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ON THE RECOVERY OF A DISCONTINUOUS WAVESPEED IN WAVE SCATTERING IN A NONHOMOGENEOUS MEDIUM

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Abstract. The wave propagation in a nonhomogeneous medium is considered in the frequency domain. An algorithm is described to recover the wavespeed when the properties of the medium change abruptly at a finite number of interfaces. The transmission coefficient is constructed from the reflection coefficient and the bound state energies. The wavespeed is obtained from the solution of a key singular integral equation that is uniquely solvable. The essential spectrum of the integral operator is studied using a class of explicitly solved examples.

1. Introduction

Consider the generalized Schrödinger equation

$$\psi''(k,x) + k^2 H(x)^2 \psi(k,x) = Q(x)\psi(k,x), \qquad x \in \mathbf{R}, \tag{1}$$

where the prime denotes differentiation with respect to the space coordinate x. This equation describes the propagation of waves in a nonhomogeneous medium where k^2 is

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energy, $1/H(x)$ is the wavespeed, and $Q(x)$ is the restoring force per unit length. Let $\mathbf{R}^- = (-\infty, 0)$, $\mathbf{R}^+ = (0, +\infty)$, and $L^1_\beta(I)$ be the Banach space of measurable complex-valued functions on I that are integrable with respect to the measure $(1+|x|)^\beta dx$. The coefficients $H(x)$ and $Q(x)$ are assumed to satisfy the following:

(H1) $H(x)$ is strictly positive and piecewise continuous with jump discontinuities at x_n for $n = 1, \cdots, N$. Here $x_1 < \cdots < x_N$.

(H2) $H(x) \to H_\pm$ as $x \to \pm\infty$, where $H_\pm$ are positive constants.

(H3) $H - H_\pm \in L^1(\mathbf{R}^\pm)$.

(H4) H' is absolutely continuous on (x_n, x_{n+1}) and $2H''H - 3(H')^2 \in L^1_1(x_n, x_{n+1})$ for $n = 0, \cdots, N$; here $x_0 = -\infty$ and $x_{N+1} = +\infty$.

(H5) $Q \in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha \in (0, 1]$.

There are two linearly independent solutions of (1), called the Jost solutions from the left and from the right, respectively, satisfying the boundary conditions

$$f_l(k,x) = \begin{cases} e^{ikH_+x} + o(1), & x \to +\infty, \\ \dfrac{1}{T_l(k)} e^{ikH_-x} + \dfrac{L(k)}{T_l(k)} e^{-ikH_-x} + o(1), & x \to -\infty, \end{cases}$$

$$f_r(k,x) = \begin{cases} \dfrac{1}{T_r(k)} e^{-iH_+x} + \dfrac{R(k)}{T_r(k)} e^{ikH_+x} + o(1), & x \to +\infty, \\ e^{-ikH_-x} + o(1), & x \to -\infty. \end{cases}$$

The functions $L(k)$ and $R(k)$ are called the reflection coefficients from the left and from the right, respectively; $T_l(k)$ and $T_r(k)$ are called the transmission coefficients from the left and from the right, respectively.

We are interested in the inverse scattering problem of determining $H(x)$ from the scattering data consisting of $R(k)$, H_+, $Q(x)$, the bound state energies, and the bound state norming constants. For a study of this inverse problem when $H(x)$ does not have any discontinuities, the reader is referred to [1]. When $H(x)$ has a finite number of jump discontinuities, Grinberg [9,10] has studied this inverse problem in the special case $Q(x) \equiv 0$. For $Q(x) \equiv 0$, the reduced reflection coefficients defined in (2) are strictly less than one in absolute value, which makes a key singular integral operator used in the inversion a strict contraction; furthermore in this case there are no bound states. When $Q(x) \not\equiv 0$ there may be bound states and the reduced reflection coefficients may be equal to -1 at $k = 0$; hence the analysis of the inverse problem becomes rather complicated.

The recovery of $H(x)$ in this general case has been studied recently [2] and the inversion is illustrated with some explicit examples [2,3]. In this article we discuss a modification of the inversion algorithm of [2] and treat a class of examples not found in [2,3].

Let us summarize briefly the inversion procedure described in [2].

1. In the upper half complex plane $\mathbf{C}^+$ the analyticity in k of the Jost solutions is established, and their Hölder continuity near $k = 0$ is proved.
2. A reduced scattering matrix is introduced as
$$\sigma(k) = \begin{bmatrix} \tau(k) & \rho(k) \\ \ell(k) & \tau(k) \end{bmatrix},$$
where
$$\tau(k) = \sqrt{\frac{H_+}{H_-}}\, T_l(k) e^{ikA} = \sqrt{\frac{H_-}{H_+}}\, T_r(k) e^{ikA},$$
$$\rho(k) = R(k) e^{2ikA_+}, \qquad \ell(k) = L(k) e^{2ikA_-}, \tag{2}$$
$$A = A_+ + A_-, \qquad A_\pm = \pm \int_0^{\pm\infty} ds\, [H_\pm - H(s)].$$
Note that $\sigma(k)$ is unitary and
$$\sigma(-k) = \overline{\sigma(k)}, \qquad k \in \mathbf{R}. \tag{3}$$
The asymptotics of $\tau(k)$, $\rho(k)$, and $\ell(k)$ as $k \to 0$ are derived. As a result, we have to distinguish between the *generic case* where $\tau(k) = ick + o(|k|^{1+\alpha})$ for some real $c \neq 0$, and the *exceptional case* where $\tau(k) = \tau(0) + o(|k|^\alpha)$ with $\tau(0) \neq 0$; here $\alpha \in (0,1)$ is the constant in (H5). In either case $\rho(k) = \rho(0) + o(|k|^\alpha)$ and $\ell(k) = \ell(0) + o(|k|^\alpha)$. In the generic case $\rho(0) = \ell(0) = -1$. When $\alpha = 1$ we have $\tau(k) = \tau(0) + O(k)$, $\rho(k) = \rho(0) + O(k)$, and $\ell(k) = \ell(0) + O(k)$.
3. In each interval (x_j, x_{j+1}) for $j = 0, 1, \cdots, N$, using the local Liouville transformation
$$y = y(x) = \int_0^x ds\, H(s), \qquad \psi(k,x) = \frac{\phi(k,y)}{\sqrt{H(x)}},$$
the generalized Schrödinger equation (1) is transformed into
$$\frac{d^2\phi(k,y)}{dy^2} + k^2\phi(k,y) = V(y)\phi(k,y),$$
where
$$V(y(x)) = \frac{2H''(x)H(x) - 3H'(x)^2}{4H(x)^4} + \frac{Q(x)}{H(x)^2}, \qquad x \in \mathbf{R} \setminus \{x_1, \cdots, x_N\}.$$

