + \dfrac{L(k)}{T_l(k)} e^{-ikH_-x} + o(1) & x \to -\infty \end{cases} \\
f_r(k,x) &= \begin{cases} \dfrac{1}{T_r(k)} e^{-ikH_+x} + \dfrac{R(k)}{T_r(k)} e^{ikH_+x} + o(1) & x \to +\infty \\ e^{-ikH_-x} + o(1) & x \to -\infty \end{cases}
\end{aligned}
\tag{2.1}
$$

where $T_l(k)$ and $T_r(k)$ are the transmission coefficients from the left and from the right, respectively, and $R(k)$ and $L(k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrix associated with (1.1) is defined by

$$\boldsymbol{S}(k) = \begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{bmatrix}.$$

For brevity, the entries of $\boldsymbol{S}(k)$ are also referred to collectively as scattering coefficients. The bound states associated with (1.1) are given by the square-integrable solutions of (1.1), and such solutions can occur only at certain discrete negative values of k^2 known as bound state energies; $k = 0$ is never a bound state.

As in [AKV95] we introduce the reduced scattering matrix

$$\sigma(k) = \begin{bmatrix} \tau(k) & \rho(k) \\ \ell(k) & \tau(k) \end{bmatrix} \tag{2.2}$$

where

$$\tau(k) = \sqrt{\frac{H_+}{H_-}} T_l(k) e^{ikA} = \sqrt{\frac{H_-}{H_+}} T_r(k) e^{ikA} \qquad \rho(k) = R(k) e^{2ikA_+} \qquad \ell(k) = L(k) e^{2ikA_-} \tag{2.3}$$

$$A_\pm = \pm \int_0^{\pm\infty} ds\, [H_\pm - H(s)] \qquad A = A_- + A_+. \tag{2.4}$$

We will refer to $\tau(k)$ as the reduced transmission coefficient and to $\rho(k)$ and $\ell(k)$ as the reduced reflection coefficients from the right and from the left, respectively. The entries of $\sigma(k)$ collectively are also referred to as reduced scattering coefficients. The matrix $\sigma(k)$ is unitary for $k \in \mathbb{R}$ and we have

$$\det\sigma(k) = \tau(k)^2 - \ell(k)\rho(k) = \frac{\tau(k)}{\tau(-k)} \tag{2.5}$$

where det denotes the matrix determinant.

As in [AKV95] we distinguish between the generic and the exceptional cases for (1.1). The generic (exceptional) case is said to occur if $\tau(0) = 0$ ($\tau(0) \neq 0$). Equivalently, the exceptional case occurs if the zero-energy Jost solutions $f_{\mathrm{l}}(0, x)$ and $f_{\mathrm{r}}(0, x)$ are linearly dependent, i.e. if we have

$$f_{\mathrm{l}}(0, x) = \gamma f_{\mathrm{r}}(0, x) \tag{2.6}$$

for some non-zero constant γ. In the generic case $f_{\mathrm{l}}(0, x)$ and $f_{\mathrm{r}}(0, x)$ are linearly independent and hence $[f_{\mathrm{l}}(0, x); f_{\mathrm{r}}(0, x)] \neq 0$, where $[f; g] = fg' - f'g$ denotes the Wronskian.

Let $\mathbb{C}^{\pm}$ denote the upper and lower half complex planes, respectively, and $\overline{\mathbb{C}^{\pm}} = \mathbb{C}^{\pm} \cup \mathbb{R}$. The following theorem proved in [AKV95] summarizes some properties of the reduced scattering coefficients that are relevant to us.

Theorem 2.1. (i) $\tau(k)$ is meromorphic in $\mathbb{C}^+$ and continuous on $\mathbb{R}$. In the generic case $\tau(k)$ vanishes linearly as $k \to 0$ in $\overline{\mathbb{C}^+}$. The bound state energies correspond to the (simple, finitely many) poles of $\tau(k)$ in $\mathbb{C}^+$, and such poles may occur only on the imaginary axis in $\mathbb{C}^+$.

(ii) $\rho(k)$ and $\ell(k)$ are continuous for $k \in \mathbb{R}$. In the generic case we have $|\rho(k)| = |\ell(k)| < 1$ for $k \neq 0$ and $\rho(0) = \ell(0) = -1$, whereas in the exceptional case we have $|\rho(k)| = |\ell(k)| < 1$ for all $k \in \mathbb{R}$.

The detailed asymptotic behaviours of $\tau(k)$, $\rho(k)$, and $\ell(k)$ as $k \to 0$ with error terms depending on α (cf (H5)) were given in [AKV95]. Using the small k-behaviour of the reduced scattering coefficients, it is possible to show that when $Q(x)$ and $\rho(k)$ are known, H_+ can be obtained from H_- and *vice versa*. This can be seen as follows. In the exceptional case we have [AKV95]

$$H_+ = \frac{\gamma^2[1 + \rho(0)]}{1 - \rho(0)} H_- \tag{2.7}$$

where γ is the constant in (2.6), and this constant is determined by $Q(x)$ alone. In the generic case we have

$$H_+ = \frac{c^2[f_{\mathrm{l}}(0, x); f_{\mathrm{r}}(0, x)]^2}{4H_-} \tag{2.8}$$

where $c := \lim_{k\to 0} \tau(k)/\mathrm{i}k$. Note that $f_{\mathrm{l}}(0, x)$ and $f_{\mathrm{r}}(0, x)$ are determined by $Q(x)$ alone, and hence their Wronskian in (2.8) is also determined by $Q(x)$ alone; furthermore we have $c^2 = \lim_{k\to 0}(1 - |\rho(k)|^2)/k^2$, while $(-1)^{\mathcal{N}-1}c > 0$, $\mathcal{N}$ being the number of bound states. Hence, c is solely determined by $\rho(k)$ and $\mathcal{N}$.

The local Liouville transformation on each interval (x_j, x_{j+1}) given by

$$y = y(x) = \int_0^x \mathrm{d}s\, H(s) \qquad \psi(k, x) = \frac{1}{\sqrt{H(x)}}\phi(k, y) \tag{2.9}$$

transforms (1.1) into the Schrödinger equation

$$\frac{\mathrm{d}^2\phi(k,y)}{\mathrm{d}y^2} + k^2\phi(k,y) = V(y)\phi(k,y) \tag{2.10}$$

where

$$V(y) = V(y(x)) = \frac{H''(x)}{2H(x)^3} - \frac{3}{4}\frac{H'(x)^2}{H(x)^4} + \frac{Q(x)}{H(x)^2}. \tag{2.11}$$

Hence $V(y)$ is defined for $y \in \mathbb{R}\setminus\{y_1, \ldots, y_N\}$, where $y_j = y(x_j)$. Since $H(x) > 0$ and has positive limits $H_\pm$ as $x \to \pm\infty$, we have $y_0 = y_0(x_0) = -\infty$ and $y_{N+1} = y(x_{N+1}) = +\infty$. Note that, since the functions $\psi(k,x)$ and $\psi'(k,x)$ are continuous at x_j, the functions $\phi(k,y)$ and $\mathrm{d}\phi(k,y)/\mathrm{d}y$ will not be continuous at y_j. From the continuity of $\psi(k,x)$ and $\psi'(k,x)$ at x_j for $j = 1, \ldots, N$, we obtain the following (internal) boundary conditions for $\phi(k,y)$:

$$\phi(k, y_j - 0) = \sqrt{q_j}\,\phi(k, y_j + 0) \tag{2.12}$$

$$\frac{\mathrm{d}\phi(k, y_j - 0)}{\mathrm{d}y} = \nu_j\phi(k, y_j + 0) + \frac{1}{\sqrt{q_j}}\frac{\mathrm{d}\phi(k, y_j + 0)}{\mathrm{d}y} \tag{2.13}$$

where

$$q_j = \frac{H(x_j - 0)}{H(x_j + 0)} \tag{2.14}$$

$$\nu_j = \frac{1}{2\sqrt{H(x_j + 0)H(x_j - 0)}}\left[\frac{H'(x_j - 0)}{H(x_j - 0)} - \frac{H'(x_j + 0)}{H(x_j + 0)}\right]. \tag{2.15}$$

It is straightforward to check that the boundary conditions (2.12)–(2.13) are self-adjoint. So we can think of (2.10) as a Schrödinger equation with potential $V(y)$ given by (2.11) on the intervals (y_j, y_{j+1}) for $j = 0, \ldots, N$ and supplemented by the boundary conditions (2.12)–(2.13) at the points y_j. As shown in the following proposition, although $V(y)$ is undefined at y_j for $j = 1, \ldots, N$, we can still associate a scattering matrix with (2.10).

Proposition 2.2. The scattering matrix for (2.10) with the boundary conditions (2.12)–(2.13) is precisely the reduced scattering matrix $\sigma(k)$ defined in (2.2).

Proof. From (2.4) and (2.9) we have

$$y(x) = H_+x - A_+ + o(1) \qquad x \to +\infty \tag{2.16}$$

$$y(x) = H_-x + A_- + o(1) \qquad x \to -\infty. \tag{2.17}$$

Hence, by using (2.16), the Jost solution from the left for (2.10) (i.e. the solution of (2.10) such that $\mathrm{e}^{-\mathrm{i}ky}\phi_\mathrm{l}(k,y) = 1 + o(1)$ as $y \to +\infty$) is given by

$$\phi_\mathrm{l}(k,y) = \frac{\sqrt{H(x)}}{\sqrt{H_+}}\mathrm{e}^{-\mathrm{i}kA_+}f_\mathrm{l}(k,x).$$

Therefore, as $y \to -\infty$, from (2.1) and (2.17) it follows that

$$\phi_\mathrm{l}(k,y) = \sqrt{\frac{H_-}{H_+}}\frac{\mathrm{e}^{-\mathrm{i}kA}}{T_\mathrm{l}(k)}\mathrm{e}^{\mathrm{i}ky} + \sqrt{\frac{H_-}{H_+}}\frac{L(k)}{T_\mathrm{l}(k)}\mathrm{e}^{\mathrm{i}k(A_- - A_+)}\mathrm{e}^{-\mathrm{i}ky} + o(1).$$

By using (2.3) we see that

$$\phi_\mathrm{l}(k,y) = \frac{1}{\tau(k)}\mathrm{e}^{\mathrm{i}ky} + \frac{\ell(k)}{\tau(k)}\mathrm{e}^{-\mathrm{i}ky} + o(1) \qquad y \to -\infty \tag{2.18}$$

and thus $\tau(k)$ is the transmission coefficient and $\ell(k)$ is the reflection coefficient from the left for (2.10). Similarly, by considering the Jost solution of (2.10) from the right, one shows that the reflection coefficient from the right is $\rho(k)$. □

Later in the paper we need to know how the (reduced) scattering matrix changes when we perform a shift $y \to y + \xi$ for a fixed $\xi \in \mathbb{R}$.

