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ASYMPTOTIC BEHAVIOUR OF DRIFT VELOCITY WITH SPATIAL DIFFUSION OF ELECTRONS

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ABSTRACT

For a rescaled linear one-dimensional transport equation with spatial dependence describing the evolution of the electron distribution in a weakly ionized host medium under the influence of a constant electric field, we derive recursively the n-th term in the Hilbert expansion and prove that the n-th order Hilbert expansion provides an $(n+1)$-th order approximation of the electron distribution function and the drift velocity under certain initial conditions. The proof is based on the perturbation formula for the mild solution of the initial-value problem and an existence result for stationary solutions in a weighted L_1-space.

1. INTRODUCTION

Let us consider the one-dimensional linear Boltzmann equation

$$\begin{aligned}&\frac{\partial}{\partial t} f(x,v,t) + v\frac{\partial}{\partial x} f(x,v,t) + a\frac{\partial}{\partial v} f(x,v,t)\\&= -\nu(v)f(x,v,t) + \int_{-\infty}^{+\infty} k(v,\hat{v})\nu(\hat{v})f(x,\hat{v},t)\,d\hat{v}, \qquad t \geq 0,\end{aligned} \tag{1.1}$$

with initial condition

$$f(x, v, 0) = g(x, v), \tag{1.2}$$

where $a = qE/m$. This equation describes the time evolution of the spatially dependent electron distribution function $f(x, v, t)$ in a weakly ionized host medium under the influence of a spatially uniform time independent electric field. In (1.1), $x \in \mathbf{R}$ and $v \in \mathbf{R}$ are the space and time variables, $t \geq 0$ is time, $a > 0$ (expressed in terms of the electron charge $-q$, the constant electric field E and the electron mass m) is the constant electrostatic acceleration, $\nu(v)$ is the collision frequency, and $k(v, v')$ is the scattering kernel. Under the hypothesis that ionization and recombination effects balance each other, $k(v, v')$ is a nonnegative measurable function such that

$$\int_{-\infty}^{+\infty} k(v, \hat{v})\, dv = 1, \qquad \hat{v} \in \mathbf{R}. \tag{1.3}$$

In this article we provide a dimensional analysis of the spatial, velocity and time variables to show the relative "importance" of each term of the evolution equation. Expressing each of these variables in physical units that are fractions of suitable intrinsic variables, we introduce the new scaled variables

$$x' = \frac{x}{x_0}, \qquad v' = \frac{v}{v_0}, \qquad t' = \frac{t}{t_0},$$

where x_0, v_0 and t_0 are reference quantities, v_0^2 being e.g. the average squared velocity of the gas particles. For a semiconductor[1] the latter is given by $k_B T/m$ with T the lattice temperature, k_B the Boltzmann constant and m the electron mass, while for an ideal monatomic gas[2] it is given by $3k_B T_0/m$ with T_0 the gas temperature.

To rewrite the evolution equation in terms of the new scaled variables we define the new function

$$F(x', v', t') = f(x_0 x', v_0 v', t_0 t') = f(x, v, t)$$

and introduce a reference electric field E_0 and a reference voltage $U_0 = m v_0^2/q$ such that

$$E' = \frac{E}{E_0}, \qquad a = \frac{qE}{m} = \frac{v_0^2 E_0}{U_0} E'.$$

One may then show[1,3] that the collision term $(Qf)(x, v, t)$ given by the right-hand side of (1.1) is rescaled as

$$Q'(F) = \frac{\ell}{v_0} Q(f) = \tau Q(f),$$

where ℓ and τ are the mean free path and the mean free time between successive collisions. Then (1.1) takes the form

$$\frac{\ell}{t_0 v_0}\frac{\partial}{\partial t'}F(x',v',t') + \frac{\ell v'}{x_0}\frac{\partial}{\partial x'}F(x',v',t') + \frac{\ell E_0 E'}{U_0}\frac{\partial}{\partial v'}F(x',v',t') = (Q'F)(x',v',t').$$

Hence, returning to the original variables x, v and t we find

$$\frac{\ell}{t_0 v_0}\frac{\partial}{\partial t}f(x,v,t) + \frac{\ell}{x_0}v\frac{\partial}{\partial x}f(x,v,t) + \frac{\ell E_0}{U_0}E\frac{\partial}{\partial v}f(x,v,t) = (Qf)(x,v,t).$$

When the electric field is strong enough, one may take the reference electric field E_0 in such a way as to make $\frac{\ell E_0}{U_0}$ comparable with 1 and hence large in comparison with the other coefficients containing the mean free path ℓ in the numerator, taking into account that $t_0 v_0 \sim x_0$. Introducing the scaling parameter ϵ proportional to the average time between successive collisions, the initial-value problem finally takes the form, with obvious meaning of the new symbols

$$\begin{aligned} &\epsilon\frac{\partial}{\partial t}f(x,v,t) + \epsilon v\frac{\partial}{\partial x}f(x,v,t) + a\frac{\partial}{\partial v}f(x,v,t) \\ &= -\nu(v)f(x,v,t) + \int_{-\infty}^{+\infty} k(v,\hat{v})\nu(\hat{v})f(x,\hat{v},t)\,d\hat{v}, \qquad t \geq 0; \end{aligned} \tag{1.4}$$

$$f(x,v,0) = g(x,v). \tag{1.5}$$

The existence of a unique solution of problem (1.4)-(1.5) has been studied by several authors,[4–7] both the present one-dimensional model and the more realistic three-dimensional model, leading to very similar results. Some of them have also studied the runaway phenomenon[8–10] that occurs when the collision process is not sufficient to prevent the electrons from being accelerated indefinitely under the influence of the electric field. In this article we give a mathematical justification of the Hilbert expansion of the solution $f(x,v,t;\epsilon)$ of Eqs. (1.4)-(1.5) to arbitrarily high order. In particular, we prove the two approximations

$$\int_{-\infty}^{+\infty}\left|f(x,v,t;\epsilon) - \sum_{j=0}^{n}\epsilon^j f_j(x,v,t)\right|(1+|v|)^k\,dx dv = O(\epsilon^{n+1}), \qquad k = 0,1, \tag{1.6}$$

as $\epsilon \downarrow 0$, for arbitrarily large $n \in \mathbf{N}$, under suitable assumptions on the initial condition. We thus generalize a result by Poupaud[6,11] on the zero-th order ($n = 0$) approximation. We remark that Poupaud has used methods going back to the work of Bardos et al.[12] As an ancillary result, we obtain an asymptotic series for the average drift velocity of the electrons, thus justifying an additive decomposition Skullerud[13] obtained for this velocity.

Since its first introduction,[14] the Hilbert expansion for the linearized Boltzmann equation has been studied extensively by Grad. He explained its scope from the physical point of view[15] and proved it to be an asymptotic expansion of solutions of the linearized Boltzmann equation with Maxwellian initial condition.[16] He also studied the initial layer effect occurring when the initial condition is not a Maxwellian.[16] A discussion of these matters as well as references may be found in Ref. 3. Recently, Banasiak and Mika[17] have obtained first order ($n = 1$) results for the neutron transport equation incorporating the initial layer effect. In this article we avoid the initial layer effect by restricting the initial condition, but we obtain results for arbitrarily high orders.

In this article we restrict ourselves to a one-dimensional model. From the physical point of view, this is a major restriction. Nevertheless, virtually all of the mathematically interesting features of the three-dimensional problem are already present in the one-dimensional case and we expect all of our derivations to go through in the three-dimensional situation.

Let us give an outline of the article. In Section 2 we study the inhomogeneous stationary problem in various Banach spaces, obtaining one-dimensional versions of some of Poupaud's results.[6,11] In Section 3 we consider the asymptotic problem, using its Hilbert expansion. In essence, at each step an inhomogeneous stationary problem is obtained with an undetermined constant which is in turn found with the help of the solvability condition on the right-hand side of the stationary equation. In Section 4 we prove the Hilbert expansion in the form (1.6) for $k = 0, 1$, which shows the Hilbert expansion to be an asymptotic series for the solution of the initial-value problem (1.4)-(1.5) as $\epsilon \downarrow 0$. The $k = 1$ expansion then yields an asymptotic expansion for the average drift velocity of the electrons.

