Nonuniqueness in inverse acoustic scattering on the line

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The generalized one-dimensional Schrödinger equation $d^2\phi/dx^2+k^2H(x)^2\phi=P(x)\phi$ is considered. The nonuniqueness is studied in the recovery of the function $P(x)$ when the scattering matrix, $H(x)$, and the bound state energies and norming constants are known. It is shown that when the reflection coefficient is unity at zero energy, there is a one-parameter family of functions $P(x)$ corresponding to the same scattering data. An explicitly solved example is provided. The construction of $H(x)$ from the scattering data is also discussed when $H(x)$ is piecewise continuous, and two explicitly solved examples are given with $H(x)$ containing a jump discontinuity.

I. INTRODUCTION

Consider the generalized Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2}+k^2H(x)^2\psi(k,x)=Q(x)\psi(k,x),\quad x\in\mathbf{R},\tag{1.1}$$

where $H(x)\to H_\pm$ and $Q(x)\to 0$ in some sense as $x\to\pm\infty$; here $H_\pm$ are some positive constants. The equation given in (1.1) is used to describe the wave propagation in a one-dimensional medium where $H(x)^{-1}$ is the wavespeed, $Q(x)$ is the restoring force per unit length, and k^2 is energy; the literature for (1.1) or related equations is enormous and it is impossible to give a complete bibliography here.[1–5] The following conditions are sufficient for the results in this paper to hold: $Q\in L_1^1(\mathbf{R})$, $H(x)$ is positive and bounded above, $H_+-H\in L^1(0,+\infty)$, $H_--H\in L^1(-\infty,0)$, and $G\in L_1^1(\mathbf{R})$, where $G(x)$ is the quantity

$$G(x)=-\frac{H''(x)}{2H(x)^2}+\frac{3}{4}\frac{H'(x)^2}{H(x)^3}-\frac{Q(x)}{H(x)}.\tag{1.2}$$

Here $L_\alpha^1(\mathbf{R})$ denotes the space of Lebesgue integrable functions on the real axis with the weight function $(1+|x|)^\alpha$, and $L^1(\mathbf{R})=L_0^1(\mathbf{R})$. In Secs. IV and V we discuss the case in which $H(x)$ has jump discontinuities at isolated points; at such points $G(x)$ does not exist.

The generalized Schrödinger equation (1.1) has two linearly independent scattering solutions, the physical solution $\psi_l(k,x)$ from the left corresponding to a plane wave sent from $x=-\infty$, and the physical solution $\psi_r(k,x)$ from the right corresponding to a plane wave sent from $x=+\infty$. These solutions satisfy the boundary conditions

$$\psi_l(k,x)=\begin{cases}T_l(k)e^{ikH_+x}+o(1), & x\to+\infty,\\ e^{ikH_-x}+L(k)e^{-ikH_-x}+o(1), & x\to-\infty,\end{cases}\tag{1.3}$$

$$\psi_r(k,x)=\begin{cases}e^{-ikH_+x}+R(k)e^{ikH_+x}+o(1), & x\to+\infty,\\ T_r(k)e^{-ikH_-x}+o(1), & x\to-\infty,\end{cases}\tag{1.4}$$

where $T_l(k)$ and $T_r(k)$ are the transmission coefficients from the left and from the right, respectively, and $R(k)$ and $L(k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrix associated with (1.1) is defined as

$$\mathbf{S}(k)=\begin{bmatrix} T_l(k) & R(k) \\ L(k) & T_r(k) \end{bmatrix}. \tag{1.5}$$

For the class of functions specified in the sentence following (1.1), there are two distinct cases to consider; in the generic case $T_l(0)=T_r(0)=0$ and $R(0)=L(0)=-1$, whereas in the exceptional case the transmission coefficients do not vanish at $k=0$. In the generic case the transmission coefficients vanish linearly as $k\to 0$. The potential $Q(x)$ in (1.1) is the only factor determining whether we have the generic case or the exceptional case. When $H(x)\equiv 1$ in (1.1), the inverse scattering problem of the recovery of $Q(x)$ from $\mathbf{S}(k)$ and the bound state energies and norming constants is well understood.[2,6–9]

In this paper we study, in the generic case, the counterpart of (1.1), namely the generalized Schrödinger equation (2.1), that corresponds to the scattering matrix

$$\mathbf{JS}(k)\mathbf{J}=\begin{bmatrix} T_l(k) & -R(k) \\ -L(k) & T_r(k) \end{bmatrix},$$

where

$$\mathbf{J}=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Note that when $H(x)\equiv 1$ in (1.1) and (2.1), the relationships between (1.1) and (2.1) and the resulting nonuniqueness are well understood;[10–13] when $H(x)\equiv 1$ the nonuniqueness in the generic case results due to the fact that there is a one-parameter family of functions $P(x)$ that corresponds to the same scattering data that consist of $\mathbf{S}(k)$, the bound state energies, and the bound state norming constants; the non-negative parameter a appearing in the potential can be fixed and the nonuniqueness can thus be removed by specifying the ratio of the physical solutions from the right and from the left, respectively, of (2.1) at $k=0$. Here we show that even when $H(x)$ is not identically equal to 1, there is a one-parameter family of functions $P(x)$ that corresponds to the same scattering data that consist of $H(x)$, $\mathbf{S}(k)$, the bound state energies, and the bound state norming constants.

In Refs. 14 and 15, when $H_{\pm}=1$ we presented a method to recover $H(x)$ from the scattering data consisting of $\mathbf{S}(k)$, $Q(x)$, and the bound state energies and norming constants. In Sec. IV we generalize that method to the case where $H_{\pm}$ are not necessarily equal to 1; in this section we also obtain $\lim_{k\to 0}\lim_{H(x)\to 1}\mathbf{S}(k)$ in terms of $\mathbf{S}(0)$ and vice versa, as these two matrices are not necessarily equal if $H_+\neq H_-$. In Sec. IV we also discuss the inverse problem of recovery of $H(x)$ when $H(x)$ is piecewise continuous and give two examples. Finally, in Sec. V we present two explicitly solved examples of construction of $H(x)$ with a jump discontinuity. Recently, Grinberg[5] presented a method to recover $H(x)$ having jump discontinuities when $Q(x)=0$; in this method a singular integral equation is solved iteratively to recover $H(x)$ from one of the reflection coefficients.