Let $y_n = y(x_n)$ for $n = 0, \cdots, N+1$. Define

$$V_{n,n+1}(y) = V(y)[\theta(y - y_n) - \theta(y - y_{n+1})], \qquad y \in \mathbf{R},$$

where $\theta(y)$ is the Heaviside function. Note that (H4) implies that $V_{n,n+1} \in L_1^1(\mathbf{R})$, and hence the scattering theory for the Schrödinger equations

$$\frac{d^2\phi_{n,n+1}(k,y)}{dy^2} + k^2\phi_{n,n+1}(k,y) = V_{n,n+1}(y)\phi_{n,n+1}(k,y) \tag{4}$$

is well understood [4,5,6]. Relating $\sigma(k)$ to the scattering matrices for (4), we obtain

$$\frac{1}{\tau(k)} = a(k)[1 + o(1)], \qquad k \to \infty \text{ in } \overline{\mathbf{C}^+}, \tag{5}$$

$$\rho(k) = -\frac{b(k)}{a(k)} + o(1), \qquad k \to \pm\infty, \tag{6}$$

$$\begin{bmatrix} a(k) & b(k) \\ b(-k) & a(-k) \end{bmatrix} = \prod_{n=1}^{N} \begin{bmatrix} \alpha_n & \beta_n e^{-2iky_n} \\ \beta_n e^{2iky_n} & \alpha_n \end{bmatrix}$$

with

$$\alpha_n = \frac{H(x_n - 0) + H(x_n + 0)}{2\sqrt{H(x_n - 0)H(x_n + 0)}}, \qquad \beta_n = \frac{H(x_n - 0) - H(x_n + 0)}{2\sqrt{H(x_n - 0)H(x_n + 0)}}.$$

Then $a(k)$ and $b(k)$ are almost periodic polynomials of the form $\sum_s \gamma_s e^{2ikc_s}$ with all γ_s real; in the case of $a(k)$, we also have $c_0 = 0$, $\gamma_0 \geq 1$, and all other $c_s > 0$. Moreover,

$$a(-k) = \overline{a(k)}, \qquad b(-k) = \overline{b(k)}, \qquad k \in \mathbf{R}, \tag{7}$$

$$|a(k)|^2 - |b(k)|^2 = 1. \tag{8}$$

4. Introducing

$$Z_l(k,y) = \sqrt{\frac{H(x)}{H_+}}\, e^{-ik(y+A_+)} f_l(k,x), \quad Z_r(k,y) = \sqrt{\frac{H(x)}{H_-}}\, e^{ik(y-A_-)} f_r(k,x), \tag{9}$$

we exploit the analyticity properties and large k-asymptotics of $Z_l(k,y)$ and $Z_r(k,y)$ to derive the Riemann-Hilbert problem

$$F_+(k,x,y) - F_-(k,x,y) = -\rho(k)e^{2iky}F_-(-k,x,y) + \frac{\rho(k)e^{2iky} - \rho(0)}{k}\,\frac{f_l(0,x)}{\sqrt{H_+}} \tag{10}$$

for $k \in \mathbf{R}$, $x \in \mathbf{R} \setminus \{x_1, \cdots, x_N\}$, and $y \in \mathbf{R} \setminus \{y_1, \cdots, y_N\}$, where $F_\pm(\pm k, x, y)$ is analytic in $k \in \mathbf{C}^+$ and vanishes as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Here x and y are real parameters. In fact,

$$F_+(k,x,y) = \frac{\tau(k)Z_r(k,y) - \tau(0)Z_r(0,y)}{k\sqrt{H(x)}}, \quad F_-(k,x,y) = \frac{Z_l(-k,y) - Z_l(0,y)}{k\sqrt{H(x)}}. \tag{11}$$

For $\alpha \in (0,1)$ and $y \in \mathbf{R} \setminus \{y_1, \cdots, y_N\}$, as $k \to 0$ in $\overline{\mathbf{C}^+}$ we have

$$Z_l(k,y) = Z_l(0,y) + o(|k|^\alpha), \qquad Z_r(k,y) = Z_r(0,y) + o(|k|^\alpha),$$

and if $\alpha = 1$ the error terms above are $O(k)$. Hence, using Plemelj's formulas [8], from (10) for $k \in \mathbf{R}$ we obtain the singular integral equation

$$\begin{aligned} F_-(k,x,y) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ds}{s + k - i0}\,\rho(-s)e^{-2isy}F_-(s,x,y) \\ = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ds}{s - k + i0}\,\frac{\rho(s)e^{2isy} - \rho(0)}{s}\,\frac{f_l(0,x)}{\sqrt{H_+}}. \end{aligned} \tag{12}$$

5. Let p be any constant such that $1 < p < 1/(1-\alpha)$ if $\alpha \in (0,1)$ and $p \in (1,\infty)$ if $\alpha = 1$. One proves that the singular integral equation (12) is uniquely solvable in the Hardy space $\mathbf{H}^p_-(\mathbf{R})$ for every right-hand side in $\mathbf{H}^p_-(\mathbf{R})$ and the solution $F_-(k,x,y)$ can be obtained by iteration. Here $\mathbf{H}^p_-(\mathbf{R})$ is the Banach space of all analytic functions on $\mathbf{C}^-$ for which $\sup_{\epsilon>0}\int_{-\infty}^{\infty} dk\,|f(k - i\epsilon)|^p$ is finite. For $p = 2$ a contraction argument suffices to prove the result, but for $p \neq 2$ the singular integral equation must first be written as a two-vector Riemann-Hilbert problem, the uniquely solvability of which can be proved by generalized Wiener-Hopf factorization [2].
6. Letting $\tilde{F}_-(k,y) = \frac{\sqrt{H_+}}{f_l(0,x)}F_-(k,x,y)$, from (9) and (11) we get the equation

$$-i\tilde{F}_-(0,y) = i\frac{\dot{f}_l(0,x)}{f_l(0,x)} + y + A_+ \tag{13}$$

relating x and y, where the overdot denotes the k-derivative. Note that $f_l(0,x)$ is known once $Q(x)$ is known and $\tilde{F}_-(k,y)$ is uniquely obtained by solving the singular integral equation

$$\begin{aligned} \tilde{F}_-(k,y) - \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ds}{s + k - i0}\,\rho(-s)e^{-2isy}\tilde{F}_-(s,y) \\ = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{ds}{s - k + i0}\,\frac{\rho(s)e^{2isy} - \rho(0)}{s}, \end{aligned} \tag{14}$$

which is obtained by dividing (12) by $\frac{f_l(0,x)}{\sqrt{H_+}}$. From (13) we then compute $y = y(x)$. Then A_+ follows from the initial condition $y(0) = 0$, and $H(x)$ follows by using $H(x) = y'(x)$. This method works only if $Q \in L^1_2(\mathbf{R})$ because we assumed the continuity of $F_-(k, x, y)$ at $k = 0$. For more general $Q(x)$ some modifications are available [2].

7. If there are bound states, the previous two steps of the inversion algorithm need to be modified. The details can be found in [2].

In this article we will describe a slightly different inversion algorithm. For simplicity it will be presented when there are no bound states. Instead of using $\rho(k)$ to solve (14) and then computing $y(x)$ from (13), we will solve (14), compute $\dot{\tau}(0)$ from $|\rho(k)|$ and then modify the stages 5 and 6 of the algorithm of [2]. The result will be an algorithm that works in the generic case for $Q \in L^1_{1+\alpha}(\mathbf{R})$ for any $\alpha \in (0, 1]$. We will also describe an inversion algorithm for the exceptional case. Finally, we will study the essential spectrum of the singular integral operator appearing in (14) in some detail.