For later use we define a number of function spaces. Letting μ be a measure on E, by $L_1(E; d\mu)$ and $L_\infty(E; d\mu)$ we mean the Banach spaces of μ-measurable functions on E that are bounded with respect to the norms $\|f\|_1 = \int_E |f(t)|\, d\mu(t)$ and $\|f\|_\infty = \mu - \operatorname{ess\,sup}_{t\in E} |f(t)|$, respectively. In this article we always deal with $E = \mathbf{R}$ or $E = \mathbf{R}^2$ and μ a suitably weighted Lebesgue measure. Next, if X is a Banach space, $\mathcal{U}$ is an interval and $j \in \mathbf{N}$, then $C^j(\mathcal{U}; X)$ denotes the Banach space of all vector functions $h : \mathcal{U} \to X$ that have j continuous strong derivatives and are bounded with respect to the norm

$$\|h\|_{C^j(\mathcal{U};X)} = \sup_{t\in\mathcal{U}} \sum_{i=0}^{j} \|h^{[i]}(t)\|_X.$$

If $X = \mathbf{C}$ we write $C^j(\mathcal{U})$ instead of $C^j(\mathcal{U}; \mathbf{C})$. Finally, if $n \in \mathbf{N}$, then by $H_1^n(\mathbf{R})$ we denote the Sobolev space[18] of all functions $g \in L_1(\mathbf{R}^2; dvdx)$ whose first n distributional derivatives belong to $L_1(\mathbf{R}^2; dvdx)$, endowed with the norm

$$\|g\|_{H_1^n(\mathbf{R})} = \sum_{r=0}^{n} \int_{-\infty}^{+\infty} |g^{[r]}(x)|\, dx.$$

2. STATIONARY PROBLEM WITH INHOMOGENEOUS TERM

In this preliminary section we study the inhomogeneous stationary problem

$$a\frac{\partial \varphi}{\partial v} + \nu(v)\varphi(v) = (K\varphi)(v) + \phi(v), \tag{2.1}$$

where

$$(K\varphi)(v) = \int_{-\infty}^{+\infty} k(v, v')\nu(v')\varphi(v')\, dv',$$

under the general assumptions of Ref. 4 and then impose some restrictions on $\nu(v)$ and K to get similar results in $L_1(\mathbf{R}; (1 + |v|)dv)$. Let us first assume that $a > 0$, $\nu(v)$ is an almost everywhere positive function in $L_{1,\mathrm{loc}}(\mathbf{R}; dv)$, and K is a positive operator from $L_1(\mathbf{R}; \nu(v)dv)$ into $L_1(\mathbf{R}; dv)$ satisfying

$$\|K\varphi\|_1 = \|\varphi\|_\nu, \qquad \varphi \geq 0 \text{ in } L_1(\mathbf{R}; dv),$$

where $\|\varphi\|_\nu = \|\nu\varphi\|_1$. Let us define[4]

$$(L\varphi)(v) = \int_{-\infty}^{v} \frac{1}{a} \exp\left\{-\frac{1}{a}\int_{v'}^{v} \nu(\hat{v})\, d\hat{v}\right\} \varphi(v')\, dv'.$$

Then L is a bounded positive operator from $L_1(\mathbf{R}; dv)$ into $L_1(\mathbf{R}; \nu(v)dv)$ satisfying

$$\|L\varphi\|_\nu \leq \|\varphi\|_1, \qquad \varphi \geq 0 \text{ in } L_1(\mathbf{R}; dv).$$

Writing

$$(\varphi, \psi) = \int_{-\infty}^{+\infty} \varphi(v)\psi(v)\, dv, \qquad (\varphi, \psi)_\nu = \int_{-\infty}^{+\infty} \varphi(v)\psi(v)\nu(v)\, dv,$$

we easily derive

$$(K\varphi, 1) = (\varphi, 1)_\nu, \qquad \varphi \in L_1(\mathbf{R}; \nu(v)dv), \tag{2.2}$$

where $1(v) \equiv 1$.

Suppose $\phi \in L_1(\mathbf{R}; dv)$. Then by a *solution* of Eq. (2.1) we mean a function $\varphi \in L_1(\mathbf{R}; \nu(v)dv)$ that is absolutely continuous on every compact real interval and satisfies (2.1). Using the arguments of the proofs of Theorems 2 and 4 of Ref. 4 we obtain

Proposition 1. *Let $\phi \in L_1(\mathbf{R}; dv)$. Then either of the following two situations occurs:*

1. $\int_{-\infty}^{+\infty} \nu(v)\, dv = +\infty$: *A function $\varphi \in L_1(\mathbf{R}; \nu(v)dv)$ is a solution of (2.1) if and only if*

$$\varphi = LK\varphi + L\phi, \tag{2.3}$$

and in that case $\varphi(-\infty) = \varphi(+\infty) = 0$.

2. $\int_{-\infty}^{+\infty} \nu(v)\, dv < +\infty$: *A function $\varphi \in L_1(\mathbf{R}; \nu(v)dv)$ is a solution of (2.1) if and only if $\varphi(-\infty) = \varphi(+\infty)$ and*

$$\varphi = LK\varphi + \varphi(\pm\infty)e + L\phi,$$

where $e(v) = \exp\left\{-\frac{1}{a}\int_{-\infty}^{v} \nu(\hat{v})\, d\hat{v}\right\}$.

Moreover, if LK maps $L_1(\mathbf{R};\nu(v)dv)$ into $L_1(\mathbf{R};dv)$ and $L\phi \in L_1(\mathbf{R};dv)$, the solutions $\varphi \in L_1(\mathbf{R};dv)$ whenever, in the second situation, $\varphi(\pm\infty) = 0$.

The two situations show a very different physical behaviour also for nontrivial sources. When the collision frequency is not integrable on $\mathbf{R}$, absorption is still significant for large values of $|v|$, and so a stationary solution can exist also when ϕ is nonnegative and large. When the collision frequency is integrable, the absorption for large values of $|v|$ is so insignificant that the source can be assigned arbitrarily (see (2.6)). Now we examine these two situations in detail.

The existence and uniqueness issue is easily resolved if

$$\int_{-\infty}^{+\infty} \nu(v)\, dv < +\infty.$$

In that case LK is a bounded positive operator on $L_1(\mathbf{R};\nu(v)dv)$ of norm strictly less than 1 and hence for each choice of equal values $\varphi(\pm\infty)$ there is a unique solution of (2.1) and this solution is nonnegative if and only if $\varphi(\pm\infty)$ and ϕ are nonnegative.

On the other hand, if

$$\int_{-\infty}^{+\infty} \nu(v)\, dv = +\infty, \tag{2.4}$$

we have

$$(L\varphi, 1)_\nu = (\varphi, 1), \qquad \varphi \in L_1(\mathbf{R};dv). \tag{2.5}$$

Therefore, by (2.2) and (2.5), a necessary condition for the existence of a solution of (2.1) is

$$(\phi, 1) = \int_{-\infty}^{+\infty} \phi(v)\, dv = 0. \tag{2.6}$$

Condition (2.4) turns out to be a necessary (but not a sufficient) condition[19] for the existence of a steady state, i.e. a nontrivial solution of Eq. (2.1) with $\phi = 0$. A sufficient condition[4] for the existence of a steady state is the weak compactness of LK on $L_1(\mathbf{R};\nu(v)dv)$. It should be noted that if there exists a steady state there is a one-dimensional subspace of steady states[1] $\varphi \in L_1(\mathbf{R};dv)$ such that $\varphi = LK\varphi$.

The next result gives the necessary and sufficient conditions for the existence of a solution of (2.1). Poupaud[6] has obtained the analogous result for the three-dimensional BGK model.

Theorem 2. *Suppose $\phi \in L_1(\mathbf{R};dv)$ and $\int_{-\infty}^{+\infty} \nu(v)\, dv = +\infty$, and let LK be a weakly compact operator on $L_1(\mathbf{R};\nu(v)dv)$. Then there exists a solution φ of (2.1) if and only if (2.6) is satisfied.*

Proof: Since LK is weakly compact on $L_1(\mathbf{R};\nu(v)dv)$, $(LK)^2$ is compact on $L_1(\mathbf{R};\nu(v)dv)$ and hence $1 - LK$ has a closed range. Thus the nonzero spectrum of LK consists of a sequence of isolated eigenvalues of finite algebraic multiplicity with zero as the only possible accumulation point. Writing

$$\mathcal{M}_\lambda = \{\varphi \in L_1(\mathbf{R};\nu(v)dv) : LK\varphi = \lambda\varphi\},$$

we have

$$L_1(\mathbf{R};\nu(v)dv) = \mathcal{M}_1 \oplus \mathcal{N}_1 \oplus \mathcal{N}_2, \tag{2.7}$$

where $\mathcal{N}_1$ is a closed invariant subspace of LK on which LK has spectral radius strictly less than 1 and $\mathcal{N}_2 = \oplus_{|\lambda|=1,\lambda\neq 1} \mathcal{M}_\lambda$. Further, $\mathcal{M}_1$ has dimension 1. Indeed, there exists[4] a unique nonnegative $\varphi_0 \in \mathcal{M}_1$ such that

$$(\varphi_0, 1) = 1.$$

Moreover, if $LK\psi = \psi + \varphi_0$ for some $\psi \in L_1(\mathbf{R};dv)$, $(LK)^n\psi = \psi + n\varphi_0$ and hence $\|\psi + n\varphi_0\|_1 = \|(LK)^n\psi\|_1 \leq \|\psi\|_1$ for all $n \in \mathbf{N}$, which is a contradiction. Therefore such ψ does not exist and $\mathcal{M}_1$ has indeed dimension 1.