The methods given in this paper as well as the nonuniqueness in the inverse problem for (2.1) are readily generalizable to the differential equation

$$\frac{d}{dx}\left[h(x)\,\frac{d\psi(k,x)}{dx}\right]+k^2H(x)^2\psi(k,x)=P(x)\psi(k,x),$$

where one recovers $P(x)$ from the scattering data consisting of $\mathbf{S}(k)$, $h(x)$, $H(x)$, and the bound state energies as well as the bound state norming constants.[16]

II. RECOVERY OF $P(x)$ AND NONUNIQUENESS

In this section in the generic case we investigate the generalized Schrödinger equation

$$\frac{d^2\phi(k,x)}{dx^2}+k^2H(x)^2\phi(k,x)=P(x)\phi(k,x), \tag{2.1}$$

corresponding to the scattering matrix $\mathbf{JS}(k)\mathbf{J}$. The physical solutions of (2.1), $\phi_l(k,x)$ from the left and $\phi_r(k,x)$ from the right, respectively, satisfy the boundary conditions

$$\phi_l(k,x)=\begin{cases} T_l(k)e^{ikH_+x}+o(1), & x\to+\infty,\\ e^{ikH_-x}-L(k)e^{-ikH_-x}+o(1), & x\to-\infty,\end{cases} \tag{2.2}$$

$$\phi_r(k,x)=\begin{cases} e^{-ikH_+x}-R(k)e^{ikH_+x}+o(1), & x\to+\infty,\\ T_r(k)e^{-ikH_-x}+o(1), & x\to-\infty.\end{cases} \tag{2.3}$$

Using the Liouville transformation $\psi(k,x)=H(x)^{-1/2}\xi(k,y)$, we associate with (1.1) the Schrödinger equation

$$\frac{d^2\xi(k,y)}{dy^2}+k^2\xi(k,y)=V(y)\xi(k,y), \tag{2.4}$$

where

$$y=y(x)=\int_0^x ds\,H(s), \tag{2.5}$$

$$V(y)=-G(x)/H(x), \tag{2.6}$$

in which $G(x)$ is the quantity defined in (1.2). Let $\sigma(k)$ denote the scattering matrix of (2.4); it is related to $\mathbf{S}(k)$ in (1.5) as

$$\sigma(k)=\begin{bmatrix}\tau(k) & \rho(k)\\ \ell(k) & \tau(k)\end{bmatrix}=\begin{bmatrix}\sqrt{\frac{H_+}{H_-}}\,T_l(k)e^{ikA} & R(k)e^{2ikA_+}\\ L(k)e^{2ikA_-} & \sqrt{\frac{H_-}{H_+}}\,T_r(k)e^{ikA}\end{bmatrix}, \tag{2.7}$$

where

$$A_\pm=\pm\int_0^{\pm\infty} ds[H_\pm-H(s)], \tag{2.8}$$

$$A=A_++A_-=\int_{-\infty}^0 ds[H_--H(s)]+\int_0^\infty ds[H_+-H(s)]. \tag{2.9}$$

We have $\mathbf{S}(-k)=\overline{\mathbf{S}(k)}$ for $k\in\mathbb{R}$, where the overbar denotes complex conjugation. From (2.7) we obtain

$$\mathbf{S}(k)=e^{-ikA}e^{-(i/2)\mathbf{J}k(A_+-A_-)}\begin{bmatrix}\sqrt{H_-} & 0\\ 0 & \sqrt{H_+}\end{bmatrix}\begin{bmatrix}\tau(k) & \rho(k)\\ \ell(k) & \tau(k)\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{H_+}} & 0\\ 0 & \frac{1}{\sqrt{H_-}}\end{bmatrix}e^{(i/2)\mathbf{J}k(A_+-A_-)},$$

and hence using (1.3) and (1.4) we obtain

$$H_+T_l(k)=H_-T_r(k), \tag{2.10}$$

$$T_r(k)T_l(-k)+R(k)R(-k)=1,$$

$$T_r(k)L(-k)+R(k)T_r(-k)=0, \tag{2.11}$$

$$T_l(k)L(-k)+R(k)T_l(-k)=0, \tag{2.12}$$

$$T_r(k)T_l(-k)+L(k)L(-k)=1.$$

As a result, $\sigma(k)$ is unitary. Note that $y-H_+x+A_+=o(1)$ as $x\to+\infty$ and $y-H_-x-A_-=o(1)$ as $x\to-\infty$. Let $\xi_l(k,y)$ and $\xi_r(k,y)$ be the physical solutions of (2.4) from the left and from the right, respectively, satisfying the boundary conditions

$$\xi_l(k,y)=\begin{cases}\tau(k)e^{iky}+o(1), & y\to+\infty,\\ e^{iky}+\ell(k)e^{-iky}+o(1), & y\to-\infty,\end{cases} \tag{2.13}$$

$$\xi_r(k,y)=\begin{cases}e^{-iky}+\rho(k)e^{iky}+o(1), & y\to+\infty,\\ \tau(k)e^{-iky}+o(1), & y\to-\infty.\end{cases} \tag{2.14}$$

Let us also define the Faddeev functions $Z_l(k,y)$ from the left and $Z_r(k,y)$ from the right, respectively, associated with (2.4) as

$$Z_l(k,y)=\frac{1}{\tau(k)}\,e^{-iky}\xi_l(k,y), \tag{2.15}$$

$$Z_r(k,y)=\frac{1}{\tau(k)}\,e^{iky}\xi_r(k,y). \tag{2.16}$$

Then $Z_l(k,y)$ satisfies

$$Z_l''(k,y)+2ikZ_l'(k,y)=V(y)Z_l(k,y), \tag{2.17}$$

and the boundary conditions

$$Z_l(k,y)=1+o(1) \quad\text{and}\quad Z_l'(k,y)=o(1), \quad y\to+\infty, \tag{2.18}$$

and $Z_r(k,y)$ satisfies

$$Z_r''(k,y)-2ikZ_r'(k,y)=V(y)Z_r(k,y), \tag{2.19}$$

and the boundary conditions

$$Z_r(k,y)=1+o(1) \quad\text{and}\quad Z_r'(k,y)=o(1), \quad y\to-\infty. \tag{2.20}$$

Similarly, let us define the Faddeev functions $m_l(k,x)$ from the left and $m_r(k,x)$ from the right, respectively, associated with (1.1) as

$$m_l(k,x)=\frac{1}{T_l(k)}\, e^{-ikH_+x}\psi_l(k,x), \tag{2.21}$$

$$m_r(k,x)=\frac{1}{T_r(k)}\, e^{ikH_-x}\psi_r(k,x). \tag{2.22}$$

Using (1.3), (1.4), (2.7), and (2.13)–(2.16), we obtain

$$m_l(k,x)=\sqrt{\frac{H_+}{H(x)}}\, e^{ikA_+-ikH_+x+iky(x)}Z_l\,(k,y(x)), \tag{2.23}$$

$$m_r(k,x)=\sqrt{\frac{H_-}{H(x)}}\, e^{ikA_-+ikH_-x-iky(x)}Z_r\,(k,y(x)). \tag{2.24}$$

