Inverse Scattering in One Dimension for a Generalized Schrödinger Equation

Tuncay Aktosun[1] *and Cornelis van der Mee*[2]

[1] Department of Mathematics, North Dakota State University
Fargo, ND 58105, USA
[2] Department of Mathematics, University of Cagliari
Via Ospedale 72, 09124 Cagliari, Italy

Abstract: The generalized one-dimensional Schrödinger equation $\frac{d^2\psi}{dx^2}+k^2H(x)^2\,\psi = Q(x)\,\psi$ is considered, where $H(x)\to 1$ and $Q(x)\to 0$ as $x\to\pm\infty$. The function $H(x)$ is recovered when the scattering matrix, $Q(x)$, the bound state energies and norming constants are known.

Keywords. Inverse scattering, Schrödinger equation, acoustic scattering

1 Introduction

Consider the generalized Schrödinger equation

$$\frac{d^2\psi(k,x)}{dx^2}+k^2H(x)^2\,\psi(k,x)=Q(x)\,\psi(k,x), \tag{1.1}$$

where $H(x)\to 1$ and $Q(x)\to 0$ in some sense as $x\to\pm\infty$. The following conditions are sufficient for the results in this paper to hold: $Q\in L^1_1(\mathbf{R})$, $H(x)$ is positive and bounded above, $1-H\in L^1(\mathbf{R})$, and $G\in L^1_1(\mathbf{R})$, where $G(x)$ is the quantity given in (2.4). Here $L^1_\alpha(\mathbf{R})$ denotes the space of Lebesgue integrable functions on the real axis with the weight function $(1+|x|)^\alpha$ and $L^1(\mathbf{R})=L^1_0(\mathbf{R})$. The physical solution of (1.1) from the left $\psi_l(k,x)$ corresponds to a plane wave sent from $x=-\infty$ and satisfies the boundary conditions

$$\psi_l(k,x)=\begin{cases} T(k)e^{ikx}+o(1), & x\to+\infty\\ e^{ikx}+L(k)e^{-ikx}+o(1), & x\to-\infty.\end{cases}$$

Similarly, the physical solution of (1.1) from the right $\psi_r(k,x)$ corresponds to a plane wave sent from $x=+\infty$ and satisfies the boundary conditions

$$\psi_r(k,x)=\begin{cases} e^{-ikx}+R(k)e^{ikx}+o(1), & x\to+\infty\\ T(k)e^{-ikx}+o(1), & x\to-\infty.\end{cases}$$

Here $T(k)$ is the transmission coefficient, $R(k)$ and $L(k)$ are the reflection coefficients from the right and from the left, respectively. The scattering matrix associated with (1.1) is defined as

$$\mathbf{S}(k) = \begin{bmatrix} T(k) & R(k) \\ L(k) & T(k) \end{bmatrix}. \tag{1.2}$$

In this paper we study an inverse scattering problem for (1.1), namely the recovery of $H(x)$ when one knows $\mathbf{S}(k)$, $Q(x)$, and the bound state energies as well as the bound state norming constants. Our purpose here is twofold. In [4] we presented a method to recover $H(x)$ when one knows $\mathbf{S}(k)$, $Q(x)$, the bound state energies and the bound state norming constants, and either of $A_{\pm}$, where

$$A_{\pm} = \int_0^{\pm\infty} dz\,[1 - H(z)]. \tag{1.3}$$

Our first goal is to show that in the method of [4] neither of $A_{\pm}$ are needed to recover $H(x)$. The same thing is true for the method of [3] to recover $H(x)$ in the differential equation

$$\frac{d}{dx}\left[a(x)\frac{d\psi(k,x)}{dx}\right] + k^2 H(x)^2\,\psi(k,x) = Q(x)\,\psi(k,x), \tag{1.4}$$

when one knows $\mathbf{S}(k)$, $a(x)$, $Q(x)$, and the bound state energies as well as the bound state norming constants; neither of $A_{\pm}$ are needed to recover $H(x)$ in (1.4).

Currently, we are developing another method of recovery of $H(x)$ that may be generalizable to the multidimensional generalized Schrödinger equation

$$\Delta\psi(k,x) + k^2 H(x)^2\,\psi(k,x) = Q(x)\,\psi(k,x), \qquad x \in \mathbf{R}^n,$$

where $n \geq 2$; this method [8] is based on the Wiener-Hopf factorization of an operator function related to the scattering matrix. Our second goal in this paper is to explore the connection between the method of [4] and a variation of the method of [8].