There exist $C > 0$ and $\epsilon \in (0,1)$ such that for any $n \in \mathbf{N}$

$$\|(LK)^n\varphi\|_\nu \leq C\epsilon^n \|\varphi\|_\nu, \qquad \varphi \in \mathcal{N}_1.$$

Therefore if $\varphi \in \mathcal{N}_1$ we have

$$|(\varphi, 1)_\nu| = |((LK)^n\varphi, 1)_\nu| \leq \|(LK)^n\varphi\|_\nu \|1\|_\infty \leq C\epsilon^n\|\varphi\|_\nu, \qquad n \in \mathbf{N},$$

and hence $(\varphi, 1)_\nu = 0$ for every $\varphi \in \mathcal{N}_1$.

If $\varphi \in \mathcal{M}_\lambda$ for some $\lambda \neq 1$, we have

$$(\varphi, 1)_\nu = \frac{(\varphi,1)_\nu - \lambda(\varphi,1)_\nu}{1-\lambda} = \frac{(\varphi - LK\varphi, 1)_\nu}{1-\lambda} = 0.$$

For general $\varphi \in L_1(\mathbf{R};\nu(v)dv)$ we have

$$\varphi = c\varphi_0 + \varphi_1 + \varphi_2, \tag{2.8}$$

where $c \in \mathbf{C}$, $\varphi_1 \in \mathcal{N}_1$ and $\varphi_2 \in \mathcal{N}_2$. Thus

$$(\varphi, 1)_\nu = c(\varphi_0, 1)_\nu + (\varphi_1, 1)_\nu + (\varphi_2, 1)_\nu = c.$$

Hence

$$\mathcal{N}_1 \oplus \mathcal{N}_2 = \{\varphi \in L_1(\mathbf{R};\nu(v)dv) : (\varphi, 1)_\nu = 0\}. \tag{2.9}$$

We may thus write $L\phi = (L\phi)_1 + (L\phi)_2$ with $(L\phi)_1 \in \mathcal{N}_1$ and $(L\phi)_2 \in \mathcal{N}_2$ if ϕ satisfies (2.6).

Finally, from (2.3) and (2.8) we get

$$\varphi_1 + \varphi_2 = LK(\varphi_1 + \varphi_2) + L\phi, \tag{2.10}$$

with $(L\phi, 1)_\nu = (\phi, 1) = 0$. Equation (2.1) can be decomposed into equations on $\mathcal{N}_1$ and $\mathcal{N}_2$ separately

$$\varphi_1 = LK\varphi_1 + (L\phi)_1, \qquad \varphi_2 = LK\varphi_2 + (L\phi)_2, \tag{2.11}$$

where $((L\phi)_2, 1)_\nu = 0$ [because $(L\phi)_2 \in \mathcal{N}_2$] and hence $((L\phi)_1, 1)_\nu = 0$. Because the spectral radius of LK on $\mathcal{N}_1$ is strictly less than 1, the first of (2.11) has a unique solution given by the absolutely convergent series

$$\varphi_1 = \sum_{n=0}^{\infty} (LK)^n (L\phi)_1,$$

whence

$$\varphi = c\varphi_0 + \sum_{n=0}^{\infty} (LK)^n (L\phi)_1 + (I - LK)^{-1}(L\phi)_2, \qquad c \in \mathbf{C}, \tag{2.12}$$

is the general solution of (2.3). ∎

It is easily shown that $\mathcal{N}_2 = \{0\}$ under the condition that $k(v, v') > 0$ for almost every (v, v'), which is satisfied for every realistic scattering model. Indeed, the spectrum of LK on the unit circle is cyclic (cf. Ref. 20, Theorem V 5.2), i.e. there exists $n \in \mathbf{N}$ such that the point spectrum of LK on the unit circle coincides with the group of n-th roots of unity. Moreover, because the eigenvalue of LK at $\lambda = 1$ is simple, all eigenvalues of LK on the unit circle are simple. Further, there exists a function $\epsilon(v)$ with $|\epsilon(v)| \equiv 1$ such that $LK(\varphi_0 \epsilon^k) = e^{2\pi k/n}\varphi_0 \epsilon^k$. In addition, under the above condition on $k(v, v')$ one has $\int_{-\infty}^{+\infty} (LK\phi)(v)h(v)\nu(v)\,dv > 0$ for all nontrivial and nonnegative $\phi \in L_1(\mathbf{R}; dv)$ and $h \in L_\infty(\mathbf{R}; dv)$, and hence $\lambda = 1$ is the only eigenvalue of LK on the unit circle.

If $\nu(v)$ is bounded, (2.4) is fulfilled and LK is weakly compact on $L_1(\mathbf{R}; dv)$, then there exists a solution $\varphi \in L_1(\mathbf{R}; dv)$ of (2.1) for given $\phi \in L_1(\mathbf{R}; dv)$ if and only if (2.6) is satisfied. Uniqueness is guaranteed if we require the solution φ to satisfy $(\varphi, 1) = 0$. In fact, once a solution φ is found, $\varphi - \{(\varphi, 1)/(\varphi_0, 1)\}\varphi_0$ satisfies this requirement. Moreover, if we write

$$\varphi = M\phi$$

for the unique solution of (2.1) satisfying $(\varphi, 1) = 0$, then, by the inverse operator theorem, M is bounded on $\{\varphi \in L_1(\mathbf{R}; dv) : (\varphi, 1) = 0\}$.

To deal with the stationary problem (2.1) on $L_1(\mathbf{R}; (1 + |v|)^n dv)$, we make the following assumptions:

A1. $\nu(v)$ is bounded and satisfies (2.4).

A2-n. L maps $L_1(\mathbf{R}; (1 + |v|)^n dv)$ into itself. This is equivalent to the requirement

$$\Lambda_n = \sup_{v \in \mathbf{R}} \frac{1}{a} \int_v^{+\infty} \left(\frac{1 + |v'|}{1 + |v|} \right)^n \exp\left(-\frac{1}{a} \int_v^{v'} \nu(\hat{v})\, d\hat{v} \right) dv' < +\infty. \tag{2.13}$$

Actually, the real number Λ_n is the norm of L as an operator on $L_1(\mathbf{R}; (1 + |v|)^n dv)$.

A3-n. K is a bounded operator on $L_1(\mathbf{R};(1+|v|)^n dv)$ such that LK is weakly compact on both $L_1(\mathbf{R};dv)$ and $L_1(\mathbf{R};(1+|v|)^n dv)$.

Then it is easy to see that the nonzero spectra of LK as operators on $L_1(\mathbf{R};dv)$ and $L_1(\mathbf{R};(1+|v|)^n dv)$ coincide, to the extent that even the multiplicities of the corresponding eigenvalues are the same. Also all (generalized) eigenvectors of LK corresponding to nonzero eigenvalues belong to the smaller space $L_1(\mathbf{R};(1+|v|)^n dv)$. In particular, $\varphi_0 \in L_1(\mathbf{R};(1+|v|)^n dv)$. In analogy with (2.7) we have

$$L_1(\mathbf{R};(1+|v|)^n dv) = \mathcal{M}_1 \oplus [\mathcal{N}_1 \cap L_1(\mathbf{R};(1+|v|)^n dv)] \oplus \mathcal{N}_2,$$

where all subspaces occurring in the direct sum are closed in $L_1(\mathbf{R};(1+|v|)^n dv)$. Given $\phi \in L_1(\mathbf{R};dv)$ we have $L\phi \in L_1(\mathbf{R};(1+|v|)^n dv)$, which. via $(L\phi,1)_\nu = (\phi,1) = 0$, implies

$$L\phi \in [\mathcal{N}_1 \cap L_1(\mathbf{R};(1+|v|)^n dv)] \oplus \mathcal{N}_2.$$

