On the Riemann–Hilbert problem for the one-dimensional Schrödinger equation

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A matrix Riemann–Hilbert problem associated with the one-dimensional Schrödinger equation is considered, and the existence and uniqueness of its solutions are studied. The solution of this Riemann–Hilbert problem yields the solution of the inverse scattering problem for a larger class of potentials than the usual Faddeev class. Some examples of explicit solutions of the Riemann–Hilbert problem are given, and the connection with ambiguities in the inverse scattering problem is established.

I. INTRODUCTION

Consider the Schrödinger equation

$$\psi''(k,x)+k^2\psi(k,x)=V(x)\psi(k,x), \tag{1.1}$$

where $x\in\mathbf{R}$ is the space coordinate and k^2 is energy. Unless otherwise stated, we assume that $V(x)$ is a real-valued potential in L_1^1, where $L_\mu^1=\{V\,|\,\int_{-\infty}^{\infty}(1+|x|)^\mu|V(x)|dx<\infty\}$.

There are two linearly independent solutions ψ_l and ψ_r of Eq. (1.1), called the physical solutions from the left and from the right, respectively, such that

$$\psi_l(k,x)=\begin{cases}T(k)e^{ikx}+o(1), & x\to+\infty\\ e^{ikx}+L(k)e^{-ikx}+o(1), & x\to-\infty,\end{cases} \tag{1.2}$$

$$\psi_r(k,x)=\begin{cases}e^{-ikx}+R(k)e^{ikx}+o(1), & x\to+\infty\\ T(k)e^{-ikx}+o(1), & x\to-\infty.\end{cases} \tag{1.3}$$

Here $T(k)$ is the transmission coefficient, and $L(k)$ and $R(k)$ are the reflection coefficients from the left and from the right, respectively. The scattering matrix $\mathbf{S}(k)$ is defined as

$$\mathbf{S}(k)=\begin{bmatrix}T(k) & R(k)\\ L(k) & T(k)\end{bmatrix}.$$

For $k\in\mathbf{R}$ the pairs $\{\psi_l(k,x),\psi_r(k,x)\}$ and $\{\psi_l(-k,x),\psi_r(-k,x)\}$ form linearly independent sets of solutions of Eq. (1.1). Furthermore, $\psi_l(-k,x)=\overline{\psi_l(k,x)}$ and $\psi_r(-k,x)=\overline{\psi_r(k,x)}$ when $k\in\mathbf{R}$, where a bar denotes complex conjugation, and we have[1]

$$\begin{bmatrix}\psi_r(k,x)\\ \psi_l(k,x)\end{bmatrix}=\mathbf{S}(k)\begin{bmatrix}\psi_l(-k,x)\\ \psi_r(-k,x)\end{bmatrix},\quad k\in\mathbf{R}. \tag{1.4}$$

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Define $m_l(k,x)=[1/T(k)]e^{-ikx}\psi_l(k,x)$ and $m_r(k,x)=[1/T(k)]e^{ikx}\psi_r(k,x)$. Then the functions $m_l(k,x)$ and $m_r(k,x)$ satisfy the equations

$$m_l''(k,x)+2ikm_l'(k,x)=V(x)m_l(k,x), \tag{1.5}$$

$$m_r''(k,x)-2ikm_r'(k,x)=V(x)m_r(k,x). \tag{1.6}$$

Let $\mathbf{J}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ and $m(k,x)=\begin{bmatrix}m_l(k,x)\\m_r(k,x)\end{bmatrix}$. Then we can write Eqs. (1.5) and (1.6) as a vector equation

$$m''(k,x)+2ik\mathbf{J}m'(k,x)=V(x)m(k,x). \tag{1.7}$$

Let $\mathbf{q}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$. Then Eq. (1.4) is equivalent to

$$m(-k,x)=\mathbf{G}(k,x)\mathbf{q}m(k,x), \quad k\in\mathbf{R}, \tag{1.8}$$

where

$$\mathbf{G}(k,x)=\begin{bmatrix}T(k) & -R(k)e^{2ikx}\\ -L(k)e^{-2ikx} & T(k)\end{bmatrix}.$$

It is known[2] that for potentials in L_1^1 the functions $m_l(k,x)$ and $m_r(k,x)$ are continuous in k in $\mathbf{R}$, can be extended analytically in k to $\mathbf{C}^+$, and $m_l(k,x)\to 1$ and $m_r(k,x)\to 1$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Here $\mathbf{C}^+$ denotes the upper-half complex plane and $\overline{\mathbf{C}^+}$ denotes its closure. Similarly, $\mathbf{C}^-$ and $\overline{\mathbf{C}^-}$ denote the lower-half complex plane and its closure, respectively. Thus $m(-k,x)$ can be extended analytically in k to $\mathbf{C}^-$. Hence, when $\mathbf{S}(k)$ is given, solving Eq. (1.8) for $m(k,x)$ becomes a Riemann–Hilbert problem. Once Eq. (1.8) is solved, the potential can be obtained from Eqs. (1.5) or (1.6) using

$$V(x)=\frac{m_l''(k,x)+2ikm_l'(k,x)}{m_l(k,x)}=\frac{m_r''(k,x)-2ikm_r'(k,x)}{m_r(k,x)}. \tag{1.9}$$

Thus, solving the Riemann–Hilbert problem (1.8) amounts to solving the inverse scattering problem for Eq. (1.1), namely, the recovery of the potential from the scattering data.

Associated with the scattering matrix $\mathbf{S}(k)$ we have the matrix

$$\mathbf{JS}(k)\mathbf{J}=\begin{bmatrix}T(k) & -R(k)\\ -L(k) & T(k)\end{bmatrix}. \tag{1.10}$$

Consider the vector Riemann–Hilbert problem associated with $\mathbf{JG}(k,x)\mathbf{J}$

$$n(-k,x)=\mathbf{JG}(k,x)\mathbf{Jq}n(k,x), \quad k\in\mathbf{R}, \tag{1.11}$$

where the solution vector $n(k,x)=\begin{bmatrix}n_l(k,x)\\n_r(k,x)\end{bmatrix}$ is sought such that, for each fixed $x\in\mathbf{R}$, $n(k,x)$ is continuous in k in $\mathbf{R}\setminus\{0\}$, can be extended analytically in k to $\mathbf{C}^+$, $n_l(k,x)\to 1$ and $n_r(k,x)\to 1$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. The exact behavior of $n(k,x)$ at $k=0$ depends on the scattering matrix $\mathbf{S}(k)$ and will be specified below. It will be shown that there always exists a potential U whose scattering matrix is $\mathbf{JS}(k)\mathbf{J}$, but U will "generically" be nonunique. The term "generic" will be made precise below. In analogy to Eq. (1.7), $n(k,x)$ obeys

$$n''(k,x)+2ik\mathbf{J}n'(k,x)=U(x)n(k,x). \tag{1.12}$$

Let $\phi_l(k,x)$ and $\phi_r(k,x)$ denote the physical solutions associated with the potential U; that is, $\phi_l(k,x)=T(k)e^{ikx}n_l(k,x)$ and $\phi_r(k,x)=T(k)e^{-ikx}n_r(k,x)$. Then from Eqs. (1.2), (1.3), and (1.10) we see that

$$\phi_l(k,x)=\begin{cases} T(k)e^{ikx}+o(1), & x\to+\infty \\ e^{ikx}-L(k)e^{-ikx}+o(1), & x\to-\infty, \end{cases} \tag{1.13}$$

$$\phi_r(k,x)=\begin{cases} e^{-ikx}-R(k)e^{ikx}+o(1), & x\to+\infty \\ T(k)e^{-ikx}+o(1), & x\to-\infty. \end{cases} \tag{1.14}$$

Define the 2×2 matrix $\mathbf{M}(k,x)$ by[3,4]

$$\mathbf{M}(k,x)=\frac{1}{2}\begin{bmatrix} m_l(k,x)+n_l(k,x) & m_l(k,x)-n_l(k,x) \\ m_r(k,x)-n_r(k,x) & m_r(k,x)+n_r(k,x) \end{bmatrix}. \tag{1.15}$$

Let

$$\hat{1}=\begin{bmatrix}1\\1\end{bmatrix} \quad\text{and}\quad \hat{e}=\begin{bmatrix}1\\-1\end{bmatrix}.$$

We then have

$$m(k,x)=\mathbf{M}(k,x)\hat{1}, \tag{1.16}$$

$$n(k,x)=\mathbf{JM}(k,x)\hat{e}. \tag{1.17}$$

We can combine Eqs. (1.8) and (1.11) into the matrix Riemann–Hilbert problem

$$\mathbf{M}(-k,x)=\mathbf{G}(k,x)\mathbf{q}\mathbf{M}(k,x)\mathbf{q}, \quad k\in\mathbf{R}, \tag{1.18}$$

where $\mathbf{M}(k,x)$ is continuous for $k\in\mathbf{R}\setminus\{0\}$ and has an analytic extension in k to $\mathbf{C}^+$, and $\mathbf{M}(k,x)\to\mathbf{I}$, the identity matrix, as $k\to\infty$ in $\overline{\mathbf{C}^+}$ for each x. Note that Eq. (1.18) is a generalization of the standard Riemann–Hilbert problem in the sense that we do not require $\mathbf{M}(k,x)$ to be continuous at $k=0$. The behavior of $\mathbf{M}(k,x)$ at $k=0$ depends on that of $T(k)$. In this article we only consider transmission coefficients for which the following dichotomy holds: (i) $T(k)=ick+o(k)$ as $k\to 0$ where c is a real nonzero constant, or (ii) $T(k)\to T(0)\neq 0$. For $V\in L_1^1$ it is well-known that this dichotomy holds.[2,5] We will refer to case (i) as the generic case and to case (ii) as the exceptional case. Then $R(0)=L(0)=-1$ in the generic case and $|R(0)|=|L(0)|<1$ in the exceptional case. We will use the terms "generic" and "exceptional" also for the potentials U if the associated potential $V(x)$ has the corresponding property. Note that in the generic case $U\notin L_1^1$ because at $k=0$ the reflection coefficients for U have the "wrong" value $+1$ instead of -1. In the exceptional case $U\in L_1^1$ if and only if $V\in L_1^1$.

The inverse scattering problem can be formulated as a matrix Riemann–Hilbert problem in the form of Eq. (1.18) or in related forms.[1,4,6,7] In this article we show that the solution of Eq. (1.18) leads to the solutions of the inverse scattering problems for the scattering matrices $\mathbf{S}(k)$ and $\mathbf{JS}(k)\mathbf{J}$ and that the solutions of these two inverse problems satisfy Newton's miracle condition,[1] namely, the potentials obtained using the solutions from the left and from the right are the same. The solution of the matrix Riemann–Hilbert problem (1.18) allows us to obtain the solution of the inverse scattering problem for a larger class of potentials than the usual Faddeev class[2] of potentials belonging to L_1^1. Since the solution of Eq. (1.18) can be singular at $k=0$ and since we merely require $\mathbf{M}(k,x)\to\mathbf{I}$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$ and not $\mathbf{M}(k,x)-\mathbf{I}\in L^2(\mathbf{R})$, we do not use a Fourier transform in k together with Marchenko integral equations. Instead, our approach relies on the Darboux transformation which allows us to relate the solutions of Eq. (1.7) to those of Eq. (1.12). Then we obtain the solution of the matrix Riemann–Hilbert problem (1.18) in terms of the solutions of the Schrödinger equation (1.1). The Darboux transformation was used by Degasperis and Sabatier[8] in the absence of bound states and when $V\in L^1_m$ with $m\geqslant 2$ in order to construct a one-parameter family of potentials U and their wave functions corresponding to the scattering matrix $\mathbf{JS}(k)\mathbf{J}$. Examples of such families were also obtained by other methods.[3,7,9] One can ask whether the potentials obtained previously represent all possible solutions of the inverse scattering problem for the scattering matrix $\mathbf{JS}(k)\mathbf{J}$. Here we show that the answer is in the affirmative, provided the associated solution of Eq. (1.18) obeys certain restrictions. In the process we extend the analysis of Degasperis and Sabatier to the case when $V\in L^1_1$ and also when bound states are present. As a further by-product of our analysis we find that the bound state norming constants for a potential $V\in L^1_1$ whose support is contained in a half line; i.e., $V(x)=0$ for $x>a_1$ or $x<a_2$, are already determined by the scattering matrix, and that in solving the inverse scattering problem for such a potential the norming constants cannot be specified arbitrarily. This answers a question raised by P. Sacks (private communication to T. Aktosun).

This article is organized as follows. In Sec. II, when $V\in L^1_1$, we study the Darboux transformation and the small k behavior of the solution $\mathbf{M}(k,x)$ of Eq. (1.18), and thus in particular we analyze the behavior of the solutions of Eq. (1.12) at $k=0$. In Sec. III we characterize the class of potentials to which U belongs whenever $V\in L^1_1$ and Eq. (1.1) has no bound states; in Sec. IV we study the case when there are bound states and also show how bound states can be added to and removed from the potential U. In Sec. V we analyze the Riemann–Hilbert problem (1.18) and establish the connection of its solutions with the solutions of the Schrödinger equations (1.7) and (1.12); in Sec. VI this analysis is extended to include the bound states. In Sec. VII the Wiener–Hopf factorization of $\mathbf{G}(k,x)$ is given in terms of the solution of the Riemann–Hilbert problem (1.18). Finally, in Sec. VIII we give some examples of explicit solutions of the Riemann–Hilbert problem (1.18) and the potentials obtained from those solutions.

II. DARBOUX TRANSFORMATION AND PROPERTIES OF *M(k,x)*

In this section we study the Darboux transformation relevant to the Riemann–Hilbert problem (1.18) and also obtain some estimates on the solutions of the Schrödinger equations (1.7) and (1.12) as $k\to 0$. It is known[2,10] that $m_l(k,x)$ and $m_r(k,x)$ satisfy

$$m_l(k,x)=1+\frac{1}{2ik}\int_x^\infty [e^{2ik(y-x)}-1]\,V(y)\,m_l(k,y)\,dy, \tag{2.1}$$

$$m_r(k,x)=1+\frac{1}{2ik}\int_{-\infty}^x [e^{2ik(x-y)}-1]\,V(y)\,m_r(k,y)\,dy. \tag{2.2}$$

Proposition 2.1: Let $V\in L^1_1$ and $k\in\overline{\mathbf{C}^+}$. Set $\sigma_\pm(x)=\pm\int_x^{\pm\infty}(1+|y|)\,|V(y)|\,dy$. With constants C_1, C_2, C_3, C_4 independent of k, we have

$$\text{(i)}\quad |m_l(k,x)-1|\leqslant C_1\sigma_+(x)\,\frac{1+\max\{-x,0\}}{1+|k|},\quad |m_r(k,x)-1|\leqslant C_2\sigma_-(x)\,\frac{1+\max\{x,0\}}{1+|k|},$$

$$\text{(ii)}\quad |m_l'(k,x)|\leqslant C_3\int_x^\infty |V(y)|\,dy,\quad x\geqslant 0,\quad |m_r'(k,x)|\leqslant C_4\int_{-\infty}^x |V(y)|\,dy,\quad x\leqslant 0,$$

$$\text{(iii)}\quad |m_l'(k,x)|,|m_r'(k,x)|\to 0\quad\text{as}\quad k\to\infty\quad\text{in}\quad \overline{C^+},\quad\text{uniformly in}\quad x.$$

Proof: It suffices to consider $m_l(k,x)$ because the proof for $m_r(k,x)$ is similar. We use the order symbols O and o in the limit as $k\to\infty$ in $\overline{\mathbf{C}^+}$ when the estimates are valid uniformly in x. The proof of (i) is given in Ref. 10 (Lemma 1, p. 130). To prove (ii) note that by Eq. (2.1)

$$m_l'(k,x)=-\int_x^\infty e^{2ik(y-x)}V(y)m_l(k,y)dy=I_1(k,x)+I_2(k,x),$$

where

$$I_1(k,x)=-\int_x^\infty e^{2ik(y-x)}V(y)[m_l(k,y)-1]dy, \tag{2.3}$$

$$I_2(k,x)=-\int_x^\infty e^{2ik(y-x)}V(y)dy.$$

Using (i), for $x\geqslant 0$ we get

$$|I_1(k,x)|\leqslant C_1\frac{\sigma_+(x)}{1+|k|}\int_x^\infty |V(y)|dy,$$

$$|I_2(k,x)|\leqslant \int_x^\infty |V(y)|dy.$$

Hence (ii) follows.