2. Construction of the reduced transmission coefficient

Before formulating an alternative to steps 5 and 6 described in Section 1, we first construct $\tau(k)$ from $|\rho(k)|$ in the absence of bound states. If there are bound states, the function $\tau(k)$ obtained in the absence of bound states should be multiplied by a suitable rational function, as will be explained below.

Starting from $|\rho(k)|$ we obtain from (3) and the unitarity of $\sigma(k)$

$$\tau(k)\tau(-k) = 1 - |\rho(k)|^2, \qquad k \in \mathbf{R}. \tag{15}$$

In the absence of bound states, $\tau(k)$ is continuous in $\overline{\mathbf{C}^+}$, and is analytic and nonzero in $\mathbf{C}^+$. However, the oscillatory behavior of $\tau(k)$ as $k \to \pm\infty$ prevents one from obtaining $\tau(k)$ directly from (15) by using (generalized) Wiener-Hopf factorization or other methods.

Using (7) and (8) we derive

$$a(k)a(-k) = \left[1 - |\rho_{as}(k)|^2\right]^{-1}, \qquad k \in \mathbf{R}, \tag{16}$$

where $\rho_{as}(k) = -\frac{b(k)}{a(k)}$ is the asymptotic part of $\rho(k)$ given in (6). The right-hand side of (16) can be determined uniquely, because, as a result of (3), $|\rho_{as}(k)|^2$ is the asymptotic

part of $|\rho(k)|^2$. Since the right-hand side of (16) is strictly positive and belongs to the Banach algebra AP^W of almost periodic functions with absolutely convergent Fourier series, it can be written as $\exp[g(k)]$ where $g \in AP^W$ and $g(-k) = g(k) = \overline{g(k)}$ for $k \in \mathbf{R}$ (cf. [7], Corollary 2 of Section 29.9). Writing $g(k) = \sum_s g_s e^{2ikd_s}$ with $d_0 = 0$ we define

$$a_+(k) = \exp\left[\frac{1}{2}g_0 + \sum_{d_s>0} g_s e^{2ikd_s}\right], \qquad a_-(k) = \exp\left[\frac{1}{2}g_0 + \sum_{d_s<0} g_s e^{2ikd_s}\right],$$

where $g_s = g_{-s}$ is real. Then $a_+(k) = a_-(-k)$, and we put $a(k) = a_+(k) = a_-(-k)$.

Define $m(k) = \tau(k)a(k)$ in the exceptional case and $m(k) = \frac{k+i}{k}\tau(k)a(k)$ in the generic case. Then $m(k)$ is continuous in $\overline{\mathbf{C}^+}$, is analytic in $\mathbf{C}^+$, approaches 1 as $k \to \infty$ in $\overline{\mathbf{C}^+}$, does not vanish, and satisfies $m(-k) = \overline{m(k)}$ for $k \in \mathbf{R}$. This function may be found uniquely by solving the Wiener-Hopf factorization problem

$$m(k)m(-k) = \varphi(k), \qquad k \in \mathbf{R}, \tag{17}$$

where $\varphi(k) = |a(k)|^2[1 - |\rho(k)|^2](k^2+1)^h/k^{2h}$ such that $h = 0$ in the exceptional case and $h = 1$ in the generic case. The reduced transmission coefficient $\tau(k)$ then follows immediately.

For the existence of the factorization (17) one needs the Hölder continuity of $\tau(k(\xi))$ when ξ belongs to the unit circle in the complex plane, where $k(\xi) = i\frac{1+\xi}{1-\xi}$ is a Möbius transformation from the unit circle onto the upper half plane. From (4.2)-(4.4) of [2] it is clear that one needs for $|\xi| = 1$ the Hölder continuity of the elements of $\sigma_{n,n+1}(k(\xi))$, which are the scattering matrices corresponding to the potentials $V_{n,n+1} \in L^1_1(\mathbf{R})$. Under the assumption $V_{n,n+1} \in L^1_{1+\alpha}(\mathbf{R})$ for some $\alpha \in (0, 1]$, one may apply the methods of [5] to the Volterra equations for the Jost solutions of (4) and then use the Wronskian relations for these Jost solutions to prove the Hölder continuity of the elements of $\sigma_{n,n+1}(k(\xi))$ for $|\xi| = 1$. Making this assumption would mean replacing (H4) by the stronger assumption

(H4′) For $n = 0, \cdots, N$ and some $\alpha \in (0, 1]$, H' is absolutely continuous on (x_n, x_{n+1}) and $2H''H - 3(H')^2 \in L^1_{1+\alpha}(x_n, x_{n+1})$.

We will not give details, since analogous arguments are available in the literature [2,5].

If there are bound state poles at $i\kappa_1, \cdots, i\kappa_{\mathcal{N}}$ in $\mathbf{C}^+$, then $\tau(k)$ is obtained uniquely from the function $\tau(k)$ computed above multiplying it by $\prod_{j=1}^{\mathcal{N}} \frac{k+i\kappa_j}{k-i\kappa_j}$.

Note that, once $a(k)$ is constructed from $|\rho(k)|$, it is also possible to construct $\tau(k)$ by using a procedure similar to that given in [5]. Let

$$(\Pi_{\pm}f)(k) = \pm\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{ds}{s-k\mp i0}f(s). \tag{18}$$

Then whenever $\log\varphi(\cdot) \in L^p(\mathbf{R})$ for some $1 < p < \infty$, we obtain

$$\tau(k) = \frac{1}{a(k)}\left(\frac{k}{k+i}\right)^h \left(\prod_{j=1}^{\mathcal{N}}\frac{k+i\kappa_j}{k-i\kappa_j}\right)\exp\left(\frac{1}{2\pi i}\int_{-\infty}^{\infty}ds\frac{\log\varphi(s)}{s-k-i0}\right).$$

This is because from (17) we get $m(k) = \exp([2\Pi_+\log(|m(\cdot)|)](k))$, which is equivalent to

$$m(k) = \exp\left(\frac{1}{\pi i}\int_{-\infty}^{\infty}ds\frac{\log|m(s)|}{s-k-i0}\right). \tag{19}$$

Since $m(k) \to 1$ as $k \to \pm\infty$, we have $|1-m(k)| \le \epsilon < 1$ and hence $|\log|m(k)|| \le \frac{|1-m(k)|}{1-\epsilon}$ for $|k| \ge N(\epsilon)$. We may show that it suffices to make the assumption (H4′) to ensure that $m(k) = 1 + o(|k|^{-\alpha})$ as $k \to \pm\infty$. Then $\log(|m(\cdot)|) \in L^p(\mathbf{R})$ for $1 < p < 1/(1-\alpha)$ if $\alpha \in (0,1)$ and $1 < p < \infty$ if $\alpha = 1$; hence the integral in (19) exists and $\log m(\cdot) \in L^p(\mathbf{R})$.