Associated with (2.4) is the Schrödinger equation corresponding to the scattering matrix $\mathbf{J}\sigma(k)\mathbf{J}$

$$\frac{d^2\eta(k,y)}{dy^2}+k^2\eta(k,y)=U(y)\eta(k,y), \tag{2.25}$$

where $\sigma(k)$ is the matrix in (2.7). In the generic case there is a one-parameter family of functions $U(y)$ depending on a parameter $a\in[0,+\infty]$, and this parameter does not appear[12,13] in the scattering matrix $\mathbf{S}(k)$. Let us define

$$\alpha(y;a)=\frac{Z_l'(0,y)+aZ_r'(0,y)}{Z_l(0,y)+aZ_r(0,y)}, \tag{2.26}$$

where $Z_l(k,y)$ and $Z_r(k,y)$ are the Faddeev functions given in (2.15) and (2.16), respectively. In terms of the solutions of (2.4), the physical solutions of (2.25), $\eta_l(k,y)$ from the left and $\eta_r(k,y)$ from the right, respectively, are given by[12,13]

$$\eta_l(k,y)=(1/ik)\,[\xi_l'(k,y)-\xi_l(k,y)\alpha(y;a)], \tag{2.27}$$

$$\eta_r(k,y)=-(1/ik)\,[\xi_r'(k,y)-\xi_r(k,y)\alpha(y;a)]. \tag{2.28}$$

We also have

$$V(y)=\alpha(y;a)^2+\alpha'(y;a), \tag{2.29}$$

$$U(y)=\alpha(y;a)^2-\alpha'(y;a). \tag{2.30}$$

In analogy with (2.15) and (2.16), the Faddeev functions associated with (2.25) are given by

$$Y_l(k,y)=\frac{1}{\tau(k)}\, e^{-iky}\eta_l(k,y), \quad Y_r(k,y)=\frac{1}{\tau(k)}\, e^{iky}\eta_r(k,y),$$

and they satisfy

$$Y_l''(k,y)+2ikY_l'(k,y)=U(y)Y_l(k,y),$$

$$Y_r''(k,y)-2ikY_r'(k,y)=U(y)Y_r(k,y).$$

From (2.27) and (2.28) we obtain

$$Y_l(k,y)=Z_l(k,y)-(i/k)[Z_l'(k,y)-Z_l(k,y)\alpha(y;a)], \tag{2.31}$$

$$Y_r(k,y)=Z_r(k,y)+(i/k)[Z_r'(k,y)-Z_r(k,y)\alpha(y;a)]. \tag{2.32}$$

It is also known[13] that

$$\alpha(y;a)=-\frac{Y_l'(0,y)+aY_r'(0,y)}{Y_l(0,y)+aY_r(0,y)}.$$

Let $n_l(k,x)$ and $n_r(k,x)$ denote the Faddeev functions from the left and from the right, respectively, associated with (2.1) corresponding to the scattering matrix $\mathbf{JS}(k)\mathbf{J}$. In terms of the physical solutions of (2.1), in analogy with (2.21) and (2.22), we have

$$n_l(k,x)=\frac{1}{T_l(k)}e^{-ikH_+x}\phi_l(k,x),\quad n_r(k,x)=\frac{1}{T_r(k)}e^{ikH_-x}\phi_r(k,x).$$

Theorem 1: The potential $P(x)$ in (2.1) depends on the parameter a and is given by

$$P(x)=-\frac{H''(x)}{2H(x)}+\frac{3}{4}\frac{H'(x)^2}{H(x)^2}+H(x)^2[\alpha(y(x);a)^2-\alpha'(y(x);a)], \tag{2.33}$$

or equivalently is given by

$$P(x)=Q(x)-2H(x)^2\alpha'(y(x);a), \tag{2.34}$$

where $Q(x)$ is the potential in (1.1) and $\alpha(y(x);a)$ is the quantity defined in (2.26), $y(x)$ is the travel time coordinate given in (2.5), and $\alpha'(y(x);a)=(d/dy)\alpha(y(x);a)$. The Faddeev functions of (2.1) are given by

$$n_l(k,x)=\sqrt{\frac{H_+}{H(x)}}\,e^{ikA_+-ikH_+x+iky(x)}Y_l(k,y(x)), \tag{2.35}$$

$$n_r(k,x)=\sqrt{\frac{H_-}{H(x)}}\,e^{ikA_-+ikH_-x-iky(x)}Y_r(k,y(x)), \tag{2.36}$$

where $Y_l(k,y(x))$ and $Y_r(k,y(x))$ are the Faddeev functions given in (2.31) and (2.32), respectively.

Proof: Using (2.6) and (2.30) it can be directly verified that the quantity $n_l(k,x)$ defined in (2.35) satisfies

$$n_l''(k,x)+2ikH_+n_l'(k,x)=k^2[H_+^2-H(x)^2]n_l(k,x)+[Q(x)-2H(x)^2\alpha'(y(x);a)]n_l(k,x),$$

and hence $T_l(k)e^{ikH_+x}n_l(k,x)$ satisfies (2.1) with $P(x)$ given in (2.33). From (1.2) and (2.29) we then obtain (2.34). Similarly, the quantity $n_r(k,x)$ defined in (2.36) satisfies

$$n_r''(k,x)-2ikH_-n_r'(k,x)=k^2[H_-^2-H(x)^2]n_r(k,x)+[Q(x)-2H(x)^2\alpha'(y(x);a)]n_r(k,x),$$

and hence $T_r(k)e^{-ikH_-x}n_r(k,x)$ satisfies (2.1) with $P(x)=Q(x)-2H(x)^2\alpha'(y(x);a)$. The Faddeev functions $Y_l(k,y(x))$ and $Y_r(k,y(x))$ satisfy

$$Y_l(k,y)=\begin{cases}1+o(1), & y\to+\infty,\\ \dfrac{1}{\tau(k)}-\dfrac{\ell(k)}{\tau(k)}e^{-2iky}+o(1), & y\to-\infty,\end{cases}$$

$$Y_r(k,x)=\begin{cases}\dfrac{1}{\tau(k)}-\dfrac{\rho(k)}{\tau(k)}e^{2iky}+o(1), & y\to+\infty,\\ 1+o(1), & y\to-\infty,\end{cases}$$

and hence $T(k)e^{ikx}n_l(k,x)$ satisfies the boundary conditions in (2.2). Similarly, $T_r(k)e^{-ikH_-x}n_r(k,x)$ satisfies the boundary conditions in (2.3). ■

In the generic case the Faddeev functions $n_l(k,x)$ and $n_r(k,x)$ depend on the parameter a whereas in the exceptional case they are independent of a; this is because in the exceptional case a is absent in $\alpha(y;a)$ in (2.26) due to the fact that $Z_l(0,y)$ and $Z_r(0,y)$ are linearly dependent. One possible way to fix the parameter a is to use

$$\lim_{k\to0}\frac{n_l(k,x)}{n_r(k,x)}=(-1)^{\mathcal{N}}a\sqrt{\frac{H_+}{H_-}}$$

because we have[13]

$$\lim_{k\to0}\frac{Y_l(k,z)}{Y_r(k,z)}=(-1)^{\mathcal{N}}a,$$

where $\mathcal{N}$ is the number of bound states.