This paper is organized as follows. In Section 2 we outline the method of [4] to recover $H(x)$ in (1.1). In Section 3 we show that in the method of [4], neither of $A_{\pm}$ are needed to recover $H(x)$, and this is illustrated by an example. In Section 4 we discuss a Riemann-Hilbert problem related to the scattering matrix of (1.1); this Riemann-Hilbert problem has infinitely many solutions, one of which leads to the solution of the inverse scattering problem for (1.1) of the recovery of $H(x)$. In Section 4 we also present a lemma that shows the equivalence of $L^1_\alpha(\mathbf{R}; dx)$ and $L^1_\alpha(\mathbf{R}; dy)$, which is useful in demonstrating the connection between (1.1) and (2.2).

2 Recovery of $H(x)$

In this section we outline the method of [4] and show that we can recover $H(x)$ when we know $\mathrm{S}(k)$, $Q(x)$, either of $A_\pm$, the bound state energies and the bound state norming constants. In the next section we will show that neither of $A_\pm$ are needed to recover $H(x)$.

Associated with (1.1) are the two Schrödinger equations

$$\frac{d^2\psi^{[0]}(k,x)}{dx^2}+k^2\psi^{[0]}(k,x)=Q(x)\,\psi^{[0]}(k,x), \tag{2.1}$$

$$\frac{d^2\phi(k,y)}{dy^2}+k^2\phi(k,y)=V(y)\,\phi(k,y), \tag{2.2}$$

where

$$y=y(x)=\int_0^x dz\,H(z), \tag{2.3}$$

$$V(y)=-\frac{G(x)}{H(x)},$$

$$G(x)=-\frac{H''(x)}{2H(x)^2}+\frac{3}{4}\frac{H'(x)^2}{H(x)^3}-\frac{Q(x)}{H(x)}. \tag{2.4}$$

Let $\boldsymbol{\sigma}(k)$ denote the scattering matrix of (2.2). We then have

$$\boldsymbol{\sigma}(k)=\begin{bmatrix}\tau(k) & \rho(k)\\ \ell(k) & \tau(k)\end{bmatrix}=\begin{bmatrix}T(k)\,e^{ikA} & R(k)\,e^{2ikA_+}\\ L(k)\,e^{2ikA_-} & T(k)\,e^{ikA}\end{bmatrix}, \tag{2.5}$$

where $A_\pm$ are the constants defined in (1.3) and

$$A=A_+ + A_- = \int_{-\infty}^{\infty} dz\,[1-H(z)].$$

Note that when $\mathrm{S}(k)$ is known, one also knows A because $\tau(k)$ in (2.5) converges to 1 as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Let $\phi_l(k,y)$ and $\phi_r(k,y)$ be the physical solutions of (2.2) from the left and from the right, respectively; we then have

$$\phi_l(k,y)=\begin{cases}\tau(k)e^{iky}+o(1), & y\to+\infty\\ e^{iky}+\ell(k)e^{-iky}+o(1), & y\to-\infty,\end{cases}$$

$$\phi_r(k,y)=\begin{cases}e^{-iky}+\rho(k)e^{iky}+o(1), & y\to+\infty\\ \tau(k)e^{-iky}+o(1), & y\to-\infty.\end{cases}$$

Let us also define the Faddeev functions $Z_l(k,y)$ from the left and $Z_r(k,y)$ from the right, respectively, associated with (2.2) as

$$Z_l(k,y)=\frac{1}{\tau(k)}e^{-iky}\,\phi_l(k,y),$$

$$Z_r(k,y) = \frac{1}{\tau(k)} e^{iky}\,\phi_r(k,y).$$

Similarly, let us define the Faddeev functions $m_l(k,x)$ from the left and $m_r(k,x)$ from the right, respectively, associated with (1.1) as

$$m_l(k,x) = \frac{1}{T(k)} e^{-ikx}\,\psi_l(k,x),$$

$$m_r(k,x) = \frac{1}{T(k)} e^{ikx}\,\psi_r(k,x).$$

We then have [4,5]

$$m_l(k,x) = \frac{1}{\sqrt{H(x)}}\, e^{ik\int_x^\infty [1-H]} Z_l(k,y(x)), \tag{2.6}$$

$$m_r(k,x) = \frac{1}{\sqrt{H(x)}}\, e^{ik\int_{-\infty}^x [1-H]} Z_r(k,y(x)). \tag{2.7}$$

Associated with (2.1) we have the scattering matrix

$$\mathbf{S}^{[0]}(k,x) = \begin{bmatrix} T^{[0]}(k) & R^{[0]}(k) \\ L^{[0]}(k) & T^{[0]}(k) \end{bmatrix}.$$

The physical solutions $\psi_l^{[0]}(k,x)$ and $\psi_r^{[0]}(k,x)$ of (2.1) from the left and from the right, respectively, satisfy

$$\psi_l^{[0]}(k,x) = \begin{cases} T^{[0]}(k)e^{ikx} + o(1), & x \to +\infty \\ e^{ikx} + L^{[0]}(k)e^{-ikx} + o(1), & x \to -\infty, \end{cases}$$

$$\psi_r^{[0]}(k,x) = \begin{cases} e^{-ikx} + R^{[0]}(k)e^{ikx} + o(1), & x \to +\infty \\ T^{[0]}(k)e^{-ikx} + o(1), & x \to -\infty. \end{cases}$$