3. An alternative inversion algorithm

In the generic case where $\tau(k) = ick + o(k)$ as $k \to 0$ for some nonzero $c \in \mathbb{R}$, from (9) and (11) we have

$$F_+(0,x,y) = \frac{icZ_r(0,y)}{\sqrt{H(x)}} = \frac{icf_r(0,x)}{\sqrt{H_-}}, \tag{20}$$

implying that $F_+(k,x,y)$ is continuous at $k=0$ for any $Q \in L^1_{1+\alpha}(\mathbf{R})$ with $\alpha > 0$. As a result, the alternative inversion algorithm outlined in this section does not require the modifications of the method of [2] mentioned in step 6 of Section 2.

Recall that $\rho(0) = -1$ in the generic case. The only information about $\tau(k)$ needed below will be the nonzero real constant $c = \lim_{k\to 0}\tau(k)/(ik)$, which is obtained using the identity $c = -\lim_{k\to 0} k^{-1}\sqrt{1-|\rho(k)|^2}$. This identity is obvious from (15), except for the negative sign on the right-hand side, which is clear from the fact that $1/\tau(k)$ is real and nonzero on the positive imaginary axis and approaches a positive constant as $-ik \to +\infty$ as seen from (5).

In order to recover $H(x)$ from $\rho(k)$ and H_+, we will use (20), which relates x and y. Solving (14) we uniquely obtain $\tilde F_-(k,y) = \frac{\sqrt{H_+}}{f_l(0,x)}F_-(k,x,y)$. Then from (5.20) in [2] and putting $\tilde F_\pm(k,y) = \frac{\sqrt{H_+}}{f_l(0,x)}F_\pm(k,x,y)$ we get

$$\tilde F_+(k,y) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{ds}{s-k-i0}\left[\frac{\rho(s)e^{2isy}-\rho(0)}{s} - \rho(s)e^{2isy}\tilde F_-(-s,y)\right], \tag{21}$$

which is valid in the generic case as well as in the exceptional case. Substituting $k=0$ in (21) and using (20) we obtain

$$-i\tilde F_+(0,y) = c\sqrt{\frac{H_+}{H_-}}\frac{f_r(0,x)}{f_l(0,x)}. \tag{22}$$

Note that the right-hand side of (22) is known when one knows $\rho(k)$, $Q(x)$, and H_+; this is because $f_l(0,x)$ and $f_r(0,x)$ are determined from $Q(x)$ alone [2] and H_- can be evaluated from H_+ as follows:

$$H_- = \begin{cases} \frac{|c|^2\,|[f_l(0,x);f_r(0,x)]|^2}{4H_+}, & \text{generic case} \\ \frac{1-\rho(0)}{\gamma^2[1+\rho(0)]}H_+, & \text{exceptional case,} \end{cases}$$

where γ is the nonzero constant defined through $f_l(0,x) = \gamma f_r(0,x)$ due to the linear dependence of $f_l(0,x)$ and $f_r(0,x)$ in the exceptional case. Thus, $y = y(x)$ is constructed using (22) and $H(x) = y'(x)$ is recovered. We remark that $\tilde F_+(0,y)$ cannot be obtained from $\tilde F_-(k,y)$ with the help of (10) if $Q \notin L^1_2(\mathbf{R})$, since in that case $\tilde F_-(k,y)$ may be discontinuous at $k=0$.

In the exceptional case (22) cannot be used to obtain $y = y(x)$ because $\tau(0) \ne 0$ in (11); moreover, the linear dependence of $f_l(0,x)$ and $f_r(0,x)$ would make it impossible to determine $y = y(x)$ using (22). Instead one finds from (9) and (11)

$$F_+(k,x,y) = \frac{\tau(k)e^{ik(y-A_-)}f_r(k,x) - \tau(0)f_r(0,x)}{k\sqrt{H_-}},$$

which implies

$$-i\tilde F_+(0,y) = \frac{\tau(0)}{\gamma}\sqrt{\frac{H_+}{H_-}}\left[y - A_- - i\frac{\dot f_r(0,x)}{f_r(0,x)} - i\frac{\dot\tau(0)}{\tau(0)}\right]. \tag{23}$$

where $Q \in L^1_2(\mathbf{R})$ is assumed and $\dot{\tau}(0)$ is found by first computing $\tau(k)$ from $|\rho(k)|$ using the method of Section 2. Knowing $\rho(k)$, $Q(x)$, and H_+, the right-hand side of (23) is known; the left-hand side is uniquely constructed by solving (14) and using (21). Then $y = y(x)$ is constructed using (23) and $H(x) = y'(x)$ is recovered. In the exceptional case with $Q \in L^1_2(\mathbf{R})$ the inversion algorithm given in [2] is more efficient than the one explained presently.

If there are bound states, the inversion algorithm presented above should be appropriately modified. We will not present the details.

4. Spectrum of the singular integral operator, and an example

In [2,3] the inversion procedure has been applied to various specific $\rho(k)$ of the form $\rho(k) = e^{ik\beta}\rho_0(k)$ where $\rho_0(k)$ is continuous in $k \in \mathbf{R}$ and tends to the same nonzero constant as $k \to \pm\infty$. The singular integral equation (14) is often exactly solvable in these simple cases, but the computations in the inversion algorithm are nontrivial. In this section we study (14) when we have

$$\rho(k) = \sum_{s=1}^{K} \rho_s e^{-ik\beta_s}, \tag{24}$$

where $\beta_1 > \cdots > \beta_K > 0$, and the coefficients $\rho_1, \cdots, \rho_K$ are real, nonzero constants. In fact, (14) has the form $F_-(k,x,y) - [\mathcal{O}_y F_-](k,x,y) = F_0(k,x,y)$, where $\mathcal{O}_y = \Pi_- \rho e^{2i(\cdot)y}\mathcal{J}$ is defined on the Hardy space $\mathbf{H}^p_-(\mathbf{R})$ $(1 < p < \infty)$. Here $\Pi_\pm$ as defined in (18) are the (bounded) complementary projections of $L^p(\mathbf{R})$ onto $\mathbf{H}^p_\pm(\mathbf{R})$ and $(\mathcal{J}f)(k) = f(-k)$. Defining $\mathcal{P}_y = \mathcal{F}\mathcal{O}_y\mathcal{F}^{-1}$ with $\rho(k)$ as in (24), where $\mathcal{F}$ denotes the Fourier transform $(\mathcal{F}g)(t) = \int_{-\infty}^{\infty} dk\, e^{itk} g(k)$, we obtain

$$(\mathcal{P}_y h)(t) = \begin{cases} \displaystyle\sum_{s=1}^{K(y)} \rho_s h(\beta_s - 2y - t), & 0 < t < \beta_{K(y)} - 2y, \\ \displaystyle\sum_{s=1}^{j-1} \rho_s h(\beta_s - 2y - t), & \beta_j - 2y < t < \beta_{j-1} - 2y,\ j = 2, \cdots, K(y), \\ 0, & t > \beta_1 - 2y, \end{cases}$$

where $K(y)$ is the largest integer j for which $\beta_j - 2y > 0$. Thus $\mathcal{P}_y$ is the zero operator if $\beta_1 - 2y \le 0$. We remark that $\mathcal{P}_y$ is a bounded operator on the Banach spaces of measurable complex-valued functions $h(t)$ such that $\mathcal{F}^{-1}h \in \mathbf{H}^p_-(\mathbf{R})$ for some $1 < p < \infty$. The elements of these Banach spaces are functions supported on $\mathbf{R}^+$, and $L^2(\mathbf{R}^+)$ is one of the spaces on which $\mathcal{P}_y$ is defined. With no loss of generality we will do the subsequent analysis on $L^2(\mathbf{R}^+)$.

