CONDITIONS FOR MATRICES RELEVANT TO POLARIZED LIGHT TRANSFER

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ABSTRACT. A review is given of conditions on an arbitrary real 4×4 matrix to transform the four-vector of Stokes parameters of an input beam linearly into that of an output beam. A distinction is made between pure Mueller matrices, weighted sums of pure Mueller matrices, and matrices satisfying the Stokes criterion. Special attention is given to polarized light transfer in planetary atmospheres.

1. INTRODUCTION

In this article we are concerned with transfer of polarized light, excluding nonlinear effects and interference phenomena. Within this context, a light beam can be described by the real column vector $\mathbf{I}=\{I,Q,U,V\}$ of Stokes parameters (Refs. 1-3). These vectors satisfy the inequality $I\geq(Q^2+U^2+V^2)^{1/2}$, because the intensity $I\geq 0$ and the degree of polarization $p=(Q^2+U^2+V^2)^{1/2}/I$ satisfies $0\leq p\leq 1$. Adding beams of light pertains to adding their respective Stokes parameters. An optical process (e.g. single scattering, refraction, reflection, absorption, but also multiple scattering) can be described by a real 4×4 matrix $\mathbf{M}$ transforming the four-vector of Stokes parameters of an input beam of polarized light into the four-vector of Stokes parameters of the corresponding output beam. Mathematically, $\mathbf{M}$ must satisfy the so-called *Stokes criterion*, i.e. it must transform real four-vectors $\mathbf{I}_0=\{I_0,Q_0,U_0,V_0\}$ satisfying $I_0\geq(Q_0^2+U_0^2+V_0^2)^{1/2}$ into real four-vectors $\mathbf{I}=\{I,Q,U,V\}$ of the same type. In general, however, there are other conditions on matrices relevant to polarized light transfer. This has been the subject of many papers [See e.g. Ref. 4]. The main goal of this paper is to give a succinct review of such conditions.

2. PURE MUELLER MATRICES

The elementary process of transforming the electric vector $\mathbf{E}_0$ of a monochromatic beam of light into the electric vector $\mathbf{E}$ of the outgoing monochromatic beam is linear and is described by the complex 2×2 Jones matrix $\mathbf{J}$: $\mathbf{E}=\mathbf{J}\mathbf{E}_0$. When converting the electric vectors to Stokes vectors, we find $\mathbf{I}=\mathbf{M}\mathbf{I}_0$ where $\mathbf{M}$ is a so-called *pure Mueller matrix*. In fact,

$$\mathbf{M}=\mathbf{M_J}=\mathbf{A}(\mathbf{J}\otimes\overline{\mathbf{J}})\mathbf{A}^{-1}, \qquad (1)$$

where $\mathbf{A}$ is a fixed unitary matrix, $\overline{\mathbf{J}}$ denotes the complex conjugate of $\mathbf{J}$, and $\otimes$ denotes the tensor product of two 2×2 matrices (Refs. 5-6; cf. Ref. 2 for expressions not involving matrices). Thus a real 4×4 matrix $\mathbf{M_J}$ is obtained from a complex 2×2 matrix $\mathbf{J}$. Taking account of an arbitrary phase in the electric vectors, one expects 9 independent relations between the elements of $\mathbf{M_J}$.

There are various ways to characterize pure Mueller matrices. Firstly, the 37 quadratic relations between their elements may be arranged in a number of pictograms allowing one to reproduce them by heart (Refs. 7-9). Secondly, it may be shown that a pure Mueller matrix $\mathbf{M}$ that is invertible, obeys the relations

$$\widetilde{\mathbf{M}}\mathbf{G}\mathbf{M}=c^2\mathbf{G}, \qquad \det\mathbf{M}=c>0, \qquad [\mathbf{M}]_{11}>0. \qquad (2)$$

Here a tilde denotes the transpose of a matrix and $\mathbf{G}=\mathrm{diag}\,(1,-1,-1,-1)$. Conversely, it can be proven that a real invertible matrix $\mathbf{M}$ is a pure Mueller matrix if Eq. (2) is satisfied (Ref. 10). It turns out that such a matrix

$\mathbf{M}$ can be obtained from a Jones matrix $\mathbf{J}$ whose determinant is $\sqrt[4]{c}$ in absolute value and that all Jones matrices pertaining to $\mathbf{M}$ have the form $e^{i\theta}\mathbf{J}$ for some phase factor $e^{i\theta}$. A third way to characterize pure Mueller matrices will be mentioned at the end of the next section.