Thus (2.10) has at least one solution in $L_1(\mathbf{R};(1+|v|)^n dv)$ and the general solution in $L_1(\mathbf{R};(1+|v|)^n dv)$ is given by (2.12). Taking

$$c = -\sum_{n=0}^{\infty} ((LK)^n L\phi, 1)/(\varphi_0, 1)$$

we obtain $\varphi = M\phi \in L_1(\mathbf{R};(1+|v|)^n dv)$ satisfying $(\varphi,1) = 0$. Moreover, by the inverse operator theorem, M is a bounded operator on $L_1(\mathbf{R};(1+|v|)^n dv)$. We have

Theorem 3. *Let $\nu(v)$ and K satisfy the conditions* (A1), (A2-n) *and* (A3-n) *for some $n \geq 0$. Then for every $\phi \in L_1(\mathbf{R};(1+|v|)^n dv)$ with $(\phi,1) = 0$ there exists a unique solution $\varphi = M\phi$ of* (2.1) *in $L_1(\mathbf{R};(1+|v|)^n dv)$ satisfying $(\varphi,1) = 0$ and*

$$\|M\phi\|_{1,n} \leq \mu_n \|\phi\|_{1,n}, \qquad \phi \in L_1(\mathbf{R};(1+|v|)^n dv), \tag{2.14}$$

for some constant μ_n, where $\|\cdot\|_{1,n}$ denotes the norm on $L_1(\mathbf{R};(1+|v|)^n dv)$.

Let us consider the BGK model where

$$k(v,v') = F_m(v)$$

for the nonnegative function $F_m(v) = (\beta/\pi)^{1/2} e^{-\beta v^2}$ with $\|F_m\|_1 = 1$, for collision frequencies satisfying (A1) and (A2-n) for every $n \geq 0$. Then (A3-n) is satisfied for every $n \geq 0$, the nonnegative homogeneous stationary solution of unit L_1-norm is given by $\varphi_0 = LF_m/\|LF_m\|_1$ and

$$M\phi = L\phi - (L\phi,1)\varphi_0. \tag{2.15}$$

For $\nu(v) \equiv \nu_0$ we easily find

$$(M\phi)(v) = (L\phi)(v) - (\phi, 1)(LF_m)(v) = (L\phi)(v) = \frac{1}{a}\int_{-\infty}^{v} e^{-\frac{\nu_0}{a}(v-v')}\phi(v')\,dv',$$

because the solvability condition for (2.3) is $(\phi, 1) = 0$.

3. ASYMPTOTIC EXPANSIONS IN POWERS OF ϵ

In this section we derive a recursive expression for the n-th term of the Hilbert expansion of Eqs. (1.4)-(1.5). For the sake of clarity, we will at first disregard the spaces to which various functions belong and disregard smoothness conditions. At the end of this section we will derive some preliminary estimates. In Section 4, where the approximation result for the n-th order Hilbert expansion is established, we will specify the regularity required for the initial condition and discuss the spaces to which all functions appearing in the expression belong. For the time being we will take derivatives with respect to x and t without worrying about regularity. In Section 4 the functions appearing in the initial condition will be required to belong to suitable Sobolev spaces, thus justifying taking these derivatives.

We look for solutions of (1.4) as power series in ϵ and consider the formal expansion[3] of $f(x, v, t)$ in powers of ϵ

$$f(x, v, t; \epsilon) = \sum_{j=0}^{\infty} \epsilon^j f_j(x, v, t) = f_0(x, v, t) + \epsilon f_1(x, v, t) + \epsilon^2 f_2(x, v, t) + \cdots. \quad (3.1)$$

Substituting (3.1) into Eq. (1.4) and equating corresponding powers of ϵ we find

$$a\frac{\partial f_0}{\partial v} + Q(f_0) = 0, \quad (3.2)_0$$

and for $j \geq 1$

$$a\frac{\partial f_j}{\partial v} + Q(f_j) = -\left\{\frac{\partial f_{j-1}}{\partial t} + v\frac{\partial f_{j-1}}{\partial x}\right\}, \quad (3.2)_j$$

where we have abbreviated the collision operator as

$$(Qh)(v) = \nu(v)h(v) - \int_{-\infty}^{+\infty} k(v, v')\nu(v')h(v')\,dv'.$$

For the sake of generality we expand the function $g(x, v)$ appearing in the initial condition (1.5), i.e.

$$g(x, v; \epsilon) = \sum_{j=0}^{\infty} \epsilon^j g_j(x, v) = g_0(x, v) + \epsilon g_1(x, v) + \epsilon^2 g_2(x, v) + \cdots, \quad (3.3)$$

although in the most common applications we have

$$g(x, v; \epsilon) = g_0(x, v), \qquad g_j(x, v) = 0, \ j \geq 1.$$

In general, the sequence of initial conditions

$$f_j(x, v, 0) = g_j(x, v) \tag{3.4$_j$}$$

is not satisfied. Notice that each one of Eqs. $(3.2)_j$ has the form

$$a\frac{\partial f_j}{\partial v} + Q(f_j) = \mathrm{Rhs}_j(x, v, t),$$

where only the right-hand side $\mathrm{Rhs}_j(x, v, t)$ depends on (x, t) and $\mathrm{Rhs}_0(x, v, t) \equiv 0$.

Let us now assume that the homogeneous stationary problem (2.1) has a nontrivial solution $\varphi_0(v)$ belonging to $L_1(\mathbf{R}; (1 + |v|)dv)$. Then we may suppose[4] that φ_0 is nonnegative and normalized in $L_1(\mathbf{R}; dv)$, i.e. $\int_{-\infty}^{+\infty} \varphi_0(v)\, dv = 1$. Moreover, the steady state's average velocity

$$\langle v_0 \rangle = \int_{-\infty}^{+\infty} v\varphi_0(v)\, dv$$

is a finite real number. Then Eq. $(3.2)_0$ has the general solution

$$f_0(x, v, t) = m_0(x, t)\varphi_0(v), \tag{3.5}$$

where $m_0(x, t)$ is a scalar not depending on v to be specified later. As a result, the average velocity

$$\langle v \rangle_0(x, t) = \frac{\int_{-\infty}^{+\infty} v f_0(x, v, t)\, dv}{\int_{-\infty}^{+\infty} f_0(x, v, t)\, dv}$$

is identical to the average velocity $\langle v_0 \rangle$ of the steady state and hence does not depend on x and t.

Next, we solve Eq. $(3.2)_1$ with $f_0(x, v, t)$ given by (3.5). This equation takes the form

$$a\frac{\partial f_1}{\partial v} + Q(f_1) = -\varphi_0(v)\left\{\frac{\partial m_0}{\partial t} + v\frac{\partial m_0}{\partial x}\right\},$$

which leads to the necessary condition for solvability

$$\int_{-\infty}^{+\infty} \varphi_0(v)\left(\frac{\partial m_0}{\partial t}(x, t) + v\frac{\partial m_0}{\partial x}(x, t)\right) dv = 0,$$

or in other words

$$\frac{\partial m_0}{\partial t}(x, t) + \langle v_0 \rangle \frac{\partial m_0}{\partial x}(x, t) = 0.$$

The solution of the latter equation is an arbitrary function h_0 of the variable $x - \langle v_0 \rangle t$, i.e.

$$m_0(x,t) = h_0(x - \langle v_0 \rangle t).$$

Using the initial condition $(3.4)_0$ we find $h_0(x) = \int_{-\infty}^{+\infty} g_0(x,v)\,dv$. Hence

$$f_0(x,v,t) = \varphi_0(v)\, h_0(x - \langle v_0 \rangle t).$$

Equation $(3.2)_1$ now takes the form

$$a\frac{\partial f_1}{\partial v} + Q(f_1) = -(v - \langle v_0 \rangle)\varphi_0(v) h_0{}'(x - \langle v_0 \rangle t), \tag{3.6}$$

where the right-hand side integrated with respect to $v \in \mathbf{R}$ yields zero. Letting $\varphi_1(v)$ be the unique solution of the stationary equation

$$a\frac{\partial \varphi_1}{\partial v} + Q(\varphi_1) = (v - \langle v_0 \rangle)\varphi_0(v)$$

in $L_1(\mathbf{R}; (1+|v|)dv)$ that satisfies $\int_{-\infty}^{+\infty} \varphi_1(v)\,dv = 0$, we obtain

$$f_1(x,v,t) = -\varphi_1(v) h_0{}'(x - \langle v_0 \rangle t) + m_1(x,t)\varphi_0(v), \tag{3.7}$$

where $m_1(x,t)$ is a v-independent constant to be determined below.