Using (i) in Eq. (2.3), it follows that $I_1(k,x)=O(1/k)$ for $x\geqslant 0$; hence in order to prove (iii), it suffices to show that $I_2(k,x)=o(1)$. Given any $\epsilon>0$, let $\widetilde{V}\in C_0^\infty$ be such that $\|V-\widetilde{V}\|_{L^1}<\epsilon$; thus $\widetilde{V}\in L^1$. Then

$$I_2(k,x)=I_3(k,x)+I_4(k,x),$$

where

$$I_3(k,x)=-\int_x^\infty e^{2ik(y-x)}[V(y)-\widetilde{V}(y)]dy,$$

$$I_4(k,x)=-\int_x^\infty e^{2ik(y-x)}\widetilde{V}(y)dy. \tag{2.4}$$

Thus, for $k\in\overline{\mathbf{C}^+}$, we have $|I_3(k,x)|\leqslant\epsilon$. From Eq. (2.4) using integration by parts we obtain

$$I_4(k,x)=\frac{1}{2ik}\widetilde{V}(x)+\int_x^\infty \frac{e^{2ik(y-x)}}{2ik}\widetilde{V}'(y)dy,$$

and since $\widetilde{V}\in C_0^\infty$, it follows that $I_4(k,x)=O(1/k)$ as $k\to\infty$. Since $\epsilon>0$ is arbitrary, assertion (iii) follows. ■

Note that the decay in k of the integral I_2 above can be arbitrarily slow. For example, let $x=0$ and $V(y)=y^{-\epsilon}[H(y)-H(y-1)]$ with $0<\epsilon<1$ where $H(y)$ is the Heaviside function. Then $V\in L_1^1$ and $I_2(k,0)\sim c_\epsilon k^{\epsilon-1}$ as $k\to\infty$, where $c_\epsilon=-\int_0^\infty e^{2iu}u^{-\epsilon}du$. For this reason we could not rely on the estimates for $m_l'(k,x)$ stated in Ref. 10 [Lemma 1, (iii) and (iv), p. 130].

For $k=0$ the Schrödinger equation (1.1) reduces to

$$\psi''(k,x)=V(x)\psi(k,x), \tag{2.5}$$

and Eqs. (2.1) and (2.2) become

$$m_l(0,x)=1+\int_x^\infty (y-x)V(y)m_l(0,y)dy, \tag{2.6}$$

$$m_r(0,x)=1+\int_{-\infty}^x (x-y)V(y)m_r(0,y)dy, \tag{2.7}$$

respectively. Moreover,

$$m_l'(0,x)=-\int_x^\infty V(y)m_l(0,y)dy, \tag{2.8}$$

$$m_r'(0,x)=\int_{-\infty}^x V(y)m_r(0,y)dy. \tag{2.9}$$

From Eqs. (2.6)–(2.9), we obtain the relations

$$m_l(0,x)=\begin{cases}1+o(1), & x\to+\infty\\ -c_lx+o(x), & x\to-\infty,\end{cases} \tag{2.10}$$

$$m_l'(0,x)=\begin{cases}o(1/x), & x\to+\infty\\ -c_l+o(1), & x\to-\infty,\end{cases} \tag{2.11}$$

$$m_r(0,x)=\begin{cases}c_rx+o(x), & x\to+\infty\\ 1+o(1), & x\to-\infty,\end{cases} \tag{2.12}$$

$$m_r'(0,x)=\begin{cases}c_r+o(1), & x\to+\infty\\ o(1/x), & x\to-\infty,\end{cases} \tag{2.13}$$

where

$$c_l=\int_{-\infty}^\infty V(y)m_l(0,y)dy \quad\text{and}\quad c_r=\int_{-\infty}^\infty V(y)m_r(0,y)dy. \tag{2.14}$$

Let $[f;g]=fg'-f'g$ denote the Wronskian. In the absence of bound states $m_l(0,x)$ and $m_r(0,x)$ are positive,[8,10] and thus we obtain

$$[m_l(0,x);m_r(0,x)]=c_l=c_r\geqslant 0. \tag{2.15}$$

Hence, $c_l=c_r=0$ if and only if $m_l(0,x)$ and $m_r(0,x)$ are linearly dependent, and by Eqs. (2.10) and (2.12) it is seen that this happens if and only if $m_l(0,x)$ and $m_r(0,x)$ are bounded. If a zero-energy solution of the Schrödinger equation is bounded but not in L^2, it is called a half bound state. It is known[2,10] that for $V\in L_1^1$ a half bound state occurs if and only if $V(x)$ is exceptional, which follows from the Wronskian

$$[f_l(k,x);f_r(k,x)]=-\frac{2ik}{T(k)} \tag{2.16}$$

by letting $k\to 0$, where $f_l(k,x)$ is the Jost solution of Eq. (1.1) from the left and $f_r(k,x)$ is that from the right. Recall[1] that the Jost solutions of Eq. (1.1) satisfy $\lim_{x\to+\infty} e^{-ikx}f_l(k,x)=1$ and $\lim_{x\to-\infty} e^{ikx}f_r(k,x)=1$, and they are related to the solutions of Eqs. (1.5) and (1.6) as

$$f_l(k,x)=e^{ikx}m_l(k,x) \quad \text{and} \quad f_r(k,x)=e^{-ikx}m_r(k,x). \tag{2.17}$$

In the generic case we define a family of solutions of Eq. (2.5) depending on a nonnegative parameter a by

$$\chi(x;a)=\begin{cases} m_l(0,x)+am_r(0,x), & 0\leqslant a<\infty \\ m_r(0,x), & a=\infty. \end{cases} \tag{2.18}$$

From Eqs. (2.10), (2.12), and the fact that $m_l(0,x)$ and $m_r(0,x)$ are positive, it follows that $\chi(x;a)>0$ and hence the set $\{\psi \mid \psi=c\chi(\cdot;a),\ c>0,\ 0\leqslant a\leqslant\infty\}$ represents all positive solutions of Eq. (2.5). Let

$$\rho(x;a)=\frac{\chi'(x;a)}{\chi(x;a)}. \tag{2.19}$$

Then from Eq. (2.5) it follows that $\rho(x;a)$ obeys the Riccati equation

$$\rho'(x;a)+\rho(x;a)^2=V(x). \tag{2.20}$$

Now let $\psi(k,x)$ be an arbitrary solution of Eq. (1.1) and define

$$\varphi(k,x;a)=\psi'(k,x)-\rho(x;a)\psi(k,x). \tag{2.21}$$

Then φ obeys the differential equation

$$\varphi''(k,x;a)+k^2\varphi(k,x;a)=U(x;a)\varphi(k,x;a), \tag{2.22}$$

where

$$U(x;a)=-\rho'(x;a)+\rho(x;a)^2. \tag{2.23}$$

From Eqs. (2.20) and (2.23), we have

$$U(x;a)=V(x)-2\rho'(x;a), \tag{2.24}$$

$$U(x;a)=2\rho(x;a)^2-V(x). \tag{2.25}$$

We will refer to the transformation from the potential $V(x)$ to the potential $U(x;a)$ by means of Eq. (2.24) as a Darboux transformation; note that this was called the limit Darboux transformation in Ref. 8. In the exceptional case instead of Eqs. (2.18) and (2.19), we define $\chi(x)=m_l(0,x)$ and $\rho(x) = m_l'(0,x)/m_l(0,x)$ because $m_l(0,x)$ and $m_r(0,x)$ are linearly dependent and thus the parameter a does not appear in Eqs. (2.18) and (2.19); the Darboux transform of $V(x)$, which is then unique, will be denoted by $U(x)$.

Proposition 2.2: In the generic case, for $\rho(x;a)$ defined in Eq. (2.19), we have

(i) if $0<a<\infty$ then $\rho(x;a)=1/x+o(1/x)$ as $x\to\pm\infty$,

(ii) if $a=0$ then

$$\rho(x;0)=\begin{cases} o(1/x), & x\to+\infty \\ 1/x+o(1/x), & x\to-\infty, \end{cases}$$

(iii) if $a=\infty$ then

$$\rho(x;\infty)=\begin{cases}1/x+o(1/x), & x\to+\infty\\ o(1/x), & x\to-\infty.\end{cases}$$

(iv) In the exceptional case, we have $\rho(x)=o(1/x)$ as $x\to\pm\infty$.

Proof: The results follow from Eqs. (2.10)–(2.13) and the definitions of $\rho(x;a)$ and $\rho(x)$. ■

Applying the Darboux transformation to the Jost solutions $f_l(k,x)$ and $f_r(k,x)$ defined in Eq. (2.17), we obtain

$$g_l(k,x;a)=\frac{1}{ik}\left[f_l'(k,x)-\rho(x;a)f_l(k,x)\right], \tag{2.26}$$

$$g_r(k,x;a)=-\frac{1}{ik}\left[f_r'(k,x)-\rho(x;a)f_r(k,x)\right]. \tag{2.27}$$

Theorem 2.3: The functions $g_l(k,x;a)$ and $g_r(k,x;a)$ given by Eqs. (2.26) and (2.27) are the Jost solutions of the Schrödinger equation with the potential $U(x;a)$; equivalently, $\phi_l(k,x;a)=T(k)g_l(k,x;a)$ and $\phi_r(k,x;a)=T(k)g_r(k,x;a)$ are the physical solutions of Eq. (2.22) satisfying Eqs. (1.13) and (1.14), and $n_l(k,x;a)=e^{-ikx}g_l(k,x;a)$ and $n_r(k,x;a)=e^{ikx}g_r(k,x;a)$ are solutions of Eq. (1.12) with the potential $U(x;a)$.

Proof: By straightforward verification using Proposition 2.2. ■

From Eqs. (2.26) and (2.27) we obtain

$$n_l(k,x;a)=\frac{1}{ik}\left[m_l'(k,x)+ikm_l(k,x)-\rho(x;a)m_l(k,x)\right], \tag{2.28}$$

$$n_r(k,x;a)=-\frac{1}{ik}\left[m_r'(k,x)-ikm_r(k,x)-\rho(x;a)m_r(k,x)\right]. \tag{2.29}$$

The next theorem gives the behavior of $n_l(k,x;a)$ and $n_r(k,x;a)$ at $k=0$.

Theorem 2.4: Let $n_l(k,x;a)$ and $n_r(k,x;a)$ be the functions defined in Eqs. (2.28) and (2.29). In the generic case as $k\to0$ in $\overline{\mathbf{C}^+}$, for $0<a<\infty$ we have

$$\lim_{k\to0} ikn_l(k,x;a)=-\frac{ac_r}{m_l(0,x)+am_r(0,x)}, \tag{2.30}$$

$$\lim_{k\to0} ikn_r(k,x;a)=-\frac{c_r}{m_l(0,x)+am_r(0,x)}, \tag{2.31}$$

for $a=0$ we have

$$\lim_{k\to0} n_l(k,x;a)=\frac{1}{m_l(0,x)}, \tag{2.32}$$

$$\lim_{k\to0} ikn_r(k,x;a)=-\frac{c_r}{m_l(0,x)}, \tag{2.33}$$

and for $a=\infty$ we have

$$\lim_{k\to 0} ikn_l(k,x;a) = -\frac{c_r}{m_r(0,x)}, \tag{2.34}$$

$$\lim_{k\to 0} n_r(k,x;a) = \frac{1}{m_r(0,x)}, \tag{2.35}$$

where c_r is the constant defined in Eq. (2.14). In the exceptional case, we have

$$\lim_{k\to 0} n_l(k,x) = \frac{1}{m_l(0,x)} \quad \text{and} \quad \lim_{k\to 0} n_r(k,x) = \frac{1}{m_r(0,x)}. \tag{2.36}$$

Proof: The proof of Eq. (2.30) is obtained by letting $k\to 0$ in Eq. (2.28) and by using Eqs. (2.15), (2.18), and (2.19). Similarly Eq. (2.31) is obtained from Eq. (2.29). From Eq. (1.5) we have

$$[m_l(k,x)m_l'(0,x) - m_l(0,x)m_l'(k,x)]' = 2ikm_l'(k,x)m_l(0,x),$$

and by Proposition 2.1 (ii), $m_l'(k,x)$ is absolutely integrable near $x=+\infty$, and hence

$$m_l(0,x)m_l'(k,x) - m_l(k,x)m_l'(0,x) = 2ik\int_x^\infty m_l'(k,y)m_l(0,y)dy. \tag{2.37}$$

Using Lebesgue's dominated convergence theorem, from Eq. (2.37) in the limit $k\to 0$, we obtain

$$m_l(0,x)m_l'(k,x) - m_l(k,x)m_l'(0,x) = ik[1-m_l(0,x)^2] + o(k). \tag{2.38}$$

Then from Eqs. (2.37), (2.38), and the fact that $\rho(x;0) = m_l'(0,x)/m_l(0,x)$, we get Eq. (2.32). The limits in Eqs. (2.33) and (2.34) follow from Eqs. (2.15), (2.18), (2.28), and (2.29). The proof of Eq. (2.35) is analogous to that of Eq. (2.32), where, instead of Eq. (2.37), we use

$$m_r(0,x)m_r'(k,x) - m_r(k,x)m_r'(0,x) = 2ik\int_{-\infty}^x m_r'(k,y)m_r(0,y)dy.$$

The proof of the two equations in (2.36) is similar to that of Eqs. (2.32) and (2.35), respectively. ■

Next we consider the zero-energy solutions of the Schrödinger equation with the potential $U(x;a)$.

Proposition 2.5: In the generic case, $k=0$ is a bound state for the potential $U(x;a)$ if $0<a<\infty$, and a half bound state if $a=0$ or $a=\infty$. In the exceptional case, $k=0$ is always a half bound state. Moreover, in either case there is a bounded zero-energy solution of the Schrödinger equation which can be chosen to be positive, and this is (apart from constant multiples) the only positive solution.

Proof: It follows from Eqs. (2.30), (2.32), and (2.34) that in the generic case

$$\eta(x;a) = \begin{cases} \dfrac{1}{m_l(0,x)+am_r(0,x)}, & 0\leqslant a<\infty \\ \dfrac{1}{m_r(0,x)}, & a=\infty, \end{cases} \tag{2.39}$$

is a positive, bounded zero-energy solution of the Schrödinger equation with the potential $U(x;a)$; when $0<a<\infty$, by Eqs. (2.10) and (2.12) this solution is in L^2 and hence $k=0$ is a bound state. When $a=0$ or $a=\infty$, $\eta(x;a)$ is bounded but not in L^2 and thus we have a half bound state. A second, linearly independent solution is given by $\tilde{\eta}(x;a)=\eta(x;a)\times\int_0^x\eta^{-2}(y;a)dy$. Using Eqs. (2.10) and (2.12), for $0<a<\infty$ we have $\tilde{\eta}(x;a)=(ac_r/3)x^2+o(x^2)$ as $x\to+\infty$ and $\tilde{\eta}(x;a)=-(c_l/3)x^2+o(x^2)$ as $x\to-\infty$, for $a=0$ we have $\tilde{\eta}(x;a)=x+o(x)$ as $x\to+\infty$ and $\tilde{\eta}(x;a)=-(c_rx^2/3)+o(x^2)$ as $x\to-\infty$, and for $a=\infty$ we have $\tilde{\eta}(x;a)=(c_rx^2/3)+o(x^2)$ as $x\to+\infty$ and $\tilde{\eta}(x;a)=x+o(x)$ as $x\to-\infty$; thus no solution that is linearly independent of $\eta(x;a)$ can be either bounded or positive. In the exceptional case, we have $\eta(x)=1/m_l(0,x)$ which is bounded, but not in L^2, and hence it is a half bound state.■

III. CHARACTERIZATION OF POTENTIALS

In this section we characterize the potentials $U(x;a)$ that arise as the Darboux transformation of some $V\in L_1^1$; a similar characterization was given in Ref. 8 for the more restrictive class when $V\in L_2^1$.

First we note that the Darboux transformation given in Eqs. (2.26) and (2.27) is invertible. Differentiating Eq. (2.26) and using Eq. (1.1), we obtain

$$f_l(k,x)=\frac{1}{ik}\,[g_l'(k,x;a)+\rho(x;a)g_l(k,x;a)],\tag{3.1}$$

and similarly from Eqs. (2.27) and (1.1), we obtain

$$f_r(k,x)=-\frac{1}{ik}\,[g_r'(k,x;a)+\rho(x;a)g_r(k,x;a)].\tag{3.2}$$

Thus we have

$$m_l(k,x)=\frac{1}{ik}\,[n_l'(k,x;a)+ikn_l(k,x;a)+\rho(x;a)n_l(k,x;a)],\tag{3.3}$$

$$m_r(k,x)=-\frac{1}{ik}\,[n_r'(k,x;a)-ikn_r(k,x;a)+\rho(x;a)n_r(k,x;a)].\tag{3.4}$$

We see that the inverse Darboux transform has the same form as the direct transform, but that $\rho(x;a)$ is replaced by $-\rho(x;a)$. With $\eta(x;a)$ given by Eq. (2.39) and $\omega(x;a)=\eta'(x;a)/\eta(x;a)$, we have $\omega(x;a)=-\rho(x;a)$, and hence

$$V(x)=\omega(x;a)^2-\omega'(x;a).\tag{3.5}$$

In Eq. (3.5) the right hand side is independent of the parameter a and thus $V(x)$ is uniquely determined by $U(x;a)$. In other words, if we know that a given $U(x;a)$ is the Darboux transform of some $V\in L_1^1$, then there is only one such V and it is given by Eq. (3.5) in terms of the zero-energy solution of the Schrödinger equation with the potential U.