Pure Mueller matrices are characteristic of optical systems which can be described by a linear transformation of the electric vector when the input beam is strictly monochromatic. They occur for scattering by a single particle (Ref. 2) and for various optical devices (Ref. 11, 4).

3. WEIGHTED SUMS OF PURE MUELLER MATRICES

We now consider matrices $\mathbf{M}$ that are averages of pure Mueller matrices. These matrices, which we shall call *weighted sums of pure Mueller matrices*, have the form

$$\mathbf{M} = \sum_{r=1}^{N} c_r \mathbf{M}_{\mathbf{J}_r}, \tag{3}$$

where $c_1, \cdots, c_N$ are nonnegative constants and $\mathbf{M}_{\mathbf{J}_r}$ is the pure Mueller matrix corresponding to the Jones matrix $\mathbf{J}_r$.

When $\mathbf{M}$ is a weighted sum of pure Mueller matrices, one readily derives the six elementary inequalities (Refs. 7, 12)

$$([\mathbf{M}]_{11} \pm [\mathbf{M}]_{22})^2 \geq ([\mathbf{M}]_{12} \pm [\mathbf{M}]_{21})^2 + ([\mathbf{M}]_{33} \pm [\mathbf{M}]_{44})^2 + ([\mathbf{M}]_{34} \mp [\mathbf{M}]_{43})^2; \tag{4}$$

$$([\mathbf{M}]_{11} \pm [\mathbf{M}]_{12})^2 \geq ([\mathbf{M}]_{21} \pm [\mathbf{M}]_{22})^2 + ([\mathbf{M}]_{31} \pm [\mathbf{M}]_{32})^2 + ([\mathbf{M}]_{41} \pm [\mathbf{M}]_{42})^2; \tag{5}$$

$$([\mathbf{M}]_{11} \pm [\mathbf{M}]_{21})^2 \geq ([\mathbf{M}]_{12} \pm [\mathbf{M}]_{22})^2 + ([\mathbf{M}]_{13} \pm [\mathbf{M}]_{23})^2 + ([\mathbf{M}]_{14} \pm [\mathbf{M}]_{24})^2, \tag{6}$$

as well as the “trace" inequality

$$\mathrm{Tr}\,\mathbf{M} = [\mathbf{M}]_{11} + [\mathbf{M}]_{22} + [\mathbf{M}]_{33} + [\mathbf{M}]_{44} \geq 0, \tag{7}$$

which follows from Eq. (4). Equations (5) and (6) are easily derived from the Stokes criterion only, but there exist matrices $\mathbf{M}$ satisfying the Stokes criterion for which Eqs. (4) and (7) are not valid [Examples are given in the next section].

A full proof algorithm for determining if a real 4×4 matrix is a weighted sum of pure Mueller matrices is due to Cloude (Refs. 13-14). Writing the Jones matrix as a linear combination of the Pauli matrices σ_r, $r = 0, 1, 2, 3$,

$$\mathbf{J} = \sum_{r=0}^{3} k_r \sigma_r = \begin{pmatrix} k_0 + k_1 & k_2 - ik_3 \\ k_2 + ik_3 & k_0 - k_1 \end{pmatrix}, \tag{8}$$

one finds

$$[\mathbf{M}_{\mathbf{J}}]_{r+1,s+1} = \sum_{t=0}^{3}\sum_{u=0}^{3} \frac{1}{2}\mathrm{Tr}(\sigma_r\sigma_t\sigma_s\sigma_u)[\mathbf{T}]_{t,u}, \tag{9}$$