III. AN EXAMPLE OF NONUNIQUENESS

In this section we present an explicitly solved example to illustrate the method given in Sec. II and obtain a one-parameter family of functions $P(x)$ corresponding to the same scattering data consisting of $H(x)$ and $\mathbf{S}(k)$.

Assume that

$$H(x)=\begin{cases}\frac{3}{2}+\frac{1}{2}e^{-x}, & x>0,\\ \frac{1}{2}+\frac{3}{2}e^{x}, & x<0.\end{cases}$$

We thus have $H_+=\frac{3}{2}$, $H_-=\frac{1}{2}$, and

$$y(x)=\begin{cases}\frac{3}{2}x+\frac{1}{2}-\frac{1}{2}e^{-x}, & x>0,\\ \frac{1}{2}x-\frac{3}{2}+\frac{3}{2}e^{x}, & x<0.\end{cases} \tag{3.1}$$

The quantities $A_\pm$ in (2.8) are then given by $A_-=-\frac{3}{2}$ and $A_+=-\frac{1}{2}$, and hence $A=-2$, where A is the constant in (2.9). Assume that $\mathbf{S}(k)$ is such that the matrix $\sigma(k)$ in (2.7) is given by

$$\tau(k)=\frac{k}{k+i},\quad \rho(k)=\ell(k)=\frac{-i}{k+i}.$$

Using the method in Ref. 11 or 13, the potential in (2.4) is obtained as

$$V(y)=2\delta(y),$$

where $\delta(y)$ is the Dirac delta function; the Faddeev functions associated with (2.4) are given by

$$Z_l(k,y)=\begin{cases}1, & y>0,\\ 1+(i/k)(1-e^{-2iky}), & y<0,\end{cases}$$

$$Z_l(k,y)=\begin{cases}1+(i/k)(1-e^{2iky}), & y>0,\\ 1, & y<0.\end{cases}$$

Using (2.26), we obtain

$$\alpha(y;a)=\begin{cases}\dfrac{2a}{1+a+2ay}, & y>0,\\[2mm] \dfrac{-2}{1+a-2y}, & y<0,\end{cases}$$

where a is an arbitrary parameter with $0<a<+\infty$. Thus, we obtain the potential in (2.25) are given by

$$U(y)=-2\delta(y)+\theta(y)\,\frac{8a^2}{(1+a+2ay)^2}+\theta(-y)\,\frac{8}{(1+a-2y)^2},$$

where $\theta(y)$ is the Heaviside function; the Faddeev functions associated with (2.25) are given by

$$Y_l(k,y)=\begin{cases}1+\dfrac{i}{k}\dfrac{2a}{1+a+2ay}, & y>0,\\[2mm] 1+\dfrac{i}{k}(1+e^{-2iky})-\dfrac{i}{k}\dfrac{2}{1+a-2y}\left[1+\dfrac{i}{k}(1-e^{-2iky})\right], & y<0,\end{cases} \tag{3.2}$$

$$Y_r(k,y)=\begin{cases}1+\dfrac{i}{k}(1+e^{2iky})-\dfrac{i}{k}\dfrac{2a}{1+a+2ay}\left[1+\dfrac{i}{k}(1-e^{2iky})\right], & y>0\\[2mm] 1+\dfrac{i}{k}\dfrac{2}{1+a-2y}, & y<0.\end{cases} \tag{3.3}$$

From (2.35) and (2.36) we obtain $n_l(k,x)$ and $n_r(k,x)$, and $P(x)$ is constructed using (2.33) and $Q(x)$ is constructed using (2.34) or using (1.2). We have

$$Q(x)=\tfrac{9}{2}\delta(x)+Q_+(x)\theta(x)+Q_-(x)\theta(-x),$$

$$P(x)=-\tfrac{7}{2}\delta(x)+\theta(x)P_+(x)+\theta(-x)P_-(x),$$

where

$$Q_+(x) = -\frac{1}{2}\frac{e^{-x}}{3+e^{-x}} + \frac{3}{4}\frac{e^{-2x}}{(3+e^{-x})^2},$$

$$Q_-(x) = -\frac{3}{2}\frac{e^{x}}{1+3e^{x}} + \frac{27}{4}\frac{e^{2x}}{(1+3e^{x})^2},$$

$$P_+(x) = -\frac{1}{2}\frac{e^{-x}}{3+e^{-x}} + \frac{3}{4}\frac{e^{-2x}}{(3+e^{-x})^2} + \frac{2a^2(3+e^{-x})^2}{(1+2a+3ax-ae^{-x})^2},$$

$$P_-(x) = -\frac{3}{2}\frac{e^{x}}{1+3e^{x}} + \frac{27}{4}\frac{e^{2x}}{(1+3e^{x})^2} + \frac{2(1+3e^{x})^2}{(4+a-x-3e^{x})^2}.$$

The Faddeev functions $n_l(k,x)$ and $n_r(k,x)$ can be computed explicitly in terms of x from (2.35) and (2.36) by using $Y_l(k,y)$ and $Y_r(k,y)$ in (3.2) and (3.3), respectively, and $y(x)$ in (3.1).

IV. RECOVERY OF $H(x)$

In this section we generalize the method of Refs. 14 and 15 in order to recover $H(x)$ where the scattering data consist of $\mathbf{S}(k)$, $Q(x)$, $H_\pm$, the bound state energies, and the bound state norming constants. We also discuss the recovery of a piecewise continuous $H(x)$ when the points of discontinuity for $H(x)$ are known. At the end of the section we study the relationship between the scattering matrices of (1.1) and of (4.5) at $k=0$, as $\mathbf{S}(0)$ and $\mathbf{S}^{[0]}(0)$ are not necessarily equal if $H_+ \neq H_-$.