The Faddeev functions $m_l^{[0]}(k,x)$ from the left and $m_r^{[0]}(k,x)$ from the right, respectively, associated with (2.1) are then defined as

$$m_l^{[0]}(k,x) = \frac{1}{T^{[0]}(k)} e^{-ikx}\,\psi_l^{[0]}(k,x),$$

$$m_r^{[0]}(k,x) = \frac{1}{T^{[0]}(k)} e^{ikx}\,\psi_r^{[0]}(k,x).$$

When $Q(x)$ is known, $m_l^{[0]}(k,x)$ and $m_r^{[0]}(k,x)$ are uniquely determined. Furthermore, it can be shown that for $k=0$, the scattering matrices $\mathbf{S}(k)$ and $\mathbf{S}^{[0]}(k)$ coincide, namely $\mathbf{S}(0) = \mathbf{S}^{[0]}(0)$. It is seen that as $k \to 0$, (1.1) and (2.1) reduce to the same equation. Hence the physical solutions of (1.1) and those of (2.1) coincide at $k=0$. Thus, the Faddeev functions associated with (1.1) and those associated with (2.1) coincide at $k=0$, and we have

$$m_l(0,x)=m_l^{[0]}(0,x), \tag{2.8}$$

$$m_r(0,x)=m_r^{[0]}(0,x). \tag{2.9}$$

Hence, from (2.6) and (2.7), one obtains

$$m_l(0,x)=\frac{1}{\sqrt{H(x)}}Z_l(0,y(x)),$$

$$m_r(0,x)=\frac{1}{\sqrt{H(x)}}Z_r(0,y(x)).$$

From (2.3) it is then seen that

$$\frac{dy}{dx}=H(x), \tag{2.10}$$

and hence one has

$$\frac{dy}{Z_l(0,y)^2}=\frac{dx}{m_l^{[0]}(0,x)^2}, \tag{2.11}$$

$$\frac{dy}{Z_r(0,y)^2}=\frac{dx}{m_r^{[0]}(0,x)^2}. \tag{2.12}$$

The first order ordinary differential equations given in (2.11) and (2.12) are both separable; their solutions with the initial condition $y(0)=0$ give us $y(x)$. Using $y(x)$ in either of the two equations

$$H(x)=\frac{Z_l(0,y(x))^2}{m_l^{[0]}(0,x)^2}, \tag{2.13}$$

$$H(x)=\frac{Z_r(0,y(x))^2}{m_r^{[0]}(0,x)^2}, \tag{2.14}$$

one then recovers $H(x)$. From (2.13) it is seen that, in order to obtain $H(x)$, one needs to know $m_l^{[0]}(0,x)$ and $Z_l(0,y(x))$. Equivalently, from (2.14) it is seen that, in order to obtain $H(x)$, one needs to know $m_r^{[0]}(0,x)$ and $Z_r(0,y(x))$. When one knows $Q(x)$, one then also knows $m_l^{[0]}(0,x)$ and $m_r^{[0]}(0,x)$. If one knows $\mathrm{S}(k)$, the information on the bound states, and either of $A_\pm$ defined in (1.3), one can then obtain $Z_l(k,y)$ and $Z_r(k,y)$ from $\sigma(k)$ given in (2.5) by solving a Riemann-Hilbert problem [4,5]. Hence, $\mathrm{S}(k)$, the bound state information, and either of $A_\pm$ are sufficient to obtain $Z_l(0,y(x))$ and $Z_r(0,y(x))$. Thus, knowing $Q(x)$, $\mathrm{S}(k)$, the bound state information, and either of $A_\pm$, one obtains $H(x)$ using the method of [4].

3 Phase Independence

Instead of using $\sigma(k)$ in (2.5), let us define

$$\sigma_b(k) = \begin{bmatrix} \tau(k) & \rho(k)\,e^{ikb} \\ \ell(k)\,e^{-ikb} & \tau(k) \end{bmatrix} = \begin{bmatrix} T(k)\,e^{ikA} & R(k)\,e^{ikb+2ikA_+} \\ L(k)\,e^{-ikb+2ikA_-} & T(k)\,e^{ikA} \end{bmatrix},$$

where b is an arbitrary real parameter. This is equivalent to shifting the phase of the reflection coefficient $\rho(k)$ in (2.5) by b. Such a phase shift causes the shift $y \mapsto y+b$ in the Schrödinger equation (2.2), and it is known that the Faddeev functions for (2.2) are then transformed according to $Z_l(k,y) \mapsto Z_l(k,y+b)$ and $Z_r(k,y) \mapsto Z_r(k,y+b)$ [1,2,11]. Then in the solution of the first order differential equations (2.11) and (2.12), the initial condition is replaced by $y(0) = -b$; as seen from (2.10), $H(x)$ is independent of the shift $y \mapsto y+b$ and hence no matter how the phase of $\rho(k)$ is chosen, we are led to the same $H(x)$ in the solution of the inverse problem for (1.1).