Next, we solve Eq. $(3.2)_2$ with $f_1(x,v,t)$ given by (3.7). This equation takes the form

$$a\frac{\partial f_2}{\partial v} + Q(f_2) = (v - \langle v_0 \rangle)\varphi_1(v) h_0{}''(x - \langle v_0 \rangle t) - \varphi_0(v)\left\{\frac{\partial m_1}{\partial t} + v\frac{\partial m_1}{\partial x}\right\}. \tag{3.8}$$

The necessary condition for solvability thus leads to the equation

$$\frac{\partial m_1}{\partial t} + \langle v_0 \rangle \frac{\partial m_1}{\partial x} = \alpha_1 h_0{}''(x - \langle v_0 \rangle t)$$

with $\alpha_1 = \int_{-\infty}^{+\infty} (v - \langle v_0 \rangle)\varphi_1(v)\,dv$ where the right-hand side is easily found from the initial condition. This equation has the solution

$$m_1(x,t) = h_1(x - \langle v_0 \rangle t) + \alpha_1 t h_0{}''(x - \langle v_0 \rangle t)$$

for some function h_1 of the one variable $x - \langle v_0 \rangle t$. Hence (3.8) reduces to

$$\begin{aligned} a\frac{\partial f_2}{\partial v} + Q(f_2) &= \{(v - \langle v_0 \rangle)\varphi_1(v) - \alpha_1\varphi_0(v)\}\; h_0{}''(x - \langle v_0 \rangle t) \\ &\quad - (v - \langle v_0 \rangle)\varphi_0(v)\{h_1{}'(x - \langle v_0 \rangle t) + \alpha_1 t h_0^{[3]}(x - \langle v_0 \rangle t)\}. \end{aligned}$$

Using the initial condition $(3.4)_1$ in combination with (3.7) we readily find $h_1(x) = \int_{-\infty}^{+\infty} g_1(x,v)\,dv$. Letting $\varphi_2(v)$ be the unique solution of the stationary equation

$$a\frac{\partial \varphi_2}{\partial v} + Q(\varphi_2) = (v - \langle v_0\rangle)\varphi_1(v) - \alpha_1\varphi_0(v)$$

in $L_1(\mathbf{R};(1+|v|)dv)$ that satisfies $\int_{-\infty}^{+\infty} \varphi_2(v)\,dv = 0$, we obtain

$$\begin{aligned} f_2(x,v,t) &= \varphi_2(v)h_0''(x - \langle v_0\rangle t) \\ &\quad - \varphi_1(v)\{h_1'(x - \langle v_0\rangle t) + \alpha_1 t h_0^{[3]}(x - \langle v_0\rangle t)\} + m_2(x,t)\varphi_0(v), \end{aligned}$$

where $m_2(x,t)$ is a v-independent constant to be determined below.

Let us now proceed as above for f_j with the help of an induction argument involving some bookkeeping. Let us start from the hypothesis

$$\begin{aligned} f_j(x,v,t) &= (-1)^j \Big[\varphi_j(v)h_0^{[j]}(x - \langle v_0\rangle t) \\ &\quad + \sum_{s=1}^{m(j)} F_{js}(v) \sum_{k=0}^{q(j,s)} \frac{t^k}{k!} G_{kjs}(x - \langle v_0\rangle t)\Big] + m_j(x,t)\varphi_0(v), \end{aligned} \tag{3.9}$$

where $j \geq 2$, $m_j(x,t)$ is some v-independent constant to be determined below, and $\int_{-\infty}^{+\infty} F_{js}(v)\,dv = 0$. Then Eq. $(3.2)_{j+1}$ reads

$$\begin{aligned} a\frac{\partial f_{j+1}}{\partial v} + Q(f_{j+1}) &= -\varphi_0(v)\left\{\frac{\partial m_j}{\partial t} + v\frac{\partial m_j}{\partial x}\right\} \\ &\quad + (-1)^{j+1}\Big[(v - \langle v_0\rangle)\varphi_j(v)h_0^{[j+1]}(x - \langle v_0\rangle t) \\ &\quad + \sum_{s=1}^{m(j)} (v - \langle v_0\rangle)F_{js}(v) \sum_{k=0}^{q(j,s)} \frac{t^k}{k!} G_{kjs}'(x - \langle v_0\rangle t) \\ &\quad + \sum_{s=1}^{m(j)} F_{js}(v) \sum_{k=1}^{q(j,s)} \frac{t^{k-1}}{(k-1)!} G_{kjs}(x - \langle v_0\rangle t)\Big] \end{aligned} \tag{3.10}$$

The necessary condition for solvability and $\int_{-\infty}^{+\infty} F_{js}(v)\,dv = 0$ thus lead to the equation

$$\begin{aligned} &\frac{\partial m_j}{\partial t} + \langle v_0\rangle \frac{\partial m_j}{\partial x} \\ &\quad = (-1)^{j+1}\left[\alpha_j h_0^{[j+1]}(x - \langle v_0\rangle t) + \sum_{s=1}^{m(j)} f_{js} \sum_{k=0}^{q(j,s)} \frac{t^k}{k!} G_{kjs}'(x - \langle v_0\rangle t)\right], \end{aligned}$$

where $\alpha_j = \int_{-\infty}^{+\infty} (v - \langle v_0 \rangle)\varphi_j(v)\,dv$ and $f_{js} = \int_{-\infty}^{+\infty} (v - \langle v_0 \rangle)F_{js}(v)\,dv$. The equation has the solution

$$\begin{aligned} m_j(x,t) = h_j(x - \langle v_0 \rangle t) + (-1)^{j+1} \Bigg[&\alpha_j t h_0^{[j+1]}(x - \langle v_0 \rangle t) \\ &+ \sum_{s=1}^{m(j)} f_{js} \sum_{k=0}^{q(j,s)} \frac{t^{k+1}}{(k+1)!} G_{kjs}{}'(x - \langle v_0 \rangle t) \Bigg] \end{aligned} \tag{3.11}$$

for some function h_j of the one variable $x - \langle v_0 \rangle t$. Using the initial condition $(3.4)_j$ in combination with (3.9) and $\int_{-\infty}^{+\infty} F_{js}(v)\,dv = 0$ we find

$$h_j(x) = \int_{-\infty}^{+\infty} g_j(x,v)\,dv.$$

Now (3.10) reduces to the equation

$$\begin{aligned} a\frac{\partial f_{j+1}}{\partial v} + Q(f_{j+1}) = &-(v - \langle v_0 \rangle)\varphi_0(v) h_j{}'(x - \langle v_0 \rangle t) \\ &+ (-1)^{j+1} \Bigg[-\alpha_j t (v - \langle v_0 \rangle)\varphi_0(v) h_0^{[j+2]}(x - \langle v_0 \rangle t) \\ &+ \{(v - \langle v_0 \rangle)\varphi_j(v) - \alpha_j \varphi_0(v)\} h_0^{[j+1]}(x - \langle v_0 \rangle t) \\ &+ \sum_{s=1}^{m(j)} \{(v - \langle v_0 \rangle)F_{js}(v) - f_{js}\varphi_0(v)\} \sum_{k=0}^{q(j,s)} \frac{t^k}{k!} G_{kjs}{}'(x - \langle v_0 \rangle t) \\ &+ \sum_{s=1}^{m(j)} F_{js}(v) \sum_{k=1}^{q(j,s)} \frac{t^{k-1}}{(k-1)!} G_{kjs}(x - \langle v_0 \rangle t) \\ &- (v - \langle v_0 \rangle)\varphi_0(v) \sum_{s=1}^{m(j)} f_{js} \sum_{k=0}^{q(j,s)} \frac{t^{k+1}}{(k+1)!} G_{kjs}{}''(x - \langle v_0 \rangle t) \Bigg]. \end{aligned} \tag{3.12}$$