From (1.1), (1.3), (1.4), (2.21), and (2.22), it is seen that we have

$$m_l''(k,x) + 2ikH_+ m_l'(k,x) = k^2[H_+^2 - H(x)^2]m_l(k,x) + Q(x)m_l(k,x), \tag{4.1}$$

$$m_r''(k,x) - 2ikH_- m_r'(k,x) = k^2[H_-^2 - H(x)^2]m_r(k,x) + Q(x)m_r(k,x), \tag{4.2}$$

$$m_l(k,x) = 1 + o(1), \quad m_l'(k,x) = o(1), \quad x \to +\infty, \tag{4.3}$$

$$m_r(k,x) = 1 + o(1), \quad m_r'(k,x) = o(1), \quad x \to -\infty. \tag{4.4}$$

When $H(x) \equiv 1$, (1.1) is reduced to the Schrödinger equation

$$\frac{d^2\psi^{[0]}(k,x)}{dx^2} + k^2\psi^{[0]}(k,x) = Q(x)\psi^{[0]}(k,x). \tag{4.5}$$

Associated with (4.5) we have the scattering matrix

$$\mathbf{S}^{[0]}(k,x) = \begin{bmatrix} T^{[0]}(k) & R^{[0]}(k) \\ L^{[0]}(k) & T^{[0]}(k) \end{bmatrix}.$$

The physical solutions $\psi_l^{[0]}(k,x)$ and $\psi_r^{[0]}(k,x)$ of (4.5) from the left and from the right, respectively, satisfy

$$\psi_l^{[0]}(k,x)=\begin{cases} T^{[0]}(k)e^{ikx}+o(1), & x\to+\infty, \\ e^{ikx}+L^{[0]}(k)e^{-ikx}+o(1), & x\to-\infty, \end{cases}$$

$$\psi_r^{[0]}(k,x)=\begin{cases} e^{-ikx}+R^{[0]}(k)e^{ikx}+o(1), & x\to+\infty, \\ T^{[0]}(k)e^{-ikx}+o(1), & x\to-\infty. \end{cases}$$

The Faddeev functions $m_l^{[0]}(k,x)$ from the left and $m_r^{[0]}(k,x)$ from the right, respectively, associated with (4.5) are defined as

$$m_l^{[0]}(k,x)=\frac{1}{T^{[0]}(k)}\,e^{-ikx}\psi_l^{[0]}(k,x),$$

$$m_r^{[0]}(k,x)=\frac{1}{T^{[0]}(k)}\,e^{ikx}\psi_r^{[0]}(k,x).$$

From (4.1)–(4.4) it is seen that they satisfy the following differential equations and the boundary conditions:

$$m_l^{[0]\prime\prime}(k,x)+2ikm_l^{[0]\prime}(k,x)=Q(x)m_l^{[0]}(k,x), \tag{4.6}$$

$$m_r^{[0]\prime\prime}(k,x)-2ikm_r^{[0]\prime}(k,x)=Q(x)m_r^{[0]}(k,x), \tag{4.7}$$

$$m_l^{[0]}(k,x)=1+o(1), \quad m_l^{[0]\prime}(k,x)=o(1), \quad x\to+\infty, \tag{4.8}$$

$$m_r^{[0]}(k,x)=1+o(1), \quad m_r^{[0]\prime}(k,x)=o(1), \quad x\to-\infty. \tag{4.9}$$

When $Q(x)$ is known, $m_l^{[0]}(k,x)$ and $m_r^{[0]}(k,x)$ are uniquely determined. The potential $Q(x)$ in (1.1) and (4.5) is the only factor determining whether we have the generic case or the exceptional case. For (4.5) the generic case occurs if $T^{[0]}(0)=0$ and the exceptional case if $T^{[0]}(0)\neq 0$.

From (4.1)–(4.4) and (4.6)–(4.9) it is seen that

$$m_l(0,x)=m_l^{[0]}(0,x) \quad \text{and} \quad m_r(0,x)=m_r^{[0]}(0,x), \quad x\in\mathbf{R}. \tag{4.10}$$

In the exceptional case $m_l^{[0]}(k,x)$ and $m_r^{[0]}(k,x)$ become linearly dependent at $k=0$; however, in the generic case $m_l^{[0]}(k,x)$ and $m_r^{[0]}(k,x)$ remain linearly independent at $k=0$. From (2.23) and (2.24) we have

$$m_l(0,x)=\sqrt{\frac{H_+}{H(x)}}\,Z_l(0,y(x)), \tag{4.11}$$

$$m_r(0,x)=\sqrt{\frac{H_-}{H(x)}}\,Z_r(0,y(x)), \tag{4.12}$$

and hence using (2.5) and (4.10)–(4.12), we obtain

$$\frac{dy}{Z_l(0,y)^2}=H_+\,\frac{dx}{m_l^{[0]}(0,x)^2}, \quad x,y\in\mathbf{R}, \tag{4.13}$$

$$\frac{dy}{Z_r(0,y)^2}=H_-\frac{dx}{m_r^{[0]}(0,x)^2},\quad x,y\in\mathbf{R}. \tag{4.14}$$

The first-order ordinary differential equations given in (4.13) and (4.14) are both separable; their solutions with the initial condition $y(0)=0$ give us $y(x)$. Using $y(x)$ in either of

$$\frac{dy}{dx}=H(x)=H_+\frac{Z_l(0,y(x))^2}{m_l^{[0]}(0,x)^2},\quad \frac{dy}{dx}=H(x)=H_-\frac{Z_r(0,y(x))^2}{m_r^{[0]}(0,x)^2}, \tag{4.15}$$

one recovers $H(x)$ for $x\in\mathbf{R}$. The corresponding Faddeev functions $m_l(k,x)$ and $m_r(k,x)$ can be constructed using (2.23) and (2.24) as

$$m_l(k,x)=\frac{m_l^{[0]}(0,x)}{Z_l(0,y(x))}e^{ikA_+-ikH_+x+iky(x)}Z_l(k,y(x)), \tag{4.16}$$

$$m_r(k,x)=\frac{m_r^{[0]}(0,x)}{Z_r(0,y(x))}e^{ikA_-+ikH_-x-iky(x)}Z_r(k,y(x)), \tag{4.17}$$

and the physical solutions of (1.1) can be constructed using (2.21), (2.22), (4.16), and (4.17).