Proposition 2.3. For any $\xi \in \mathbb{R}$, let $V(y;\xi) = V(y+\xi)$. Consider (2.10) with $V(y)$ replaced by $V(y;\xi)$ and boundary conditions of the form (2.12)–(2.13) at the points $y_j - \xi$, where the numerical values of q_j and μ_j are independent of ξ. Then the scattering coefficients for $V(y)$ and $V(y;\xi)$ are related by

$$\tau(k;\xi) = \tau(k) \qquad \rho(k;\xi) = \mathrm{e}^{2\mathrm{i}k\xi}\rho(k) \qquad \ell(k;\xi) = \mathrm{e}^{-2\mathrm{i}k\xi}\ell(k). \tag{2.19}$$

Proof. The Jost solution from the left associated with $V(y;\xi)$ is given by $\phi_{\mathrm{l}}(k, y;\xi) = \mathrm{e}^{-\mathrm{i}k\xi}\phi_{\mathrm{l}}(k, y+\xi)$. Then (2.19) is obtained by using (2.3), (2.4), and (2.18). □

Let $V_{j,j+1}(y)$ be the potential defined by

$$V_{j,j+1}(y) = \begin{cases} V(y) & y \in (y_j, y_{j+1}) \\ 0 & \text{elsewhere.} \end{cases} \tag{2.20}$$

As a consequence of hypothesis (H4) we have

$$V_{j,j+1} \in L^1_1(\mathbb{R}) \qquad j = 0, \ldots, N. \tag{2.21}$$

Let $g_{l;j,j+1}(k, y)$ and $g_{r;j,j+1}(k, y)$ denote the Jost solutions from the left and right, respectively, associated with the potential $V_{j,j+1}(y)$. Then the functions defined by

$$\eta_{j,j+1}(k, x) = \frac{1}{\sqrt{H(x)}} g_{l;j,j+1}(k, y) \qquad \xi_{j,j+1}(k, x) = \frac{1}{\sqrt{H(x)}} g_{r;j,j+1}(k, y) \tag{2.22}$$

become solutions of (1.1). Let us introduce the matrices

$$\Gamma_{j,j+1}(k, x) = \begin{bmatrix} \eta_{j,j+1}(k, x) & \xi_{j,j+1}(k, x) \\ \eta'_{j,j+1}(k, x) & \xi'_{j,j+1}(k, x) \end{bmatrix} \qquad j = 0, \ldots, N$$

$$\mathcal{G}(k) = \prod_{n=1}^{N} \Gamma_{n-1,n}(k, x_n - 0)^{-1}\Gamma_{n,n+1}(k, x_n + 0). \tag{2.23}$$

Let $t_{j,j+1}(k)$, $r_{j,j+1}(k)$, and $l_{j,j+1}(k)$ denote the scattering coefficients for the potential $V_{j,j+1}(y)$. It was shown in [AKV95] that

$$\frac{1}{\tau(k)} = \frac{1}{t_{0,1}(k)}[\,1 \quad 0\,]\mathcal{G}(k)\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{t_{N,N+1}(k)}[\,0 \quad 1\,]\mathcal{G}(k)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \tag{2.24}$$

$$\frac{\ell(k)}{\tau(k)} = \begin{bmatrix} \dfrac{l_{0,1}(k)}{t_{0,1}(k)} & 1 \end{bmatrix}\mathcal{G}(k)\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\frac{\rho(k)}{\tau(k)} = \begin{bmatrix} 1 & \dfrac{r_{N,N+1}(k)}{t_{N,N+1}(k)} \end{bmatrix}\mathcal{G}(k)^{-1}\begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{2.25}$$

Moreover,

$$\det\Gamma_{n,n+1}(k, x) = -\frac{2\mathrm{i}k}{t_{n,n+1}(k)} \qquad \det\mathcal{G}(k) = \frac{t_{0,1}(k)}{t_{N,N+1}(k)}.$$

Let

$$\alpha_n = \frac{1}{2}\left(\sqrt{q_n} + \frac{1}{\sqrt{q_n}}\right) \qquad \beta_n = \frac{1}{2}\left(\sqrt{q_n} - \frac{1}{\sqrt{q_n}}\right) \tag{2.26}$$

$$E(k, x_n) = \begin{bmatrix} \alpha_n & \beta_n\mathrm{e}^{-2\mathrm{i}ky_n} \\ \beta_n\mathrm{e}^{2\mathrm{i}ky_n} & \alpha_n \end{bmatrix} \tag{2.27}$$

with q_n as in (2.14); let us also define $a(k)$ and $b(k)$ by

$$\begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix} = \prod_{n=1}^{N} E(k, x_n). \tag{2.28}$$

From (2.26)–(2.27) we see that

$$|a(k)|^2 - |b(k)|^2 = 1 \qquad k \in \mathbb{R}. \tag{2.29}$$

Let AP^W (almost periodic functions with Wiener norm) stand for the algebra of all complex-valued functions $f(k)$ on $\mathbb{R}$ which are of the form $f(k) = \sum_{j=-\infty}^{\infty} f_j \mathrm{e}^{\mathrm{i}k\lambda_j}$, where $f_j \in \mathbb{C}$ and $\lambda_j \in \mathbb{R}$ for all j and $\sum_j |f_j| < \infty$. By letting $k \to \infty$ in (2.23) and using (2.24) and (2.25) we obtain the following theorem proved in [AKV95].

Theorem 2.4. We have

$$\frac{1}{\tau(k)} = a(k)[1 + o(1)] \qquad k \to \infty \text{ in } \overline{\mathbb{C}^+} \tag{2.30}$$

$$\rho(k) = \frac{-b(k)}{a(k)} + o(1) \qquad k \to \pm\infty \text{ in } \mathbb{R}. \tag{2.31}$$

Moreover, $|a(k)| \geqslant 1$ on $\overline{\mathbb{C}^+}$ and the functions $a(k)$, $b(k)$, $1/a(k)$, and $b(k)/a(k)$ belong to AP^W.

3. Scattering data and uniqueness

The motivation for this section comes from some observations made in [AKV95] concerning the uniqueness of solutions to the inverse problem. The question of uniqueness is closely related to the choice of an appropriate set of scattering data. Recall our assumption that there are no bound states. We will show that in the generic case the scattering data consisting of $Q(x)$ and a reduced reflection coefficient uniquely determine $H(x)$; in the exceptional case either H_+ or H_- must be specified in addition to $Q(x)$ and a reduced reflection coefficient to determine $H(x)$ uniquely. There is no loss of generality in using $\rho(k)$ as the reduced scattering coefficient in the scattering data, and without further mentioning it we will simply use $\rho(k)$; one can easily modify the proofs if $\ell(k)$ is used instead of $\rho(k)$ in the scattering data. We will also give the appropriate modification if one uses a reflection coefficient instead of a reduced reflection coefficient in the scattering data; it then turns out that in the generic case when (3.41) fails one also must include either H_+ or H_- in the scattering data. Since the proofs essentially remain the same whether one uses $R(k)$ or $L(k)$ as the reflection coefficient, without loss of generality we will state and prove our results by using only $R(k)$.

We recall that in the absence of bound states the inversion procedure described in [AKV95] requires two key ingredients: the potential $Q(x)$ and the reduced reflection coefficient $\rho(k)$ (or, alternatively, $R(k)$). In the exceptional case one also needs to know H_+ in order to determine $H(x)$ uniquely. For example, consider the scattering data given by $Q(x) = 0$ and $\rho(k) = \rho_0$, where $\rho_0 \in (-1, 1)$ is a constant. Corresponding to this set of data we have

$$H(x) = \begin{cases} H_+ & x > 0 \\ \dfrac{1-\rho_0}{1+\rho_0} H_+ & x < 0 \end{cases} \tag{3.1}$$

and hence a one-parameter family of functions $H(x)$ corresponds to the same scattering data. In general, in the exceptional case no conditions on H_+ arise during the inversion procedure, and hence one always ends up with a one-parameter family of functions $H(x)$, parametrized by H_+. However, the parameter H_+ will generally not be a multiplicative factor in $H(x)$ as in (3.1). The proof that, in the exceptional case, there exists a one-parameter family of functions $H(x)$ depending on H_+, having the same $\rho(k)$, and satisfying (H1)–(H4), was not given in [AKV95]; it will be given here in theorem 3.2.

On the other hand, in the generic case, we learned from example 6.2 in [AKV95] that H_+ is not a free parameter as in the exceptional case but is determined by $\rho(k)$ and $Q(x)$. We have since realized that this is generally true in the generic case, and we will prove this fact in theorem 3.1. It is possible to modify the inversion procedure of [AKV95] and use the reflection coefficient $R(k)$ instead of the reduced reflection coefficient $\rho(k)$ in the scattering data. Somewhat surprisingly, it then turns out that in the generic case there is one special situation, where H_+ also becomes a free parameter; this special case occurs when (3.41) fails and it will be described in theorem 3.3.

We will first show that in the absence of bound states the scattering data appropriate for the unique solution of the inverse problem associated with (1.1) are:

(i) in the generic case: $\{Q(x), \rho(k)\}$;

(ii) in the exceptional case: $\{Q(x), \rho(k), H_+\}$.

In preparation of the proof of our first theorem we recall some results from [AKV95]. The function $Q(x)$ enters into our formalism through the zero-energy Jost solution $f_l(0,x)$ and its k-derivative $\dot{f}_l(0,x)$. These two functions satisfy the following integral equations:

$$f_l(0,x) = 1 + \int_x^\infty \mathrm{d}z\,(z-x)Q(z)f_l(0,z) \tag{3.2}$$

$$\dot{f}_l(0,x) = \mathrm{i}H_+x + \int_x^\infty \mathrm{d}z\,(z-x)Q(z)\dot{f}_l(0,z). \tag{3.3}$$