There is no loss of generality if we take $y = 0$ and put $\mathcal{P} = \mathcal{P}_y$. Then the eigenvalue equation $\lambda h - \mathcal{P}h = 0$ has the form

$$\begin{cases} \lambda h(t) - \displaystyle\sum_{s=1}^{K} \rho_s h(\beta_s - t) = 0, & 0 < t < \beta_K, \\ \lambda h(t) - \displaystyle\sum_{s=1}^{j-1} \rho_s h(\beta_s - t) = 0, & \beta_j < t < \beta_{j-1}, \quad j = 2, \cdots, K, \\ \lambda h(t) = 0, & t > \beta_1. \end{cases} \tag{25}$$

One easily sees that zero is an eigenvalue of $\mathcal{P}$ with corresponding eigenspace $\mathcal{K}_0 = \{h \in L^2(0,\infty) : h(t) = 0 \text{ for } 0 < t < \beta_1\}$. Excluding its last line, (25) may be viewed as a set of linear relations between the values of h at $t, \beta_1 - t, \cdots, \beta_K - t$ with t in one of the intervals $(0,\beta_K), \cdots, (\beta_2, \beta_1)$.

We now consider the K transformations $t \mapsto \beta_j - t$ with domain and range $[0,\beta_j]$ and construct the smallest subset of $[0,\beta_1]$ that is closed with respect to these K transformations and containing the set $\{0, \beta_K, \cdots, \beta_1\}$. Let this subset be denoted by $\Gamma = \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_M = \beta_1\}$. Then Γ is finite if $K = 1$ (when $\Gamma = \{0, \beta_1\}$) and $K = 2$ (when Γ can be determined explicitly; see Example 3 below), or if $\beta_1, \cdots, \beta_K$ are integer multiples of the same positive number β_0 (so that Γ is contained in $\{j\beta_0 : j = 0, 1, 2, \cdots\} \cap [0, \beta_1]$). For example, this is the case when $\beta_1, \cdots, \beta_K$ are rationals. If $K = 3$ and $\beta_1, \beta_2, \beta_3$ are not integer multiples of the same positive number, Γ may be either finite or infinite. So far we do not have necessary and sufficient conditions on $\beta_1, \cdots, \beta_K$ in order that Γ be finite, but in the sequel we will assume Γ to be finite unless stated otherwise.

We now consider the M intervals $[0, \gamma_1]$, $[\gamma_1, \gamma_2]$, $\cdots$, $[\gamma_{M-1}, \gamma_M]$ as the vertices of the undirected graph $\mathbf{G}$ where two intervals are connected if and only if there exists a transformation $t \mapsto \beta_j - t$ mapping the first interval onto the second (and conversely). Then $\mathbf{G}$ has a finite number, say r, of connected components $\mathbf{G}_1, \cdots, \mathbf{G}_r$ whose vertices we denote by $I_{j,1}, \cdots, I_{j,m(j)}$ $(j = 1, \cdots, r)$. The left endpoints of $I_{j,1}, \cdots, I_{j,m(j)}$ $(j = 1, \cdots, r)$ are in increasing order as are the left endpoints of $I_{1,1}, \cdots, I_{r,1}$. Thus the left endpoint of $I_{1,1}$ is zero. Note that the intervals occurring as the vertices of the same connected component of $\mathbf{G}$ must have the same length, because the transformations $t \mapsto \beta_j - t$ $(j = 1, \cdots, K)$ are measure preserving.

Only the transformations $t \mapsto \beta_s - t$ with $s = 1, \cdots, n(j)$ map the interval $I_{j,1}$ into $[0, \beta_1]$, and hence only for those s the transformations $t \mapsto \beta_s - t$ can be applied. Suppose $\beta_s - t \in I_{j,p(j,s)}$ for $s = 1, \cdots, n(j)$ whenever $t \in I_{j,1}$; then $p(j, n(j)) < \cdots < p(j,2) < p(j,1)$ where $n(j) \leq m(j)$. Moreover, the transformation $t \mapsto \beta_s - t$ maps $I_{j,q}$ into $I_{j,p(j,s)+1-q}$ if $q = 1, \cdots, p(j,s)$, and is undefined on $I_{j,p(j,s)+1}, \cdots, I_{j,m(j)}$. Define

$$\sigma_i^j = \begin{cases} \rho_s, & i = p(j,s), \quad s = 1, \cdots, n(j), \\ 0, & \text{otherwise.} \end{cases}$$

Put $I_{j,s} = [a_s^{j,l}, a_s^{j,r}]$, $t_s^{j,l} = a_s^{j,l} - a_1^{j,l} + t$, and $t_s^{j,r} = a_s^{j,r} + a_1^{j,l} - t$, where $t \in I_{j,1}$. Then for every $t \in I_{j,1}$, one obtains the linear system of equations

$$H_{m(j)}(\sigma_{m(j)}^j, \cdots, \sigma_1^j)\mathbf{h}^j(t) = \lambda \mathbf{h}^j(t), \tag{26}$$

where $\mathbf{h}^j(t)$ is the column vector of length $2m(j)$ whose transpose is

$$\mathbf{h}^j(t)^T = (h(t_1^{j,l}), h(t_1^{j,r}), \cdots, h(t_{m(j)}^{j,l}), h(t_{m(j)}^{j,r})),$$

and $H_m(q_1, \cdots, q_m)$ is the Hankel matrix of order $2m$ defined by

$$H_m(q_1, \cdots, q_m) = \left[\begin{array}{cccccccccc} 0 & q_m & 0 & q_{m-1} & \cdots & 0 & q_2 & 0 & q_1 \\ q_m & 0 & q_{m-1} & 0 & & q_2 & 0 & q_1 & 0 \\ 0 & q_{m-1} & 0 & q_{m-2} & \cdots & 0 & q_1 & & \\ q_{m-1} & 0 & q_{m-2} & 0 & & q_1 & 0 & & O \\ & \vdots & & \vdots & \cdots & & O & & O \\ 0 & q_1 & & & & & & & \\ q_1 & 0 & & O & \cdots & & O & & O \end{array}\right].$$