We will illustrate this phase independence by an explicit example. Let $Q(x)$ in (1.1) be given by

$$Q(x) = \sqrt{3}\,\delta(x) - \theta(x)\frac{2\sqrt{3}e^x}{(1+\sqrt{3})^2}, \tag{3.1}$$

where $\delta(x)$ is the Dirac delta function and $\theta(x)$ is the Heaviside function. The Faddeev function from the right corresponding to $Q(x)$ is given by

$$m_r^{[0]}(k,x) = 1, \qquad x \le 0,$$

$$m_r^{[0]}(k,x) = \frac{k+i}{k+i/2}\left[1 + \frac{i}{k-i/2}\frac{1}{1+\sqrt{3}e^x}\right]$$
$$- \frac{\sqrt{3}i/2}{k-i/2}e^{2ikx}\left[1 - \frac{i}{k+i/2}\frac{1}{1+\sqrt{3}e^x}\right], \qquad x \ge 0.$$

Thus, we have

$$m_r^{[0]}(0,x) = 1, \qquad x \le 0,$$
$$m_r^{[0]}(0,x) = (2+\sqrt{3})\frac{\sqrt{3}e^x - 1}{\sqrt{3}e^x + 1}, \qquad x \ge 0.$$

Consider the scattering matrix $\mathbf{S}(k)$ in (1.2) with

$$T(k) = \tfrac{k+i}{k+2i}e^{ik(1+3\sqrt{3})},$$

$$R(k) = \tfrac{\sqrt{3}\,i}{k+2i}e^{4ik(\sqrt{3}+1)}, \quad L(k) = \tfrac{\sqrt{3}\,i}{k+2i}\tfrac{k+i}{k-i}e^{2ik(\sqrt{3}-1)}. \tag{3.2}$$

Note that the constant A in this example is seen to be $A = -1-3\sqrt{3}$, which is obtained from the large k asymptotics of $T(k)$. Let us define

$$\tau_a(k) = \frac{k+i}{k+2i}, \qquad \rho_a(k) = \frac{\sqrt{3}\,i}{k+2i}e^{ika}, \qquad \ell_a(k) = \frac{\sqrt{3}\,i}{k+2i}\frac{k+i}{k-i}e^{-ika}, \tag{3.3}$$

where a is an arbitrary real parameter. The Faddeev function from the right corresponding to the scattering matrix in (3.3) can be obtained by solving a Riemann-Hilbert problem [5,6], and we have

$$Z_r(k,y;a) = \frac{k+2i}{k+i} + \frac{\sqrt{3}i}{k+i} e^{2ik(y+a)}, \qquad y \geq -a, \tag{3.4}$$

$$Z_r(k,y;a) = 1 + \frac{i}{k+i} \frac{2}{\sqrt{3}e^{-2(y+a)} - 1}, \qquad y \leq -a. \tag{3.5}$$

We then have

$$Z_r(0,y;a) = 2 + \sqrt{3}, \qquad y \geq -a, \tag{3.6}$$

$$Z_r(0,y;a) = \frac{\sqrt{3} + e^{2a+2y}}{\sqrt{3} - e^{2a+2y}}, \qquad y \leq -a. \tag{3.7}$$

From (2.14) we obtain

$$H(x) = \left(\frac{\sqrt{3} + e^{2a+2y}}{\sqrt{3} - e^{2a+2y}} \right)^2, \qquad x \leq 0,\ y \leq -a, \tag{3.8}$$

$$H(x) = \left(\frac{\sqrt{3}e^x + 1}{\sqrt{3}e^x - 1} \right)^2, \qquad x \geq 0,\ y \geq -a. \tag{3.9}$$

In solving (2.12) we need to use the initial condition $y(0) = -a$; we then obtain

$$x = y + a - \frac{2\sqrt{3}}{\sqrt{3}+1} + \frac{2\sqrt{3}}{\sqrt{3} + e^{2y+2a}}, \qquad x \leq 0,\ y \leq -a, \tag{3.10}$$

$$y = x - a + 2\frac{\sqrt{3}+1}{\sqrt{3}-1} - 2\frac{\sqrt{3}e^x + 1}{\sqrt{3}e^x - 1}, \qquad x \geq 0,\ y \geq -a. \tag{3.11}$$