where $[\mathbf{T}]_{t,u} = k_t\overline{k_u}$. Generalizing the linear transformation (9) to arbitrary real 4×4 matrices, one obtains a one-to-one correspondence between the real 4×4 matrices $\mathbf{M}$ and the hermitian matrices $\mathbf{T}$. It turns out that $\mathbf{M}$ is a weighted sum of pure Mueller matrices if and only if its so-called *coherency matrix* $\mathbf{T}$ has only nonnegative eigenvalues. If $\mathbf{M}$ is a weighted sum of pure Mueller matrices or a limit of such a sum, we can use an orthonormal basis of eigenvectors of $\mathbf{T}$ to obtain the so-called target decomposition of $\mathbf{M}$ as a weighted sum of at most four pure Mueller matrices: $\mathbf{M} = \sum_{r=0}^{3} c_r \mathbf{M}_{\mathbf{J}_r}$. Moreover, $\mathbf{M}$ is a pure Mueller matrix if and only if $\mathbf{T}$ has exactly one positive and three zero eigenvalues. Matrices unitarily equivalent to the coherency matrix (apart from a factor $\sqrt{2}$) are obtained if one writes the Jones matrix as a linear combination of the four 2×2 matrices having one element 1 and three zero elements (cf. Refs. 15-16).

4. MATRICES SATISFYING THE STOKES CRITERION

A pure Mueller matrix always satisfies the Stokes criterion (Ref. 7) and the same thing is true for a weighted sum of pure Mueller matrices. Conversely, a real 4×4 matrix that satisfies the Stokes criterion is not always a pure Mueller matrix or a weighted sum of pure Mueller matrices. Examples of this are $\mathrm{diag}\,(1,-1,-1,-1)$ and $\mathrm{diag}\,(1,1,1,-1)$, as may be readily verified. Consequently, it is not surprising that conditions have been derived (Refs. 10, 17-19) for real 4×4 matrices which are solely based on their property of satisfying the Stokes criterion. It has been shown, for instance, that in such a case an eigenvalue test can be conducted which is based on the eigenvalues of the matrix $\mathbf{G}\widetilde{\mathbf{M}}\mathbf{G}\mathbf{M}$ (Ref. 10).

5. LIGHT SCATTERING BY PARTICLES AND SURFACES

The considerations above are quite general. In practice, one may first try to use the physical context of a problem to establish whether one deals with a pure Mueller matrix or with a weighted sum of pure Mueller matrices. If no firm conclusion can be drawn, one can at least use the conditions valid for a matrix satisfying the Stokes criterion.

As an illustration we first consider the single scattering of light by a gas containing many independently scattering particles each of which is characterized by a pure Mueller matrix (Ref. 2, Sec. 4.22; also Ref. 20, Sec. 8.8). The differences between the phases of the waves scattered by the individual particles in a non-forward direction at a distance from the scattering location large in comparison to wavelength are randomly distributed and change rapidly during one experiment, so that the interference phenomena expressed by the addition of the electric vectors are not noticed in practice. Consequently, when considering the single scattering of an input beam by an infinitesimal volume element of independently scattering particles, one must add the Stokes vectors of the output beams produced upon scattering by the constituent particles to find the output beam produced upon scattering by the volume element. As a result, the volume element is a linear optical system described by a weighted sum of pure Mueller matrices.

Secondly, when a plane monochromatic wave is reflected by a randomly rough surface, the detector receives reflected light from various points at the surface whose phases are approximatedly distributed at random. Consequently, one may add the Stokes parameters of the light beams arriving from the individual surface elements to find the Stokes vector describing the light received by the detector. The resulting reflection matrix then is a weighted sum of pure Mueller matrices.

Our third illustration is provided by the problem of multiple scattering of polarized light in a planetary atmosphere containing gases and many independently scattering particles which is bounded below by a reflecting surface (Ref. 21). This may be reduced to a large number of single scattering problems. Let us partition the atmosphere-surface system into small volume and surface elements with each such element acting as an independent single scatterer and having a so-called scattering matrix $\mathbf{F} = \mathbf{F}(\omega_{\mathrm{in}}, \omega_{\mathrm{out}})$ depending on the direction ω_{in} of the input beam and the direction ω_{out} of the output beam. Then the multiple scattering problem is essentially a bookkeeping problem where one keeps track of the input and output beams for each constituent part and the linear relation between the two. When taking account of all constituent parts and light paths, the Stokes vector of the light present at a certain location in a certain direction depends linearly on the Stokes vectors of all of the input beams through a polarization matrix that is a weighted sum of the Mueller matrices of the constituent volume and surface elements and therefore is a weighted sum of pure Mueller matrices.

The above reasoning implies that many matrices appearing in studies of multiple scattering in planetary atmospheres (Refs. 1, 3, 21-22), such as reflection and transmission matrices, matrices describing the internal radiation field and Green's function matrices, are weighted sums of pure Mueller matrices.

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