Incidentally, (3.3) shows that $\dot{f}_l(0,x)$ is also a zero-energy solution of (1.1) and is linearly independent of $f_l(0,x)$, since it grows as $x \to +\infty$. From (3.2) and (3.3), the estimates

$$|f_l(0,x)| \leqslant (1+\max\{0,-x\})\exp\left[\int_{-\infty}^\infty \mathrm{d}z\,(1+|z|)|Q(z)|\right] \tag{3.4}$$

$$|\dot{f}_l(0,x)| \leqslant H_+(1+|x|)\exp\left[\int_{-\infty}^\infty \mathrm{d}z\,(1+|z|)^2|Q(z)|\right] \tag{3.5}$$

follow by iteration. Since $Q \in L_2^1(\mathbb{R})$, from (3.3) and (3.5) we conclude that

$$\dot{f}_l(0,x) = \mathrm{i}H_+x + o(1) \qquad x \to +\infty. \tag{3.6}$$

Since we assume that there are no bound states, we have $f_l(0,x) > 0$ for all $x \in \mathbb{R}$. On letting $x \to -\infty$ in (3.2) and using (3.4) we find

$$f_l(0,x) = -c_l x + d_l + \epsilon_l(x) \qquad x \to -\infty \tag{3.7}$$

where

$$c_l = \int_{-\infty}^\infty \mathrm{d}z\,Q(z)f_l(0,z) \tag{3.8}$$

$$d_l = 1 + \int_{-\infty}^\infty \mathrm{d}z\,z\,Q(z)f_l(0,z) \tag{3.9}$$

$$\epsilon_l(x) = -\int_{-\infty}^x \mathrm{d}z\,(z-x)Q(z)f_l(0,z). \tag{3.10}$$

From (2.15) of [AKV93] it follows that $c_l = [f_l(0,x); f_r(0,x)] > 0$. The detailed asymptotics stated in (3.7) will be needed at the end of this section. We denote by $\dot{f}_{l,1}(0,x)$ the unique solution of (3.3) for $H_+ = 1$. The ratios defined by

$$G(x) = -\mathrm{i}\frac{\dot{f}_l(0,x)}{f_l(0,x)} \qquad G_1(x) = -\mathrm{i}\frac{\dot{f}_{l,1}(0,x)}{f_l(0,x)} \tag{3.11}$$

will play an important role in the sequel. By (3.3) we have

$$\dot{f}_l(0,x) = H_+\dot{f}_{l,1}(0,x) \qquad G(x) = H_+G_1(x) \tag{3.12}$$

and (cf (2.27) in [AKV95])

$$G_1'(x) = \frac{1}{f_l(0,x)^2} > 0. \tag{3.13}$$

Moreover, using (3.6) and (3.10) we obtain

$$G_1(x) = x + o(1) \qquad x \to +\infty. \tag{3.14}$$

We now return to the inversion method of [AKV95]. The solution of the inverse problem leads to the following implicit equation (cf (5.24) in [AKV95]):

$$y + A_+ + \tilde{X}(0,y) = H_+G_1(x) \tag{3.15}$$

where $y = y(x)$ is the function defined in (2.9) and $\tilde{X}(k,y)$ is the solution of the singular integral equation

$$\tilde{X}(k,y) = \tilde{X}_0(k,y) + (\mathcal{O}_y\tilde{X})(k,y) \tag{3.16}$$

with

$$\tilde{X}_0(k,y) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{s-k+\mathrm{i}0}\,\frac{\rho(s)\mathrm{e}^{2\mathrm{i}sy}-\rho(0)}{s} \tag{3.17}$$

$$(\mathcal{O}_y\tilde{X})(k,y) = \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\frac{\mathrm{d}s}{s+k-\mathrm{i}0}\rho(-s)\mathrm{e}^{-2\mathrm{i}sy}\tilde{X}(s,y). \tag{3.18}$$

Note that the function $\tilde{X}(k,y)$ is related to the solution $X(k,x,y)$ of (5.21) in [AKV95] by

$$X(k,x,y) = -\mathrm{i}\tilde{X}(k,y)\frac{f_l(0,x)}{\sqrt{H_+}}.$$

The existence and uniqueness of solutions of (3.16) in the Hardy spaces $\boldsymbol{H}^p_-(\mathbb{R})$ with $1 < p < \infty$ was proved in [AKV95]. Recall that the Hardy spaces $\boldsymbol{H}^p_\pm(\mathbb{R})$ are the spaces of analytic functions $F(k)$ on $\mathbb{C}^\pm$ for which $\sup_{\epsilon>0}\int_{-\infty}^{\infty}\mathrm{d}k|F(k\pm\mathrm{i}\epsilon)|^p$ is finite. The constant A_+ in (3.15) is determined uniquely by the condition $y(0) = 0$, i.e.

$$A_+ = H_+G_1(0) - \tilde{X}(0,0) \tag{3.19}$$

and thus (3.15) can be written as

$$y + \tilde{X}(0,y) = H_+[G_1(x) - G_1(0)] + \tilde{X}(0,0). \tag{3.20}$$

Theorem 3.1. For a given set of scattering data, if a solution $H(x)$ of the inverse problem exists, then it is unique.

Proof. In the exceptional case, $f_l(0,x) \to \gamma$ as $x \to -\infty$, where γ is the constant defined in (2.6); since $f_l(0,x)$ is bounded and strictly positive, using (3.13) we conclude that the range of $G(x)$ is the whole real line. In the generic case, by using (3.7), (3.11), and (3.13), we see that

$$\lim_{x\to-\infty} G_1(x) = G_1(-\infty) = G_1(0) - \int_{-\infty}^{0} \mathrm{d}z\, \frac{1}{f_l(0,z)^2} \tag{3.21}$$

is finite. Therefore, in the generic case, by using (3.12) and (3.13), we see that the range of $G(x)$ is the interval $(H_+G_1(-\infty), +\infty)$. A solution $y(x)$ of (3.15) is assumed to exist and $y(x)$ is monotonically increasing; hence the left-hand side of (3.15) must also be monotonically increasing as a function of y. In fact, by differentiating (3.20) and using (3.13) and $\mathrm{d}y/\mathrm{d}x = H(x)$, we see that the function $\tilde{X}(0,y)$ is continuously differentiable except possibly at the points $y_j = y(x_j)$, and

$$\frac{\mathrm{d}[y+\tilde{X}(0,y)]}{\mathrm{d}y} = \frac{H_+}{H(x)\,f_l(0,x)^2} > 0 \qquad y \in \mathbb{R}\setminus\{y_1,\ldots,y_N\}.$$

Since the ranges of both sides of (3.15) must be equal and $\lim_{x\to\pm\infty} y(x) = \pm\infty$, we conclude that in the exceptional case

$$\lim_{y\to\pm\infty} [y+\tilde{X}(0,y)] = \pm\infty.$$

In the generic case we have

$$\lim_{y\to+\infty} [y+\tilde{X}(0,y)] = +\infty$$

and from (3.20) and (3.21) we conclude that the limit

$$w := \lim_{y\to-\infty} [y+\tilde{X}(0,y)] \tag{3.22}$$

exists and is finite, and that

$$w = H_+[G_1(-\infty) - G_1(0)] + \tilde{X}(0,0). \tag{3.23}$$

Hence, solving (3.23) for H_+ we obtain

$$H_+ = \frac{w - \tilde{X}(0,0)}{G_1(-\infty) - G_1(0)} \tag{3.24}$$

which shows that in the generic case H_+ is determined uniquely by $\rho(k)$ and $Q(x)$. This is the reason why we do not include H_+ in the scattering data for the generic case. In the exceptional case both sides of (3.15) have infinite range and hence there is no restriction on H_+ arising from the implicit equation (3.15). From the monotonicity of the two sides of (3.15) it is clear that (3.15) is uniquely solvable for $y(x)$. The constant A_+, the function $\tilde{X}(0,y)$, and, in the generic case, the value of H_+ are determined uniquely by the scattering data. Hence the proof is complete. □

In the rest of this section we will obtain some further results on the function $\tilde{X}(0,y)$. The first piece of information comes from the fact that the two expressions for A_+, (2.4) and (3.19), must agree. Let us temporarily denote the constant in (2.4) by $A_+^{(1)}$ and the constant in (3.19) by $A_+^{(2)}$. Then

$$A_+^{(1)} = \int_0^x \mathrm{d}s\,[H_+ - H(s)] + \int_x^\infty \mathrm{d}s\,[H_+ - H(s)] = H_+x - y(x) + o(1) \tag{3.25}$$

where $o(1)$ stands for terms that go to zero as $x \to +\infty$. Replacing A_+ by $A_+^{(2)}$ in (3.15), from (3.14), (3.15), and (3.25) we obtain

$$A_+^{(1)} = A_+^{(2)} + \tilde{X}(0, y) + o(1).$$

Hence, $A_+^{(1)} = A_+^{(2)}$ if and only if

$$\lim_{y\to+\infty} \tilde{X}(0, y) = 0. \tag{3.26}$$

This amounts to a condition on $\rho(k)$. For example, (3.26) holds if $\rho(k)$ has an analytic continuation into $\mathbb{C}^+$ and obeys the estimate $|\rho(k)| \leqslant c_1 \exp(c_2 \operatorname{Im} k)$ on $\overline{\mathbb{C}^+}$ for some constants c_1 and c_2. This follows from (3.16)–(3.18) by contour integration and iteration. The examples discussed in [AKV95] have this property.

The fact that in the exceptional case the constant H_+ is not restricted by (3.15) suggests that it is a free parameter in the sense described in the introduction to this section. The next theorem will make this notion precise. We will distinguish a particular function $H_0(x)$ satisfying (H1)–(H4) and denote its reduced scattering matrix by $\sigma_0(k)$ and the corresponding solution $\tilde{X}(0, y)$ of (3.16) by $\tilde{X}_0(0, y)$. A subscript zero will be used also on other quantities to indicate that they are associated with $H_0(x)$; e.g. we will write $\sigma_0(k)$, $\rho_0(k)$, and $A_{0,+}$ for the quantities $\sigma(k)$, $\rho(k)$, and A_+ defined in (2.2)–(2.4), etc. Then we consider (3.15) with $\tilde{X}_0(0, y)$ in place of $\tilde{X}(0, y)$, but on the right-hand side we leave $H_+ > 0$ and view it as a parameter (so H_+ need not be equal to $H_{0,+}$); in other words, we consider

$$y + A_+ + \tilde{X}_0(0, y) = H_+ G_1(x). \tag{3.27}$$

In the following, the quantities that do not carry a subscript zero are associated with the solution $y(x)$ of (3.27) for a given H_+. The following theorem shows that, in the exceptional case, there is a one-parameter family of functions $H(x)$ with parameter H_+, which corresponds to the same scattering data $\{Q(x), \rho(k)\}$.

Theorem 3.2. Suppose we are in the exceptional case with $Q \in L_2^1(\mathbb{R})$ and that $H_0(x)$ obeys (H1)–(H4). Then for any $H_+ > 0$, the function $H(x) = y'(x)$, where $y(x)$ is the solution of (3.27), also obeys (H1)–(H4). Moreover, $\sigma(k) = \sigma_0(k)$.