Letting $\varphi_{j+1}(v)$, $\hat{F}_{js}(v)$ and $H_{js}(v)$ be the unique solutions of the stationary equations

$$a\frac{\partial \varphi_{j+1}}{\partial v} + Q(\varphi_{j+1}) = (v - \langle v_0 \rangle)\varphi_j(v) - \alpha_j \varphi_0(v); \;\; j = 1, 2, \ldots;$$

$$a\frac{\partial \hat{F}_{js}}{\partial v} + Q(\hat{F}_{js}) = (v - \langle v_0 \rangle)F_{js}(v) - f_{js}\varphi_0(v); \;\; s = 1, \ldots, m(j);$$

$$a\frac{\partial H_{js}}{\partial v} + Q(H_{js}) = F_{js}(v),$$

in $L_1(\mathbf{R};(1+|v|)dv)$ that have zero integrals for $v \in \mathbf{R}$, we obtain

$$\begin{aligned} f_{j+1}(x,v,t) &= -\varphi_1(v)\Big[h_j{}'(x-\langle v_0\rangle t)+(-1)^{j+1}\alpha_j t h_0^{[j+2]}(x-\langle v_0\rangle t) \\ &+(-1)^{j+1}\sum_{s=1}^{m(j)} f_{js}\sum_{k=0}^{q(j,s)}\frac{t^{k+1}}{(k+1)!}G_{kjs}{}''(x-\langle v_0\rangle t)\Big] \\ &+(-1)^{j+1}\Big[\varphi_{j+1}(v)h_0^{[j+1]}(x-\langle v_0\rangle t)+\sum_{s=1}^{m(j)}\hat{F}_{js}(v)\sum_{k=0}^{q(j,s)}\frac{t^k}{k!}G_{kjs}{}'(x-\langle v_0\rangle t) \\ &+\sum_{s=1}^{m(j)}H_{js}(v)\sum_{k=1}^{q(j,s)}\frac{t^{k-1}}{(k-1)!}G_{kjs}(x-\langle v_0\rangle t)\Big]+m_{j+1}(x,t)\varphi_0(v), \end{aligned} \tag{3.13}$$

where $m_{j+1}(x,t)$ is a v-independent constant to be determined. Hence $f_{j+1}(x,v,t)$ has the form (3.9) with j replaced by $j+1$, which completes the induction argument.

Before giving a description of how the functions $F_{js}(v)$ look like, we study the number $N(j)$ of terms in the expansion of $f_j(x,v,t)$. From (3.9) and (3.13) one finds immediately

$$N(j)=2+\sum_{s=1}^{m(j)} q(j,s);$$

$$N(j+1)=4+\sum_{s=1}^{m(j)} q(j,s)+\sum_{s=1}^{m(j)}\{q(j,s)-1\}+\sum_{s=1}^{m(j)} q(j,s)=3N(j)-m(j)-2.$$

For low values of j we have $N(0)=1$, $N(1)=2$, $N(2)=4$, $N(3)=9$, $N(4)=21$, $N(5)=53$ and $N(6)=140$ so that

$$N(j)\le C_k\cdot 3^j,\qquad j\ge k, \tag{3.14}$$

where C_k is a suitable constant. However, to find a recurrence relation that allows one to calculate $N(j)$ for every j without evaluating $f_j(x,v,t)$ one has to sort the terms appearing in (3.9) according to their factors $(t^k/k!)$. Writing $p(j,k)$ for the number of terms having a factor $(t^k/k!)$, by inspection we obtain for $j=2$ $p(2,0)=p(2,1)=1$ and for $j=3$ $p(3,0)=4$, $p(3,1)=2$ and $p(3,2)=1$. Comparing (3.9) with (3.12) we find the recurrence relations for $j>2$

$$p(j+1,k)=\begin{cases} 2+p(j,0)+p(j,1), & k=0\\ p(j,k-1)+p(j,k)+p(j,k+1), & 1\le k\le j-2\\ p(j,j-2)+p(j,j-1), & k=j-1\\ 1, & k=j.\end{cases}$$

Hence the total number of terms in $f_j(x,v,t)$ is

$$N(j)=2+\sum_{k=0}^{j-1} p(j,k).$$

Starting from the first known values $p(2,0) = p(2,1) = 1$, it is now clear how we have evaluated $N(j)$ for $j \leq 6$. As an ancillary result all functions $G_{kjs}(z)$ have the form $h_p^{[q]}(z)$, i.e. the q-th derivative of h_p, where the highest derivative of $h_r(z)$ appearing in $f_j(x,v,t)$ is the $(2j-2r-1)$-th, occurring as one of the functions $G_{kjs}(z)$ with $k = j-r-1$, with r ranging from 0 to $j-1$. More precisely, $G_{kjs}(x) = h_r^{[j+k-r]}(x)$ for some $0 \leq r \leq j-k-1$.

Let us now discuss the general form of the functions $F_{js}(v)$ as well as upper bounds for their norms in $L_1(\mathbf{R}; (1+|v|)^r dv)$, putting

$$\gamma = \sup_{v\in\mathbf{R}} \frac{|v - \langle v_0\rangle|}{1+|v|} = \max(1, |\langle v_0\rangle|).$$

We have in fact five ways to generate $F_{j+1,s}(v)$ from some $F_{js}(v)$, namely as either $\hat{F}_{js}(v)$, $H_{js}(v)$, $-f_{js}\varphi_1(v)$, or, for $k=0$, $-\varphi_1(v)$ or $-\alpha_j\varphi_1(v)$. Using Theorem 3 for $n=r$ repeatedly we have

$$\|-\varphi_1\|_{1,r} \leq \mu_r \|(v - \langle v_0\rangle)\varphi_0\|_{1,r} \leq \gamma\mu_r\|\varphi_0\|_{1,r+1}; \tag{3.15a}$$

$$\|-\alpha_j\varphi_1\|_{1,r} \leq |\alpha_j|\gamma\mu_r\|\varphi_0\|_{1,r+1} \leq \gamma^2\mu_r\|\varphi_0\|_{1,r+1}\|\varphi_j\|_{1,1},$$

which, in combination with the estimate

$$\|\varphi_j\|_{1,q} \leq 2\mu_q\|(v-\langle v_0\rangle)\varphi_{j-1}\|_{1,q} \leq 2\gamma\mu_q\|\varphi_{j-1}\|_{1,q+1},$$

yields

$$\begin{aligned}\|-\alpha_j\varphi_1\|_{1,r} &\leq \gamma^2\mu_r\|\varphi_0\|_{1,r+1}\cdot(2\gamma)^j\mu_1\mu_2\cdots\mu_j\|\varphi_0\|_{1,j+1} \\ &= 2^j\gamma^{j+2}(\mu_1\mu_2\cdots\mu_j)\mu_r\|\varphi_0\|_{1,r+1}\|\varphi_0\|_{1,j+1}.\end{aligned} \tag{3.15b}$$

The other three basic estimates are given recursively. Indeed,

$$\|\hat{F}_{js}\|_{1,r} \leq 2\mu_r\|(v-\langle v_0\rangle)F_{js}\|_{1,r} \leq 2\gamma\mu_r\|F_{js}\|_{1,r+1}; \tag{3.15c}$$

$$\|H_{js}\|_{1,r} \leq \mu_r\|F_{js}\|_{1,r}; \tag{3.15d}$$

$$\|-f_{js}\varphi_1\|_{1,r} \leq |f_{js}|\cdot\|\varphi_1\|_{1,r} \leq \gamma\|F_{js}\|_{1,1}\|\varphi_1\|_{1,r} \leq \gamma^2\mu_r\|\varphi_0\|_{1,r+1}\|F_{js}\|_{1,1}. \tag{3.15e}$$

As a result, every $\|F_{js}\|_{1,r}$ has a compound of constants involving various norms of φ_0 as its upper bound, provided, of course, conditions (A1), (A2-n) and (A3-n) are satisfied for $n = j+1$.

Let us study $f_j(x,v,t=0)$. We readily find

$$\begin{aligned}
f_0(x,v,0) &= h_0(x)\varphi_0(v),\\
f_1(x,v,0) &= h_1(x)\varphi_0(v) - h_0{}'(x)\varphi_1(v),\\
f_2(x,v,0) &= h_2(x)\varphi_0(v) - h_1{}'(x)\varphi_1(v) + h_0{}''(x)\varphi_2(v),\\
f_3(x,v,0) &= h_3(x)\varphi_0(v) - h_2{}'(x)\varphi_1(v) + h_1{}''(x)\varphi_2(v) - h_0^{[3]}(x)\{\varphi_3(v) - \alpha_1\psi_1(v)\},
\end{aligned}$$

where $\psi_1(v)$ is the solution of Eq. (2.1) with $\phi = \varphi_1$ satisfying $(\psi_1, 1) = 0$, while for $j \geq 2$

$$f_j(x,v,0) = h_j(x)\varphi_0(v) + (-1)^j \left[\sum_{s=1}^{m(j)} G_{0js}(x) F_{js}(v) + h_0^{[j]}(x)\varphi_j(v) \right],$$

where $G_{0js}(x) = h_r^{[j-r]}(x)$ for some $0 \leq r \leq j-1$. Hence

$$f_j(x,v,0) = h_j(x)\varphi_0(v) + \sum_{r=0}^{j-1} (-1)^{j-r} h_r^{[j-r]}(x)\phi_{jr}(v) \tag{3.16}$$

for certain $\phi_{jr} \in L_1(\mathbf{R}; dv)$. As a result,

$$f_j(\cdot,\cdot,0) \in L_1(\mathbf{R}; dx) \otimes \mathcal{H}^j,$$

where $\mathcal{H}^j$ is a $(j+1)$-dimensional space of functions of v only. Note that $\mathcal{H}_j \subset \mathcal{H}_{j+1}$ and that $f_j(\cdot,\cdot,t) \in L_1(\mathbf{R}; dx) \otimes \mathcal{H}^j$ for every $t \geq 0$. As a result, the initial condition $(3.4)_j$ is generally not satisfied, although always

$$\int_{-\infty}^{+\infty} f_j(x,v,0)\, dv = \int_{-\infty}^{+\infty} g_j(x,v)\, dv = h_j(x), \qquad x \in \mathbf{R}.$$