Since y appears as $y + a$ in (3.4)-(3.11), we see that no matter what a is, we get the same $H(x)$. Once $H(x)$ is obtained, we can then evaluate $A_\pm$ by using (1.3) to find

$$A_+ = -2\sqrt{3} - 2, \qquad A_- = 1 - \sqrt{3}.$$

4 Riemann-Hilbert Problem

Associated with the scattering matrix $\mathbf{S}(k)$, we define another matrix

$$\mathbf{G}(k,x) = e^{i\mathbf{J}kx} \mathbf{J}\mathbf{S}(k)\mathbf{J} e^{-i\mathbf{J}kx} = \begin{bmatrix} T(k) & -R(k)e^{2ikx} \\ -L(k)e^{-2ikx} & T(k) \end{bmatrix},$$

where $\mathbf{J} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Let us define the vectors $m(k,x)$ and $m^{[0]}(k,x)$ in terms of the Faddeev functions for (1.1) and (2.1), respectively, as

$$m(k,x) = \begin{bmatrix} m_l(k,x) \\ m_r(k,x) \end{bmatrix}, \qquad m^{[0]}(k,x) = \begin{bmatrix} m_l^{[0]}(k,x) \\ m_r^{[0]}(k,x) \end{bmatrix}.$$

It is known that $m(k,x)$ is analytic in $k \in \mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}$, that $m(0,x) = m^{[0]}(0,x)$, and when $H(x) \leq 1$ the vector $m(k,x)$ remains bounded as $k \to \infty$ in $\overline{\mathbf{C}^+}$. Furthermore, we have

$$m(-k,x) = \mathbf{G}(k,x)\,\mathbf{q}\,m(k,x), \qquad k \in \mathbf{R}, \tag{4.1}$$

where $\mathbf{q} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. As $k \to \pm\infty$, the quantities $\mathbf{G}(k,x)$ and $m(k,x)$ do not converge to constant limits, and hence the standard techniques on Riemann-Hilbert problems [9,10] cannot be applied to (4.1). In terms of $\mathbf{S}(k)$ of (1.2) or $\boldsymbol{\sigma}(k)$ of (2.5), let us define

$$\Lambda(k,x) = e^{ikA}\mathbf{G}(k,x) = e^{i\mathbf{J}k(x+\gamma)}\mathbf{J}\boldsymbol{\sigma}(k)\mathbf{J}e^{-i\mathbf{J}k(x+\gamma)}, \tag{4.2}$$

where $\gamma = \frac{1}{2}A_- - \frac{1}{2}A_+$.

To solve (4.1) one needs a variant of the Wiener-Hopf factorization of the matrix function $\Lambda(k,x)$ in (4.2) with special symmetry properties [6,7]; such a special factorization of the matrix function $\Lambda(k,x)$ can be given in terms of the solutions of the Schrödinger equation (2.2) [6,7]; that factorization can also be obtained by solving the matrix Riemann-Hilbert problem

$$\mathbf{P}(-k,x) = \Lambda(k,x)\,\mathbf{q}\,\mathbf{P}(k,x)\,\mathbf{q}, \qquad k \in \mathbf{R}, \tag{4.3}$$

where given $\Lambda(k,x)$, we seek the matrix function $\mathbf{P}(k,x)$ such that it is analytic for $k \in \mathbf{C}^+$, continuous for $k \in \overline{\mathbf{C}^+} \setminus \{0\}$, $\mathbf{P}(k,x) \to \mathbf{I}$ as $k \to \infty$ in $\overline{\mathbf{C}^+}$, where $\mathbf{I}$ is the identity matrix. Furthermore, in the exceptional case, i.e. when $T(0) \neq 0$, $\mathbf{P}(k,x)$ is continuous at $k=0$ whereas in the generic case, i.e. when $T(0)=0$, it has a $1/k$-singularity at $k=0$. When (1.1) has no bound states, the matrix Riemann-Hilbert problem (4.3) has a unique solution in the exceptional case and there is a one-parameter family of solutions in the generic case. If there are $\mathcal{N}$ bound states then the matrix Riemann-Hilbert problem (4.3) has a $(2\mathcal{N})$-parameter family of solutions in the exceptional case and $(2\mathcal{N}+\infty)$-parameter family of solutions in the generic case [7]. The special factorization of $\Lambda(k,x)$ given by

$$\Lambda(k,x) = \mathbf{P}(-k,x)\,\mathbf{q}\,\mathbf{P}(k,x)^{-1}\,\mathbf{q}, \tag{4.4}$$

exists whenever $G \in L^1_1(\mathbf{R})$ and $H(x)$ is positive and bounded above, where $G(x)$ is the quantity defined in (2.4); equivalent conditions for the existence of the special factorization of $\Lambda(k,x)$ can also be given directly in terms of $\mathbf{S}(k)$, and the following lemma will be useful for that purpose.

LEMMA 4.1 Assume $1-H \in L^1(\mathbf{R})$. Then $V \in L^1_\alpha(\mathbf{R};dy)$ if and only if $G \in L^1_\alpha(\mathbf{R};dx)$, where $V(y)$ is the potential in (2.2) and $G(x)$ is the function in (2.4).