Proof. We will first verify (H1)–(H4) in the order (H2), (H3), (H1), and (H4). Differentiating (3.27) with respect to x and using $G_1'(x) = 1/f_l(0, x)^2$ (cf (3.13)) we get

$$\tilde{X}_0'(0, y) = \frac{H_+}{H(x) f_l(0, x)^2} - 1 \tag{3.28}$$

where the prime on $\tilde{X}_0(0, y)$ denotes the y-derivative. Replacing in (3.28) $H(x)$ and H_+ by $H_0(x)$ and $H_{0,+}$, respectively, and letting $x \to \pm\infty$, we obtain

$$\lim_{y\to+\infty} \tilde{X}_0'(0, y) = 0 \qquad \lim_{y\to-\infty} \tilde{X}_0'(0, y) = \frac{H_{0,+}}{H_{0,-}\gamma^2} - 1 \tag{3.29}$$

where we have also used (2.6). Since $H_0(x)$ is bounded and bounded away from zero by (H1) and (H2), we see from (3.28) that $\tilde{X}_0'(0, y)$ must obey an estimate of the form

$$0 < C_1 \leqslant 1 + \tilde{X}_0'(0, y) \leqslant C_2 < \infty \tag{3.30}$$

for some constants C_1 and C_2. Now return to (3.28) with an arbitrary $H(x)$. By using (3.29) and (3.30) we conclude that $H(x)$ must approach finite limits as $x \to \pm\infty$; in particular $\lim_{x\to+\infty} H(x) = H_+$. Moreover, from $x \to -\infty$, we obtain

$$\frac{H_+}{H_-} = \frac{H_{0,+}}{H_{0,-}} \tag{3.31}$$

i.e. the ratio H_+/H_- is the same for all solutions of (3.27). This shows that $H(x)$ obeys (H2). In order to deal with (H3) we recall that from (3.2) and the assumption $Q \in L^1_2(\mathbb{R})$ it follows that (cf [DT79], lemma 1, p 130)

$$1 - f_l(0,\cdot) \in L^1(\mathbb{R}^+) \qquad \gamma - f_l(0,\cdot) \in L^1(\mathbb{R}^-). \tag{3.32}$$

Now write (3.28) as

$$\tilde{X}_0'(0,y) = \frac{H_+ - H(x)}{H(x) f_l(0,x)^2} + \frac{1 - f_l(0,x)^2}{f_l(0,x)^2}. \tag{3.33}$$

Since $H_0(x)$ obeys (H3), using (3.32) we have $\tilde{X}_0'(0,\cdot) \in L^1(\mathbb{R}^+)$. Using (3.33) we see that $H - H_+ \in L^1(\mathbb{R}^+)$. Similarly, when $x < 0$ we write

$$\tilde{X}_0'(0,y) - \frac{H_+}{H_-\gamma^2} + 1 = \frac{H_+}{H_-}\left[\frac{H_- - H(x)}{H(x) f_l(0,x)^2} + \frac{\gamma^2 - f_l(0,x)^2}{f_l(0,x)^2\gamma^2}\right]$$

and, since for $H(x) = H_0(x)$ the right-hand side is in $L^1(\mathbb{R}^-)$, the left-hand side must be in $L^1(\mathbb{R}^-)$. Hence $H - H_- \in L^1(\mathbb{R}^-)$, i.e $H(x)$ obeys (H3). Next we consider (H1). Solving (3.28) for $H(x)$ we obtain

$$H(x) = \frac{H_+}{f_l(0,x)^2[1 + \tilde{X}_0'(0,y)]}. \tag{3.34}$$

The points $x_{0,1}, \ldots, x_{0,N}$ where $H_0(x)$ has discontinuities determine, via (2.12), the points $y_1, \ldots, y_N$, where $\tilde{X}_0'(0,y)$ has discontinuities. Then, for an arbitrary $H(x)$ the discontinuities x_j are given by $y_j = \int_0^{x_j} \mathrm{d}s\, H(s)$. Thus the number of discontinuities is the same for all functions $H(x)$ given by (3.34). The estimate (3.30) guarantees that $H(x)$ is bounded from above and bounded away from zero. Thus $H(x)$ obeys (H1). The verification of (H4) and $\rho(k) = \rho_0(k)$ will be done together, by using the Liouville transformation given in (2.9)–(2.11). By differentiating (3.34), after lengthy calculations, we obtain for the potential $V(y)$ in (2.11)

$$V(y) = \frac{3}{4}\frac{\tilde{X}_0''(0,y)^2}{[1 + \tilde{X}_0'(0,y)]^2} - \frac{\tilde{X}_0'''(0,y)}{2[1 + \tilde{X}_0'(0,y)]} \qquad y \in \mathbb{R}\setminus\{y_1, \ldots, y_N\}. \tag{3.35}$$

The boundary conditions at y_j are given by (2.12)–(2.13) with

$$q_j = \frac{1 + \tilde{X}_0'(0, y_j + 0)}{1 + \tilde{X}_0'(0, y_j - 0)} \tag{3.36}$$

$$\nu_j = \frac{1}{2}\left[\frac{\tilde{X}_0''(0, y_j + 0)}{[1 + \tilde{X}_0'(0, y_j + 0)]^2} - \frac{\tilde{X}_0''(0, y_j - 0)}{[1 + \tilde{X}_0'(0, y_j - 0)]^2}\right] \times\sqrt{1 + \tilde{X}_0'(0, y_j + 0)}\sqrt{1 + \tilde{X}_0'(0, y_j - 0)}. \tag{3.37}$$

Since $H_0(x)$ satisfies (H4), $V(y)$ satisfies (2.21) and, in turn, this implies that $H(x)$ satisfies (H4). The essential point of (3.35)–(3.37) is that $V(y)$ and the boundary conditions depend only on $\rho_0(k)$ and not on H_+. Therefore, the scattering matrix for (3.35) does not depend on H_+, i.e. by proposition 2.2 we have $\sigma(k) = \sigma_0(k)$. □

We remark that in the case when $Q(x) = 0$ we can obtain $H(x)$ from $H_0(x)$ by a scaling transformation, namely

$$H(x) = \frac{H_+}{H_{0,+}} H_0(H_+ x / H_{0,+}).$$

Furthermore, from (2.7) and (3.29) it follows that

$$\lim_{y\to-\infty} \tilde{X}_0'(0, y) = \frac{2\rho_0(0)}{1-\rho_0(0)}$$

which complements (3.26).

Next we discuss the extensions of theorems 3.1 and 3.2 to the case when $R(k)$ is known instead of $\rho(k)$ as part of the scattering data. In view of (2.3), (3.15) assumes the form

$$y + A_+ + \tilde{X}_1(0, y + A_+) = H_+ G_1(x). \tag{3.38}$$

Here $\tilde{X}_1(0, y)$ is the function obtained by solving (3.16) with $\rho(k)$ replaced by $R(k)$. We have used the fact that, if in (3.16)–(3.18) we replace $\rho(k)$ by $\rho(k)\mathrm{e}^{2\mathrm{i}k\theta}$ with $\theta \in \mathbb{R}$, then this amounts to a shift $y \to y + \theta$; in our case $\theta = A_+$. The constant A_+ is determined by

$$A_+ + \tilde{X}_1(0, A_+) = H_+ G_1(0). \tag{3.39}$$

Since both sides of (3.38) are monotonically increasing functions of their respective variables, A_+ is determined uniquely. In the generic case, we let (cf (3.22))

$$w_0 = \lim_{z\to-\infty} [z + \tilde{X}_1(0, z)].$$

Then, by (3.38), H_+ is given as

$$H_+ = \frac{w_0}{G_1(-\infty)} \tag{3.40}$$

provided that

$$G_1(-\infty) \neq 0. \tag{3.41}$$

If $G_1(-\infty) = 0$, then H_+ remains undetermined. Note that if $G_1(-\infty) = 0$, we must also have $w_0 = 0$ in order for (3.38) to be solvable for y as a function of x. We will show below that if $G_1(-\infty) = 0$, then H_+ is a free parameter as in the exceptional case. It is interesting to see that in the construction of $H(x)$ from $R(k)$ one may encounter this special situation which does not arise if one starts from $\rho(k)$ (the denominator in (3.24) is never zero). Hence, if $\rho(k)$ is replaced by $R(k)$, then the scattering data should be redefined as follows.

(1) In the generic case:
   (a) if $G_1(-\infty) \neq 0$: $\{Q(x), R(k)\}$;
   (b) if $G_1(-\infty) = 0$: $\{Q(x), R(k), H_+\}$.
(2) In the exceptional case: $\{Q(x), R(k), H_+\}$.

Theorem 3.3. Suppose that $Q \in L_2^1(\mathbb{R})$ and that there are no bound states. Then the solution of the inverse problem with the above scattering data is unique. Moreover, in the generic case with $G_1(-\infty) = 0$ and in the exceptional case, the constant H_+ is a free parameter in the sense that for any choice of $H_+ > 0$, the function $H(x)$ resulting from the solution of (3.38) corresponds to the same reflection coefficient $R(k)$.

Proof. The uniqueness follows as in the proof of theorem 3.1 from the monotonicity of both sides of (3.38). The proof that in the exceptional case H_+ is a free parameter and that the reflection coefficient does not depend on H_+ is similar as in the proof of theorem 3.2, the only difference being that the potential $V(y)$ in (3.35) is now replaced by $V(y + A_+)$. If $\rho_0(k)$ denotes the reduced reflection coefficient for a particular function $H_0(x)$, then, by proposition 2.3 with $\xi = A_+ - A_{0,+}$,

$$\rho(k) = \mathrm{e}^{2\mathrm{i}k(A_+ - A_{0,+})}\rho_0(k)$$

and thus, by (2.3),

$$R(k) = R_0(k). \tag{3.42}$$

It remains to deal with the generic case when $G_1(-\infty) = 0$. We first show that for any H_+ the function $H(x)$ arising from the solution of (3.38) obeys (H2) and (H3). It suffices to consider $x < 0$, since for $x > 0$ the reasoning is the same as in the case of theorem 3.2; there is no difference between the generic and exceptional cases when $x > 0$. Since in the generic case, $f_l(0, x)$ does not approach a finite limit as $x \to -\infty$, the arguments based on (3.28) have to be refined. We first describe the idea behind the proof and then fill in the technical details. Again we assume that there is a solution $y_0(x)$ with a corresponding function $H_0(x)$ obeying (H1)–(H4). We can think of $H_0(x)$ as the function $H(x)$ that, via its reflection coefficient, determines $\tilde{X}_1(0, y)$ in (3.38). Now define $\eta = \eta(x)$ such that

$$H_+ G_1(\eta) = H_{0,+} G_1(x). \tag{3.43}$$

Due to the monotonicity of $G_1(x)$, $\eta(x)$ is uniquely determined, and it satisfies $\eta(x) \to -\infty$ as $x \to -\infty$. In order to avoid possible confusion we mention that the relevant function is $y(\eta)$ and not $y(\eta(x))$; in fact, we have $y(\eta(x)) = y_0(x)$. From (3.38) and (3.43) we have

$$y(\eta) + A_+ + \tilde{X}_1(0, y(\eta) + A_+) = y_0(x) + A_{0,+} + \tilde{X}_1(0, y_0(x) + A_{0,+}).$$

Consequently, by the monotonicity of the function $z + \tilde{X}_1(0, z)$, we conclude that

$$y(\eta) + A_+ = y_0(x) + A_{0,+}. \tag{3.44}$$

Differentiating (3.44) with respect to x we obtain

$$y_0'(x) = y'(\eta)\eta'(x). \tag{3.45}$$

Let us assume for the moment that $\eta'(x)$ has a limit as $x \to -\infty$. Then $y'(\eta)$ also has a limit as $\eta \to -\infty$, which we call H_-, and (3.45) implies that

$$\lim_{x\to-\infty} \eta'(x) = \frac{H_{0,-}}{H_-}. \tag{3.46}$$

Moreover, we can write

$$y'(\eta) - H_- = \frac{1}{\eta'(x)}[y_0'(x) - H_{0,-}] + \frac{H_-}{\eta'(x)}\left[\frac{H_{0,-}}{H_-} - \eta'(x)\right].$$

This suggests that in order to verify (H3) for $H(x)$ we must show that

$$\frac{H_{0,-}}{H_-} - \eta' \in L_1(\mathbb{R}^-) \tag{3.47}$$

since the difference $y_0'(x) - H_{0,-}$ satisfies (H3) by assumption.