Let $W\in L_1^1$ and $\epsilon_j=0$, 1 with $j=1$, 2. We define $\mathcal{U}$ to be the family of potentials U having the form

$$U(x)=\epsilon_1\frac{2}{x^2+1}H(x)+\epsilon_2\frac{2}{x^2+1}H(-x)+W(x)\tag{3.6}$$

and satisfying the following two conditions:

(i) the Schrödinger equation with the potential U has no negative-energy bound states,

(ii) $k=0$ is either a bound state or a half bound state of the Schrödinger equation.

Proposition 3.1: Let $U\in\mathscr{U}$. Then $k=0$ is a bound state of the Schrödinger equation if and only if $\epsilon_1=\epsilon_2=1$.

Proof: Suppose that $U\in\mathscr{U}$ with $\epsilon_1=\epsilon_2=1$ and consider the interval $x>1$. Recall that the Schrödinger equation $\phi''=(2/x^2)\phi$ has the two linearly independent solution x^{-1} and x^2. Define

$$Q(x)=U(x)-\frac{2}{x^2}=-\frac{2}{x^2(x^2+1)}+W(x), \quad x>1, \tag{3.7}$$

and note that $Q\in L_1^1(1,\infty)$. Considering $Q(x)$ as a perturbation of the potential $2/x^2$, we can use variation of parameters to construct a decaying solution $u_l(x)$ of the equation

$$\phi''=U(x)\phi, \quad x\in\mathbf{R}, \tag{3.8}$$

such that

$$u_l(x)=\frac{1}{x}+\frac{1}{3}\int_x^\infty\left[\frac{y^2}{x}-\frac{x^2}{y}\right]Q(y)u_l(y)dy, \quad x>1. \tag{3.9}$$

To see that $u_l(x)$ is well-defined for $x>1$, we define $u_l^{(0)}(x)=0$ and

$$u_l^{(n+1)}(x)=\frac{1}{x}+\frac{1}{3}\int_x^\infty\left[\frac{y^2}{x}-\frac{x^2}{y}\right]Q(y)u_l^{(n)}(y)dy, \quad n=0,1,2,..., \tag{3.10}$$

and set

$$\Delta^{(n+1)}(x)=x\,|u_l^{(n+1)}(x)-u_l^{(n)}(x)|.$$

Then from Eq. (3.10), we obtain

$$\Delta^{(n+1)}(x)<\frac{1}{3}\int_x^\infty y|Q(y)|\Delta^{(n)}(y)dy,$$

and hence by iteration,

$$\Delta^{(n+1)}(x)<\frac{1}{3^n n!}\left(\int_x^\infty y\left|Q(y)\right|dy\right)^n.$$

Thus $u_l(x)=\lim_{n\to\infty}u^{(n)}(x)$ exists and we have

$$x|u_l(x)|<e^{(1/3)\int_x^\infty y|Q(y)|dy}, \quad x>1. \tag{3.11}$$

It follows from Eqs. (3.9) and (3.11) that $u_l(x)=1/x+o(1/x)$ as $x\to+\infty$ and hence $u_l\in L^2(1,\infty)$. Any solution of Eq. (3.8) that is linearly independent of $u_l(x)$ grows like cx^2 as $x\to+\infty$ with $c\neq0$. Similarly, since $\epsilon_2=1$ there is a unique solution $u_r(x)$ of Eq. (3.8) obeying $u_r(x)=1/x+o(1/x)$ as $x\to-\infty$ and any other linearly independent solution grows quadratically as $x\to-\infty$. Assumption (ii) therefore implies that $u_l(x)$ and $u_r(x)$ must be linearly dependent. Hence $u_l(x)=O(1/x)$ as $x\to\pm\infty$ and so $u_l\in L^2(\mathbf{R})$. Thus $k=0$ is a bound state.

Conversely, suppose that $U\in\mathscr{U}$ and $k=0$ is a bound state. If $\epsilon_1=0$, then, since $U(x)=W(x)$ for $x>0$ by Eq. (3.8) and $W(x)\in L_1^1$, there are two linearly independent solutions of Eq. (3.11) which are asymptotic to 1 and x as $x\to+\infty$, respectively. Hence no nontrivial linear combination of these solutions can lie in L^2, contradicting the assumption that $k=0$ is a bound state. Similarly, $\epsilon_2=0$ is ruled out. Hence we must have $\epsilon_1=\epsilon_2=1$. ■

Theorem 3.2: We have the following:

(i) Assume $V\in L_1^1$ without bound states. If V is generic, then $U(\cdot;a)\in\mathscr{U}$ for every $a\in[0,\infty]$. In fact, if $0<a<\infty$ then $\epsilon_1=\epsilon_2=1$, if $a=0$ then $\epsilon_1=0$, $\epsilon_2=1$, and if $a=\infty$ then $\epsilon_1=1$, $\epsilon_2=0$. If V is exceptional, then $U\in\mathscr{U}$ with $\epsilon_1=\epsilon_2=0$.

(ii) For every $U\in\mathscr{U}$, there is a unique $V\in L_1^1$ without bound states such that U is related to V as in Eq. (2.24) for a unique value of a.

Proof: For $x>1$, from Eqs. (2.6), (2.7), and (2.14), we have

$$m_l(0,x)=1+s_l(x), \tag{3.12}$$

$$m_r(0,x)=c_rx+s_r(x), \tag{3.13}$$

where

$$s_l(x)=\int_x^\infty (y-x)V(y)m_l(0,y)dy, \tag{3.14}$$

$$s_r(x)=1-x\int_x^\infty V(y)m_r(0,y)dy-\int_{-\infty}^x yV(y)m_r(0,y)dy. \tag{3.15}$$

When $0<a<\infty$, our first goal is to show that $\rho(x;a)^2-1/x^2\in L_1^1(1,\infty)$, where $\rho(x;a)$ is the quantity in Eq. (2.19). Define

$$r(x)=\rho(x;a)-\frac{1}{x}.$$

From Eqs. (2.18), (2.19), (3.12)–(3.15), we obtain

$$r(x)=\frac{1}{ac_rx^2}\left[\frac{xs_l'(x)-s_l(x)+axs_r'(x)-as_r(x)-1}{1+b(x)}\right], \tag{3.16}$$

where

$$b(x)=\frac{1+s_l(x)+as_r(x)}{ac_rx}. \tag{3.17}$$

Note that $1+b(x)$ is strictly positive and continuous on $(1,\infty)$ because we have

$$1+b(x)=\frac{m_l(0,x)+am_r(0,x)}{ac_r x}.$$

From Eqs. (3.12) and (3.14) we have $s_l(x)=o(1)$ as $x\to+\infty$. To estimate the second integral in Eq. (3.15), with $x>1$ and $0<\epsilon<1$, note that

$$\int_{-\infty}^x |y|\,|V(y)|\,|m_r(0,y)|dy=\int_{-\infty}^{x^\epsilon} |y|\,|V(y)|\,|m_r(0,y)|dy+\int_{x^\epsilon}^x y|V(y)|\,|m_r(0,y)|dy. \tag{3.18}$$

Using $|m_r(0,y)|<Cy$ for $y>1$, we have

$$\int_{x^\epsilon}^{x} y|V(y)||m_r(0,y)|dy \leqslant Cx\int_{x^\epsilon}^{\infty} y|V(y)|dy. \tag{3.19}$$

Since $V\in L_1^1$, the last integral in Eq. (3.19) behaves like $o(1)$ as $x\to+\infty$ and hence from Eqs. (3.18) and (3.19) we see that

$$\int_{-\infty}^{x} |y||V(y)||m_r(0,y)|dy=o(x), \quad x\to+\infty. \tag{3.20}$$

Using Eqs. (2.12), (3.15), and

$$|s_r(x)|\leqslant 1+x\int_x^\infty |V(y)||m_r(0,y)|dy+\int_{-\infty}^{x^\epsilon} |y||V(y)||m_r(0,y)|dy$$
$$+\int_{x^\epsilon}^{\infty} y|V(y)||m_r(0,y)|dy,$$

we obtain $s_r(x)=o(x)$ as $x\to+\infty$, and hence from Eq. (3.17) we have

$$b(x)=o(1), \quad x\to+\infty. \tag{3.21}$$

From Eq. (3.14) we have

$$xs_l'(x)=-x\int_x^\infty V(y)m_l(0,y)dy,$$

which implies

$$|xs_l'(x)|\leqslant\int_x^\infty y|V(y)||m_l(0,y)|dy,$$

$$xs_l'(x)=o(1), \quad x\to+\infty. \tag{3.22}$$

From Eq. (3.15) we have

$$xs_r'(x)-s_r(x)=-1+\int_{-\infty}^{x} yV(y)m_r(0,y)dy. \tag{3.23}$$

By Eqs. (3.16), (3.21), (3.22), and (3.23) there is a constant C such that

$$|r(x)|\leqslant C\left[\frac{1}{x^2}+\frac{1}{x^2}\int_{-\infty}^{x} |y||V(y)||m_r(0,y)|dy\right], \quad x>1.$$

Integration by parts yields

$$\int_1^\infty \left(x^{-2}\int_{-\infty}^{x} |y||V(y)||m_r(0,y)|dy\right)dx=\left[-\frac{1}{x}\int_{-\infty}^{x} |y||V(y)||m_r(0,y)|dy\right]_1^\infty$$
$$+\int_1^\infty |V(x)||m_r(0,x)|dx. \tag{3.24}$$

By Eq. (3.20) the first term on the right hand side of Eq. (3.24) vanishes at the upper limit. Hence $r\in L^1(1,\infty)$. Since $\rho(x;a)^2-1/x^2=[\rho(x;a)+(1/x)]r(x)$ and $\rho(x;a)=O(1/x)$ as $x\to+\infty$, it follows that $\rho(x;a)^2-1/x^2\in L_1^1(1,\infty)$. From Eq. (2.25), writing

$$U(x)=\frac{2}{x^2+1}+2\left[\rho^2(x;a)-\frac{1}{x^2}\right]+2\left[\frac{1}{x^2}-\frac{1}{x^2+1}\right]-V(x)$$

and using the fact that $x^{-2}-(x^2+1)^{-1}\in L_1^1(1,\infty)$, we see that, for $x>1$, $U(x;a)$ is of the form (3.6) with $\epsilon_1=1$. A similar analysis for $x<-1$ shows that $\epsilon_2=1$.

For $a=0$, using $m_l(0,x)\to 1$ as $x\to+\infty$, from Eqs. (2.8) and (2.19) we obtain

$$\rho(x;0)^2\leqslant C\left[\int_x^\infty V(y)m_l(0,y)dy\right]^2,\quad x>1.$$

Then

$$\int_1^\infty x\rho(x;0)^2dx\leqslant C\left[\int_1^\infty y|V(y)||m_l(0,y)|dy\right]\left[\int_1^\infty |V(y)||m_l(0,y)|dy\right],$$

and so $\rho(\cdot;0)^2\in L_1^1(1,\infty)$; thus by Eq. (2.25), $U(\cdot;0)\in L_1^1(1,\infty)$ and hence $\epsilon_1=0$. When $x<-1$ we proceed as in the case $0<a<\infty$ and find that $U(x;0)$ is of the form Eq. (3.6) with $\epsilon_2=1$. When $a=\infty$ the proof is analogous to that for $a=0$ and we find that $\epsilon_1=1$ and $\epsilon_2=0$. In the exceptional case, from Eqs. (2.25), (3.6), and Proposition 2.2 (iv), we see that $U\in\mathscr{U}$ with $\epsilon_1=\epsilon_2=0$. Thus the proof of (i) is complete.

Now let us prove (ii). Let $U\in\mathscr{U}$ be given and suppose that $\epsilon_1=\epsilon_2=1$. Let $u_l(x)$ denote the zero-energy solution given in Eq. (3.9). We have

$$u_l(x)=\frac{1}{x}+h(x), \tag{3.25}$$

where, for $x>1$ we have

$$h(x)=\frac{1}{3}\int_x^\infty\left[\frac{y^2}{x}-\frac{x^2}{y}\right]Q(y)u_l(y)dy. \tag{3.26}$$

Let

$$\theta(x)=\frac{u_l'(x)}{u_l(x)}. \tag{3.27}$$

Since U does not support any negative-energy bound states, $u_l(x)\neq 0$ (Theorem on p. 94 of Ref. 8), and hence $\theta(x)$ in Eq. (3.27) is well-defined. Then, from Eq. (3.25) we have

$$\theta(x)=-\frac{1}{x}+\frac{1}{x}\left[\frac{x^2h'(x)+xh(x)}{1+xh(x)}\right].$$

Using Eq. (3.26) we obtain

$$x^2h'(x)+xh(x)=-x^3\int_x^\infty Q(y)y^{-1}u_l(y)dy.$$

Put

$$t(x)=\theta(x)+\frac{1}{x}=\frac{1}{x}\left[\frac{x^2h'(x)+xh(x)}{1+xh(x)}\right]. \tag{3.28}$$

Note that from Eq. (3.11) we have $u_l(y)=O(1/y)$ as $y\to+\infty$, and from Eqs. (3.11) and (3.26) we have $h(x)=o(1/x)$ as $x\to+\infty$; furthermore, from Eq. (3.25) we see that $1+xh(x)$ is continuous and bounded. Hence we get

$$|t(x)|\leqslant C\int_x^\infty |Q(y)|dy, \quad x>1. \tag{3.29}$$

Thus $t\in L^1(1,\infty)$ because $Q\in L_1^1(1,\infty)$. As a result, from Eq. (3.28) we have $\theta(x)=O(1/x)$ as $x\to+\infty$, and thus using Eq. (3.29) and the fact that $\theta(x)^2-(1/x^2)=[\theta(x)-(1/x)]t(x)$, we conclude

$$\theta(x)^2-\frac{1}{x^2}\in L_1^1(1,\infty). \tag{3.30}$$

A similar analysis on $(-\infty,-1)$ shows that $\theta(x)^2-(1/x^2)\in L_1^1(-\infty,-1)$. On $(-1,1)$, $\theta(x)$ is bounded. Thus using Eq. (3.7) and defining $V(x)=2\theta(x)^2-U(x)$, we obtain $V\in L_1^1(\mathbf{R})$. The potential V has no bound states because U has no negative-energy bound states and the transmission coefficients for U and V are equal to each other. A comparison with Eq. (2.24) and the explanation given following Eq. (3.5) show that there is a unique value of the parameter $0<a<\infty$ such that U is the Darboux transform of V. With appropriate modifications the above proof also works when one or both of ϵ_1 and ϵ_2 are zero. If $\epsilon_1=0$, $\epsilon_2=1$ ($\epsilon_1=1$, $\epsilon_2=0$), then one finds $a=0$ ($a=\infty$) in accordance with (i). If $\epsilon_1=\epsilon_2=0$, then $V\in L_1^1$ is exceptional and the parameter a plays no role. ■

Returning to the matrix $\mathbf{M}(k,x)$, assuming there are no bound states, we note that by Eq. (1.15)

$$\det\mathbf{M}(k,x)=\tfrac{1}{2}[m_l(k,x)n_r(k,x)+m_r(k,x)n_l(k,x)].$$

Hence from Eqs. (2.16), (2.28), and (2.29) we obtain

$$\det\mathbf{M}(k,x)=\frac{1}{T(k)}. \tag{3.31}$$

Alternatively, this relation can also be deduced directly from Eq. (1.18). By taking the determinant of both sides of Eq. (1.18) and using $\det\mathbf{G}(k,x)=\det\mathbf{S}(k)=T(k)/T(-k)$, we obtain

$$T(-k)\det\mathbf{M}(-k,x)=T(k)\det\mathbf{M}(k,x), \quad k\in\mathbf{R}. \tag{3.32}$$

Since in the generic case $T(k)=O(k)$ as $k\to 0$, in both the generic and exceptional cases, $T(k)$ $\det\mathbf{M}(k,x)$ is analytic in $\mathbf{C}^+$, continuous in $\overline{\mathbf{C}^+}$, and converges to 1 as $k\to\infty$ in $\overline{\mathbf{C}^+}$. In Eq. (3.32) we have a scalar Riemann–Hilbert problem, where knowing $T(k)$ we seek $\det\mathbf{M}(k,x)$. By virtue of Eq. (3.32), $T(k)\det\mathbf{M}(k,x)$ has an analytic continuation to $\mathbf{C}^-$. Therefore, by Liouville's theorem, we conclude that $T(k)\det\mathbf{M}(k,x)\equiv 1$ on $\mathbf{C}$, which gives Eq. (3.31). So $\mathbf{M}(k,x)$ is invertible in $\overline{\mathbf{C}^+}\setminus\{0\}$ and we have that

$$\mathbf{M}(k,x)^{-1}=\frac{T(k)}{2}\begin{bmatrix} m_r(k,x)+n_r(k,x) & n_l(k,x)-m_l(k,x) \\ n_r(k,x)-m_r(k,x) & m_l(k,x)+n_l(k,x)\end{bmatrix}.$$

Since in the generic case $T(k)$ vanishes linearly as $k\to 0$ and thus cancels the $1/k$ singularities of $n_l(k,x)$ and $n_r(k,x)$, we see that in both the generic and exceptional cases $\mathbf{M}(k,x)^{-1}$ is continuous at $k=0$. When there are bound states $T(k)$ is not analytic in $\mathbf{C}^+$, and Eq. (3.31) is no longer valid.