PROOF: From (2.3) and (1.3) we obtain the identity

$$y = x - A_+ + \int_x^\infty [1-H] = x + A_- - \int_{-\infty}^x [1-H],$$

and hence we have

$$x - \beta \leq y \leq x + \beta$$

for a suitable $\beta \geq 0$. Then

$$1+|y| \leq \frac{1+|x|+\beta}{1+|x|}(1+|x|) \leq (1+\beta)(1+|x|),$$

$$1+|x| \leq \frac{1+|y|+\beta}{1+|y|}(1+|y|) \leq (1+\beta)(1+|y|),$$

so that

$$\int_{-\infty}^{\infty} dy\,(1+|y|)^\alpha |V(y)| \leq (1+\beta)^\alpha \int_{-\infty}^{\infty} dx\,(1+|x|)^\alpha |G(x)|,$$

$$\int_{-\infty}^{\infty} dx\,(1+|x|)^\alpha |G(x)| \leq (1+\beta)^\alpha \int_{-\infty}^{\infty} dy\,(1+|y|)^\alpha |V(y)|.$$

Hence the spaces $L^1_\alpha(\mathbf{R}; dx)$ and $L^1_\alpha(\mathbf{R}; dy)$ coincide. ∎

Using (4.2) and (4.4) in (4.1), we obtain

$$\mathbf{P}(-k,x)^{-1}\,m(-k,x) = e^{-iAk}\mathbf{q}\,\mathbf{P}(k,x)^{-1}\,m(k,x). \qquad k \in \mathbf{R}. \tag{4.5}$$

From (4.5) at $k=0$ we have

$$\mathbf{P}(-0,x)^{-1}\,m(0,x) = \mathbf{q}\,\mathbf{P}(0,x)^{-1}\,m(0,x),$$

and hence

$$\mathbf{P}(0,x)^{-1}\,m(0,x) = \mathbf{P}(0,x)^{-1}\,m^{[0]}(0,x) = c(x)\hat{1}, \tag{4.6}$$

where $\hat{1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $c(x)$ is a scalar function determined by $Q(x)$, $\mathbf{S}(k)$, and the bound state norming constants. It is possible to evaluate $c(x)$ explicitly. For simplicity, let us assume that there are no bound states. Let a denote an arbitrary real parameter in $[0,\infty]$. Let us also define $p_l(k,x)$ and $p_r(k,x)$ as in

$$\mathbf{P}(k,x)\hat{1} = \begin{bmatrix} p_l(k,x) \\ p_r(k,x) \end{bmatrix}.$$

Both in the generic and exceptional cases, $p_l(k,x)$ and $p_r(k,x)$ are well defined and correspond to the Faddeev functions from the left and from the right, respectively, for the scattering matrix $\mathbf{J}\Lambda(k,0)\mathbf{J}$.

Using (2.8), (2.9), Theorem 2.3 of [7], and (4.8) of [5], in the exceptional case we obtain

$$\mathbf{P}(0,x)^{-1} = \frac{p_l(0,x)\,p_r(0,x)}{p_l(0,x)^2+p_r(0,x)^2}\begin{bmatrix} \frac{1+p_r(0,x)^2}{p_r(0,x)} & \frac{1-p_l(0,x)^2}{p_l(0,x)} \\ \frac{1-p_r(0,x)^2}{p_r(0,x)} & \frac{1+p_l(0,x)^2}{p_l(0,x)} \end{bmatrix},$$

and after using $\frac{p_r(0,x)}{p_l(0,x)} = \frac{m_r^{[0]}(0,x)}{m_l^{[0]}(0,x)}\frac{1+R(0)}{T(0)}$, we find

$$\mathbf{P}(0,x)^{-1}\,m(0,x) = \frac{m_l^{[0]}(0,x)}{p_l(0,x)}\hat{1} = \frac{m_r^{[0]}(0,x)}{p_r(0,x)}\hat{1}, \tag{4.7}$$

and in the generic case we obtain

$$\mathbf{P}(0,x)^{-1} = \frac{1}{p_l(0,x) + a\,p_r(0,x)} \begin{bmatrix} 1 & a \\ 1 & a \end{bmatrix},$$

$$\mathbf{P}(0,x)^{-1}\, m(0,x) = \frac{m_l^{[0]}(0,x) + a\, m_r^{[0]}(0,x)}{p_l(0,x) + a\,p_r(0,x)} \hat{1}. \tag{4.8}$$

Hence, from (4.6) and (4.8) we see that

$$c(x) = \frac{m_l^{[0]}(0,x) + a\, m_r^{[0]}(0,x)}{p_l(0,x) + a\,p_r(0,x)}. \tag{4.9}$$

Note that both the numerator and the denominator in $c(x)$ are strictly positive [7]. In the generic case $c(x)$ depends on the parameter a whereas in the exceptional case $c(x)$ is independent of a because in the generic case the Faddeev functions from the left and from the right are linearly independent at $k = 0$ whereas in the exceptional case they are linearly dependent at $k = 0$.