Next we turn to the justification of the steps leading to (3.47) and of (3.47) itself. Integrating (3.13) and using $G_1(-\infty) = 0$ and (3.7)–(3.10), we obtain

$$G_1(x) = \int_{-\infty}^{x} \frac{\mathrm{d}z}{f_l(0, z)^2} = -\frac{1}{c_l^2 x} - \frac{d_l}{c_l^3 x^2} + \frac{2}{c_l^3}\int_{-\infty}^{x} \mathrm{d}z\, \frac{\epsilon_l(z)}{z^3} + O\left(\frac{1}{x^3}\right) \qquad x \to -\infty. \tag{3.48}$$

Let

$$\varphi(x) = \int_{-\infty}^{x} \mathrm{d}z\, \frac{\epsilon_l(z)}{z^3}. \tag{3.49}$$

Using (3.48) in (3.43) we have

$$\eta(x) = \frac{H_+}{H_{0,+}} x + O(1) \qquad x \to -\infty \tag{3.50}$$

and then using (3.50) we obtain

$$\eta(x) = \frac{H_+}{H_{0,+}} x + \frac{d_1(H_{0,+} - H_+)}{c_1 H_{0,+}} + \frac{2H_+ x}{c_1 H_{0,+}} [x\varphi(x) - \eta\varphi(\eta)] + O\left(\frac{1}{x}\right) \qquad x \to -\infty. \tag{3.51}$$

We have left an η-dependent term on the right-hand side in order to combine it with another term later. From (3.51) we obtain

$$\frac{c_1\eta - d_1}{c_1 x - d_1} = \frac{H_+}{H_{0,+}} + \frac{2H_+}{H_{0,+}} \frac{x[x\varphi(x) - \eta\varphi(\eta)]}{c_1 x - d_1} + O\left(\frac{1}{x^2}\right) \qquad x \to -\infty. \tag{3.52}$$

Differentiating (3.43) and using (3.13) we get

$$\eta'(x) = \frac{H_{0,+}}{H_+} \frac{f_1(0,\eta)^2}{f_1(0,x)^2} = \frac{H_{0,+}}{H_+} \frac{(c_1\eta - d_1)^2}{(c_1 x - d_1)^2} \frac{[1 - \epsilon_1(\eta)/(c_1\eta - d_1)]^2}{[1 - \epsilon_1(x)/(c_1 x - d_1)]^2}. \tag{3.53}$$

Expanding the right-hand side of (3.53) with the help of (3.51) and (3.52), we find

$$\eta'(x) = \frac{H_+}{H_{0,+}} + \frac{2H_+}{c_1 H_{0,+}} \left[2x\varphi(x) + \frac{\epsilon_1(x)}{x} - \left(2\eta\varphi(\eta) + \frac{\epsilon_1(\eta)}{\eta}\right)\right] + O\left(\frac{1}{x^2}\right) \qquad x \to -\infty. \tag{3.54}$$

From (3.54) we see that

$$\lim_{x\to-\infty} \eta'(x) = \frac{H_+}{H_{0,+}} \tag{3.55}$$

and hence the steps leading to (3.46) have been justified. Moreover, comparing (3.46) and (3.55) we obtain

$$H_{0,-}H_{0,+} = H_- H_+. \tag{3.56}$$

By (3.54) and (3.56), in order to verify (3.47) it suffices to show that

$$I(x) := 2x\varphi(x) + \frac{\epsilon_1(x)}{x} \in L^1(\mathbb{R}^-).$$

Using (3.10), (3.49), and integration by parts we obtain

$$I(x) = x \int_{-\infty}^{x} \mathrm{d}s \, \frac{1}{s^2} \int_{-\infty}^{s} \mathrm{d}z \, Q(z) f_1(0,z).$$

Hence

$$|I(x)| \leqslant \int_{-\infty}^{x} \mathrm{d}z \, |Q(z)| |f_1(0,z)|$$

and thus after another integration by parts, we obtain

$$\int_{-\infty}^{0} \mathrm{d}x \, |I(x)| \leqslant \int_{-\infty}^{0} \mathrm{d}z \, |z| |Q(z)| |f_1(0,z)| < \infty.$$

This proves (H3). Property (H1) is clear from (3.45). The verification of (H4) and of $R(k) = R_0(k)$ is done as in the exceptional case. □

We remark that, in addition to (3.42), we have from (2.3) and (2.19)

$$L(k) = e^{-2ik(A-A_0)} L_0(k).$$

Moreover, in the exceptional case, using (3.31) we get

$$T_l(k) = e^{ik(A_0-A)} T_{0,l}(k) \qquad T_r(k) = e^{ik(A_0-A)} T_{0,r}(k)$$

and in the generic case, using (3.56), we have

$$T_l(k) = \frac{H_{0,+}}{H_+} e^{ik(A_0-A)} T_{0,l}(k) \qquad T_r(k) = \frac{H_+}{H_{0,+}} e^{ik(A_0-A)} T_{0,r}(k).$$

Hence unlike theorem 3.2, the scattering matrices $\boldsymbol{S}(k)$ are not the same for all potentials $H(x)$ resulting from the solution of (3.38).

The following example illustrates the case $G_1(-\infty) = 0$, in which case H_+ needs to be specified as part of the scattering data in order to obtain $H(x)$ uniquely.

Example 3.4. In order to avoid lengthy formulae we assume that $Q(x) = \delta(x-1)$, where δ denotes the Dirac delta function. This $Q(x)$ does not satisfy (H5), but it can be approximated by $Q(x)$ that do, without affecting the conclusions of the example. For the reflection coefficient we take

$$R(k) = \frac{1+ik}{-1+3ik} e^{-4ik}. \tag{3.57}$$

Such reflection coefficients were considered in [AKV95] (example 6.2, with $\mu = 1/3$, $\xi = 1/3$ and $\beta = -4$), where we solved (3.16). We have

$$\tilde{X}_1(0,y) = \begin{cases} 0 & y > 2 \\ \dfrac{(y-2)(y-1)}{3-y} & y < 2 \end{cases}$$

$$f_l(0,x) = \begin{cases} 1 & x > 1 \\ 2-x & x < 1 \end{cases}$$

$$-i\dot{f}_{l,1}(0,x) = \begin{cases} x & x > 1 \\ 1 & x < 1. \end{cases}$$

Therefore

$$G_1(x) = \begin{cases} x & x > 1 \\ \dfrac{1}{2-x} & x < 1. \end{cases}$$

Thus we are in the generic case with $G_1(-\infty) = 0$ and we also have $w_0 = 0$. From (3.39) we obtain $A_+ = H_+/2$ if $H_+ \geqslant 4$ and $A_+ = 3 - (4/H_+)$ if $H_+ < 4$. Solving (3.38) we obtain

(i) if $H_+ > 2$:

$$H(x) = \begin{cases} H_+ & x > 1 \\ \dfrac{H_+}{(2-x)^2} & \dfrac{4-H_+}{2} < x < 1 \\ \dfrac{2}{H_+} & x < \dfrac{4-H_+}{2} \end{cases}$$

(ii) if $0 < H_+ < 2$:

$$H(x) = \begin{cases} H_+ & x > \dfrac{2}{H_+} \\ \dfrac{2}{H_+ x^2} & 1 < x < \dfrac{2}{H_+} \\ \dfrac{2}{H_+} & x < 1 \end{cases}$$

(iii) if $H_+ = 2$:

$$H(x) = \begin{cases} 2 & x > 1 \\ 1 & x < 1. \end{cases}$$

The functions $H(x)$ all have the same reflection coefficient given by (3.57), independently of the value of H_+. Note that the product of $\lim_{x\to-\infty} H(x)$ and H_+ does not depend on H_+ (cf (3.56)).

4. An algorithm to recover discontinuities

In this section we first show how certain characteristic quantities associated with the discontinuities of $H(x)$ can be recovered knowing only the leading asymptotic behaviour of $\rho(k)$ for large k. Among these quantities are the number N of discontinuities, the values y_j, and the ratios q_j given by (2.14). Later in the section we will study the recovery of these quantities when the large k-asymptotics of $R(k)$ is used instead of the asymptotics of $\rho(k)$. The algorithm described here can be applied to the (reduced) reflection coefficient associated with (1.1) satisfying (H1)–(H5), and it is not restricted to sectionally constant $H(x)$. We will see that, in the special case when $H(x)$ is sectionally constant, our algorithm recovers $H(x)$ exactly.

According to (2.31) the leading term of $\rho(k)$ as $k \to \infty$ is given by

$$\rho_{\mathrm{as}}(k) = -\frac{b(k)}{a(k)}$$

and from (2.26)–(2.28) we see that $\rho_{\mathrm{as}}(k)$ is completely determined by N, y_j, and q_j for $j = 1, \dots, N$. In particular, $Q(x)$ has no influence on $\rho_{\mathrm{as}}(k)$. In order to recover the locations $x_1, \dots, x_N$ of the discontinuities, further information about $H(x)$ is required. For example, if $H(x)$ is known to be piecewise constant, given $\rho_{\mathrm{as}}(k)$ and either H_+ or H_-, the points $x_1, \dots, x_N$ can be determined uniquely; hence in this special case, $H(x)$ itself is recovered uniquely by our algorithm. As we will see in section 5, when $\rho(k) = \rho_{\mathrm{as}}(k)$, the product $H(x) f_l(0, x)^2$ is a piecewise constant function; in that case our algorithm also yields $H(x)$ in terms of $\rho(k)$ and either H_+ or H_-.