In the absence of bound states the parameter a can be recovered from Eqs. (2.30) and (2.31) via

$$\lim_{k\to 0}\frac{n_l(k,x;a)}{n_r(k,x;a)}=a,\quad 0\leqslant a\leqslant\infty, \tag{3.33}$$

or using Theorem 2.4 via

$$\lim_{k\to 0}k\mathbf{M}(k,x;a)=\begin{cases}\dfrac{ic_r}{2\chi(x;a)}\begin{bmatrix}a & -a\\ -1 & 1\end{bmatrix}, & 0\leqslant a<\infty\\[2ex] \dfrac{ic_r}{2\chi(x;\infty)}\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}, & a=\infty,\end{cases}$$

where $\chi(x;a)$ is the function defined in Eq. (2.18) and c_r is the constant in Eq. (2.15). If $0<a<\infty$, then $k=0$ is a bound state for the potential $U(x;a)$. Moreover, the parameter a is then related to the norming constant of the zero-energy solution $\eta(x;a)$ in Eq. (2.39). From Eqs. (2.15) and (2.39) we have $\eta(x;a)^2=-[1/ac_r][m_l(0,x)\eta(x;a)]'$, and thus using Eqs. (2.10) and (2.12) we obtain

$$\frac{1}{\int_{-\infty}^{\infty}\eta(x;a)^2dx}=ac_r. \tag{3.34}$$

The next theorem summarizes the main results about the Riemann–Hilbert problem (1.18) obtained thus far.

Theorem 3.3: Suppose that $V\in L_1^1$ without bound states. In the generic case the Riemann–Hilbert problem (1.18) has a one-parameter family of solutions $\mathbf{M}(k,x;a)$ such that $m(k,x)=\mathbf{M}(k,x;a)\hat{1}$ is a solution of Eq. (1.7) with the potential $V(x)$ and $n(k,x;a)=\mathbf{J}\mathbf{M}(k,x;a)\hat{e}$ is a solution of Eq. (1.12) with the potential $U(x;a)$, where $U\in\mathscr{U}$ but $U\notin L_1^1$. Moreover, $\mathbf{M}(k,x;a)$ has a $1/k$ singularity at $k=0$ and $\mathbf{M}(k,x;a)$ is invertible for $k\in\overline{\mathbf{C}^+}\setminus\{0\}$. In the exceptional case, the solution $\mathbf{M}(k,x)$ of Eq. (1.18) does not depend on the parameter a and $U\in L_1^1$; furthermore, $\mathbf{M}(k,x)$ is continuous at $k=0$ and invertible for $k\in\overline{\mathbf{C}^+}$. In both the generic and exceptional cases, $\mathbf{M}(k,x)^{-1}$ is continuous at $k=0$.

IV. BOUND STATES

Now we turn to the case when the potential $V(x)$ supports $\mathscr{N}$ bound states with energies $-\beta_1^2,\ldots,-\beta_{\mathscr{N}}^2$ where $\beta_{\mathscr{N}}>\cdots>\beta_1>0$. Then $T(k)$ has simple poles at $k=i\beta_j$ $(j=1,\ldots,\mathscr{N})$. Define

$$\check{\mathbf{S}}(k)=\left(\prod_{j=1}^{\mathscr{N}}\frac{k-i\beta_j}{k+i\beta_j}\right)\mathbf{J}^{\mathscr{N}}\mathbf{S}(k)\mathbf{J}^{\mathscr{N}}, \tag{4.1}$$

or equivalently,

$$\check{T}(k)=\left(\prod_{j=1}^{\mathscr{N}}\frac{k-i\beta_j}{k+i\beta_j}\right)T(k), \tag{4.2}$$

$$\check{R}(k)=(-1)^{\mathcal{N}}\left(\prod_{j=1}^{\mathcal{N}}\frac{k-i\beta_j}{k+i\beta_j}\right)R(k), \tag{4.3}$$

$$\check{L}(k)=(-1)^{\mathcal{N}}\left(\prod_{j=1}^{\mathcal{N}}\frac{k-i\beta_j}{k+i\beta_j}\right)L(k). \tag{4.4}$$

The potential corresponding to $\check{\mathbf{S}}(k)$ will be denoted by $\check{V}(x)$ and its Jost solutions by $\check{f}_l(k,x)$ and $\check{f}_r(k,x)$. The Schrödinger equation with the potential $\check{V}$ has no bound states since $\check{T}(k)$ has no poles in $\mathbf{C}^+$. Let

$$\kappa_j=\left(\int_{-\infty}^{\infty}f_l(i\beta_j,x)^2dx\right)^{-1} \tag{4.5}$$

be the norming constant associated with the bound state $-\beta_j^2$ of V, and let

$$\alpha_j=\frac{2(-1)^{j-1}\beta_j}{\kappa_j}\left(\prod_{l\neq j}^{\mathcal{N}}\frac{\beta_l+\beta_j}{\beta_l-\beta_j}\right)\check{T}(i\beta_j),\quad j=1,...,\mathcal{N}. \tag{4.6}$$

Note that $\check{T}(i\beta)>0$ for $\beta\geq 0$ because $\check{T}(k)$ is nonzero and continuous in $\mathbf{C}^+$, $\check{T}(i\beta)$ is real, and $\check{T}(k)\to 1$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. Thus, in particular $\check{T}(i\beta_j)>0$ for $j=1,...,\mathcal{N}$. Hence, from Eq. (4.6) it is seen that $\alpha_j>0$. It is known (Theorem 6, p. 176 of Ref. 10) that V can be obtained from $\check{V}$ inductively by defining

$$V^{[0]}(x)=\check{V}(x),$$

$$V^{[j]}(x)=V^{[j-1]}(x)-2\frac{d^2}{dx^2}\log v_j(x),\quad j=1,...,\mathcal{N}, \tag{4.7}$$

where

$$v_j(x)=f_l^{[j-1]}(i\beta_j,x)+\alpha_j f_r^{[j-1]}(i\beta_j,x),\quad j=1,...,\mathcal{N},$$

and $f_l^{[j]}(k,x)$ and $f_r^{[j]}(k,x)$ being the Jost solutions associated with the potential $V^{[j]}(x)$, and

$$f_l^{[0]}(i\beta_j,x)=\check{f}_l(i\beta_j,x),\quad f_r^{[0]}(i\beta_j,x)=\check{f}_r(i\beta_j,x),$$

and $V^{[\mathcal{N}]}(x)=V(x)$. Note that $v_j(x)>0$ because $-\beta_j^2<-\beta_{j-1}^2$ and hence (Ref. 10) $f_l^{[j-1]}(i\beta_j,x)>0$ and $f_r^{[j-1]}(i\beta_j,x)>0$. It follows that $V^{[j]}(x)$ is a potential having j bound states with energies $-\beta_1^2,...,-\beta_j^2$. For more details and proofs we refer the reader to Ref. 10. The bound states can also be added or removed one at a time by Newton's method.[1,11]

Instead of obtaining V from $\check{V}$ by adding a bound state one at a time, it is possible to go directly from $\check{V}$ to V as follows. Define

$$w_j(x)=(-1)^{j+1}\check{f}_l(i\beta_j,x)+\alpha_j\check{f}_r(i\beta_j,x), \tag{4.8}$$

where α_j is given in Eq. (4.6). Let $\Omega[w_1,...,w_{\mathcal{N}},\check{f}_l(k,x)]$ denote the determinant of the $(\mathcal{N}+1)\times(\mathcal{N}+1)$ matrix with entries

$$\Omega_{2j-1,s}(k,x)=\begin{cases}\beta_s^{2j-2}w_s(x), & s=1,...,\mathcal{N}\\ (-1)^{j-1}k^{2j-2}\check{f}_l(k,x), & s=\mathcal{N}+1,\end{cases} \tag{4.9}$$

$$\Omega_{2j,s}(k,x)=\begin{cases}\beta_s^{2j-2}w_s'(x), & s=1,...,\mathcal{N}\\ (-1)^{j-1}k^{2j-2}\check{f}_l'(k,x), & s=\mathcal{N}+1.\end{cases} \tag{4.10}$$

Let $\Omega[w_1,...,w_{\mathcal{N}}]$ be the principal minor of $\Omega[w_1,...,w_{\mathcal{N}},\check{f}_l(k,x)]$. Then

$$V(x)=\check{V}(x)-2\frac{d^2}{dx^2}\log\Omega[w_1,...,w_{\mathcal{N}}], \tag{4.11}$$

$$f_l(k,x)=(-i)^{\mathcal{N}}\left(\prod_{j=1}^{\mathcal{N}}\frac{1}{k+i\beta_j}\right)\frac{\Omega[w_1,...,w_{\mathcal{N}},\check{f}_l(k,x)]}{\Omega[w_1,...,w_{\mathcal{N}}]}, \tag{4.12}$$

$$f_r(k,x)=i^{\mathcal{N}}\left(\prod_{j=1}^{\mathcal{N}}\frac{1}{k+i\beta_j}\right)\frac{\Omega[w_1,...,w_{\mathcal{N}},\check{f}_r(k,x)]}{\Omega[w_1,...,w_{\mathcal{N}}]}. \tag{4.13}$$

Similar relations hold for the potential $U(x)$ if we define

$$U^{[0]}(x)=\check{U}(x), \tag{4.14}$$

$$U^{[j]}(x)=U^{[j-1]}(x)-2\frac{d^2}{dx^2}\log y_j(x), \quad j=1,...,\mathcal{N}, \tag{4.15}$$

where

$$y_j(x)=g_l^{[j-1]}(i\beta_j,x)+\ell_j g_r^{[j-1]}(i\beta_j,x), \quad j=1,...,\mathcal{N}, \tag{4.16}$$

with $0<\ell_j<\infty$, $U^{[\mathcal{N}]}(x) = U(x)$, $g_l^{[j]}(k,x)$ and $g_r^{[j]}(k,x)$ are the Jost solutions associated with the potential $U^{[j]}(x)$, and $g_l^{[0]}(k,x)=\check{g}_l(k,x)$ and $g_r^{[0]}(k,x)=\check{g}_r(k,x)$ are the Jost solutions associated with $\check{U}(x)$. Note that $y_j(x)$ satisfies the Schrödinger equation

$$y_j''=[\beta_j^2+U^{[j-1]}(x)]y_j, \tag{4.17}$$

and that $y_j(x)>0$ since $g_l^{[j-1]}(i\beta_j,x)>0$ and $g_r^{[j-1]}(i\beta_j,x)>0$. To see this, we first note that any nontrivial solution of Eq. (4.17) can have at most one zero (Theorem 2.c of Ref. 12). Suppose that $g_l^{[j-1]}(i\beta_j,x)$ has a zero at $x=x_0$. Pick any $x_1<x_0$ and let $h_1(x)$ be a nontrivial solution of Eq. (4.17) such that $h_1(x_1)=0$. Then $h_1(x)$ is linearly independent of $g_l^{[j-1]}(i\beta_j,x)$. Thus $h_1(x)\neq 0$ for $x\neq x_1$ and $h_1(x) = ce^{\beta_j x} + o(e^{\beta_j x})$ as $x\to+\infty$ with $c\neq 0$. For $x>x_1$ define $h_2(x)=h_1(x)\int_x^\infty h_1(y)^{-2}dy$. Then $h_2(x)$ is a solution of Eq. (4.17) and $h_2(x) = [1/2c\beta_j]e^{-\beta_j x} + o(e^{-\beta_j x})$ as $x\to+\infty$. Therefore $g_l^{[j-1]}(i\beta_j,x)=(2c\beta_j)\ h_2(x)$. Since $h_2(x_0)\neq 0$ and $g_l^{[j-1]}(i\beta_j,x_0)=0$, we have a contradiction. Hence $g_l^{[j-1]}(i\beta_j,x)$ has no zeros and so $g_l^{[j-1]}(i\beta_j,x)>0$. Similarly one sees that $g_r^{[j-1]}(i\beta_j,x)>0$. In Eq. (4.14) $\check{U}$ is a Darboux transform of $\check{V}$, where in the generic case the parameter a needed to uniquely specify $\check{U}$ has been fixed. Hence a will be suppressed in our notation.

Theorem 4.1: The potentials $U^{[j]}(x)$ for $j=0,...,\mathcal{N}$ are of the form (3.6), and the values of ϵ_1 and ϵ_2 are the same for all j. Moreover, $k=0$ is a bound state (half bound state) for $U^{[j]}(x)$, $j=1,...,\mathcal{N}$ if and only if $k=0$ is a bound state (half bound state) for $\check{U}(x)$.

Proof: From Theorem 3.2 we know that $\check{U}$ is of the form (3.6). Let us use induction and assume that $U^{[j-1]}(x)$ is of the form (3.6) with $\epsilon_1=1$. From Eqs. (4.15) and (4.16) we have

$$U^{[j]}(x)=U^{[j-1]}(x)-2\frac{y_j''(x)}{y_j(x)}+2\left[\frac{y_j'(x)}{y_j(x)}\right]^2=-U^{[j-1]}(x)-2\beta_j^2+2\left[\frac{y_j'(x)}{y_j(x)}\right]^2. \tag{4.18}$$

We have

$$\frac{y_j'(x)}{y_j(x)}=\frac{[e^{-\beta_j x}y_j(x)]'}{e^{-\beta_j x}y_j(x)}+\beta_j. \tag{4.19}$$

From the proof of Theorem 3.2 we know that the Schrödinger equation with the potential $U^{[j-1]}(x)$ has a zero-energy solution $u_l^{[j-1]}(x)$ which is asymptotic to $1/x$ as $x\to+\infty$. Hence there is an x_0 such that $u_l^{[j-1]}(x)>0$ for $x>x_0$. In analogy to Eq. (3.27), let

$$\omega_j(x)=\frac{u_l^{[j-1]\prime}(x)}{u_l^{[j-1]}(x)},\quad x>x_0,$$

and define

$$p_j(x)=y_j'(x)-\omega_j(x)y_j(x),\quad x>x_0. \tag{4.20}$$

Then $p_j(x)$ is a "local" Darboux transform of $y_j(x)$ on the interval $x_0<x<\infty$. As seen from Eqs. (2.21), (2.22), (2.23), and (2.25), for $x>x_0$, $p_j(x)$ satisfies the Schrödinger equation $\psi''-\beta_j^2\psi=\tilde{V}_j(x)\psi$, where

$$\tilde{V}_j(x)=2\omega_j(x)^2-U^{[j-1]}(x)=\omega_j(x)^2-\omega_j'(x)\in L_1^1(x_0,\infty). \tag{4.21}$$

Defining $\tilde{V}_j(x)=0$ when $x<x_0$, we can extend $p_j(x)$ to all of $\mathbf{R}$ as a solution of the Schrödinger equation with the potential $\tilde{V}_j(x)$. Then we have

$$p_j(x)=a_1e^{-\beta_j x}\tilde{m}_l^{[j]}(i\beta_j,x)+a_2e^{\beta_j x}\tilde{m}_r^{[j]}(i\beta_j,x),$$

where a_1 and a_2 are suitable constants and $\tilde{m}_l^{[j]}$ and $\tilde{m}_r^{[j]}$ are the solutions of Eqs. (1.5) and (1.6) with the potential $\tilde{V}_j$. We have (Lemma 6, p. 156 of Ref. 10) that $\tilde{m}_l^{[j]\prime}(i\beta_j,x)$, $\tilde{m}_r^{[j]\prime}(i\beta_j,x)\to 0$ as $x\to\pm\infty$ and $\tilde{m}_l^{[j]\prime}(i\beta_j,\cdot)$, $\tilde{m}_r^{[j]\prime}(i\beta_j,\cdot)\in L_1^1(\mathbf{R})$. Therefore, we obtain

$$[e^{-\beta_j x}p_j(x)]'\to 0,\quad x\to\pm\infty, \tag{4.22}$$

$$[e^{-\beta_j x}p_j(x)]'\in L_1^1(\mathbf{R}). \tag{4.23}$$

From Eqs. (4.17), (4.20), and (4.21), we obtain

$$[e^{-\beta_j x}p_j(x)]'+[\omega_j(x)+\beta_j][e^{-\beta_j x}y_j(x)]'=\omega_j(x)^2e^{-\beta_j x}y_j(x),$$

and hence

$$\frac{[e^{-\beta_j x}y_j(x)]'}{e^{-\beta_j x}y_j(x)}=\frac{\omega_j(x)^2}{\omega_j(x)+\beta_j}-\frac{[e^{-\beta_j x}p_j(x)]'}{[\omega_j(x)+\beta_j][e^{-\beta_j x}y_j(x)]}$$

$$=\frac{\omega_j(x)^2}{\beta_j}-\frac{\omega_j(x)^3}{\beta_j[\omega_j(x)+\beta_j]}+q(x),\quad x>x_0, \tag{4.24}$$

where

$$q(x)=-\frac{[e^{-\beta_j x}p_j(x)]'}{[\omega_j(x)+\beta_j][e^{-\beta_j x}y_j(x)]}.$$