Thus $H_m(q_1, \cdots, q_m)$ may be viewed is an $m \times m$ matrix whose (i,j)-element is

$$\begin{bmatrix} 0 & q_{m+2-i-j} \\ q_{m+2-i-j} & 0 \end{bmatrix}$$

if $2 \leq i + j \leq m + 1$ and the 2×2 zero matrix if $i + j > m + 1$. The Hankel matrix appearing in the linear system (26) has the form

$$H_{m(j)}(\rho_1, 0, \cdots, 0, \rho_2, 0, \cdots, 0, \rho_3, \cdots, \rho_{n(j)}, 0, \cdots, 0), \tag{27}$$

where the string of length $m(j)$ occurring as the argument of $H_{m(j)}$ has ρ_1 as its first element and contains all of the numbers $\rho_1, \cdots, \rho_{n(j)}$ in exactly this order interrupted by $m(j) - n(j)$ zeros. As a result, as none of the coefficients $\rho_1, \cdots, \rho_K$ vanish, the nonzero eigenvalue spectrum of $\mathcal{P}$ consists precisely of the eigenvalues of the matrices given by (27) where $j = 1, \cdots, r$; here we note that the matrices in (27) are all nonsingular. Moreover, since the coefficients $\rho_1, \cdots, \rho_K$ are real, the Hankel matrices in (27) are all real symmetric and hence their eigenvalues are real. Further, the identity

$$J_m H_m(q_1, \cdots, q_m) = -H_m(q_1, \cdots, q_m) J_m,$$

where J_m is the unitary and selfadjoint matrix

$$J_m = \underbrace{\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \oplus \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}}_{m \text{ times}},$$

implies that the eigenvalues occur in $\pm$ pairs. Since the matrices in (27) are real symmetric, their eigenvalues span the complete $2\sum_{j=1}^{r} m(j)$-dimensional space. As a result, all of the eigenspaces of $\mathcal{P}$ corresponding to nonzero eigenvalues are closed and infinite dimensional subspaces of $\mathcal{A} = \{h \in L^2(0,\infty) : h(t) = 0 \text{ for } t > \beta_1\}$ which add up to all of $\mathcal{A}$. Thus the spectrum of $\mathcal{P}$ consists of a finite number of eigenvalues of infinite multiplicity, all of which are real and consist of zero and $\pm$ pairs.

To reduce the effort in computing the nonzero eigenvalues of $\mathcal{P}$, we observe that

$$P_m^{\pm} H_m(q_1, \cdots, q_m) = H_m(q_1, \cdots, q_m) P_m^{\pm},$$

where $P_m^{\pm}$ are the complementary projections defined by

$$P_m^{\pm} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix} \oplus \frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix} \oplus \cdots \oplus \frac{1}{2}\begin{bmatrix} 1 & \pm 1 \\ \pm 1 & 1 \end{bmatrix}}_{m \text{ times}}.$$

Then $H_m(q_1, \cdots, q_m) P_m^{\pm} = \pm K_m^{(2)}(q_1, \cdots, q_m) P_m^{\pm}$, where

$$K_m^{(2)}(q_1, \cdots, q_m) = \left[\begin{array}{cccccccccc} q_m & 0 & q_{m-1} & 0 & \cdots & q_2 & 0 & q_1 & 0 \\ 0 & q_m & 0 & q_{m-1} & & 0 & q_2 & 0 & q_1 \\ q_{m-1} & 0 & q_{m-2} & 0 & \cdots & q_1 & 0 & & \\ 0 & q_{m-1} & 0 & q_{m-2} & & 0 & q_1 & & O \\ & \vdots & & \vdots & \cdots & & O & & O \\ q_1 & 0 & & & & & & & \\ 0 & q_1 & & O & \cdots & & O & & O \end{array}\right],$$

which may be viewed as the tensor product of the $m \times m$ Hankel matrix

$$K_m(q_1, \cdots, q_m) = \begin{bmatrix} q_m & q_{m-1} & \cdots & q_2 & q_1 \\ q_{m-1} & q_{m-2} & \cdots & q_1 & 0 \\ \vdots & \vdots & \cdots & 0 & 0 \\ q_1 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

and the 2×2 identity matrix. Thus the eigenvalues of $H_m(q_1, \cdots, q_m)$ are exactly the $\pm$ pairs $\pm\lambda_1, \cdots, \pm\lambda_m$ where $\lambda_1, \cdots, \lambda_m$ are the eigenvalues of $K_m(q_1, \cdots, q_m)$. Hence the calculation of the nonzero eigenvalues of $\mathcal{P}$ may be reduced to the calculation of the eigenvalues of r Hankel matrices of orders $m(1), \cdots, m(r)$.

Let us work out some explicit examples and the general theory for $K = 2$ (Example 3).

1. $K = 3$, $\beta_1 = 6$, $\beta_2 = 2$ and $\beta_3 = 1$. Then $\Gamma = \{0, 1, 2, 4, 5, 6\}$. Then the vertices of $\mathbf{G}$ are the intervals $[0,1]$, $[1,2]$, $[2,4]$, $[4,5]$, and $[5,6]$, and $\mathbf{G}$ has the form:

$$\mathbf{G}_1: \begin{array}{ccc} [0,1]^\bullet & \Longleftrightarrow & [1,2] \\ \Updownarrow & & \Updownarrow \\ [5,6] & & [4,5] \end{array} \qquad \mathbf{G}_2: \quad [2,4]^\bullet,$$

where $\bullet$ denotes the existence of a connection between a vertex and itself. We then get the linear systems

$$\begin{bmatrix} \lambda & -\rho_3 & 0 & -\rho_2 & 0 & 0 & 0 & -\rho_1 \\ -\rho_3 & \lambda & -\rho_2 & 0 & 0 & 0 & -\rho_1 & 0 \\ 0 & -\rho_2 & \lambda & 0 & 0 & -\rho_1 & 0 & 0 \\ -\rho_2 & 0 & 0 & \lambda & -\rho_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\rho_1 & \lambda & 0 & 0 & 0 \\ 0 & 0 & -\rho_1 & 0 & 0 & \lambda & 0 & 0 \\ 0 & -\rho_1 & 0 & 0 & 0 & 0 & \lambda & 0 \\ -\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} h(t) \\ h(1-t) \\ h(t+1) \\ h(2-t) \\ h(t+4) \\ h(5-t) \\ h(t+5) \\ h(6-t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for $t \in [0,1]$, and

$$\begin{bmatrix} \lambda & -\rho_1 \\ -\rho_1 & \lambda \end{bmatrix} \begin{bmatrix} h(t) \\ h(6-t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

for $t \in [2,4]$. Thus the nonzero eigenvalues of $\mathcal{P}$ are the eigenvalues of the Hankel matrices $H_4(\rho_1, 0, \rho_2, \rho_3)$ and $H_1(\rho_1)$. These eigenvalues occur in pairs $\pm\lambda$ where λ is one of the eigenvalues of the Hankel matrices $K_4(\rho_1, 0, \rho_2, \rho_3)$ and $K_1(\rho_1)$.