We first observe that $|a(k)|^2 = 1/(1 - |\rho_{\mathrm{as}}(k)|^2)$ and that one can construct $a(k)$ from $|a(k)|$ as described in [AKV94]. Hence $b(k) = -\rho_{\mathrm{as}}(k)a(k)$ is also known. Therefore, we may assume that $a(k)$ and $b(k)$ are known separately. Note that in an application $\rho_{\mathrm{as}}(k)$ might not initially be available as the ratio $-b(k)/a(k)$ with given functions $a(k)$ and $b(k)$. However, as indicated in theorem 2.4, $\rho_{\mathrm{as}}(k)$ is almost periodic, and hence from a given $\rho(k)$ we can always find $\rho_{\mathrm{as}}(k)$ in the form

$$\rho_{\mathrm{as}}(k) = \sum_{n=-\infty}^{\infty} \rho_n \mathrm{e}^{\mathrm{i}k\eta_n}$$

with $\sum_{n=-\infty}^{\infty}|\rho_n|<\infty$, $\eta_n\in\mathbb{R}$, and

$$\rho_n=\lim_{L\to+\infty}\frac{1}{2L}\int_{-L}^{L}\mathrm{d}k\,\rho(k)\mathrm{e}^{-\mathrm{i}k\eta_n}.$$

Note that ρ_n are real because of the symmetry $\rho_{\mathrm{as}}(-k)=\overline{\rho_{\mathrm{as}}(k)}$ for $k\in\mathbb{R}$. For later reference we list the expressions for $a(k)$ and $b(k)$ when $N=1,2,3,4$.

If $N=1$:

$$a(k)=\alpha_1 \qquad \mathrm{e}^{2\mathrm{i}ky_1}b(k)=\beta_1. \tag{4.1}$$

If $N=2$:

$$a(k)=\alpha_1\alpha_2+\beta_1\beta_2\mathrm{e}^{2\mathrm{i}k(y_2-y_1)} \tag{4.2}$$

$$\mathrm{e}^{2\mathrm{i}ky_2}b(k)=\alpha_1\beta_2+\beta_1\alpha_2\mathrm{e}^{2\mathrm{i}k(y_2-y_1)}. \tag{4.3}$$

If $N=3$:

$$a(k)=\alpha_1\alpha_2\alpha_3+\beta_1\beta_2\alpha_3\mathrm{e}^{2\mathrm{i}k(y_2-y_1)}+\alpha_1\beta_2\beta_3\mathrm{e}^{2\mathrm{i}k(y_3-y_2)}+\beta_1\alpha_2\beta_3\mathrm{e}^{2\mathrm{i}k(y_3-y_1)} \tag{4.4}$$

$$\mathrm{e}^{2\mathrm{i}ky_3}b(k)=\alpha_1\alpha_2\beta_3+\beta_1\beta_2\beta_3\mathrm{e}^{2\mathrm{i}k(y_2-y_1)}+\alpha_1\beta_2\alpha_3\mathrm{e}^{2\mathrm{i}k(y_3-y_2)}+\beta_1\alpha_2\alpha_3\mathrm{e}^{2\mathrm{i}k(y_3-y_1)}. \tag{4.5}$$

If $N=4$:

$$\begin{aligned}a(k)=\alpha_1\alpha_2\alpha_3\alpha_4&+\beta_1\beta_2\alpha_3\alpha_4\mathrm{e}^{2\mathrm{i}k(y_2-y_1)}+\alpha_1\beta_2\beta_3\alpha_4\mathrm{e}^{2\mathrm{i}k(y_3-y_2)}+\beta_1\alpha_2\beta_3\alpha_4\mathrm{e}^{2\mathrm{i}k(y_3-y_1)}\\&+\alpha_1\alpha_2\beta_3\beta_4\mathrm{e}^{2\mathrm{i}k(y_4-y_3)}+\alpha_1\beta_2\alpha_3\beta_4\mathrm{e}^{2\mathrm{i}k(y_4-y_2)}+\beta_1\alpha_2\alpha_3\beta_4\mathrm{e}^{2\mathrm{i}k(y_4-y_1)}\\&+\beta_1\beta_2\beta_3\beta_4\mathrm{e}^{2\mathrm{i}k(y_4-y_3+y_2-y_1)}\end{aligned} \tag{4.6}$$

$$\begin{aligned}\mathrm{e}^{2\mathrm{i}ky_4}b(k)=\alpha_1\alpha_2\alpha_3\beta_4&+\beta_1\beta_2\alpha_3\beta_4\mathrm{e}^{2\mathrm{i}k(y_2-y_1)}+\alpha_1\beta_2\beta_3\beta_4\mathrm{e}^{2\mathrm{i}k(y_3-y_2)}+\beta_1\alpha_2\beta_3\beta_4\mathrm{e}^{2\mathrm{i}k(y_3-y_1)}\\&+\alpha_1\alpha_2\beta_3\alpha_4\mathrm{e}^{2\mathrm{i}k(y_4-y_3)}+\alpha_1\beta_2\alpha_3\alpha_4\mathrm{e}^{2\mathrm{i}k(y_4-y_2)}+\beta_1\alpha_2\alpha_3\alpha_4\mathrm{e}^{2\mathrm{i}k(y_4-y_1)}\\&+\beta_1\beta_2\beta_3\alpha_4\mathrm{e}^{2\mathrm{i}k(y_4-y_3+y_2-y_1)}.\end{aligned} \tag{4.7}$$

The expressions for $a(k)$ and $b(k)$ when N is arbitrary can be obtained from (2.28); the expression for $a(k)$ was also given in [Gr91]. In the following we will make certain statements about the general form of $a(k)$ and $b(k)$. These can all be easily proved by induction on N noting the fact that in view of (2.28) adding a discontinuity on the right corresponds to a multiplication from the right by a known matrix. For example, we note that $a(k)$ and $b(k)$ are exponential polynomials having at most 2^{N-1} non-zero terms. Note also that for a given N, $\mathrm{e}^{2\mathrm{i}ky_N}b(k)$ is obtained from $a(k)$ and *vice versa* by interchanging α_N and β_N. We write

$$k)=a_0+\sum_n a_n\mathrm{e}^{2\mathrm{i}k\lambda_n} \tag{4.8}$$

$$b(k)=\sum_n b_n\mathrm{e}^{2\mathrm{i}k\xi_n}$$

where $a_0>0$ and a_n, b_n are non-zero real constants. It is evident from (4.1)–(4.7) and can also be proved by induction on N that $\lambda_n>0$, and that the ξ_n are of the form λ_n-y_N.

Next we list the steps of the algorithm allowing us to recover N, y_j, and q_j from $\rho_{\mathrm{as}}(k)$. Recall that $a(k)$ and $b(k)$ are known when $\rho_{\mathrm{as}}(k)$ is given.

(1) From $b(k)$, we obtain y_N as $y_N=-\min_n\xi_n$. Note that the coefficient of that exponential term is $\alpha_1\alpha_2\cdots\alpha_{N-1}\beta_N$.

(2) The constant term in $a(k)$ is equal to $\alpha_1\alpha_2\cdots\alpha_{N-1}\alpha_N$.

(3) From the ratio of the coefficients in steps (1) and (2) above, we obtain β_N/α_N and hence

$$q_N=\frac{1+\beta_N/\alpha_N}{1-\beta_N/\alpha_N}.$$

(4) We construct the matrix $E(k, x_N)$ defined in (2.27) by using y_N and q_N.

(5) From (2.27), we obtain the matrix $E(k, x_N)^{-1}$ and then define

$$\begin{bmatrix} a^{[N-1]}(k) & b^{[N-1]}(k) \\ b^{[N-1]}(-k) & a^{[N-1]}(-k) \end{bmatrix} := \begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix} E(k, x_N)^{-1}.$$

Note that $a^{[N-1]}(k)$, when not constant, has again the form (4.8) with $a_0 > 0$ and a_n, b_n non-zero real constants, but with fewer terms.

(6) We replace $a(k)$ and $b(k)$ by $a^{[N-1]}(k)$ and $b^{[N-1]}(k)$, respectively, and repeat steps (1)–(5). This results in functions $a^{[N-2]}(k)$ and $b^{[N-2]}(k)$. We repeat the procedure until the matrix in step (5) no longer contains any exponential terms on the diagonal, i.e. until we arrive at the matrix $E(k, x_1)$. From it we find y_1 and q_1.

Note that it is possible to determine y_1 right after step (1) of the algorithm as follows. From (4.8) we obtain $y_N - y_1$ as $y_N - y_1 = \max_n \lambda_n$; note that there is a unique term for which this maximum occurs and that the coefficient in front of this exponential term is $\beta_1\alpha_2 \cdots \alpha_{N-1}\beta_N$; hence, having obtained y_N from step (1), we also have y_1. This determination of y_1 can help us to check the correctness of the computations since y_1 is also determined as explained in step (6). There are also ways to speed up the algorithm if further information on $H(x)$ is available. For example, if $Q(x) = 0$, H_+ is given, and $H(x)$ is known to be piecewise constant, then theorem 5.1 implies that $\rho(k) = \rho_{\mathrm{as}}(k)$. We can therefore use $\rho(0) = -b(0)/a(0)$ in (2.7) with $\gamma = 1$ to determine $H_- = H(x_1 - 0)$. Then we can use the fact that under a reflection $x \to -x$ the function $a(k)$ remains invariant, whereas the function $b(k)$ changes to $-b(-k)$. Using this property, we can determine q_1 at the same time we determine q_N. Then, we can carry out the algorithm by working from both ends. As we will see in section 5, whenever $\rho(k) = \rho_{\mathrm{as}}(k)$, our algorithm gives us $H(x)$ in terms of $\rho(k)$ and H_+.

In the following example we illustrate the above algorithm (without the improvements mentioned in the previous paragraph). Assuming $H(x)$ is piecewise constant and H_+ is given, we also determine the values $x_1, \ldots, x_N$.