Since $y_j(x)>0$, we can find $\delta>0$ such that $e^{-\beta_j x}y_j(x) > \delta$ for all x. This follows from Eq. (4.16) and standard asymptotic results since, by Eq. (3.6), $U^{[j-1]}\in L^1$, $g_r^{[j-1]}(i\beta_j,x)>0$ and $e^{-\beta_j x}g_r^{[j-1]}(i\beta_j,x) \to 1$ as $x\to-\infty$ and $g_r^{[j-1]}(i\beta_j,x) = ce^{\beta_j x} + o(e^{\beta_j x})$ as $x\to+\infty$ for some $c>0$. Hence, by Eqs. (4.22) and (4.23), we have $q\in L_1^1(x_0,\infty)$ and that $q(x)\to 0$ as $x\to+\infty$. Using Eqs. (4.24) in (4.19), we obtain

$$\left[\frac{y_j'(x)}{y_j(x)}\right]^2=2\omega_j^2(x)+\beta_j^2+z(x), \tag{4.25}$$

where the remainder $z\in L_1^1(x_0,\infty)$. Substituting Eq. (4.25) in Eq. (4.18) and using Eqs. (3.30) and (3.6), it follows that on the interval $x>x_0$, $U^{[j]}(x)$ is of the form (3.6) and that $\epsilon_1=1$. Thus ϵ_1 does not change if we add a bound state. Next, returning to the beginning of the proof, we assume that $\epsilon_1=0$ for $U^{[j-1]}(x)$, which means that $U^{[j-1]}\in L_1^1(x_0,\infty)$. Then, for $x>x_0$ the proof is essentially the same as that in the L_1^1 case (Theorem 2, p. 167 of Ref. 10). We find that $\omega_j\in L_1^1(x_0,\infty)$ and $\omega_j(x)\to 0$ as $x\to+\infty$. Thus $y_j'/y_j \in L_1^1(x_0,\infty)$ and hence $U^{[j]}\in L_1^1$, and $\epsilon_1=0$. A similar analysis can be given on an interval $x<x_0$ where x_0 is sufficiently negative. It follows that ϵ_2 does not change even if we add a bound state. This proves the assertions concerning the form of $U^{[j]}(x)$. As for the behavior near $k=0$, we note that the Jost solutions corresponding to $U^{[j]}(x)$ are given by (Theorem 2, p. 167 of Ref. 10)

$$g_l^{[j]}(k,x)=\frac{-i}{k+i\beta_j}\left[g_l^{[j-1]\prime}(k,x)-\frac{y_j'(x)}{y_j(x)}g_l^{[j-1]}(k,x)\right], \tag{4.26}$$

$$g_r^{[j]}(k,x)=\frac{i}{k+i\beta_j}\left[g_r^{[j-1]\prime}(k,x)-\frac{y_j'(x)}{y_j(x)}g_r^{[j-1]}(k,x)\right].$$

Suppose that the parameter a that specifies $\check{U}$ satisfies $0<a<\infty$. By Theorem 3.2 and Proposition 3.1 this means that $\epsilon_1=\epsilon_2=1$ and that $k=0$ is a bound state for $\check{U}$. Then we claim that the limits

$$\lim_{k\to 0} ikg_l^{[j]}(k,x)=\mu_l^{[j]}(x) \quad \text{and} \quad \lim_{k\to 0} ikg_r^{[j]}(k,x)=\mu_r^{[j]}(x) \tag{4.27}$$

exist, and that $\mu_l^{[j]}(x)$ and $\mu_r^{[j]}(x)$ are nontrivial zero-energy solutions of Eq. (3.8) with potential $U^{[j]}(x)$. The relations (4.27) may also be differentiated. Moreover, $\mu_l^{[j]}(x)=-1/x+o(1/x)$ as $x\to+\infty$ and $\mu_r^{[j]}(x)=1/x+o(1/x)$ as $x\to-\infty$. For $j=0$ this is clear from Eqs. (2.30) and (2.31). Note that $y_j'(x)/y_j(x) \to \pm\beta_j$ as $x\to\pm\infty$ and hence the asymptotic behavior of $g_l^{[j]}(k,x)$ is determined by that of $g_l^{[j-1]}(k,x)$, and similarly for $g_r^{[j]}(k,x)$. Using Wronskians, we find that $[\mu_l^{[j]}(x);\mu_r^{[j]}(x)]=-[\mu_l^{[j-1]}(x);\mu_r^{[j-1]}(x)]$. From Eqs. (2.30) and (2.31), we know that $[\mu_l^{[0]}(x);\mu_r^{[0]}(x)]=0$, and hence the induction gives us $[\mu_l^{[j]}(x);\mu_r^{[j]}(x)]=0$ for all $j=0,\ldots,\mathcal{N}$ and so $\mu_l^{[j]}(x)$ and $\mu_r^{[j]}(x)$ are linearly dependent. Therefore, $k=0$ is a bound state for $U^{[j]}(x)$ if and only if it is a bound state for $\check{U}$. In a similar way, it can be shown that if $a=0$ ($a=\infty$) or in the exceptional case, then $k=0$ is a half bound state for $U^{[j]}(x)$, $j=0,\ldots,\mathcal{N}$. ∎

The relations Eqs. (3.33) and (3.34) can be generalized to the case with bound states. Since $\mu_l^{[j]}(x)$ and $\mu_r^{[j]}(x)$ are linearly dependent we can set $\mu_l^{[j]}(x)=b_j\mu_r^{[j]}(x)$. In the generic case, from Eqs. (2.30) and (2.31) we have $b_0=a$; then by induction we obtain $b_j=-b_{j-1}$ and hence

$$\lim_{k\to 0}\frac{n_l(k,x;a)}{n_r(k,x;a)}=\lim_{k\to 0}\frac{\mu_l^{[\mathcal{N}]}(x)}{\mu_r^{[\mathcal{N}]}(x)}=(-1)^{\mathcal{N}}a.$$

Furthermore, if $0<a<\infty$, setting $\rho_j(x)=y_j'(x)/y_j(x)$, from Eqs. (4.26) and (4.27) we have

$$\int_{-\infty}^{\infty}\mu_l^{[j]}(x)^2\,dx=\frac{1}{\beta_j^2}\int_{-\infty}^{\infty}[\mu_l^{[j-1]\prime}(x)-\rho_j\mu_l^{[j-1]}(x)][\mu_l^{[j-1]\prime}(x)-\rho_j\mu_l^{[j-1]}(x)]dx,$$

and hence, by using integration by parts, this is equal to

$$-\frac{1}{\beta_j^2}\int_{-\infty}^{\infty}\mu_l^{[j-1]}(x)[\mu_l^{[j-1]\prime\prime}(x)-\rho_j'\mu_l^{[j-1]}(x)-\rho_j^2\mu_l^{[j-1]}(x)]dx$$

$$=-\frac{1}{\beta_j^2}\int_{-\infty}^{\infty}\mu_l^{[j-1]}(x)^2[U^{[j-1]}(x)-\rho_j'-\rho_j^2]dx$$

$$=\int_{-\infty}^{\infty}\mu_l^{[j-1]}(x)^2\,dx.$$

Now $\mu_l^{[0]}(x)=-ac_r\eta(x;a)$ by Eqs. (2.30) and (2.31), and so, by Eq. (3.34)

$$\frac{1}{\int_{-\infty}^{\infty}\mu_l^{[\mathcal{N}]}(x)^2\,dx}=ac_r.$$

Later we will need the expressions corresponding to Eqs. (4.8)–(4.13) for the potentials $U(x)$. In analogy with Eq. (4.5) we introduce the norming constants

$$\nu_j=\left(\int_{-\infty}^{\infty}g_l(i\beta_j,x)^2\,dx\right)^{-1},$$

and in analogy with Eq. (4.6) we define

$$\ell_j=\frac{2(-1)^{j-1}\beta_j}{\nu_j}\left(\prod_{l\neq j}^{\mathcal{N}}\frac{\beta_l+\beta_j}{\beta_l-\beta_j}\right)\check{T}(i\beta_j),\quad j=1,\ldots,\mathcal{N}. \tag{4.28}$$

Suppressing the parameter a, in analogy with Eq. (4.8) we then define

$$z_j(x)=(-1)^{j+1}\check{g}_l(i\beta_j,x)+\ell_j\check{g}_r(i\beta_j,x). \tag{4.29}$$

In Eqs. (4.9) and (4.10) replacing $\check{f}_l$, $\check{f}_r$, w_s by $\check{g}_l$, $\check{g}_r$, z_s, respectively, we obtain

$$U(x)=\check{U}(x)-2\frac{d^2}{dx^2}\log\Omega[z_1,\ldots,z_{\mathcal{N}}], \tag{4.30}$$

$$g_l(k,x)=(-i)^{\mathcal{N}}\left(\prod_{j=1}^{\mathcal{N}}\frac{1}{k+i\beta_j}\right)\frac{\Omega[z_1,\ldots,z_{\mathcal{N}},\check{g}_l(k,x)]}{\Omega[z_1,\ldots,z_{\mathcal{N}}]}, \tag{4.31}$$

$$g_r(k,x)=i^{\mathcal{N}}\left(\prod_{j=1}^{\mathcal{N}}\frac{1}{k+i\beta_j}\right)\frac{\Omega[z_1,\ldots,z_{\mathcal{N}},\check{g}_r(k,x)]}{\Omega[z_1,\ldots,z_{\mathcal{N}}]}. \tag{4.32}$$

Theorem 4.2: Suppose that the scattering matrix in the Riemann–Hilbert problem (1.18) corresponds to a potential $V\in L_1^1$ with $\mathcal{N}$ bound states. Then, in the exceptional case there is a $(2\mathcal{N})$-parameter family of solutions of Eq. (1.18), the parameters being the norming constants $\kappa_1,...,\kappa_{\mathcal{N}}$, $\nu_1,...,\nu_{\mathcal{N}}$. In the generic case, there is a $(2\mathcal{N}+1)$-parameter family of solutions of Eq. (1.18), where the additional parameter is a.

Proof: The scattering matrix in Eq. (1.18) uniquely determines the potential $\check{V}$ whose scattering matrix $\check{S}(k)$ is in Eq. (4.1). Then using the positive constants α_j in Eq. (4.8) and ℓ_j in Eq. (4.29) for $j=1,...,\mathcal{N}$, we construct the potentials $V(x) = V^{[\mathcal{N}]}(x)$ as in Eq. (4.11) and $U(x) = U^{[\mathcal{N}]}(x)$ as in Eq. (4.30). In the generic case the additional parameter a is already present in $U^{[0]}(x)$. The corresponding Jost solutions given in Eqs. (4.12), (4.13), (4.31), and (4.32) then determine the solutions $m(k,x)$ of Eq. (1.7) and $n(k,x)$ of Eq. (1.12). Hence the solution of Eq. (1.18) is given as in Eq. (1.15). ■

We conclude this section with a remark about the determinant of $\mathbf{M}(k,x)$ when there are bound states. From Eq. (3.32) we obtain

$$\prod_{j=1}^{\mathcal{N}} (k^2+\beta_j^2)T(-k)\det \mathbf{M}(-k,x)=T(k)\det \mathbf{M}(k,x) \prod_{j=1}^{\mathcal{N}} (k^2+\beta_j^2).$$

By Liouville's theorem, both sides must be equal to a polynomial of degree $2\mathcal{N}$ in k, where the leading term has coefficient 1 and the remaining terms have real coefficients depending on x. Moreover, by Eq. (3.32) this polynomial is even. Thus we have

$$\det \mathbf{M}(k,x)=\frac{k^{2\mathcal{N}}+\Sigma_{j=0}^{\mathcal{N}-1} c_j(x)k^{2j}}{T(k)\Pi_{j=1}^{\mathcal{N}} (k^2+\beta_j^2)}.$$

Hence, det $\mathbf{M}(k,x)$ vanishes at the zeros of the numerator, at least $\mathcal{N}$ of which must lie in $\overline{\mathbf{C}^+}$.

V. SOLUTION OF THE RIEMANN–HILBERT PROBLEM

Theorems 3.3 and 4.2 guarantee the existence of solutions of the Riemann–Hilbert problem (1.18) when the underlying scattering matrix comes from a potential in L_1^1. This raises the question whether the solutions found there constitute all solutions that can be associated with a Schrödinger equation. *A priori* we do not want to restrict the potentials $V(x)$ and $U(x)$ that can possibly arise as a result of such a solution of Eq. (1.18), except, of course, for minimal requirements which insure that the differential equations (1.7) and (1.12) along with the transmission and reflection coefficients are well-defined. In particular we will not *a priori* restrict the rate of decay of the potentials. Furthermore, we will require that $\mathbf{M}(k,x)=O(1/k)$ as $k\to 0$, as it is the case for the solutions constructed in the previous section. If this condition is weakened, the problem gets much more complicated and we will not consider it here. The smoothness of $\mathbf{M}(k,x)$ in the variable x will not be specified at the outset, but of course we will naturally have to make some assumptions if we want to associate $\mathbf{M}(k,x)$ with a Schrödinger equation. From a practical point of view, since the solution of the Riemann–Hilbert problem allows us to solve the inverse scattering problem by recovering $V(x)$ from $m(k,x)$, it is of interest to know whether the two components of $m(k,x)$ automatically yield the same potential. The condition that according to Eq. (1.9) both components of $m(k,x)$ must lead to the same potential is known as Newton's "miracle" condition.[1] A similar question can be asked with regard to $n(k,x)$ and $U(x)$.

Theorem 5.1: Suppose that the matrix $\mathbf{G}(k,x)$ in Eq. (1.18) is associated with a potential $V\in L_1^1$ with no bound states. Let $\mathbf{M}(k,x)$ be any solution of Eq. (1.18) such that

(i) $\mathbf{M}(k,x)$ is analytic in $\mathbf{C}^+$ and continuous in $\overline{\mathbf{C}^+}\setminus\{0\}$ for each x,

(ii) $\mathbf{M}(k,x)\to\mathbf{I}$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$ for every x, and $\mathbf{M}(-k,x) = \overline{\mathbf{M}(k,x)}$ for $k\in\mathbf{R}$,

(iii) $\mathbf{M}(k,x)=O(k^{-1})$ as $k\to 0$ in $\overline{\mathbf{C}^+}$ for every x.

We then have the following:

(a) Let $V(x)$ be generic. Then $m(k,x) = \mathbf{M}(k,x)\hat{1}$ satisfies Eq. (1.7) with $V(x)$ as the potential. Furthermore, there exists a potential $U_l(x)$ corresponding to $n_l(k,x)$ if and only if there exists a potential $U_r(x)$ corresponding to $n_r(k,x)$ and then $U_l(x) = U_r(x) = U(x;a)$ for some $a \in [0,\infty]$, where $U(x;a)$ is a Darboux transform of $V(x)$.

(b) Let $V(x)$ be exceptional. Then there exists a potential $V_l(x)$ corresponding to $m_l(k,x)$ if and only if there exists a potential $V_r(x)$ corresponding to $m_r(k,x)$ and then $V_l(x) = V_r(x) = V(x)$. Furthermore, there exists a potential $U_l(x)$ corresponding to $n_l(k,x)$ if and only if there exists a potential $U_r(x)$ corresponding to $n_r(k,x)$ and then $U_l(x) = U_r(x) = U(x)$ is unique. The associated solution $\mathbf{M}(k,x)$ of Eq. (1.18) is continuous at $k=0$.