The physical solutions of (1.1) and their derivatives are required to be continuous in x. [If $Q(x)$ contains any delta function singularities, then the derivatives of the physical solutions are required to have jump discontinuities to account for these singularities; for example, if $Q(x)$ has a delta function singularity $c_1\delta(x-x_1)$ at x_1 of strength c_1, then we require that the physical solutions satisfy $\psi'(k,x_1+)-\psi'(k,x_1-)=c_1\psi(k,x_1)$.] In case $H(x)$ has a jump discontinuity at some x value, say x_0, the physical solutions of (1.1) and their derivatives are still required to be continuous at x_0; in that case, however, as seen from (4.16) and (4.17), the Faddeev functions $Z_l(k,y(x))$ and $Z_r(k,y(x))$ have jump discontinuities at $y(x_0)$ and their derivatives contain delta function singularities at $y(x_0)$. In that case the quantity $G(x)$ in (1.2) is not well defined at x_0 because $H'(x)$ has a delta function singularity at x_0. However, the method of recovery of $H(x)$ given above does not actually use the derivatives of the Faddeev functions $Z_l(k,y(x))$ and $Z_r(k,y(x))$, but only $Z_l(0,y)$ and $Z_r(0,y)$ are used to construct $H(x)$. Thus, the construction method outlined above remains valid even if $H(x)$ has a jump discontinuity at x_0. The construction of $Z_l(k,y)$ and $Z_r(k,y)$ can be achieved by solving the Riemann–Hilbert problem[14,17]

$$\begin{bmatrix} Z_l(-k,y) \\ Z_r(-k,y)\end{bmatrix}=\begin{bmatrix} \tau(k) & -\rho(k)e^{2iky} \\ -\ell(k)e^{-2iky} & \tau(k)\end{bmatrix}\begin{bmatrix} Z_r(k,y) \\ Z_l(k,y)\end{bmatrix},\quad k\in\mathbf{R}, \tag{4.18}$$

where given the matrix $\sigma(k)$, for each y one seeks $Z_l(k,y)$ and $Z_r(k,y)$ such that they are analytic in k for $k\in C^+$, the upper-half complex plane, for $y>y(x_0)$ we require $Z_l(k,y)\to 1$ and for $y<y(x_0)$ we require $Z_r(k,y)\to 1$ as $k\to\infty$ in $\overline{C^+}$, the closure of the upper-half complex plane. In case there are more points of jump discontinuity in $H(x)$ than one, as seen from (4.16) and (4.17), we then require the continuity in y of $Z_l(k,y)/Z_l(0,y)$, $Z_r(k,y)/Z_r(0,y)$, $Z_l(0,y)^2(d/dy)[e^{iky}Z_l(k,y)/Z_l(0,y)]$, and $Z_r(0,y)^2(d/dy)[e^{-iky}Z_r(k,y)/Z_r(0,y)]$, which is equivalent to the continuity in x of the physical solutions of (1.1) and of the derivatives of those solutions. Once the solution of (4.18) is obtained, $H(x)$ can be constructed as outlined above.

In a standard Riemann–Hilbert problem, $\tau(k)\to 1$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$, and $\rho(k)$, $\ell(k)\to 0$ as $k\to\pm\infty$. In general, when $H(x)$ has discontinuities the Riemann–Hilbert problem given in (4.18) is a nonstandard problem. For example, when $Q(x)=0$ and

$$H(x)=\begin{cases}1, & x<0\\ 2, & 0<x<1\\ 1, & x>1,\end{cases}$$

we have

$$\tau(k)=\frac{8e^{-ik}}{9e^{-ik}-e^{3ik}},\quad \ell(k)=\frac{3e^{3ik}-3e^{-ik}}{9e^{-ik}-e^{3ik}},\quad \rho(k)=\frac{3e^{-ik}-3e^{-5ik}}{9e^{-ik}-e^{3ik}},$$

and consequently the corresponding Riemann–Hilbert problem is a nonstandard one. When the existence of $H(x)$ is assured, even in the nonstandard case, the corresponding Riemann–Hilbert problem (4.18) must have a solution that leads to the reconstruction of $H(x)$ via (4.15). However, in the nonstandard case there does not yet exist a theory on the existence and uniqueness of the solutions of (4.18). In the next section we will present two examples dealing with (4.18) corresponding to discontinuous $H(x)$.

The next theorem shows that the functions in (4.16) and (4.17) can still be used to obtain the physical solutions of (1.1) even when $H(x)$ has jump discontinuities.

Theorem 2: Relax the conditions on $Q(x)$ and $H(x)$ stated in the first paragraph of the Introduction by allowing $Q(x)$ to have delta function singularities at isolated points and $H(x)$ to have jump discontinuities at isolated points; let x_j denote a point where $Q(x)$ has a delta function singularity or $H(x)$ has a jump discontinuity or both. Let $y(x)$ be the continuous solution of the differential equation (4.13) with the initial condition $y(0)=0$, and let $y_j=y(x_j)$. Let $Z_l(k,y)$ be the solution of (2.17) on each interval (y_j,y_{j+1}) with the potential $V(y)$ related to $H(x)=dy/dx$ as in (1.2) and (2.6). Choose the boundary conditions on $Z_l(k,y)$ such that (2.18) is satisfied and such that at each discontinuity of $H(x)$ we have

$$\frac{Z_l(k,y_j+)}{Z_l(0,y_j+)}=\frac{Z_l(k,y_j-)}{Z_l(0,y_j-)},\tag{4.19}$$

and at each point where $Q(x)$ has a delta function singularity or $H(x)$ has a jump discontinuity we have

$$Z_l(0,y_j+)^2\frac{d}{dy}\left[e^{iky}\frac{Z_l(k,y)}{Z_l(0,y)}\right]_{y=y_j+}=Z_l(0,y_j-)^2\frac{d}{dy}\left[e^{iky}\frac{Z_l(k,y)}{Z_l(0,y)}\right]_{y=y_j-}.\tag{4.20}$$

Then the physical solution of (1.1) from the left is given by

$$\psi_l(k,x)=T_l(k)\,\frac{m_l^{[0]}(0,x)}{Z_l(0,y(x))}\,e^{ikA_+ +iky(x)}Z_l(k,y(x)).\tag{4.21}$$

Similarly, let $Z_r(k,y)$ be the solution of (2.19) on each interval (y_j,y_{j+1}) such that (2.20) is satisfied and such that at each discontinuity of $H(x)$ we have

$$\frac{Z_r(k,y_j+)}{Z_r(0,y_j+)}=\frac{Z_r(k,y_j-)}{Z_r(0,y_j-)},$$

and at each point where $Q(x)$ has a delta function singularity or $H(x)$ has a jump discontinuity we have

$$Z_r(0,y_j+)^2 \frac{d}{dy}\left[e^{-iky}\frac{Z_r(k,y)}{Z_r(0,y)}\right]_{y=y_j+} = Z_r(0,y_j-)^2 \frac{d}{dy}\left[e^{-iky}\frac{Z_r(k,y)}{Z_r(0,y)}\right]_{y=y_j-}.$$

Then the physical solution of (1.1) from the right is given by

$$\psi_r(k,x) = T_r(k)\,\frac{m_r^{[0]}(0,x)}{Z_r(0,y(x))}\,e^{ikA_- - iky(x)}Z_r(k,y(x)).$$