Let us define

$$r(k,x) = \frac{i}{k}\frac{1}{c(x)}[\mathbf{P}(k,x)^{-1}\, m(k,x) - \mathbf{P}(0,x)^{-1}\, m^{[0]}(0,x)]. \tag{4.10}$$

We can then write (4.1) as

$$-\,r(-k,x) = e^{-iAk}\mathbf{q}\, r(k,x) + \frac{i}{k}[e^{-iAk} - 1]\hat{1}, \qquad k \in \mathbf{R}. \tag{4.11}$$

Note that for $\pm A \geq 0$ we have

$$\int_0^{\pm A} dt\, e^{-ikt} = \frac{i}{k}[e^{\mp iAk} - 1]. \tag{4.12}$$

If $H(x) \leq 1$, in which case $A \geq 0$, we have the vector $r(\cdot, x)$ belonging to the Hardy space $\mathbf{H}_2^+(\mathbf{R})$ and hence its Fourier transform has support on the half axis, namely,

$$r(k,x) = \int_0^\infty dt\, \zeta(t,x)\, e^{ikt}. \tag{4.13}$$

Using (4.12) and (4.13) in (4.11) we also see that $\zeta(t,x) = 0$ when $t > A$, and hence

$$\zeta(t,x) = \begin{bmatrix} \omega(t,x) \\ -\omega(A-t,x) - 1 \end{bmatrix}, \tag{4.14}$$

where $\omega(\cdot, x) \in L^2(0,A)$ and is otherwise arbitrary. In terms of $\omega(t,x)$ and $\mathbf{P}(k,x)$, from (4.7), (4.10), and (4.12) we then obtain

$$m(k,x) = c(x)\,\mathbf{P}(k,x) \left\{ \frac{k}{i} \int_0^A dt \begin{bmatrix} \omega(t,x) \\ -\omega(A-t,x) \end{bmatrix} e^{ikt} + \begin{bmatrix} 1 \\ e^{ikA} \end{bmatrix} \right\}. \tag{4.15}$$

If $H(x) \geq 1$, in which case $A \leq 0$, we have $e^{-ikA} r(\cdot, x) \in \mathbf{H}_2^+(\mathbf{R})$ and hence its Fourier transform has support on the half line, namely,

$$s(k,x) = e^{-ikA} r(k,x) = \int_0^\infty dt\, \xi(t,x)\, e^{ikt}, \tag{4.16}$$

In this case, from (4.11) we obtain

$$-s(-k,x) = e^{iAk} \mathbf{q}\, s(k,x) + \frac{i}{k}[1 - e^{iAk}]\hat{1}, \qquad k \in \mathbf{R}. \tag{4.17}$$

Using (4.12) and (4.16) in (4.17), we see that $\xi(t,x) = 0$ when $t > -A$, and

$$\xi(t,x) = \begin{bmatrix} \varsigma(t,x) \\ -\varsigma(-A-t,x)+1 \end{bmatrix},$$

where $\varsigma(\cdot,x) \in L^2(0,-A)$ and is otherwise arbitrary. In terms of $\varsigma(t,x)$ and $\mathbf{P}(k,x)$, from (4.10) and (4.16) we obtain

$$m(k,x) = c(x)\,\mathbf{P}(k,x) \left\{ \frac{k}{i} e^{ikA} \int_0^{-A} dt \begin{bmatrix} \varsigma(t,x) \\ -\varsigma(-A-t,x) \end{bmatrix} e^{ikt} + \begin{bmatrix} 1 \\ e^{ikA} \end{bmatrix} \right\},$$

or equivalently

$$m(k,x) = c(x)\,\mathbf{P}(k,x) \left\{ \frac{k}{i} \int_0^{-A} dt \begin{bmatrix} \varsigma(-A-t,x) \\ -\varsigma(t,x) \end{bmatrix} e^{-ikt} + \begin{bmatrix} 1 \\ e^{ikA} \end{bmatrix} \right\}. \tag{4.18}$$

If $1 - H(x)$ has mixed sign, then one can modify the above procedure by using an analog of the method outlined in Section 10 of [**8**].

As seen from (4.15) and (4.18), the general solution of (4.1) by the above method is not unique; in order to solve the inverse scattering problem of the recovery of $H(x)$, among all the solutions of (4.1), either we must pick the one that will give us the Faddeev functions of (1.1) or we need to find appropriate restrictions on the solutions of (4.1) so that they will lead to $H(x)$. Currently, we are working on this problem.