Proof: (a) Let $\mathbf{M}_0(k,x;b)$ denote a particular solution of Eq. (1.18) as constructed in Sec. III, that is, $m_0(k,x) := \mathbf{M}_0(k,x;b)\hat{1}$ obeys Eq. (1.7) and $n_0(k,x;b) := \mathbf{J}\mathbf{M}_0(k,x;b)\hat{e}$ obeys Eq. (1.12) with the potential $U(x;b)$ for some $b \in [0,\infty]$. Consider b to be fixed. Then $\mathbf{M}_0(k,x;b)$ provides us with a matrix factorization of $\mathbf{G}(k,x)$, namely,

$$\mathbf{G}(k,x) = \mathbf{M}_0(-k,x;b)[\mathbf{q}\mathbf{M}_0(k,x;b)^{-1}\mathbf{q}], \tag{5.1}$$

where the first factor has an analytic continuation to $\overline{\mathbf{C}^-}\setminus\{0\}$ and the second factor (in brackets) has an analytic extension to $\overline{\mathbf{C}^+}\setminus\{0\}$. In the generic case, the factor $\mathbf{M}_0(-k,x;b)$ has a $1/k$ singularity at $k=0$ while the second factor is continuous at $k=0$. In the exceptional case, both factors are continuous at $k=0$. Let $\mathbf{M}(k,x)$ be an arbitrary solution of Eq. (1.18). Then we can write Eq. (1.18) as

$$\mathbf{M}_0(-k,x;b)^{-1}\mathbf{M}(-k,x) = \mathbf{q}\mathbf{M}_0(k,x;b)^{-1}\mathbf{M}(k,x)\mathbf{q}. \tag{5.2}$$

Since $\mathbf{M}_0(k,x;b)^{-1}$ is continuous at $k=0$ by Theorem 3.3, both sides of (5.2) are of $O(1/k)$ as $k \to 0$. Hence, by a variant of Liouville's theorem, both sides of Eq. (5.2) must be equal to $(i/k)\mathbf{A}(x) + \mathbf{I}$ for some matrix $\mathbf{A}(x)$. By Eq. (5.2) and assumption (ii), we have that $\mathbf{q}\mathbf{A}(x)\mathbf{q} = -\mathbf{A}(x)$ and $\overline{\mathbf{A}(x)} = \mathbf{A}(x)$. Therefore, $\mathbf{A}(x)$ must be of the form

$$\mathbf{A}(x) = \begin{bmatrix} A(x) & B(x) \\ -B(x) & -A(x) \end{bmatrix},$$

where $A(x)$ and $B(x)$ are real functions. Thus, by Eqs. (1.16), (1.17), and (5.2), we have

$$m(k,x) = m_0(k,x) - \frac{i\zeta(x)}{k}\mathbf{J}n_0(k,x;b), \tag{5.3}$$

where $\zeta(x) = A(x) + B(x)$ and

$$n(k,x;b) = n_0(k,x;b) - \frac{i\tau(x)}{k}\mathbf{J}m_0(k,x), \tag{5.4}$$

with $\tau(x) = A(x) - B(x)$. First take $V(x)$ to be generic. Then we conclude that $\zeta(x) = 0$ in view of assumption (iii) and the fact that $n_0(k,x;b)$ has a $1/k$ singularity at $k=0$. Hence $m(k,x) = m_0(k,x)$ and the first part of assertion (a) is proved.

To prove the second part of (a), we let

$$\mathbf{U}(x) = \begin{bmatrix} U_l(x) & 0 \\ 0 & U_r(x) \end{bmatrix},$$

and suppose that $n(k,x)=\mathbf{J}\mathbf{M}(k,x)\hat{e}$ is a solution of Eq. (1.12) with the matrix potential $\mathbf{U}(x)$, namely,

$$n''(k,x;b)+2ik\mathbf{J}n'(k,x;b)=\mathbf{U}(x)n(k,x;b). \tag{5.5}$$

Since $n_l(k,x)\to 1$ as $x\to+\infty$ and $n_r(k,x)\to 1$ as $x\to-\infty$, we must have that $\tau(x)\to 0$ as $x\to\pm\infty$. Set $U_b(x)=U(x;b)$ and $\rho_b(x)=\rho(x;b)$, where $\rho(x;b)$ is defined in Eq. (2.19). Substituting Eq. (5.4) in Eq. (5.5) yields

$$U_b n_0-\frac{i}{k}\tau''\mathbf{J}m_0-\frac{2i}{k}\tau'\mathbf{J}m_0'-\frac{i}{k}\tau\mathbf{J}Vm_0+2\tau'm_0=\mathbf{U}n_0-\frac{i}{k}\tau\mathbf{U}\mathbf{J}m_0. \tag{5.6}$$

Using that $m_0(k,x)\to\hat{1}$ and $n_0(k,x;b)\to\hat{1}$ as $k\to+\infty$, from Eq. (5.6) we obtain

$$(U_b+2\tau')\mathbf{I}=\mathbf{U}. \tag{5.7}$$

Hence $U_l(x)=U_r(x)=U_b(x)+2\tau'$. Set $U(x)=U_l(x)$. From Eqs. (2.28) and (2.29) we have

$$n_0-m_0=\frac{1}{ik}\mathbf{J}[m_0'-\rho_b m_0], \tag{5.8}$$

and from Proposition 2.1, we have

$$m_0'(k,x)\to 0 \quad\text{and}\quad n_0'(k,x;b)\to 0, \quad k\to+\infty. \tag{5.9}$$

Using Eqs. (5.7), (5.8), and (5.9) in Eq. (5.6), we obtain

$$2\tau'\rho_b+\tau''+\tau V=\tau[U_b+2\tau'].$$

By using Eq. (2.24) we get

$$2\tau'\rho_b+\tau''=-2\tau\rho_b'+2\tau'\tau,$$

and hence by integration, we have

$$\tau'=-2\tau\rho_b+\tau^2+c, \tag{5.10}$$

where c denotes the integration constant. Since $\tau\to 0$ as $x\to\pm\infty$, we have $\tau'\to 0$ and thus $c=0$. It then follows from Eq. (5.10) that $\rho_b(x)-\tau(x)$ satisfies the Riccati equation

$$(\rho_b-\tau)'+(\rho_b-\tau)^2=\rho_b'+\rho_b^2=V.$$

Therefore $\rho_b-\tau$ must be of the form ψ'/ψ where $\psi>0$ is a solution of Eq. (2.5). In other words, $\rho_b-\tau=\rho_a$ for some a with $0<a\leqslant\infty$, and by using Eq. (5.7), $U=U_b+2\rho_b'-2\rho_a'=V-2\rho_a'=U_a$. Part (a) is proved.

In the exceptional case we cannot immediately conclude from Eq. (5.3) that $\zeta(x)=0$ since $n_0(k,x;b)$ is continuous at $k=0$. Letting

$$\mathbf{V}(x)=\begin{bmatrix} V_l(x) & 0 \\ 0 & V_r(x) \end{bmatrix},$$

and substituting Eq. (5.3) in

$$m''(k,x)+2ik\mathbf{J}m'(k,x)=\mathbf{V}(x)m(k,x),$$

and using an argument similar to that following Eq. (5.5), we obtain

$$(V+2\zeta')\mathbf{I}=\mathbf{V}, \tag{5.11}$$

$$\zeta'=2\zeta\rho+\zeta^2+c, \tag{5.12}$$

where c is the integration constant and $\rho(x) = m'_{0,l}(0,x)/m_{0,l}(0,x)$ with $m_{0,l}(k,x)$ being the first component of the vector $m_0(k,x)$. From Eq. (5.11) we obtain $V_l(x)=V_r(x)$. As in the case of Eq. (5.10), since $\zeta(x)\to 0$ and $\rho(x)\to 0$ as $x\to\pm\infty$, we must have $c=0$. Then Eq. (5.12) becomes a Bernoulli equation and apart from the trivial solution, its general solution is given by

$$\zeta(x)=\frac{m_{0,l}(0,x)^2}{\tilde{c}-\int_0^x m_{0,l}(0,y)^2\,dy}, \tag{5.13}$$

where $\tilde{c}$ is an arbitrary constant. Since $m_{0,l}(0,x)$ approaches nonzero limits as $x\to\pm\infty$, the denominator in Eq. (5.13) has a zero. Hence $\zeta(x)=0$ is the only acceptable solution of Eq. (5.12) because otherwise $m(k,x)$ given in Eq. (5.3) cannot be continuous in x. This proves the assertion concerning $m(k,x)$ of part (b). As for $n(k,x)$, an argument similar to that used in the generic case again shows that $\rho_b-\tau=\rho_a$ for some a. However, since $\rho_a=\rho_b$ for any a in the exceptional case, we have $\tau=0$. Hence $U(x)$ is uniquely given by $U(x)=U_b(x)$. In the exceptional case since $\tau(x)=\zeta(x)=0$, it follows from Eqs. (5.3) and (5.5) that any solution $\mathbf{M}(k,x)$ of Eq. (1.18) satisfying (i)–(iii) must be equal to $\mathbf{M}_0(k,x)$; hence in the exceptional case (i)–(iii) imply that $\mathbf{M}(k,x)$ is continuous at $k=0$. Thus the proof is complete. ■

Now we turn to the case when V has $\mathcal{N}$ bound states with energies $\beta^2_{\mathcal{N}} < \cdots < -\beta^2_1$. In analogy with Eq. (1.18), we introduce the Riemann–Hilbert problem

$$\check{\mathbf{M}}(-k,x)=\check{\mathbf{G}}(k,x)\mathbf{q}\check{\mathbf{M}}(k,x)\mathbf{q},\quad k\in\mathbf{R}, \tag{5.14}$$

where

$$\check{\mathbf{G}}(k,x)=e^{i\mathbf{J}kx}\mathbf{J}\check{\mathbf{S}}(k)\mathbf{J}e^{-i\mathbf{J}kx}=\begin{bmatrix}\check{T}(k) & -\check{R}(k)e^{2ikx}\\ -\check{L}(k)e^{-2ikx} & \check{T}(k)\end{bmatrix},$$

with $\check{\mathbf{S}}(k)$, $\check{T}(k)$, $\check{R}(k)$, and $\check{L}(k)$ as in Eqs. (4.1)–(4.4). In (5.14) we look for $\check{\mathbf{M}}(k,x)$ which is continuous in k for $k\in\mathbf{R}\setminus\{0\}$ and has an analytic extension in k to $\mathbf{C}^+$ such that $\mathbf{M}(k,x)\to\mathbf{I}$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$ for each x.

Put

$$\check{\mathbf{M}}(k,x)=\frac{1}{2}\begin{bmatrix}\check{m}_l(k,x)+\check{n}_l(k,x) & \check{m}_l(k,x)-\check{n}_l(k,x)\\ \check{m}_r(k,x)-\check{n}_r(k,x) & \check{m}_r(k,x)+\check{n}_r(k,x)\end{bmatrix}, \tag{5.15}$$

so that

$$\check{m}(k,x)=\check{\mathbf{M}}(k,x)\hat{1}, \tag{5.16}$$

$$\check{n}(k,x)=\mathbf{J}\check{\mathbf{M}}(k,x)\hat{e}. \tag{5.17}$$

Then $\check{m}(k,x)$ and $\check{n}(k,x)$ are solutions of Eqs. (1.7) and (1.12) with the potentials $\check{V}(x)$ and $\check{U}(x)$, respectively. The parameter a will be suppressed. By Eqs. (3.31) and (4.2), we have

$$\det \check{\mathbf{M}}(k,x)=\frac{1}{\check{T}(k)}=\left(\prod_{j=1}^{\mathcal{N}}\frac{k+i\beta_j}{k-i\beta_j}\right)\frac{1}{T(k)},$$

and hence $\check{\mathbf{M}}(k,x)$ is invertible for $k\in\overline{\mathbf{C}^+}\setminus\{0\}$. The solution of the Riemann–Hilbert problem (5.14) provides us, as in Eq. (5.1), with a factorization of $\check{\mathbf{G}}(k,x)$, namely,

$$\check{\mathbf{G}}(k,x)=\check{\mathbf{M}}(-k,x)[\mathbf{q}\check{\mathbf{M}}(k,x)^{-1}\mathbf{q}],\quad k\in\mathbf{R}. \tag{5.18}$$

In the generic case we can without loss of generality assume that $0<a<\infty$, since the only purpose of introducing $\check{\mathbf{M}}(k,x)$ is to obtain a factorization of the form (5.18). Note that $\mathbf{G}(k,x)$ and $\check{\mathbf{G}}(k,x)$ are related by

$$\mathbf{G}(k,x)=\left(\prod_{j=1}^{\mathcal{N}}\frac{k+i\beta_j}{k-i\beta_j}\right)\mathbf{J}^{\mathcal{N}}\check{\mathbf{G}}(k,x)\mathbf{J}^{\mathcal{N}}. \tag{5.19}$$

It is convenient to define

$$\mathbf{N}(k,x)=\mathbf{J}^{\mathcal{N}}\check{\mathbf{M}}(k,x)\mathbf{J}^{\mathcal{N}}. \tag{5.20}$$

Inserting Eq. (5.19) in Eq. (1.18) and using Eqs. (5.18) and (5.20), we obtain

$$\left(\prod_{j=1}^{\mathcal{N}}(k-i\beta_j)\right)\mathbf{N}(-k,x)^{-1}\mathbf{M}(-k,x)=\left(\prod_{j=1}^{\mathcal{N}}(k+i\beta_j)\right)\mathbf{q}\mathbf{N}(k,x)^{-1}\mathbf{M}(k,x)\mathbf{q},\quad k\in\mathbf{R}. \tag{5.21}$$

As in the case with no bound states, we impose the condition that

$$\mathbf{M}(k,x)=O(1/k)\quad\text{as}\quad k\to 0\quad\text{in}\quad\overline{\mathbf{C}^+}. \tag{5.22}$$

By Theorem 3.3, $\check{\mathbf{M}}(k,x)^{-1}$ and thus $\mathbf{N}(k,x)^{-1}$ are continuous at $k=0$. Hence, as in the proof of Theorem 5.1, using Liouville's theorem we conclude that both sides of Eq. (5.21) must be equal to a matrix function in k of the form

$$\mathbf{P}_{\mathcal{N}}(k,x)=k^{\mathcal{N}}\left[\mathbf{I}+\sum_{j=1}^{\mathcal{N}+1}\left(\frac{i}{k}\right)^j\mathbf{A}_j(x)\right]. \tag{5.23}$$

Thus, by Eqs. (5.21) and (5.23) we obtain

$$\mathbf{M}(k,x)=\frac{k^{\mathcal{N}}}{\Pi_{j=1}^{\mathcal{N}}(k+i\beta_j)}\mathbf{N}(k,x)\left[\mathbf{I}+\sum_{j=1}^{\mathcal{N}+1}\left(\frac{i}{k}\right)^j\mathbf{q}\mathbf{A}_j(x)\mathbf{q}\right]. \tag{5.24}$$

Since $\mathbf{N}(-k,x)=\overline{\mathbf{N}(k,x)}$ and $\mathbf{M}(-k,x)=\overline{\mathbf{M}(k,x)}$ for $k\in\mathbf{R}$, from Eqs. (5.21) and (5.23) we conclude that the matrices $\mathbf{A}_j(x)$ are real and that $\mathbf{A}_j(x)=(-1)^j\mathbf{q}\mathbf{A}_j(x)\mathbf{q}$. In other words, $\mathbf{A}_j(x)$ is of the form

$$\mathbf{A}_j(x)=\begin{bmatrix}A_j(x) & B_j(x)\\ B_j(x) & A_j(x)\end{bmatrix},\quad j\quad\text{even}, \tag{5.25}$$

$$\mathbf{A}_j(x)=\begin{bmatrix}A_j(x) & B_j(x)\\ -B_j(x) & -A_j(x)\end{bmatrix},\quad j\quad\text{odd}, \tag{5.26}$$

where $A_j(x)$ and $B_j(x)$ are real functions. Let $\gamma_j(x)=(-1)^j[A_j(x)+B_j(x)]$ and $\theta_j(x)=(-1)^j[A_j(x)-B_j(x)]$. Then, using Eqs. (5.25) and (5.26) in Eq. (5.24) we obtain

$$m(k,x)=\mathbf{M}(k,x)\hat{1}=\frac{k^{\mathscr{N}}}{\Pi_{j=1}^{\mathscr{N}}(k+i\beta_j)}\mathbf{N}(k,x)\left[\hat{1}+\sum_{j=1}^{\mathscr{N}+1}\left(\frac{i}{k}\right)^j\gamma_j(x)\mathbf{J}^j\hat{1}\right], \tag{5.27}$$

$$n(k,x)=\mathbf{J}\mathbf{M}(k,x)\hat{e}=\frac{k^{\mathscr{N}}}{\Pi_{j=1}^{\mathscr{N}}(k+i\beta_j)}\mathbf{J}\mathbf{N}(k,x)\left[\hat{e}+\sum_{j=1}^{\mathscr{N}+1}\left(\frac{i}{k}\right)^j\theta_j(x)\mathbf{J}^j\hat{e}\right]. \tag{5.28}$$

Looking at Eq. (5.27) as $k\to 0$, using Eqs. (5.20) and (5.17), we see that

$$m(k,x)=i\left(\prod_{j=1}^{\mathscr{N}}\beta_j\right)^{-1}\gamma_{\mathscr{N}+1}(x)\mathbf{J}^{\mathscr{N}-1}k^{-1}\check{n}(k,x)+O(1/k).$$

In the generic case, since $\check{n}(k,x)$ has a $1/k$ singularity at $k=0$, Eq. (5.22) is violated unless

$$\gamma_{\mathscr{N}+1}(x)=0. \tag{5.29}$$

In the exceptional case, Eq. (5.22) is not violated even if we assume $\gamma_{\mathscr{N}+1}(x)\neq 0$. However, we will assume Eq. (5.29) also in the exceptional case due to the reason given following the proof of Proposition 6.3. The situation is different with respect to $\theta_{\mathscr{N}+1}(x)$; from Eq. (5.28) we see that $\theta_{\mathscr{N}+1}(x)$ can be nonzero without violating Eq. (5.22).