If $g(x,v)$ does not depend on ϵ and hence $g_j(x,v) = 0$ for $j \geq 1$, we have $h_j(x) = 0$ for $j \geq 1$, so that the expressions for $f_j(x,v,t)$ simplify for $j \geq 1$. In particular, in this case

$$f_j(x,v,0) = (-1)^j h_0^{[j]}(x)\phi_{j0}(v)$$

for a particular function $\phi_{j0} \in L_1(\mathbf{R}; dv)$.

4. GENERALIZED DIFFUSION APPROXIMATION

Truncating the formal Hilbert expansions (3.1) and (3.3) we introduce

$$F_n(x,v,t;\epsilon) = \sum_{j=0}^{n} \epsilon^j f_j(x,v,t).$$

We then easily derive

$$\begin{aligned}
&\epsilon\left\{\frac{\partial F_n}{\partial t} + v\frac{\partial F_n}{\partial x}\right\} + a\frac{\partial F_n}{\partial v} + Q(F_n) \\
&= \epsilon \sum_{j=0}^{n} \epsilon^j \left\{\frac{\partial f_j}{\partial t} + v\frac{\partial f_j}{\partial x}\right\} + \sum_{j=0}^{n} \epsilon^j \left\{a\frac{\partial f_j}{\partial v} + Q(f_j)\right\} \\
&= -\epsilon^{n+1}\left\{a\frac{\partial f_{n+1}}{\partial v} + Q(f_{n+1})\right\}.
\end{aligned}$$

Hence, introducing the error $D_n(x,v,t;\epsilon) = f(x,v,t;\epsilon) - F_n(x,v,t;\epsilon)$ in the n-th order approximation and subtracting the above equation from (1.4) we obtain the initial-value problem

$$\begin{aligned}&\epsilon\left\{\frac{\partial D_n}{\partial t} + v\frac{\partial D_n}{\partial x}\right\} + a\frac{\partial D_n}{\partial v} + Q(D_n)\\ &= \epsilon^{n+1}\left\{a\frac{\partial f_{n+1}}{\partial v} + Q(f_{n+1})\right\};\end{aligned} \tag{4.1}$$

$$D_n(x,v,0;\epsilon) = g(x,v;\epsilon) - \sum_{j=0}^{n} \epsilon^j g_j(x,v). \tag{4.2}$$

The initial-value problem (1.4)-(1.5) is uniquely solvable in the following sense.[4-5] There exists a strongly continuous positive contraction semigroup $\{S_{[\epsilon]}(t)\}_{t\geq 0}$ on $L_1(\mathbf{R}^2;dvdx)$ such that for every $g \in L_1(\mathbf{R}^2;dvdx)$ there exists a unique solution of Eqs. (1.4)-(1.5) given by $f(x,v,t;\epsilon) = (S_{[\epsilon]}(t)g)(x,v)$, and this solution is nonnegative if g is nonnegative. Moreover, since $\nu(v)$ is assumed bounded, the L_1-norm is preserved, i.e. $\|S_{[\epsilon]}(t)g\|_1 = \|g\|_1$ if $g \geq 0$ in $L_1(\mathbf{R}^2;dvdx)$. Using the variation of constants formula we get for the unique solution of (4.1)-(4.2)

$$\begin{aligned}\frac{D_n(x,v,t;\epsilon)}{\epsilon^{n+1}} &= \left[S_{[\epsilon]}(t)\,\frac{g(\epsilon) - \sum_{j=0}^{n}\epsilon^j g_j}{\epsilon^{n+1}}\right](x,v)\\ &+ \int_0^t \left[S_{[\epsilon]}(t-\tau)\left\{a\frac{\partial f_{n+1}(\tau)}{\partial v} + Q(f_{n+1}(\tau))\right\}\right](x,v)\,d\tau,\end{aligned} \tag{4.3}$$

where $g(\epsilon)(x,v) = g(x,v;\epsilon)$, so that in the norm of $L_1(\mathbf{R};dv)$

$$\frac{\|D_n(\cdot,\cdot,t;\epsilon)\|_1}{\epsilon^{n+1}} \leq \frac{\|g(\epsilon) - \sum_{j=0}^{n}\epsilon^j g_j\|_1}{\epsilon^{n+1}} + t\sup_{\tau\in[0,t]}\left\|a\frac{\partial f_{n+1}(\tau)}{\partial v} + Q(f_{n+1}(\tau))\right\|_1. \tag{4.4}$$

On the other hand, if one were to derive the analogous estimate in $L_1(\mathbf{R}^2;(1+|v|)\,dvdx)$ one must use the fact that under suitable hypotheses on the collision operator $\{S_{[\epsilon]}(t)\}_{t\geq 0}$ is a strongly continuous semigroup on $L_1(\mathbf{R}^2;(1+|v|)dvdx)$ satisfying $\|S_{[\epsilon]}(t)g\|_{1,1} \leq C(1+t)\|g\|_{1,1}$ for all $g \in L_1(\mathbf{R}^2;(1+|v|)dvdx)$. Here C is some constant. Instead of (4.4) one then finds the estimate

$$\begin{aligned}\frac{\|D_n(\cdot,\cdot,t;\epsilon)\|_{1,1}}{\epsilon^{n+1}} &\leq C(1+t)\frac{\|g(\epsilon) - \sum_{j=0}^{n}\epsilon^j g_j\|_{1,1}}{\epsilon^{n+1}}\\ &+ C(t+\frac{1}{2}t^2)\sup_{\tau\in[0,t]}\left\|a\frac{\partial f_{n+1}(\tau)}{\partial v} + Q(f_{n+1}(\tau))\right\|_{1,1}.\end{aligned} \tag{4.5}$$

The first term on the right-hand side of (4.4) [(4.5), respectively] generally does not remain bounded as $\epsilon \downarrow 0$. This is only the case if $g(\cdot,\cdot,\epsilon) \in C^n(\mathcal{U};L_1(\mathbf{R}^2;dvdx))$

$[g(\cdot,\cdot,\epsilon) \in C^n(\mathcal{U}; L_1(\mathbf{R}^2; (1+|v|)dvdx))$, respectively] and $g_0, \cdots, g_n$ are the first $n+1$ coefficients in the Taylor series expansion of $g(\cdot,\cdot,\epsilon)$ at $\epsilon = 0$ [Cf. the vector version of Theorem 5.15 of Ref. 21]. In other words, the initial conditions $(3.4)_j$ should be satisfied for $j = 0, 1, \cdots, n$. For $n = 0$ this means requiring $g_0(x,v) = \tilde{h}_0(x)\varphi_0(v)$ for some $\tilde{h}_0 \in L_1(\mathbf{R}; dx)$. For $n = 1$ we should require $g_0(x,v) = \tilde{h}_0(x)\varphi_0(v)$ and $g_1(x,v) = \tilde{h}_1(x)\varphi_0(v) - \tilde{h}_0'(x)\varphi_1(v)$ for certain $\tilde{h}_0 \in H_1^1(\mathbf{R})$ and $\tilde{h}_1 \in L_1(\mathbf{R}; dx)$. For $n = 2$ we must require $g_0(x,v) = \tilde{h}_0(x)\varphi_0(v)$, $g_1(x,v) = \tilde{h}_1(x)\varphi_0(v) - \tilde{h}_0'(x)\varphi_1(v)$, and $g_2(x,v) = \tilde{h}_2(x)\varphi_0(v) - \tilde{h}_1'(x)\varphi_1(v) + \tilde{h}_0''(x)\varphi_2(v)$ for certain $\tilde{h}_0 \in H_1^2(\mathbf{R})$, $\tilde{h}_1 \in H_1^1(\mathbf{R})$ and $\tilde{h}_2 \in L_1(\mathbf{R}; dx)$. For $n \geq 3$ the requirements on $g_0, \cdots, g_n$ become more complicated. If $g(x,v)$ does not depend on ϵ, all of these requirements simplify considerably.