Next we will illustrate the method outlined in this section on the example used in Section 3 with the scattering matrix given in (3.2). Corresponding to the potential $Q(x)$ in (3.1), we have

$$\mathbf{S}^{[0]}(k) = \begin{bmatrix} \frac{k+i/2}{k+i} & -\frac{\sqrt{3}i/2}{k+i}\frac{k+i/2}{k-i/2} \\ -\frac{\sqrt{3}i/2}{k+i} & \frac{k+i/2}{k+i} \end{bmatrix}.$$

Solving the Riemann-Hilbert problem (4.3) by the method of [**1,7**], we obtain

$$\mathbf{P}(k,x) = \begin{bmatrix} 1 & 0 \\ \frac{\sqrt{3}i}{k+i} e^{2ikx+3ik+\sqrt{3}ik} & \frac{k+2i}{k+i} \end{bmatrix}, \qquad x \geq -\frac{1}{2}(3+\sqrt{3}).$$

Thus from (4.7) we have

$$c(x) = \frac{\sqrt{3}e^x - 1}{\sqrt{3}e^x + 1}, \qquad x \geq 0.$$

In this example, the Faddeev functions for (1.1) are given by

$$m_l(k,x) = \frac{\sqrt{3}e^x - 1}{\sqrt{3}e^x + 1} e^{-\frac{4ik}{\sqrt{3}e^x - 1}}, \qquad x \geq 0,$$

$$m_r(k,x) = \frac{k+2i}{k+i}\frac{\sqrt{3}e^x - 1}{\sqrt{3}e^x + 1} e^{-ik(1+3\sqrt{3})} e^{4ik/(\sqrt{3}e^x-1)}$$
$$+ \frac{\sqrt{3}i}{k+i}\frac{\sqrt{3}e^x - 1}{\sqrt{3}e^x + 1} e^{ik(\sqrt{3}+3)} e^{2ikx} e^{-4ik/(\sqrt{3}e^x-1)}, \qquad x \geq 0.$$

The vector in (4.10) is given by

$$r(k,x) = \frac{i}{k}\begin{bmatrix} e^{-4ik/(\sqrt{3}e^x-1)} - 1 \\ e^{4ik/(\sqrt{3}e^x-1)} e^{-ik(1+3\sqrt{3})} - 1 \end{bmatrix}.$$

From (4.13) and (4.14), we then obtain

$$\varsigma(t,x) = \theta(t - 1 - 3\sqrt{3}) - \theta(t + \frac{4}{\sqrt{3}e^x - 1} - 1 - 3\sqrt{3}), \qquad x \geq 0. \qquad (4.19)$$

Although an arbitrary $\varsigma(t,x)$ leads us to a solution of (4.11), the solution of (4.11) that gives us the Faddeev function for (1.1) through (4.18) must be chosen with $\varsigma(t,x)$ as in (4.19). The quantities corresponding to negative values of x can be obtained in a similar way, but they will not be listed here.

Acknowledgment. The research leading to this article was supported in part by the National Science Foundation under grant DMS-9217627.

References

1. T. Aktosun, *Perturbations and stability of the Marchenko inversion method*, thesis, Indiana University, Bloomington, unpublished (1986).
2. T. Aktosun, *Marchenko inversion for perturbations: I*, Inverse Problems **3**, 523–553 (1987).
3. T. Aktosun, *Scattering and inverse scattering for a second order differential equation*, J. Math. Phys. **34**, 1619–1634 (1993).
4. T. Aktosun, M. Klaus, and C. van der Mee, *Inverse scattering in 1-D nonhomogeneous media and recovery of the wavespeed*, J. Math. Phys. **33**, 1395–1402 (1992).
5. T. Aktosun, M. Klaus, and C. van der Mee, *Scattering and inverse scattering in one-dimensional nonhomogeneous media*, J. Math. Phys. **33**, 1717–1744 (1992).

6. T. Aktosun, M. Klaus, and C. van der Mee, *Explicit Wiener-Hopf factorization for certain non-rational matrix functions*, Integr. Equat. Oper. Th. **15**, 879–900 (1992).
7. T. Aktosun, M. Klaus, and C. van der Mee, *On the Riemann-Hilbert problem for the one-dimensional Schrödinger equation*, J. Math. Phys., in press (1993).
8. T. Aktosun and C. van der Mee, *Scattering and inverse scattering for the 1-D Schrödinger equation with energy-dependent potentials,* J. Math. Phys. **32**, 2786–2801 (1991).
9. F. D. Gakhov, *Boundary Value Problems* (Pergamon, Oxford, 1966) [Fizmatgiz, Moscow, 1963 (Russian)].
10. N. I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, 1953) [Nauka, Moscow, 1946 (Russian)].
11. R. G. Newton, *Inverse scattering. I.*, J. Math. Phys. **21**, 493–505 (1980).