Using Eqs. (5.16), (5.17), (5.20), (5.27), (5.28), and (5.29), when $\mathscr{N}$ is even we have

$$m(k,x)=\frac{k^{\mathscr{N}}}{\Pi_{j=1}^{\mathscr{N}}(k+i\beta_j)}\left[\check{m}(k,x)\left(1+\sum_{j=1}^{\mathscr{N}/2}\left(\frac{i}{k}\right)^{2j}\gamma_{2j}(x)\right)+\mathbf{J}\check{n}(k,x)\sum_{j=1}^{\mathscr{N}/2}\left(\frac{i}{k}\right)^{2j-1}\gamma_{2j-1}(x)\right], \tag{5.30}$$

$$n(k,x)=\frac{k^{\mathscr{N}}}{\Pi_{j=1}^{\mathscr{N}}(k+i\beta_j)}\left[\check{n}(k,x)\left(1+\sum_{j=1}^{\mathscr{N}/2}\left(\frac{i}{k}\right)^{2j}\theta_{2j}(x)\right)+\mathbf{J}\check{m}(k,x)\sum_{j=0}^{\mathscr{N}/2}\left(\frac{i}{k}\right)^{2j+1}\theta_{2j+1}(x)\right], \tag{5.31}$$

and when $\mathscr{N}$ is odd we have

$$m(k,x)=\frac{k^{\mathscr{N}}}{\Pi_{j=1}^{\mathscr{N}}(k+i\beta_j)}\left[\check{n}(k,x)\left(1+\sum_{j=1}^{(\mathscr{N}-1)/2}\left(\frac{i}{k}\right)^{2j}\gamma_{2j}(x)\right)\right.$$
$$\left.+\mathbf{J}\check{m}(k,x)\sum_{j=0}^{(\mathscr{N}-1)/2}\left(\frac{i}{k}\right)^{2j+1}\gamma_{2j+1}(x)\right], \tag{5.32}$$

$$n(k,x)=\frac{k^{\mathscr{N}}}{\Pi_{j=1}^{\mathscr{N}}(k+i\beta_j)}\left[\check{m}(k,x)\left(1+\sum_{j=1}^{(\mathscr{N}+1)/2}\left(\frac{i}{k}\right)^{2j}\theta_{2j}(x)\right)\right.$$
$$\left.+\mathbf{J}\check{n}(k,x)\sum_{j=0}^{(\mathscr{N}-1)/2}\left(\frac{i}{k}\right)^{2j+1}\theta_{2j+1}(x)\right]. \tag{5.33}$$

In Eq. (5.32) the first summation is zero when $\mathscr{N}=1$. We now insert Eqs. (5.30) and (5.32) into Eq. (1.7) and insert Eqs. (5.31) and (5.33) into Eq. (1.12). We also use Eqs. (2.28), (2.29), (3.3), and (3.4) to replace $\check{n}(k,x)$ and $\check{n}'(k,x)$ by their equivalents in terms of

$\check{m}(k,x)$ and $\check{m}'(k,x)$ and apply Eq. (2.24) in subsequent calculations. Since $\check{m}(k,x)=\hat{1}+o(1)$ and $\check{m}'(k,x)=o(1)$ as $k\to\pm\infty$, we can then separate the terms of $O(1/k^n)$ and $o(1/k^n)$ for $n\geqslant 0$. Hence we obtain for $j\geqslant 1$ the following necessary and sufficient conditions for $m(k,x)$ and $n(k,x)$ to be solutions of Eqs. (1.7) and (1.12), respectively. In Eqs. (5.36), (5.37), (5.43), and (5.44) the upper (lower) sign refers to $\mathcal{N}$ even (odd).

$$V(x)=\begin{cases}\check{V}(x)-2\gamma_1'(x), & \mathcal{N}\text{ even}\\ \check{U}(x)-2\gamma_1'(x), & \mathcal{N}\text{ odd,}\end{cases} \tag{5.34}$$

$$U(x)=\begin{cases}\check{U}(x)-2\theta_1'(x), & \mathcal{N}\text{ even}\\ \check{V}(x)-2\theta_1'(x), & \mathcal{N}\text{ odd.}\end{cases} \tag{5.35}$$

$$-2\gamma_{2j}'+\gamma_{2j-1}''+2\gamma_1'\gamma_{2j-1}\mp 2(\gamma_{2j-1}\check{\rho})'=0, \tag{5.36}$$

$$-2\theta_{2j+1}'+\theta_{2j}''+2\theta_1'\theta_{2j}\mp 2\theta_{2j}'\check{\rho}=0, \tag{5.37}$$

$$\gamma_{2j}''+2\gamma_1'\gamma_{2j}-2\gamma_{2j+1}'+[2\gamma_1'\gamma_{2j-1}+\gamma_{2j-1}''-2(\gamma_{2j-1}\check{\rho})']\check{\rho}=0,\quad \mathcal{N}\text{ even,} \tag{5.38}$$

$$\gamma_{2j-1}''+2\gamma_1'\gamma_{2j-1}-2\gamma_{2j}'+2\gamma_{2j-1}\check{\rho}'+[2\gamma_1'\gamma_{2j-2}+\gamma_{2j-2}''-2\gamma_{2j-2}'\check{\rho}]\check{\rho}=0,\quad \mathcal{N}\text{ odd,} \tag{5.39}$$

$$\theta_{2j-1}''+2\theta_1'\theta_{2j-1}-2\theta_{2j}'+2\theta_{2j-1}\check{\rho}'+[2\theta_1'\theta_{2j-2}+\theta_{2j-2}''-2\theta_{2j-2}'\check{\rho}]\check{\rho}=0,\quad \mathcal{N}\text{ even,} \tag{5.40}$$

$$\theta_{2j}''+2\theta_1'\theta_{2j}-2\theta_{2j+1}'+[2\theta_1'\theta_{2j-1}+\theta_{2j-1}''-2(\theta_{2j-1}\check{\rho})']\check{\rho}=0,\quad \mathcal{N}\text{ odd,} \tag{5.41}$$

where $\check{\rho}=\check{\chi}'/\check{\chi}$ in analogy with Eq. (2.19) and

$$\check{\chi}(x)=\check{m}_l(0,x)+a\check{m}_r(0,x), \tag{5.42}$$

in analogy with Eq. (2.18). Using Eq. (5.36), we can reduce Eqs. (5.38) and (5.39) to

$$-2\gamma_{2j+1}'+\gamma_{2j}''+2\gamma_1'\gamma_{2j}\pm 2\gamma_{2j}'\check{\rho}=0, \tag{5.43}$$

and using Eq. (5.37), we can reduce Eqs. (5.40) and (5.41) to

$$-2\theta_{2j}'+\theta_{2j-1}''+2\theta_1'\theta_{2j-1}\pm 2(\theta_{2j-1}\check{\rho})'=0. \tag{5.44}$$

In Eqs. (5.36), (5.37), (5.43), and (5.44) we can let j range from 1 to $+\infty$ if we assume that $\gamma_j(x)=0$ for $j>\mathcal{N}$ and $\theta_j(x)=0$ for $j>\mathcal{N}+1$. Note that this convention is consistent with Eq. (5.29). Since $m_l(k,x)$, $n_l(k,x)\to 1$ as $x\to+\infty$, $m_r(k,x)$, $n_r(k,x)\to 1$ as $x\to-\infty$, and also $m_l'(k,x)$, $n_l'(k,x)\to 0$ as $x\to+\infty$ and $m_r'(k,x)$, $n_r'(k,x)\to 0$ as $x\to-\infty$, we see from Eqs. (5.30)–(5.33) that the following boundary conditions must be satisfied:

$$\theta_{\mathcal{N}+1}(\pm\infty)=0, \tag{5.45}$$

and for $j=1,\ldots,\mathcal{N}$,

$$\theta_j(+\infty)=\gamma_j(+\infty)=\sum_{i_1<\cdots<i_j}\beta_{i_1}\cdots\beta_{i_j}, \tag{5.46}$$

$$\theta_j(-\infty)=\gamma_j(-\infty)=(-1)^j \sum_{i_1<\cdots<i_j} \beta_{i_1}\cdots\beta_{i_j}, \tag{5.47}$$

$$\gamma_j'(x),\theta_j'(x)\to 0 \quad \text{as} \quad x\to\pm\infty. \tag{5.48}$$

From Eqs. (2.30), (2.31), (5.31), and (5.33), we find that

$$\mu_l(x)=\lim_{k\to 0} ikn_l(k,x)=\frac{1}{\Pi_{j=1}^{\mathscr{N}}\beta_j}\left[-\frac{ac_r\theta_{\mathscr{N}}(x)}{\check{\chi}(x)}-\check{m}_l(0,x)\theta_{\mathscr{N}+1}(x)\right], \tag{5.49}$$

$$\mu_r(x)=\lim_{k\to 0} ikn_r(k,x)=\frac{(-1)^{\mathscr{N}}}{\Pi_{j=1}^{\mathscr{N}}\beta_j}\left[-\frac{c_r\theta_{\mathscr{N}}(x)}{\check{\chi}(x)}+\check{m}_r(0,x)\theta_{\mathscr{N}+1}(x)\right], \tag{5.50}$$

where $\mu_l(x)$ and $\mu_r(x)$ are solutions of $\psi''=U(x)\psi$, and $\check{\chi}$ is the quantity in Eq. (5.42). From Eqs. (2.15), (5.49), and (5.50) we obtain

$$[\mu_l;\mu_r]=\frac{c_r(-1)^{\mathscr{N}}}{[\Pi_{j=1}^{\mathscr{N}}\beta_j]^2}\,[\theta_{\mathscr{N}}'\theta_{\mathscr{N}+1}-\theta_{\mathscr{N}}\theta_{\mathscr{N}+1}'-\theta_{\mathscr{N}+1}^2-2\check{p}\theta_{\mathscr{N}}\theta_{\mathscr{N}+1}]. \tag{5.51}$$

The Wronskian in Eq. (5.51) is independent of x, and using Eqs. (5.45)–(5.48) we obtain

$$\theta_{\mathscr{N}}'\theta_{\mathscr{N}+1}-\theta_{\mathscr{N}}\theta_{\mathscr{N}+1}'-\theta_{\mathscr{N}+1}^2-2\check{p}\theta_{\mathscr{N}}\theta_{\mathscr{N}+1}=0, \tag{5.52}$$

and thus $\mu_l(x)$ and $\mu_r(x)$ are linearly dependent. Letting

$$w(x)=\frac{\theta_{\mathscr{N}}(x)}{\theta_{\mathscr{N}+1}(x)}, \tag{5.53}$$

from Eq. (5.52) we obtain

$$w'-2\check{p}w=1. \tag{5.54}$$

For $0<a<\infty$, $\check{\chi}(x)$ grows linearly as $x\to\pm\infty$, and we can write the solution of Eq. (5.54) in the form

$$w(x)=\check{\chi}(x)^2\left[\int_{-\infty}^{x}\check{\chi}(y)^{-2}\,dy+c\right], \tag{5.55}$$

where c is an arbitrary constant. Note that Eq. (5.52) is satisfied also when $\theta_{\mathscr{N}+1}(x)$ vanishes identically.

Proposition 5.2: $\gamma_{\mathscr{N}+1}(x)=\theta_{\mathscr{N}+1}(x)=0$ if and only if $\mathbf{N}(k,x)^{-1}\mathbf{M}(k,x)$ is continuous at $k=0$.

Proof: From Eq. (5.24) it is seen that the continuity of $\mathbf{N}(k,x)^{-1}\mathbf{M}(k,x)$ at $k=0$ is equivalent to the continuity of the matrix $\mathbf{P}_{\mathscr{N}}(k,x)$ in Eq. (5.23), which holds if and only if $\mathbf{A}_{\mathscr{N}+1}(x)=0$. This is equivalent to $\gamma_{\mathscr{N}+1}(x)=\theta_{\mathscr{N}+1}(x)=0$. ■

VI. FURTHER ANALYSIS WITH BOUND STATES

In this section we construct the solutions of the Riemann–Hilbert problem (1.18) by using the norming constants for the bound states.

Proposition 6.1: For the solutions $n(k,x)$ corresponding to the potential $U(x)$ in Eq. (4.30), we have that $\mathbf{N}(k,x)^{-1}\mathbf{M}(k,x)$ is continuous at $k=0$ and hence $\theta_{\mathscr{N}+1}(x)=0$.

Proof: By Proposition 5.2 it suffices to show that the solution $\mathbf{M}(k,x)$ of the Riemann–Hilbert problem (1.18) associated with $U(x)$ has the property that $\mathbf{N}(k,x)^{-1}\mathbf{M}(k,x)$ is continuous at $k=0$. Considering the (1,1) entry, from Eqs. (1.15), (3.31), (5.15), and (5.20) we have

$$[\mathbf{N}(k,x)^{-1}\mathbf{M}(k,x)]_{11}=\frac{\check{T}(k)}{4}[A_1(k,x)+A_2(k,x)+A_3(k,x)], \tag{6.1}$$

where

$$A_1=\check{m}_r m_l\mp\check{m}_l m_r,\quad A_2=\check{n}_r m_l+\check{m}_r n_l\pm\check{n}_l m_r\pm\check{m}_l n_r,\quad A_3=\check{n}_r n_l\mp\check{n}_l n_r,$$

and the upper (lower) sign applies if $\mathcal{N}$ is even (odd). Clearly, the terms $\check{T}(k)A_1(k,x)$ and $\check{T}(k)A_2(k,x)$ are continuous at $k=0$. If $\mathcal{N}$ is even, in terms of $g=e^{i\mathbf{J}kx}n$ and $\check{g}=e^{i\mathbf{J}kx}\check{n}$ and using Eqs. (4.31) and (4.32), we obtain

$$\check{n}_r n_l-\check{n}_l n_r=\check{g}_r g_l-\check{g}_l g_r=\left(\prod_{j=1}^{\mathcal{N}}\frac{1}{k+i\beta_j}\right)\frac{(-i)^{\mathcal{N}}}{\Omega[z_1,...,z_{\mathcal{N}}]}\{\check{g}_r\Omega[z_1,...,z_{\mathcal{N}},\check{g}_l]-\check{g}_l\Omega[z_1,...,z_{\mathcal{N}},\check{g}_r]\},$$

where z_j is as in Eq. (4.29). Using Eqs. (4.9) and (4.10), expanding the determinants $\Omega[z_1,...,z_{\mathcal{N}},\check{g}_l]$ and $\Omega[z_1,...,z_{\mathcal{N}},\check{g}_r]$ by their last columns and using the fact that $\check{g}_l(k,x)$, $\check{g}_r(k,x)$, and their derivatives are of $O(1/k)$ near $k=0$, we obtain

$$\Omega[z_1,...,z_{\mathcal{N}},\check{g}_l(k,x)]=\check{g}_l(k,x)D_1(x)-\check{g}_l'(k,x)D_2(x)+O(k),$$

where we have defined

$$D_1(x)=\begin{vmatrix} z_1' & \cdots & z_{\mathcal{N}}' \\ \vdots & & \vdots \\ \beta_1^{\mathcal{N}}z_1 & \cdots & \beta_{\mathcal{N}}^{\mathcal{N}}z_{\mathcal{N}}\end{vmatrix}\quad\text{and}\quad D_2(x)=\begin{vmatrix} z_1 & \cdots & z_{\mathcal{N}} \\ \vdots & & \vdots \\ \beta_1^{\mathcal{N}}z_1 & \cdots & \beta_{\mathcal{N}}^{\mathcal{N}}z_{\mathcal{N}}\end{vmatrix}.$$

Similarly, as $k\to0$ we obtain

$$\Omega[z_1,...,z_{\mathcal{N}},\check{g}_r(k,x)]=\check{g}_r(k,x)D_1(x)-\check{g}_r'(k,x)D_2(x)+O(k),$$

and hence

$$A_3(k,x)=\left(\prod_{j=1}^{\mathcal{N}}\frac{1}{k+i\beta_j}\right)\frac{(-i)^{\mathcal{N}}D_2(x)}{\Omega[z_1,...,z_{\mathcal{N}}]}[\check{g}_l(k,x)\check{g}_r'(k,x)-\check{g}_r(k,x)\check{g}_l'(k,x)]+O(k).$$

As in Eq. (2.16), we have

$$\check{g}_l(k,x)\check{g}_r'(k,x)-\check{g}_r(k,x)\check{g}_l'(k,x)=[\check{g}_l(k,x);\check{g}_r(k,x)]=-\frac{2ik}{\check{T}(k)},$$

and hence $\check{T}(k)A_3(k,x)$ in Eq. (6.1) is $o(1)$ as $k\to0$. The argument for $\mathcal{N}$ odd and for the other entries of the matrix $\mathbf{N}(k,x)^{-1}\mathbf{M}(k,x)$ is similar. Thus the proof is complete. ■

Theorem 6.2: The Eqs. (5.36), (5.37), (5.43), and (5.44), with the boundary conditions (5.45)–(5.48), have exactly one solution satisfying $\theta_{\mathcal{N}+1}(x)=0$. The potentials V and U obtained from Eqs. (5.34) and (5.35), respectively, are of the form of the potentials $V^{[\mathcal{N}]}$ and $U^{[\mathcal{N}]}$ in Eqs. (4.7) and (4.15).