Theorem 4. *Let the conditions* (A1), (A2-(n+1)) *and* (A3-(n+1)) *be satisfied. In addition, let us make the following assumptions:*

1. *As a function of* ϵ, $g(\cdot,\cdot,\epsilon) \in C^n(\mathcal{U}; L_1(\mathbf{R}^2; dvdx))$ *on some interval* $\mathcal{U} = [0, \zeta)$;
2. *For* $j = 0, 1, \cdots, n$ *the initial conditions* $(3.4)_j$ *are fulfilled;*
3. *For* $0 \leq r \leq n$, h_r *belongs to* $H_1^{2n-2r+1}(\mathbf{R})$.

Then as $\epsilon \downarrow 0$

$$\|f(\cdot,\cdot,t;\epsilon) - \sum_{j=0}^{n} \epsilon^j f_j(\cdot,\cdot,t)\|_1 = O(\epsilon^{n+1}), \tag{4.6}$$

uniformly on compact time intervals.

Proof: From (3.12) we easily find

$$\begin{aligned}
\left\| a\frac{\partial f_{n+1}}{\partial v} + Q(f_{n+1})\right\|_1 &\leq \|(v - \langle v_0\rangle)\varphi_0\|_1 \|h_n'\|_1 + |\alpha_n t| \|(v - \langle v_0\rangle)\varphi_0\|_1 \|h_0^{[n+2]}\|_1 \\
&+ \|(v - \langle v_0\rangle)\varphi_n - \alpha_n\varphi_0\|_1 \|h_0^{[n+1]}\|_1 \\
&+ \sum_{s=1}^{m(n)} \|(v - \langle v_0\rangle)F_{ns} - f_{ns}\varphi_0\|_1 \sum_{k=0}^{q(n,s)} \frac{t^k}{k!} \|G_{kns}'\|_1 \\
&+ \sum_{s=1}^{m(n)} \|F_{ns}\|_1 \sum_{k=1}^{q(n,s)} \frac{t^{k-1}}{(k-1)!} \|G_{kns}\|_1 \\
&+ \|(v - \langle v_0\rangle)\varphi_0\|_1 \sum_{s=1}^{m(n)} |f_{ns}| \sum_{k=0}^{q(n,s)} \frac{t^{k+1}}{(k+1)!} \|G_{kns}''\|_1.
\end{aligned} \tag{4.7}$$

Using (3.15a)-(3.15e) one obtains an upper bound involving the norms $\|\varphi_0\|_{1,1}$, $\|\varphi_n\|_{1,1}$, $\|F_{ns}\|_{1,1}$ and $\|F_{ns}\|_1$, the norms $\|h_n'\|_1$, $\|h_0^{[n+2]}\|_1$, $\|G_{kns}'\|_1$, $\|G_{kns}\|_1$ and $\|G_{kns}''\|_1$, as well as factors $(t^k/k!)$ for $0 \leq k \leq n+1$. To mention some of the details, when checking the regularity of the functions G_{kns}, notice that every such function is of the type $h_p^{[q]}(\cdot)$, the q-th derivative of $h_p(\cdot)$, where $0 \leq q \leq 2n - 2r + 1$ with $k = n - r$ and $0 \leq r \leq n$. Further, note also that $q(n,s)$ does not exceed n. Next, when inspecting the spaces to which the functions $(v -$

$\langle v_0\rangle)\varphi_0$, $(v - \langle v_0\rangle)\varphi_n - \alpha_n\varphi_0$, $(v - \langle v_0\rangle)F_{ns} - f_{ns}\varphi_0$ and F_{ns} belong, note that $(v - \langle v_0\rangle)\varphi_0 \in L_1(\mathbf{R}; dv)$ whenever $\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)dv)$, $(v - \langle v_0\rangle)\varphi_n - \alpha_n\varphi_0 \in L_1(\mathbf{R}; dv)$ whenever $\varphi_n \in L_1(\mathbf{R}; (1 + |v|)dv)$ which is true whenever $\varphi_0 \in L_1(\mathbf{R}; (1+|v|)^{n+1}dv)$, $F_{ns} \in L_1(\mathbf{R}; dv)$ whenever $\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)^n dv)$, and $(v-\langle v_0\rangle)F_{ns} - f_{ns}\varphi_0 \in L_1(\mathbf{R}; dv)$ whenever $F_{ns} \in L_1(\mathbf{R}; (1 + |v|)dv)$ which is true whenever $\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)^{n+1}dv)$. Thus, indeed, we must require $\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)^{n+1}dv)$. Summarizing,

$$\left\| a\frac{\partial f_{n+1}}{\partial v} + Q(f_{n+1})\right\|_1 \leq 3N(n)\gamma^2 M_1^2 M_2 \max\left\{\frac{t^k}{k!} : 0 \leq k \leq n+1\right\},$$

where M_1 is the maximum of the norms in the group of quantities F_{ns} and M_2 is the maximum of the norms in the group of quantities G_{kns}. Note that the number of terms in the sum equals $3N(n) - m(n) - 3$, which is bounded above by $3N(n)$. Consequently, the expression (4.7) is bounded uniformly in t on compact time intervals.

From the vector version of Theorem 5.15 of Ref. 21, the differentiability requirement on $g(\cdot,\cdot,\epsilon)$ and the initial conditions $(3.4)_j$ for $j = 0, 1, \cdots, n$, it follows easily that $\|g(\epsilon) - \sum_{j=0}^n \epsilon^j g_j\|_1 = O(\epsilon^{n+1})$ as $\epsilon \downarrow 0$, which completes the proof. ∎

If one employs (4.5) instead of (4.4) and modifies the regularity assumptions of Theorem 4, we obtain the following result.

Theorem 5. *Let the conditions* (A1), (A2-(n+2)) *and* (A3-(n+2)) *be satisfied. In addition, let us make the following assumptions:*

1. *As a function of ϵ, $g(\cdot,\cdot,\epsilon) \in C^n(\mathcal{U}; L_1(\mathbf{R}^2; (1 + |v|)dvdx))$ on some interval $\mathcal{U} = [0, \zeta)$;*
2. *For $j = 0, 1, \cdots, n$ the initial conditions $(3.4)_j$ are fulfilled;*
3. *For $0 \leq r \leq n$, h_r belongs to $H_1^{2n-2r+1}(\mathbf{R})$.*

Then as $\epsilon \downarrow 0$

$$\|f(\cdot,\cdot,t;\epsilon) - \sum_{j=0}^n \epsilon^j f_j(\cdot,\cdot,t)\|_{1,1} = O(\epsilon^{n+1}), \tag{4.8}$$

uniformly on compact time intervals.

Proof: The proof of Theorem 4 can be repeated almost verbatim with the following two modifications. First, one employs (4.5) instead of (4.4), which is a nonessential alteration. Secondly, note that $(v-\langle v_0\rangle)\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)dv)$ whenever $\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)^2dv)$, $(v - \langle v_0\rangle)\varphi_n - \alpha_n\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)dv)$ whenever $\varphi_n \in L_1(\mathbf{R}; (1 + |v|)^2dv)$ which is true whenever $\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)^{n+2}dv)$, $F_{ns} \in L_1(\mathbf{R}; (1 + |v|)dv)$ whenever $\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)^{n+1}dv)$, and $(v - \langle v_0\rangle)F_{ns} - f_{ns}\varphi_0 \in L_1(\mathbf{R}; (1 + |v|)dv)$ whenever $F_{ns} \in L_1(\mathbf{R}; (1 + |v|)^2dv)$ which is true whenever $\varphi_0 \in L_1(\mathbf{R}; (1+|v|)^{n+2}dv)$, so that we must require $\varphi_0 \in L_1(\mathbf{R}; (1+|v|)^{n+2}dv)$. Nothing changes in the discussion of the functions G_{kjs}. ∎

Theorem 5 implies a Hilbert expansion result on the drift velocity. The only thing one has to do is to integrate the Hilbert expansion with respect to the (signed) measure vdv, assuming that $g(x,v;\epsilon) \geq 0$ and has unit norm in $L_1(\mathbf{R}^2; dvdx)$. Thus we obtain

Corollary 6. *Under the conditions of Theorem 5 we have as* $\epsilon \downarrow 0$

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v \left\{ f(x,v,t;\epsilon) - \sum_{j=0}^{n} \epsilon^j f_j(x,v,t) \right\} dxdv = O(\epsilon^{n+1}).$$

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