Proof: We will only give the proof related to the solution from the left as the proof related to the solution from the right can be obtained in a similar way. Using (2.8), (2.18), and (4.8), we see that $\psi_l(k,x)$ defined in (4.21) satisfies the boundary condition at $x=+\infty$ stated in (1.3). From (4.19) and the continuity of $m_l^{[0]}(0,x)$ we are assured that $\psi_l(k,x)$ defined in (4.21) is continuous in x for $x\in\mathbf{R}$. Taking the x derivative of both sides of (4.21) and using (4.15), we obtain

$$\frac{1}{T_l(k)}\,\psi_l'(k,x) = \frac{m_l^{[0]\prime}(0,x)}{Z_l(0,y(x))}\,e^{ikA_+ + iky(x)}Z_l(k,y(x))$$

$$+H_+\,\frac{Z_l(0,y(x))^2}{m_l^{[0]}(0,x)}\,\frac{d}{dy}\left[e^{ikA_+ + iky(x)}\,\frac{Z_l(k,y)}{Z_l(0,y)}\right], \tag{4.22}$$

and hence, using (4.19), (4.20), (4.22), and the continuity of $m_l^{[0]}(0,x)$ and of $m_l^{[0]\prime}(0,x)$, we are assured that $\psi_l'(k,x)$ is continuous in x for $x\in\mathbf{R}$. From (4.22), using (2.17), (4.6), and (4.15) we obtain $\psi_l''(k,x) = [Q(x) - k^2H(x)^2]\psi_l(k,x)$, and hence (1.1) is satisfied in each interval (x_j,x_{j+1}).

Note that if x_j is a delta function singularity in $Q(x)$, we simply replace the continuity requirement for $\psi_l'(k,x)$ at $x=x_j$ by $\psi_l'(k,x_j+) - \psi_l'(k,x_j-) = p_j\psi_l(k,x_j)$, where p_j is the coefficient of the delta function at $x=x_j$ in $Q(x)$. Using (4.22) and $m_l^{[0]\prime}(0,x_j+) - m_l^{[0]\prime}(0,x_j-) = p_j m_l^{[0]}(0,x_j)$ one can verify that the condition in (4.21) also covers this case. ■

As explained in Ref. 15 one does not need to know $A_\pm$ in order to recover $H(x)$ by the above method. When the phase of the reflection coefficient $\rho(k)$ in (2.7) is shifted by Δ, the travel time coordinate defined in (2.5) is shifted as $y\mapsto y+\Delta$ in the Schrödinger equation (2.4) and the Faddeev functions for (2.4) are then transformed[7,11] as $Z_l(k,y)\mapsto Z_l(k,y+\Delta)$ and $Z_r(k,y)\mapsto Z_r(k,y+\Delta)$. Then in the solution of the first-order differential equations (4.13) and (4.14), the initial condition is replaced by $y(0)=-\Delta$; since $H(x)=dy/dx$, $H(x)$ is independent of the shift $y\mapsto y+\Delta$ and hence no matter how the phase of $\rho(k)$ is chosen, we are led to the same $H(x)$ in the solution of the inverse problem for (1.1).

We will end this section by exploring the relationship between $\mathbf{S}(0)$ and $\mathbf{S}^{[0]}(0)$, where $\mathbf{S}(k)$ and $\mathbf{S}^{[0]}(k)$ are the scattering matrices of (1.1) and (4.5), respectively. From (1.3) and (1.4) we obtain

$$\begin{bmatrix}\psi_l(k,x)\\ \psi_r(k,x)\end{bmatrix} = \begin{bmatrix}T_l(k) & L(k)\\ R(k) & T_r(k)\end{bmatrix}\begin{bmatrix}\psi_r(-k,x)\\ \psi_l(-k,x)\end{bmatrix},\quad k\in\mathbf{R}. \tag{4.23}$$

Note that the Faddeev functions $m_l^{[0]}(k,x)$ and $m_r^{[0]}(k,x)$ of (4.5) satisfy the analog of the Riemann–Hilbert problem given in (4.18); we have[7,11,13]

$$\begin{bmatrix}m_l^{[0]}(-k,x)\\ m_r^{[0]}(-k,x)\end{bmatrix} = \begin{bmatrix}T^{[0]}(k) & -R^{[0]}(k)e^{2ikx}\\ -L^{[0]}(k)e^{-2ikx} & T^{[0]}(k)\end{bmatrix}\begin{bmatrix}m_r^{[0]}(k,x)\\ m_l^{[0]}(k,x)\end{bmatrix},\quad k\in\mathbf{R}. \tag{4.24}$$

In the generic case, $\mathbf{S}(0)=\mathbf{S}^{[0]}(0)$, and in fact we have $R(0)=L(0)=R^{[0]}(0)=L^{[0]}(0)=-1$ and $T_l(0)=T_r(0)=T^{[0]}(0)=0$. In the exceptional case, if $H_+=H_-$, we have $\mathbf{S}(0)=\mathbf{S}^{[0]}(0)$; in fact, in this case we have $R(0)=-L(0)=R^{[0]}(0)=-L^{[0]}(0)$, which can be obtained from (4.10), (4.23), and (4.24). However, if $H_+\neq H_-$, we no longer have $\mathbf{S}(0)=\mathbf{S}^{[0]}(0)$. The next proposition gives the relationship between $\mathbf{S}(0)$ and $\mathbf{S}^{[0]}(0)$ in the exceptional case when H_+ and H_- are not necessarily equal to each other.

Proposition 3: In the exceptional case we have

$$L^{[0]}(0)=\frac{[H_++H_-]L(0)+[H_+-H_-]}{[H_++H_-]+[H_+-H_-]L(0)}, \tag{4.25}$$

$$R^{[0]}(0)=\frac{[H_++H_-]R(0)-[H_+-H_-]}{[H_++H_-]-[H_+-H_-]R(0)}, \tag{4.26}$$

$$T^{[0]}(0)=\frac{2H_+T_l(0)}{[H_++H_-]+[H_+-H_-]L(0)}=\frac{2H_-T_r(0)}{[H_++H_-]-[H_+-H_-]R(0)}. \tag{4.27}$$

Proof: From (2.11) or (2.12) we have $R(0)=-L(0)$ even when H_+ and H_- are not necessarily equal to each other. Hence (4.26) can be obtained from (4.25). So let us prove (4.25) first. From (4.10) as $x\to-\infty$, we obtain

$$\frac{1+L(0)}{T_l(0)}=\frac{1+L^{[0]}(0)}{T^{[0]}(0)}, \tag{4.28}$$

and from (4.10) as $x\to+\infty$, we obtain

$$\frac{1+R(0)}{T_r(0)}=\frac{1+R^{[0]}(0)}{T^{[0]}(0)}. \tag{4.29}$$

Hence, using (4.10), (4.28), and (4.29) we obtain (4.25). Using (4.25), (4.28), and (4.29) we obtain (4.27). Note that when $H_+=H_-$, (4.25)–(4.27) give us $\mathbf{S}(0)=\mathbf{S}^{[0]}(0)$. ■