2. $K = 2$, $\beta_1 = \sqrt{2}$, $\beta_2 = 1$. Then $\Gamma = \{0, 3-2\sqrt{2}, \sqrt{2}-1, 2-\sqrt{2}, 2\sqrt{2}-2, 1, 3\sqrt{2}-3, \sqrt{2}\}$. Then the vertices of $\mathbf{G}$ are the intervals $[0, 3-2\sqrt{2}]$, $[3-2\sqrt{2}, \sqrt{2}-1]$, $[\sqrt{2}-1, 2-\sqrt{2}]$,

$[2-\sqrt{2}, 2\sqrt{2}-2]$, $[2\sqrt{2}-2, 1]$, $[1, 3\sqrt{2}-3]$ and $[3\sqrt{2}-3, \sqrt{2}]$, and $\mathbf{G}$ has the following form:

$$\mathbf{G}_1: \begin{array}{ccc} [0, 3-2\sqrt{2}] & \Longleftrightarrow & [2\sqrt{2}-2, 1] \\ \Updownarrow & & \Updownarrow \\ [3\sqrt{2}-3, \sqrt{2}] & & [\sqrt{2}-1, 2-\sqrt{2}]^\bullet \end{array}$$

$$\mathbf{G}_2: \begin{array}{ccc} [1, 3\sqrt{2}-3] & \Longleftrightarrow & [3-2\sqrt{2}, \sqrt{2}-1] \\ & & \Updownarrow \\ & & [2-\sqrt{2}, 2\sqrt{2}-2]^\bullet \end{array}$$

We then get the linear systems

$$\begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & -\rho_2 & 0 & -\rho_1 \\ 0 & \lambda & 0 & 0 & -\rho_2 & 0 & -\rho_1 & 0 \\ 0 & 0 & \lambda & -\rho_2 & 0 & -\rho_1 & 0 & 0 \\ 0 & 0 & -\rho_2 & \lambda & -\rho_1 & 0 & 0 & 0 \\ 0 & -\rho_2 & 0 & -\rho_1 & \lambda & 0 & 0 & 0 \\ -\rho_2 & 0 & -\rho_1 & 0 & 0 & \lambda & 0 & 0 \\ 0 & -\rho_1 & 0 & 0 & 0 & 0 & \lambda & 0 \\ -\rho_1 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} h(t) \\ h(3-2\sqrt{2}-t) \\ h(t+\sqrt{2}-1) \\ h(2-\sqrt{2}-t) \\ h(t+2\sqrt{2}-2) \\ h(1-t) \\ h(t+3\sqrt{2}-3) \\ h(\sqrt{2}-t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for $t \in [0, 3-2\sqrt{2}]$, and

$$\begin{bmatrix} \lambda & 0 & 0 & -\rho_2 & 0 & -\rho_1 \\ 0 & \lambda & -\rho_2 & 0 & -\rho_1 & 0 \\ 0 & -\rho_2 & \lambda & -\rho_1 & 0 & 0 \\ -\rho_2 & 0 & -\rho_1 & \lambda & 0 & 0 \\ 0 & -\rho_1 & 0 & 0 & \lambda & 0 \\ -\rho_1 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} h(t) \\ h(2-\sqrt{2}-t) \\ h(t+\sqrt{2}-1) \\ h(1-t) \\ h(t+2\sqrt{2}-2) \\ h(\sqrt{2}-t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for $t \in [3-2\sqrt{2}, \sqrt{2}-1]$. Thus the nonzero eigenvalues of $\mathcal{P}$ are the eigenvalues of the Hankel matrices $H_4(\rho_1, \rho_2, 0, 0)$ and $H_3(\rho_1, \rho_2, 0)$. These eigenvalues occur in pairs $\pm\lambda$ where λ is one of the eigenvalues of the Hankel matrices $K_4(\rho_1, \rho_2, 0, 0)$ and $K_3(\rho_1, \rho_2, 0)$.

3. $K = 2$, $(\beta_2/\beta_1) \notin \mathbf{Q}$, where $\mathbf{Q}$ denotes the set of rational numbers. Let n be the largest integer such that $n(\beta_1 - \beta_2) \le \beta_1$. Then the $2n+2$ numbers $j(\beta_1 - \beta_2)$, $\beta_1 - j(\beta_1 - \beta_2)$ $(j = 0, 1, \cdots, n)$ are different and $\Gamma = \{0 = \gamma_0 < \gamma_1 < \cdots < \gamma_{2n+1}\}$ where $\gamma_{2j} = j(\beta_1 - \beta_2)$ and $\gamma_{2j+1} = \beta_1 - (n-j)(\beta_1 - \beta_2)$. Put $I_r = [\gamma_{r-1}, \gamma_r]$. Then $t \mapsto \beta_1 - t$ maps I_{2j} into $I_{2n-2j+2}$ and I_{2j+1} into $I_{2n-2j+1}$, while $t \mapsto \beta_2 - t$ maps I_{2j} into I_{2n-2j} and I_{2j+1} into $I_{2n-2j-1}$, where we note that $t \mapsto \beta_2 - t$ cannot be applied to I_{2n} and I_{2n+1}. Hence the connected components of the graph $\mathbf{G}$ are exactly

the two complementary subgraphs whose vertices are $I_1, \cdots, I_{2n+1}$ and $I_2, \cdots, I_{2n}$, respectively. The two linear systems (26) reduce to

$$H_{n+1}(\rho_1, \rho_2, 0, \cdots, 0)\mathbf{h}_{\mathrm{odd}}(t) = \lambda \mathbf{h}_{\mathrm{odd}}(t), \qquad t \in [0, \beta_1 - n(\beta_1 - \beta_2)], \tag{28}$$

$$H_n(\rho_1, \rho_2, 0, \cdots, 0)\mathbf{h}_{\mathrm{even}}(t) = \lambda \mathbf{h}_{\mathrm{even}}(t), \qquad t \in [\beta_1 - n(\beta_1 - \beta_2), \beta_1 - \beta_2], \tag{29}$$

where

$$\mathbf{h}_{\mathrm{odd}}(t)^T = (h(t), h(\gamma_1 - t), h(\gamma_2 + t), h(\gamma_3 - t), h(\gamma_4 + t), \cdots, h(\gamma_{2n} + t), h(\gamma_{2n+1} - t)),$$

$$\mathbf{h}_{\mathrm{even}}(t)^T = (h(t), h(\gamma_3 - t), h(\gamma_2 + t), h(\gamma_5 - t), h(\gamma_4 + t), \cdots, h(\gamma_{2n-2} + t), h(\gamma_{2n+1} - t)).$$

Thus the nonzero eigenvalues of $\mathcal{P}$ are the eigenvalues of the Hankel matrices $H_{n+1}(\rho_1, \rho_2, 0, \cdots, 0)$ and $H_n(\rho_1, \rho_2, 0, \cdots, 0)$. These eigenvalues occur in pairs $\pm\lambda$ where λ is one of the eigenvalues of the Hankel matrices $K_{n+1}(\rho_1, \rho_2, 0, \cdots, 0)$ and $K_n(\rho_1, \rho_2, 0, \cdots, 0)$. Finally, if $(\beta_2/\beta_1) \in \mathbf{Q}$, some of the numbers $j(\beta_1 - \beta_2)$, $\beta_1 - j(\beta_1 - \beta_2)$ $(j = 0, 1, \cdots, n)$ may coincide. One must then remove from the Hankel matrices the rows and columns corresponding to the intervals reducing to a single point, which leads to a simplification of (28) and (29). For instance, if $(\beta_1/\beta_2) = 2$, Eqs. (28) and (29) simplify to the single equation

$$H_2(\rho_1, \rho_2)\mathbf{h}(t) = \lambda \mathbf{h}(t), \qquad t \in [0, \beta_2],$$

where $\mathbf{h}(t)^T = (h(t), h(\beta_2 - t), h(\beta_2 + t), h(\beta_1 - t))$.