Example 4.1. Assume $a(k) = f_1(k)/1008\sqrt{2}$ and $b(k) = \mathrm{e}^{-76\mathrm{i}k} f_2(k)/1008\sqrt{2}$, where

$$f_1(k) = 1625 + 130\mathrm{e}^{6\mathrm{i}k} - 50\mathrm{e}^{42\mathrm{i}k} + 25\mathrm{e}^{44\mathrm{i}k} - 25\mathrm{e}^{48\mathrm{i}k} + 2\mathrm{e}^{50\mathrm{i}k} - 130\mathrm{e}^{86\mathrm{i}k} - 65\mathrm{e}^{92\mathrm{i}k}$$
$$f_2(k) = 325 + 26\mathrm{e}^{6\mathrm{i}k} - 10\mathrm{e}^{42\mathrm{i}k} + 125\mathrm{e}^{44\mathrm{i}k} - 5\mathrm{e}^{48\mathrm{i}k} + 10\mathrm{e}^{50\mathrm{i}k} - 650\mathrm{e}^{86\mathrm{i}k} - 325\mathrm{e}^{92\mathrm{i}k}.$$

From the term $325\mathrm{e}^{-76\mathrm{i}k}/1008\sqrt{2}$ in $b(k)$, we see that

$$-2y_N = -76 \qquad \alpha_1 \cdots \alpha_{N-1}\beta_N = \frac{325}{1008\sqrt{2}}$$

and from the constant term in $a(k)$ we obtain

$$\alpha_1 \cdots \alpha_{N-1}\alpha_N = \frac{1625}{1008\sqrt{2}}.$$

Hence $y_N = 38$ and $\beta_N/\alpha_N = 325/1625 = 1/5$, and so $q_N = 3/2$. Next we construct $E(k, x_N) = \begin{bmatrix} \alpha_N & \beta_N \mathrm{e}^{-2\mathrm{i}k y_N} \\ \beta_N \mathrm{e}^{2\mathrm{i}k y_N} & \alpha_N \end{bmatrix}$ by using (2.26) with $n = N$ and obtain $\alpha_N = \frac{5}{2\sqrt{6}}$, $\beta_N = \frac{1}{2\sqrt{6}}$, and thus

$$E(k, x_N) = \frac{1}{2\sqrt{6}} \begin{bmatrix} 5 & \mathrm{e}^{-76\mathrm{i}k} \\ \mathrm{e}^{76\mathrm{i}k} & 5 \end{bmatrix}.$$

Then we form the matrix $\begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix} E(k, x_N)^{-1}$ and execute step (6). We obtain $y_{N-1} = 16$ and $q_{N-1} = 7/6$. A further repetition gives $y_{N-2} = -5$ and $q_{N-2} = 3/7$. We also obtain

$$\prod_{n=1}^{N-3} E(k, x_n) = \frac{1}{\sqrt{24}} \begin{bmatrix} 5 & -\mathrm{e}^{16\mathrm{i}k} \\ -\mathrm{e}^{-16\mathrm{i}k} & 5 \end{bmatrix}.$$

Since there are no exponential terms on the diagonal, this must be the matrix $E(k, x_1)$. Hence $N = 4$, $y_1 = -8$, and $q_1 = 2/3$.

If we further assume that $H_+ = 4$ and that $H(x)$ is piecewise constant, using $H(x_4 + 0) = H_+ = 4$ and $q_4 = 2/3$, we obtain $H(x_4 - 0) = H(x_3 + 0) = 6$; then using $q_3 = 7/6$ we obtain $H(x_3 - 0) = H(x_2 + 0) = 7$; from $q_2 = 3/7$ we have $H(x_2 - 0) = H(x_1 + 0) = 3$; finally, using $q_1 = 2/3$ we obtain $H(x_1 - 0) = H_- = 2$. Hence we have

$$H(x) = \begin{cases} 2 & y < -8 \\ 3 & -8 < y < -5 \\ 7 & -5 < y < 16 \\ 6 & 16 < y < 38 \\ 4 & y > 38. \end{cases}$$

Finally, by using $y_j = \int_0^{x_j} \mathrm{d}s\, H(s)$, we obtain $x_1 = -12/7$, $x_2 = -5/7$, $x_3 = 16/7$, and $x_4 = 125/21$.

Let us also note that in the special case when $H(x)$ is piecewise constant, it is possible, under certain restrictions, to recover $H(x)$ by using an algorithm described in Ware and Aki [WA69], which is known as the Goupillaud solution. However, that algorithm is only applicable when $H(x)$ is piecewise constant and each interval in which $H(x)$ is constant can be divided into a number of subintervals such that the travel times through these subintervals are the same. Because of the latter restriction, the Goupillaud method is unable to exactly recover a piecewise constant $H(x)$ even in simple cases. For example, when $H(x) = 2$ for $x \in (0, 1)$ and $H(x) = 3$ for $x \in (1, 1 + \pi)$, it is impossible to subdivide these intervals into a finite number of subintervals for which the travel times are the same, since we cannot find integers p, q such that $2/p = 3\pi/q$. In our opinion, the Goupillaud solution is more suitable as an approximate inversion method for a continuous non-homogeneous medium modeled by a layered medium with a suitably chosen number of layers. By contrast, the method described in this paper can be used in the special case of piecewise constant $H(x)$ without such limitations.

An algorithm for finding N, q_j, and the differences $T_j := y_{j+1} - y_j$ for $j = 1, \ldots, N-1$ from $|a(k)|$ alone was given in [Gr90, Gr91]. Below we will only refer to [Gr91], where a more detailed account was given. In [Gr91] there is the additional restriction that T_i/T_j has to be irrational whenever $i \neq j$; this restriction implies that the exponents of the 2^{N-1} terms in $a(k)$ are all different. Without this restriction some of the terms in $a(k)$ or $b(k)$ may have the same exponential factors and hence the number of distinct terms in $a(k)$ or $b(k)$ may be less than 2^{N-1}, in which case the algorithm of [Gr91] cannot lead to a unique $H(x)$. Since the coefficients of the exponential terms in $b(k)$ may be positive as well as negative, it is possible that sometimes the number of terms in $a(k)$ may be different from that in $b(k)$. In general, one cannot even obtain N from $|a(k)|$ and H_+ alone (or even from $a(k)$ and H_+ alone). Our algorithm described in this section does not have such restrictions; furthermore, it only involves simple algebraic matrix operations and hence it is easy to implement.

In the following example we show that one cannot even determine N from $a(k)$ and H_+ alone.

Example 4.2. Let

$$a_1(k) = \frac{45}{16\sqrt{6}} - \frac{5}{16\sqrt{6}}\mathrm{e}^{24\mathrm{i}k} \qquad b_1(k) = \frac{15}{16\sqrt{6}}\mathrm{e}^{-24\mathrm{i}k} - \frac{1}{2\sqrt{6}}\mathrm{e}^{-12\mathrm{i}k} - \frac{15}{16\sqrt{6}}.$$

Assuming that $H(x) = H_1(x)$ is piecewise constant and $H_+ = 3/2$, our algorithm gives us

$$H_1(x) = \begin{cases} 1 & x < 0 \\ 2 & 0 < x < 3 \\ 3 & 3 < x < 5 \\ \frac{3}{2} & x > 5. \end{cases}$$

We have $\alpha_1 = \frac{3}{2\sqrt{2}}$, $\alpha_2 = \frac{5}{2\sqrt{6}}$, $\alpha_3 = \frac{3}{2\sqrt{2}}$, $\beta_1 = -\frac{1}{2\sqrt{2}}$, $\beta_2 = -\frac{1}{2\sqrt{6}}$, and $\beta_3 = \frac{1}{2\sqrt{2}}$. Note that in this example $N = 3$, $y_1 = 0$, $y_2 = 6$, and $y_3 = 12$. Also, $\beta_1\beta_2\alpha_3 = -\alpha_1\beta_2\beta_3$, and thus the second and third terms on the right-hand side of (4.4) cancel. Now let

$$a_2(k) = a_1(k) \qquad b_2(k) = \frac{-4 - \sqrt{241}}{16\sqrt{6}}\mathrm{e}^{-24\mathrm{i}k} + \frac{-4 + \sqrt{241}}{16\sqrt{6}}.$$

Assuming again that $H(x) = H_2(x)$ is piecewise constant and $H_+ = 3/2$, our algorithm gives us

$$H_2(x) = \begin{cases} 1 & x < 0 \\ \dfrac{25 - \sqrt{241}}{16} = 0.5922\ldots & 0 < x < \frac{1}{2}(25 + \sqrt{241}) = 20.2620\ldots \\ \dfrac{3}{2} & x > \frac{1}{2}(25 + \sqrt{241}). \end{cases}$$

Note that now $N = 2$, but that $a(k)$ is the same for $H_1(x)$ and $H_2(x)$.

Finally, we discuss the modifications needed in our algorithm when $R_{\mathrm{as}}(k)$ is known instead of $\rho_{\mathrm{as}}(k)$. By $R_{\mathrm{as}}(k)$ we mean the almost periodic part of $R(k)$ given by (cf (2.3))

$$R_{\mathrm{as}}(k) = \rho_{\mathrm{as}}(k)\mathrm{e}^{-2\mathrm{i}kA_+}. \tag{4.9}$$

In order to deal with the factor $\mathrm{e}^{-2\mathrm{i}kA_+}$, we consider for a moment the shifted functions $Q(x;\kappa) = Q(x+\kappa)$ and $H(x;\kappa) = H(x+\kappa)$. It follows that the corresponding potential $V(y;\kappa)$ in (2.11) satisfies $V(\cdot;\kappa) = V(\cdot + y(\kappa))$. Therefore, proposition 2.3 implies that

$$\rho(k;\kappa) = \exp\left[2\mathrm{i}k\int_0^\kappa \mathrm{d}s\, H(s)\right]\rho(k)$$

and thus

$$\rho_{\mathrm{as}}(k;\kappa) = \exp\left[2\mathrm{i}k\int_0^\kappa \mathrm{d}s\, H(s)\right]\rho_{\mathrm{as}}(k). \tag{4.10}$$

Now choose κ_0 such that

$$\int_0^{\kappa_0} \mathrm{d}s\, H(s) = -A_+. \tag{4.11}$$

Then we see from (4.9)–(4.11) that $\rho_{\mathrm{as}}(k;\kappa_0) = R_{\mathrm{as}}(k)$. Hence we can use the algorithm with $R_{\mathrm{as}}(k)$ because it can be viewed as the reduced reflection coefficient associated with $Q(x;\kappa_0)$ and $H(x;\kappa_0)$. Note that the parameters q_j and N are invariant under the shift, and so are the differences $y_j - y_i$. The values $x_{\kappa_0,j}$ and $y_{\kappa_0,j}$, where $H(x;\kappa_0)$ and $V(y;\kappa_0)$ have discontinuities, are given by

$$x_{\kappa_0,j} = x_j - \kappa_0 \qquad y_{\kappa_0,j} = y_j - \int_0^{\kappa_0} \mathrm{d}s\, H(s).$$

Thus it is the values of N, q_j, and $y_{\kappa_0,j}$ that we obtain as a result of applying the algorithm to $R_{\mathrm{as}}(k)$. Of course, if $\rho(k)$, $Q(x)$, and H_+ are known, then A_+ is also known (cf (3.19) and (3.39)), and hence κ_0 can be determined from (4.11).

If $H(x)$ is known to be piecewise constant, then given $R_{\mathrm{as}}(k)$ and H_+, our algorithm allows us to determine $H(x;\kappa_0)$, including the points $x_{\kappa_0,1},\dots,x_{\kappa_0,N}$. Furthermore, we can find κ_0 by using (2.4), (4.11), and the fact that $H(x) = H(x-\kappa_0;\kappa_0)$. This leads to the equation $\kappa_0 = -A_{\kappa_0,+}/H_+$, where $A_{\kappa_0,+}$ is given by (2.4) in terms of $H(x;\kappa_0)$. Consequently, we can completely determine a piecewise constant $H(x)$ from $R_{\mathrm{as}}(k)$ and H_+.