Note that, using (4.25)–(4.27), we can also express $\mathbf{S}(0)$ in terms of $\mathbf{S}^{[0]}(0)$ in the exceptional case as follows:

$$L(0)=\frac{[H_++H_-]L^{[0]}(0)-[H_+-H_-]}{[H_++H_-]-[H_+-H_-]L^{[0]}(0)},$$

$$R(0)=\frac{[H_++H_-]R^{[0]}(0)+[H_+-H_-]}{[H_++H_-]+[H_+-H_-]R^{[0]}(0)},$$

$$T_l(0)=\frac{2H_-T^{[0]}(0)}{[H_++H_-]-[H_+-H_-]L^{[0]}(0)},$$

$$T_r(0)=\frac{2H_+T^{[0]}(0)}{[H_++H_-]+[H_+-H_-]R^{[0]}(0)}.$$

V. EXAMPLES IN RECOVERY OF $H(x)$

In this section we illustrate the method of Sec. IV by two explicitly solved examples, and recover $H(x)$ from the scattering data. In the first example $H(x)$ is continuous although it has different asymptotics as $x\rightarrow\pm\infty$. In the second example we give the construction of $H(x)$ with a jump discontinuity.

As a first example, assume $H_+=2$, $H_-=1$, $Q(x)=0$. Since $Q(x)=0$, we are in the exceptional case. Consider the scattering matrix $\mathbf{S}(k)$ in (1.5) with

$$T_l(k)=\tfrac{2}{3},\quad T_r(k)=\tfrac{4}{3},\quad L(k)=-\tfrac{1}{3},\quad R(k)=\tfrac{1}{3},$$

so that $A_+=A_-=A=0$. The matrix $\sigma(k)$ in (2.7) is then formed from

$$\tau(k)=\frac{2\sqrt{2}}{3},\quad \rho(k)=\frac{1}{3},\quad \ell(k)=-\frac{1}{3}.$$

In this case the Riemann–Hilbert problem in (4.18) has the solution $Z_l(k,y)=1$ for $y>0$ and $Z_r(k,y)=1$ for $y<0$. Thus, from (4.13) and (4.14) we obtain

$$dy=2dx,\quad x,y>0,$$

$$dy=dx,\quad x,y<0,$$

and hence using $y(0)=0$, we have

$$y(x)=\begin{cases}2x, & x\geqslant 0,\\ x, & x\leqslant 0,\end{cases}$$

and thus we obtain $H(x)=1$ for $x<0$ and $H(x)=2$ for $x>0$. Using (4.16) and (4.17) the physical solutions of (1.1) are constructed as

$$\psi_l(k,x)=\begin{cases}\tfrac{2}{3}e^{2ik}, & x\geqslant 0,\\ e^{ikx}-\tfrac{1}{3}e^{-ikx}, & x\leqslant 0,\end{cases}$$

$$\psi_r(k,x)=\begin{cases}e^{-2ikx}+\tfrac{1}{3}e^{2ikx}, & x\geqslant 0,\\ \tfrac{4}{3}e^{-ikx}, & x\leqslant 0,\end{cases}$$

where we have used (4.23).

As a second example, assume $H_\pm=1$ and

$$Q(x)=\sqrt{3}\delta(x)-\frac{2\sqrt{3}e^x}{(1+\sqrt{3}e^x)^2}\,\theta(x).$$

Thus we have $m_r^{[0]}(0,x)=1$ for $x\leqslant 0$ and $m_l^{[0]}(0,x)=(\sqrt{3}e^x-1)/(\sqrt{3}e^x+1)$ for $x\geqslant 0$. Consider the scattering matrix $\mathbf{S}(k)$ in (1.5) with

$$T_l(k)=T_r(k)=\frac{1}{2}\,e^{2ik(\sqrt{3}+1)},\quad L(k)=-\frac{\sqrt{3}}{2},\quad R(k)=\frac{\sqrt{3}}{2}\,e^{4ik(\sqrt{3}+1)},$$

so that $A_+=A=-2(\sqrt{3}+1)$ and $A_-=0$. The matrix $\sigma(k)$ in (2.7) is formed from

$$\tau(k)=\frac{1}{2},\quad \ell(k)=-\frac{\sqrt{3}}{2},\quad \rho(k)=\frac{\sqrt{3}}{2}.$$

The solution of the Riemann–Hilbert problem in (4.18) is given by $Z_l(k,y)=1$ for $y>0$ and $Z_r(k,y)=1$ for $y<0$. From (4.13) and (4.14) we obtain

$$dy=\left(\frac{\sqrt{3}e^x+1}{\sqrt{3}e^x-1}\right)^2 dx,\quad x,y>0, \tag{5.1}$$

$$dy=dx,\quad x,y<0. \tag{5.2}$$

Integrating (5.1) and (5.2) with the initial condition $y(0)=0$, we obtain

$$y(x)=\begin{cases} x+2\dfrac{\sqrt{3}+1}{\sqrt{3}-1}-2\dfrac{\sqrt{3}e^x+1}{\sqrt{3}e^x-1}, & x\geqslant 0,\\ x, & x\leqslant 0.\end{cases}$$

Thus we have

$$H(x)=\frac{dy}{dx}=\begin{cases}\left(\dfrac{\sqrt{3}e^x+1}{\sqrt{3}e^x-1}\right)^2, & x>0\\ 1, & x\leqslant 0.\end{cases}$$

Using (4.16) and (4.17) the physical solutions of (1.1) are constructed as

$$\psi_l(k,x)=\begin{cases}\dfrac{1}{2}\dfrac{\sqrt{3}e^x-1}{\sqrt{3}e^x+1}e^{ik[x+2(\sqrt{3}+1)/(\sqrt{3}-1)-2(\sqrt{3}e^x+1)/(\sqrt{3}e^x-1)]}, & x\geqslant 0,\\ e^{ikx}-\dfrac{\sqrt{3}}{2}e^{-ikx}, & x<0,\end{cases}$$

$$\psi_r(k,x)=\begin{cases}\dfrac{\sqrt{3}e^x-1}{\sqrt{3}e^x+1}e^{2ik(\sqrt{3}+1)}\Big[e^{-ik[x+2(\sqrt{3}+1)/(\sqrt{3}-1)-2(\sqrt{3}e^x+1)/(\sqrt{3}e^x-1)]}\\ \quad +\dfrac{\sqrt{3}}{2}e^{ik[x+2(\sqrt{3}+1)/(\sqrt{3}-1)-2(\sqrt{3}e^x+1)/(\sqrt{3}e^x-1)]}\Big], & x\geqslant 0,\\ \frac{1}{2}e^{2ik(\sqrt{3}+1)-ikx}, & x\leqslant 0.\end{cases}$$

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