4. $K = 3$. If $\beta_1 = \sqrt{17}$, $\beta_2 = 2$ and $\beta_3 = 1$, we have $\Gamma = \{0, 1, 2, \sqrt{17} - 2, \sqrt{17} - 1, \sqrt{17}\}$. Then the vertices of $\mathbf{G}$ are the intervals $[0,1]$, $[1,2]$, $[2, \sqrt{17} - 2]$, $[\sqrt{17} - 2, \sqrt{17} - 1]$, $[\sqrt{17} - 1, \sqrt{17}]$, and $[\sqrt{17} - 1, \sqrt{17}]$, and $\mathbf{G}$ has the following form:

$$\mathbf{G}_1: \quad \begin{array}{ccc} [0,1]^* & \Longleftrightarrow & [1,2] \\ \Updownarrow & & \Updownarrow \\ [\sqrt{17}-1, \sqrt{17}] & & [\sqrt{17}-2, \sqrt{17}-1] \end{array}$$

$$\mathbf{G}_2: \quad [2, \sqrt{17} - 2]^*.$$

It then follows that the nonzero eigenvalues of $\mathcal{P}$ occur in pairs $\pm\lambda$ where λ is one of the eigenvalues of the Hankel matrices $K_4(\rho_1, 0, \rho_2, \rho_3)$ and $K_1(\rho_1)$. If $\beta_1 = 2$, $\beta_2 = \sqrt{2}$, and $\beta_3 = 1$, then Γ contains the numbers ξ_n $(n = 0, 1, 2, \cdots)$ where $\xi_0 = \sqrt{2} - 1$ and

$$\xi_{n+1} = \begin{cases} \sqrt{2} - \xi_n, & n \text{ odd}, \\ c_n - \xi_n, & n \text{ even}, \end{cases}$$

with, for n even, $c_n = 1$ if $\xi_n \in (0,1)$ and $c_n = 2$ if $\xi_n \in (1,2)$. Then $\xi_n = -x_n + y_n\sqrt{2}$ if n is even and $\xi_n = x_n - y_n\sqrt{2}$ if n is odd, where x_n and y_n are positive integers satisfying $x_{n+1} = x_n$ and $y_{n+1} = y_n + 1$ if n is odd and $x_{n+1} = x_n + c_n$ and $y_{n+1} = y_n$ if n is even. Thus the numbers ξ_n $(n = 0, 1, 2, \cdots)$ are all different and hence Γ is an infinite set.

5. Conclusions

Let us first discuss the structure of the spectrum of the singular integral operator $\mathcal{O}_y$ when the asymptotic part $\rho_{as}(k) = -\frac{b(k)}{a(k)}$ of $\rho(k)$ in (6) is an almost periodic polynomial and the set Γ associated with $\rho_{as}(k)$ is finite. Then we may write the integral operator in (12) and (14) in the form

$$\mathcal{O}_y = \mathcal{O}_{\mathrm{as},y} + \mathcal{O}_{\mathrm{rem},y}. \tag{30}$$

Here the terms on the right-hand side of (30) are defined as is $\mathcal{O}_y$, but with $\rho(k)$ replaced by $\rho_{\mathrm{as}}(k)$ and $\rho_{\mathrm{rem}}(k) = \rho(k) - \rho_{\mathrm{as}}(k)$, respectively. The analysis of the spectral properties of $\mathcal{O}_{\mathrm{as},y}$ can be given as in Section 4 when $\rho(k)$ has the form (24). Note that $\mathcal{O}_{\mathrm{as},y}$ reduces to zero when $\rho(k)$ is continuous for $k \in \mathbf{R}$ and vanishes as $k \to \pm\infty$. The operator $\mathcal{O}_{\mathrm{rem},y}$ is compact on each of the Hardy spaces $\mathbf{H}^p_-(\mathbf{R})$ $(1 < p < \infty)$, as a result of the Hartman-Wintner theorem ([11], Corollary 4.10). For an independent proof of its compactness see Proposition 7.4 of [2]. As a result, when $\rho(k)$ has the form (24), the spectrum of $\mathcal{O}_y$ is made of a finite set of real numbers, consisting of zero and $\pm$ pairs, and an at most countable number of real eigenvalues of finite multiplicity which can only accumulate at this finite set. The eigenvalues of finite multiplicity need not occur in $\pm$ pairs, as shown by Example 6.2 in [2].

Since the eigenvalues of $\mathcal{O}_y$ of infinite multiplicity do not depend on the choice of the Hardy space $\mathbf{H}^p_-(\mathbf{R})$ $(1 < p < \infty)$, a simple Fredholm argument suffices to prove that the eigenvalues of finite multiplicity as well as their multiplicities do not depend on the choice of the Hardy space either. When $\rho_{\mathrm{as}}(k)$ is an almost periodic polynomial and the set Γ pertaining to $\rho_{\mathrm{as}}(k)e^{2iky}$ is finite, the spectrum of $\mathcal{O}_y$ is independent of the Hardy space used.

We have no detailed results on the spectrum of $\mathcal{O}_y$ if either $\rho_{\mathrm{as}}(k)$ has infinitely many terms or the set Γ pertaining to $\rho_{\mathrm{as}}(k)e^{2iky}$ is infinite.

Next, let us discuss the impact Example 3 might have on the solution of the singular integral equation (14). Let $\rho(k) = \rho_1 e^{-ik\beta_1} + \rho_2 e^{-ik\beta_2}$ where $\beta_1 > \beta_2$ and $\rho_1, \rho_2 \in \mathbf{R} \setminus \{0\}$. Then we must apply the results of this section as if $\rho(k)$ were replaced by $\rho_y(k) = \rho_1 e^{-ik(\beta_1 - 2y)} + \rho_2 e^{-ik(\beta_2 - 2y)}$. If $y < (\beta_2/2)$ and $n(y)$ is the largest integer not exceeding $\frac{\beta_1 - 2y}{\beta_1 - \beta_2}$, and $\frac{\beta_1 - 2y}{\beta_2 - 2y} \notin \mathbf{Q}$, the spectrum of $\mathcal{O}_y$ is a pure point spectrum consisting of zero and the pairs $\pm\lambda$ where λ is one of the eigenvalues of the Hankel matrices $K_{n(y)+1}(\rho_1, \rho_2, 0, \cdots, 0)$ and $K_{n(y)}(\rho_1, \rho_2, 0, \cdots, 0)$, i.e. $\mathcal{O}_y$ generically has exactly $4n(y) + 3$ eigenvalues. Thus as $y \to -\infty$ while $\frac{\beta_1 - 2y}{\beta_2 - 2y} \notin \mathbf{Q}$, the number of eigenvalues of $\mathcal{O}_y$ increases approximately in proportion to $|y|$. Hence as $y \to -\infty$ the complexity involved in solving the inverse problem increases indefinitely. We expect a similar phenomenon to occur for arbitrary $\rho(k)$ having an almost periodic part with at least three terms.

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