5. Inversion for almost periodic reflection coefficients

In this section, when there are no bound states, we characterize those functions $H(x)$ that satisfy (H1)–(H4) and whose scattering coefficients are almost periodic functions of k. We first determine the functions $H(x)$ for which

$$\rho(k) = \rho_{\mathrm{as}}(k) = -\frac{b(k)}{a(k)}. \tag{5.1}$$

By theorem 2.4, if (5.1) holds, then $\rho_{\mathrm{as}} \in \mathrm{AP}^W$ due to theorem 2.4. Conversely, if $\rho \in \mathrm{AP}^W$, then $\rho + b/a \in \mathrm{AP}^W$. Then the $o(1)$-term in (2.31) must be identically zero and (5.1) follows. In other words, $\rho(k) = \rho_{\mathrm{as}}(k)$ if and only if $\rho \in \mathrm{AP}^W$. Next note that when $\rho(k) = \rho_{\mathrm{as}}(k)$, by using (2.29) and the unitarity of the matrix $\sigma(k)$, we have $1/|\tau(k)| = |a(k)|$. Hence $\tau(0) \neq 0$, and we are automatically in the exceptional case. Using $\tau(-k) = \overline{\tau(k)}$ and $a(-k) = \overline{a(k)}$ for $k \in \mathbb{R}$, we obtain

$$\tau(k)a(k) = \frac{1}{\tau(-k)a(-k)} \qquad k \in \mathbb{R}. \tag{5.2}$$

Note that $a(k)$ and $1/\tau(k)$ are analytic in $\overline{\mathbb{C}^+}$, and in the absence of bound states $\tau(k)$ is analytic in $\overline{\mathbb{C}^+}$. Thus, in the absence of bound states, using (2.30), (5.2), and Liouville's theorem, we conclude that

$$\tau(k) = \frac{1}{a(k)}. \tag{5.3}$$

Furthermore, by using (2.5), (2.29), (5.1), and (5.3) we see that

$$\ell(k) = \ell_{\mathrm{as}}(k) = \frac{\overline{b(k)}}{a(k)} = -\overline{\rho(k)}\,\frac{\overline{a(k)}}{a(k)} \in \mathrm{AP}^W.$$

A close inspection of the origins of the $o(1)$-terms in (2.30) and (2.31) suggests that the condition $V(y) = 0$ for $y \in \mathbb{R}\setminus\{y_1,\dots,y_N\}$ will be part of any necessary and sufficient conditions for (5.1) to be valid. So it is natural to investigate this in more detail. If $V_{j,j+1}(y)$ given in (2.20) vanishes, then we have $g_{l,j,j+1}(k,y) = \mathrm{e}^{\mathrm{i}ky}$ and $g_{r,j,j+1}(k,y) = \mathrm{e}^{-\mathrm{i}ky}$ in (2.22), and therefore

$$\Gamma_{n-1,n}(k, x_n - 0) = \begin{bmatrix} \frac{1}{\sqrt{H(x_n-0)}}\mathrm{e}^{\mathrm{i}ky_n} & \frac{1}{\sqrt{H(x_n-0)}}\mathrm{e}^{-\mathrm{i}ky_n} \\ \mathrm{i}k\sqrt{H(x_n-0)}\mathrm{e}^{\mathrm{i}ky_n} - \frac{H'(x_n-0)}{2H(x_n-0)^{3/2}}\mathrm{e}^{\mathrm{i}ky_n} & -\mathrm{i}k\sqrt{H(x_n-0)}\mathrm{e}^{-\mathrm{i}ky_n} - \frac{H'(x_n-0)}{2H(x_n-0)^{3/2}}\mathrm{e}^{-\mathrm{i}ky_n} \end{bmatrix}$$

$$\Gamma_{n,n+1}(k, x_n + 0) = \begin{bmatrix} \frac{1}{\sqrt{H(x_n+0)}} e^{ik y_n} & \frac{1}{\sqrt{H(x_n+0)}} e^{-ik y_n} \\ ik\sqrt{H(x_n+0)} e^{ik y_n} - \frac{H'(x_n+0)}{2H(x_n+0)^{3/2}} e^{ik y_n} & -ik\sqrt{H(x_n+0)} e^{-ik y_n} - \frac{H'(x_n+0)}{2H(x_n+0)^{3/2}} e^{-ik y_n} \end{bmatrix}.$$

Hence

$$\Gamma_{n-1,n}(k, x_n - 0)^{-1}\Gamma_{n,n+1}(k, x_n + 0) = E(k, x_n) + \frac{\nu_n}{2ik} B(k, x_n) \tag{5.4}$$

where $E(k, x_n)$ is the matrix defined in (2.27), ν_n is given by (2.15), and

$$B(k, x_n) = \begin{bmatrix} 1 & e^{-2ik y_n} \\ -e^{2ik y_n} & -1 \end{bmatrix}.$$

Note that $\nu_n = 0$ if and only if $H'(x)/H(x)$ is continuous at x_n. Hence if $H'(x)/H(x)$ is continuous at each x_n, then by (2.25) and (5.4), we have $\tau(k) = 1/a(k)$ and $\rho(k) = -b(k)/a(k)$. So $V_{n,n+1}(y) = 0$ for $n = 0, \ldots, N$ and the continuity of $H'(x)/H(x)$ are sufficient for (5.1) to hold. Now we ask what functions $H(x)$ lead to $V_{n,n+1}(y) = 0$, or equivalently

$$\frac{H''(x)}{2H(x)^3} - \frac{3}{4}\frac{H'(x)^2}{H(x)^4} + \frac{Q(x)}{H(x)^2} = 0 \qquad x \in (x_n, x_{n+1}) \qquad n = 0, \ldots, N. \tag{5.5}$$

Substituting $H(x) = c/h(x)^2$ in (5.5) we obtain

$$h''(x) = Q(x)h(x) \tag{5.6}$$

that is, $h(x)$ can be any zero-energy solution of (1.1). Since we allow $H(x)$ to be discontinuous, we are looking for different solutions on each interval (x_j, x_{j+1}). For $x > x_N$ we must choose $h(x)$ proportional to $f_l(0, x)$; this is because $f_r(0, x)$ is a constant multiple of $f_l(0, x)$ and there are no other linearly independent solutions of (5.6) that remain bounded as $x \to +\infty$. Hence, in order to have $H(x)$ approaching a positive limit as $x \to +\infty$, we need to choose

$$H(x) = \frac{H_+}{f_l(0, x)^2} \qquad x > x_N.$$

Let us now impose the condition that $H'(x)/H(x)$ be continuous at each x_n. Beginning with x_N, since $H'(x)/H(x) = -2f_l'(0, x)/f_l(0, x)$, we see that the logarithmic derivative of $f_l(0, x)$ has to be continuous at x_N. Therefore, on (x_{N-1}, x_N), the solution $H(x)$ of (5.5) must be a constant multiple of $1/f_l(0, x)^2$. Arguing similarly on every interval (x_n, x_{n+1}) we obtain recursively

$$H(x) = \frac{h_{n,n+1}}{f_l(0, x)^2} \qquad x \in (x_n, x_{n+1}) \qquad n = 0, 1, \ldots, N \tag{5.7}$$

where $h_{N,N+1} = H_+$ and

$$h_{n-1,n} = \frac{H(x_n - 0)}{H(x_n + 0)} h_{n,n+1} = q_n h_{n,n+1} \qquad n = 1, \ldots, N. \tag{5.8}$$

So we have constructed a class of functions $H(x)$ for which (5.1) holds. Note that the algorithm described in section 4 can be used to obtain $h_{n,n+1}$ in terms of H_+ and $\rho_{as}(k)$. The next theorem shows that any $H(x)$ satisfying (H1)–(H4) and for which $\rho(k) = \rho_{as}(k)$ must be given by (5.7).

Theorem 5.1. In the absence of bound states, for a given $Q(x)$, consider all $H(x)$ satisfying (H1)–(H4). Then, $\rho(k) = \rho_{as}(k)$ if and only $H(x)$ is of the form (5.7).

Proof. If $H(x)$ is of the form (5.7), then from (2.11)–(2.15) we see that $\nu_n = 0$ for $n = 1, \ldots, N$, and $V(y) = 0$ everywhere except at y_n. Hence (5.1) follows from (2.25) and (5.4). Conversely, suppose (5.1) holds. From $\rho_{\mathrm{as}}(k)$ by means of the algorithm in section 4 we can determine the parameters N, y_j, and q_j of $H(x)$ for $j = 1, \ldots, N$. Then we use H_+ and the parameters q_j in (5.8) to construct a function $H(x)$ that is of the form (5.7). We then know the form of the function $H(x)$ on each interval (y_n, y_{n+1}) and all we need to do is find the points $x_1, \ldots, x_N$ that correspond to $y_1, \ldots, y_N$, respectively. If $N = 1$ and $y_1 = 0$, then $x_1 = 0$. If $N = 1$ and $y_1 \neq 0$, then we can proceed as in the case $N \geqslant 2$. If $N \geqslant 2$, then at least $N - 1$ of the points $y_1, \cdots, y_N$ must be non-zero. If at least one of these is positive, we can pick the smallest of them, say y_p. Then x_p is uniquely determined by

$$\frac{y_p}{h_{p-1,p}} = \int_0^{x_p} \frac{\mathrm{d}z}{f_l(0,z)^2} \tag{5.9}$$

and we recursively determine $x_{p+1}, \ldots, x_N$ using

$$\frac{y_{p+1} - y_p}{h_{p,p+1}} = \int_{x_p}^{x_{p+1}} \frac{\mathrm{d}z}{f_l(0,z)^2}.$$

Similarly, we can determine $x_{p-1}, x_{p-2}, \ldots, x_1$. If all y_j are non-positive, then we pick the one with smallest absolute value that is non-zero (either y_N or y_{N-1}) and find the corresponding x_j by using the appropriate integral of the form (5.9). We know that for the resulting function $H(x)$, (5.1) is satisfied, because $V(y) = 0$ for $y \in \mathbb{R} \setminus \{y_1, \ldots, y_N\}$ and $H'(x)/H(x)$ is continuous. By construction, this $H(x)$ is uniquely determined by $Q(x)$, $\rho(k)$, and H_+. Hence, by theorem 3.1, it is the only possible $H(x)$ for which (5.1) holds. □

We remark that we can prove an analogue of theorem 5.1 when $\rho(k) = \rho_{\mathrm{as}}(k)$ is replaced by $R(k) = R_{\mathrm{as}}(k)$, by arguing as in the proof of theorem 5.1 using (2.3) and (4.10).

Acknowledgments

The first author is grateful for the hospitality and support from the IMA at the University of Minnesota, where this work was completed. This material is based upon work supported by the National Science Foundation under grant DMS-9217627. This work was performed under the auspices of CNR-GNFM and partially supported by the research project 'Nonlinear problems in analysis and its physical, chemical, and biological applications: analytical, modeling, and computational aspects', of the Italian Ministry of Higher Education and Research (MURST).

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