Proof: By Proposition 6.1, a solution satisfying $\theta_{\mathcal{N}+1}(x)=0$ exists. So it suffices to prove the uniqueness. Then it follows that $V(x)$ and $U(x)$ must be of the form (4.7) and (4.15) with $j=\mathcal{N}$, respectively. Define the ratios

$$c_s=\frac{f_r(i\beta_s,x)}{f_l(i\beta_s,x)}, \tag{6.2}$$

$$d_s=\frac{g_r(i\beta_s,x)}{g_l(i\beta_s,x)}, \tag{6.3}$$

which are related to the constants α_s and ℓ_s of Eqs. (4.6) and (4.28), respectively, by

$$c_s=\frac{(-1)^{\mathcal{N}-s}}{\alpha_s}, \tag{6.4}$$

$$d_s=\frac{(-1)^{\mathcal{N}-s}}{\ell_s}. \tag{6.5}$$

Note that Eq. (6.4) can be obtained by using Eqs. (4.2), (4.6), and the relation (pp. 286–287 of Ref. 13)

$$\kappa_s^{-1}=\frac{i}{c_s}\left[\frac{d}{dk}\left(\frac{1}{T(k)}\right)\right]_{k=i\beta_s}. \tag{6.6}$$

Similarly one can derive Eq. (6.5). Inserting Eqs. (5.30)–(5.33) with $\theta_{\mathcal{N}+1}(x)=0$ in Eqs. (6.2) and (6.3), for even $\mathcal{N}$ we obtain

$$c_se^{-2\beta_sx}=\frac{\check{m}_r(i\beta_s,x)[1+\Sigma_{j=1}^{\mathcal{N}/2}\gamma_{2j}(x)/\beta_s^{2j}]-\check{n}_r(i\beta_s,x)\Sigma_{j=1}^{\mathcal{N}/2}\gamma_{2j-1}(x)/\beta_s^{2j-1}}{\check{m}_l(i\beta_s,x)[1+\Sigma_{j=1}^{\mathcal{N}/2}\gamma_{2j}(x)/\beta_s^{2j}]+\check{n}_l(i\beta_s,x)\Sigma_{j=1}^{\mathcal{N}/2}\gamma_{2j-1}(x)/\beta_s^{2j-1}}, \tag{6.7}$$

$$d_se^{-2\beta_sx}=\frac{\check{n}_r(i\beta_s,x)[1+\Sigma_{j=1}^{\mathcal{N}/2}\theta_{2j}(x)/\beta_s^{2j}]-\check{m}_r(i\beta_s,x)\Sigma_{j=0}^{(\mathcal{N}-2)/2}\theta_{2j+1}(x)/\beta_s^{2j+1}}{\check{n}_l(i\beta_s,x)[1+\Sigma_{j=1}^{\mathcal{N}/2}\theta_{2j}(x)/\beta_s^{2j}]+\check{m}_l(i\beta_s,x)\Sigma_{j=0}^{(\mathcal{N}-2)/2}\theta_{2j+1}(x)/\beta_s^{2j+1}}, \tag{6.8}$$

and for $\mathcal{N}$ odd we obtain

$$c_se^{-2\beta_sx}=\frac{\check{n}_r(i\beta_s,x)[1+\Sigma_{j=1}^{(\mathcal{N}-1)/2}\gamma_{2j}(x)/\beta_s^{2j}]-\check{m}_r(i\beta_s,x)\Sigma_{j=0}^{(\mathcal{N}-1)/2}\gamma_{2j+1}(x)/\beta_s^{2j+1}}{\check{n}_l(i\beta_s,x)[1+\Sigma_{j=1}^{(\mathcal{N}-1)/2}\gamma_{2j}(x)/\beta_s^{2j}]+\check{m}_l(i\beta_s,x)\Sigma_{j=0}^{(\mathcal{N}-1)/2}\gamma_{2j+1}(x)/\beta_s^{2j+1}}, \tag{6.9}$$

$$d_se^{-2\beta_sx}=\frac{\check{m}_r(i\beta_s,x)[1+\Sigma_{j=1}^{(\mathcal{N}-1)/2}\theta_{2j}(x)/\beta_s^{2j}]-\check{n}_r(i\beta_s,x)\Sigma_{j=0}^{(\mathcal{N}-1)/2}\theta_{2j+1}(x)/\beta_s^{2j+1}}{\check{m}_l(i\beta_s,x)[1+\Sigma_{j=1}^{(\mathcal{N}-1)/2}\theta_{2j}(x)/\beta_s^{2j}]+\check{n}_l(i\beta_s,x)\Sigma_{j=0}^{(\mathcal{N}-1)/2}\theta_{2j+1}(x)/\beta_s^{2j+1}}. \tag{6.10}$$

If we view the constants c_s, $s=1,...,\mathcal{N}$ as given, then Eqs. (6.7) and (6.9) each constitute a system of $\mathcal{N}$ equations for the $\mathcal{N}$ unknowns $\gamma_1(x),...,\gamma_{\mathcal{N}}(x)$. Similarly, if the constants d_s $s=1,...,\mathcal{N}$ are given, then Eqs. (6.8) and (6.10) each constitute a system for the unknowns $\theta_1(x),...,\theta_{\mathcal{N}}(x)$. Now let $\mathcal{N}$ be even and consider the system in Eq. (6.7). Define

$$u_s(x)=\sum_{j=1}^{\mathcal{N}/2}\frac{\gamma_{2j}(x)}{\beta_s^{2j}},$$

$$v_s(x)=\sum_{j=1}^{\mathcal{N}/2}\frac{\gamma_{2j-1}(x)}{\beta_s^{2j-1}}.$$

Then from Eq. (6.7), we get

$$c_s=\frac{\check{f}_r(i\beta_s,x)[1+u_s(x)]-\check{g}_r(i\beta_s,x)v_s(x)}{\check{f}_l(i\beta_s,x)[1+u_s(x)]+\check{g}_l(i\beta_s,x)v_s(x)},\quad s=1,\ldots,\mathcal{N}. \tag{6.11}$$

Therefore, by using Eqs. (2.26) and (2.27), we can convert Eq. (6.11) into the system

$$\sum_{j=1}^{\mathcal{N}}\mathbf{D}_{s,j}(x)\gamma_j(x)=h_s(x),$$

where

$$\mathbf{D}_{s,2j-1}(x)=\frac{1}{\beta_s^{2j}}[h_s'(x)-\check{\rho}(x)h_s(x)], \tag{6.12}$$

$$\mathbf{D}_{s,2j}(x)=-\frac{1}{\beta_s^{2j}}h_s(x), \tag{6.13}$$

with $\check{\rho}(x)$ being the quantity defined before Eq. (5.42) and

$$h_s(x)=c_s\check{f}_l(i\beta_s,x)-\check{f}_r(i\beta_s,x).$$

Let $\mathbf{D}(x)$ denote the matrix with entries given by Eqs. (6.12) and (6.13). We want to show that det $\mathbf{D}(x)\neq 0$. Define the matrix $\mathbf{E}(x)$ with entries

$$\mathbf{E}_{2j-1,s}(x)=-\mathbf{D}^T_{s,\mathcal{N}-2j+2}(x)=\beta_s^{-\mathcal{N}+2j-2}h_s(x), \tag{6.14}$$

$$\mathbf{E}_{2j,s}(x)=\mathbf{D}^T_{s,\mathcal{N}-2j+1}(x)=\beta_s^{\mathcal{N}+2j-2}[h_s'(x)-\check{\rho}(x)h_s(x)], \tag{6.15}$$

where the superscript T denotes the matrix transpose. Then det $\mathbf{D}(x)=$det $\mathbf{E}(x)$ by using standard properties of determinants. We see that when we compute det $\mathbf{E}(x)$ the terms containing $\check{\rho}(x)$ can be dropped without changing the value of the determinant. Comparing the entries of $\mathbf{E}(x)$ with those of the matrix Ω in Eqs. (4.9) and (4.10), we obtain

$$\det\mathbf{E}(x)=\left(\prod_{j=1}^{\mathcal{N}}\beta_j^{-\mathcal{N}}\right)\Omega[h_1,\ldots,h_{\mathcal{N}}], \tag{6.16}$$

where $\Omega[h_1,\ldots,h_{\mathcal{N}}]$ is the determinant defined following Eq. (4.8). We have (Ref. 10) $\Omega[w_1,\ldots,w_{\mathcal{N}}]>0$, and hence det $\mathbf{D}(x)\neq 0$. Similarly, again for even $\mathcal{N}$, considering the system (6.8), we let

$$p_s(x)=\sum_{j=1}^{\mathcal{N}/2}\frac{\theta_{2j}(x)}{\beta_s^{2j}},\quad q_s(x)=\sum_{j=0}^{(\mathcal{N}-2)/2}\frac{\theta_{2j+1}(x)}{\beta_s^{2j+1}}.$$

Then, by using Eqs. (3.1) and (3.2) we obtain

$$\sum_{j=1}^{\mathcal{N}} \mathbf{F}_{s,j}(x)\theta_j(x)=r_s(x),$$

where

$$\mathbf{F}_{s,2j-1}(x)=\frac{1}{\beta_s^{2j}}\,[r_s'(x)+\check{\rho}(x)r_s(x)],$$

$$\mathbf{F}_{s,2j}(x)=-\frac{1}{\beta_s^{2j}}\,r_s(x),$$

$$r_s(x)=\check{g}_r(i\beta_s,x)-d_s\,\check{g}_l(i\beta_s,x).$$

Proceeding as in Eqs. (6.14)–(6.16) one obtains det $\mathbf{F}(x)\neq 0$.

The proof when $\mathcal{N}$ is odd is similar to that when $\mathcal{N}$ is even. Thus, in both cases we find that given the constants c_s and d_s, there are unique solutions $\{\gamma_1(x),...,\gamma_{\mathcal{N}}(x)\}$ and $\{\theta_1(x),...,\theta_{\mathcal{N}}(x)\}$, respectively, of the systems (6.7)–(6.10). ■

We now consider the possibility that $\theta_{\mathcal{N}+1}(x)\neq 0$. Thus far, in the generic case, we have only studied this problem in detail under the assumption that $w(x)$ in Eq. (5.55) has no zeros. From Eqs. (2.39) and (3.34) it follows that the values of the integral $\int_{-\infty}^{x}\check{\chi}(y)^{-2}\,dy$ lie in $[0,1/(ac_r)]$. Hence in order for $w(x)$ not to vanish, c in Eq. (5.55) must be in $(-\infty,-1/(ac_r)]\cup[0,\infty)$. Later we will comment on the case when $w(x)$ has a zero. In order to emphasize the dependence of $\check{\chi}$ on the parameter a, we will write $\check{\chi}(x;a)=\check{\chi}(x)=\check{m}_l(0,x)+a\check{m}_r(0,x)$. Define

$$\xi(x)=\check{\chi}(x;a)\left[\int_{-\infty}^{x}\check{\chi}(y;a)^{-2}\,dy+c\right]. \tag{6.17}$$

Then $\xi(x)$ and $\check{\chi}(x;a)$ are two linearly independent solutions of $\psi''=\check{V}(x)\psi$. Using Eqs. (2.10), (2.12), (2.39), and (3.34) we obtain

$$\xi(x)=\begin{cases}\left(\dfrac{1}{c_r}+ac\right)\check{m}_r(0,x)+o(x), & x\to+\infty\\ c\check{m}_l(0,x)+o(x), & x\to-\infty.\end{cases}$$

Hence $\xi(x)$ is of the form $\xi(x)=c\check{\chi}(x;b)$, where

$$b=a+\frac{1}{c_r c}. \tag{6.18}$$

Note that the value $b=0$ corresponds to $c=-1/(ac_r)$ and the value $b=\infty$ corresponds to $c=0$; note also that b can take any non-negative value except a. Letting

$$\check{\rho}(x;b)=\frac{\xi'(x)}{\xi(x)},$$

and using Eqs. (5.55) and (6.17), we obtain

$$\check{\rho}(x;b)=\check{\rho}(x;a)+\frac{1}{w(x)}. \tag{6.19}$$

Proposition 6.3: Suppose that $V(x)$ is generic and that $\{\theta_1(x),...,\theta_{\mathcal{N}+1}(x)\}$ is a solution of Eqs. (5.37) and (5.44) such that Eqs. (5.53) holds with $w(x)\neq 0$ for all x. Then the potential $U(x)$ is of the form (4.30). The associated potential $\check{U}$ is equal to $\check{U}(x;b)$ where b is given by Eq. (6.18).

Proof: If $\mathcal{N}$ is even, define

$$\tilde{\theta}_{2j}(x)=\theta_{2j}(x), \quad j=1,...,\frac{\mathcal{N}}{2}, \tag{6.20}$$

$$\tilde{\theta}_{2j+1}(x)=\theta_{2j+1}(x)-\frac{\theta_{2j}(x)}{w(x)}, \quad j=0,...,\frac{\mathcal{N}}{2}-1, \tag{6.21}$$

where $\theta_0(x)=1$. If $\mathcal{N}$ is odd, define

$$\tilde{\theta}_{2j+1}(x)=\theta_{2j+1}(x), \quad j=0,...,\frac{\mathcal{N}-1}{2}, \tag{6.22}$$

$$\tilde{\theta}_{2j}(x)=\theta_{2j}(x)-\frac{\theta_{2j-1}(x)}{w(x)}, \quad j=1,...,\frac{\mathcal{N}-1}{2}. \tag{6.23}$$

Note that $\tilde{\theta}_{\mathcal{N}+1}(x)$ is not defined and that $\tilde{\theta}_j(x)$ obeys the boundary conditions (5.46), (5.47), and (5.48) since $1/w(x)\to 0$ as $x\to\pm\infty$. For $\mathcal{N}$ even, let

$$n(k,x;b)=\frac{k^{\mathcal{N}}}{\Pi_{j=1}^{\mathcal{N}}(k+i\beta_j)}\left[\check{n}(k,x;b)\left(1+\sum_{j=1}^{\mathcal{N}/2}\left(\frac{i}{k}\right)^{2j}\tilde{\theta}_{2j}(x)\right)\right.$$
$$\left.+\mathbf{J}\check{m}(k,x)\sum_{j=0}^{(\mathcal{N}-2)/2}\left(\frac{i}{k}\right)^{2j+1}\tilde{\theta}_{2j+1}(x)\right], \tag{6.24}$$

and for $\mathcal{N}$ odd, let

$$n(k,x;b)=\frac{k^{\mathcal{N}}}{\Pi_{j=1}^{\mathcal{N}}(k+i\beta_j)}\left[\check{m}(k,x)\left(1+\sum_{j=1}^{(\mathcal{N}-1)/2}\left(\frac{i}{k}\right)^{2j}\tilde{\theta}_{2j}(x)\right)\right.$$
$$\left.+\mathbf{J}\check{n}(k,x;b)\sum_{j=0}^{(\mathcal{N}-1)/2}\left(\frac{i}{k}\right)^{2j+1}\tilde{\theta}_{2j+1}(x)\right]. \tag{6.25}$$

In other words, $n(k,x;b)$ is of the form (5.31) and (5.32), respectively, where the parameter of the Darboux transformation is b, $\theta_j(x)$ is replaced by $\tilde{\theta}_j(x)$ and $\tilde{\theta}_{\mathcal{N}+1}(x)=0$. Let $n(k,x)$ denote the solution of Eq. (1.12) associated with the given solution $\{\theta_1(x),...,\theta_{\mathcal{N}+1}(x)\}$ of Eqs. (5.37) and (5.44). By substituting Eqs. (6.20)–(6.23) in Eqs. (6.24) and (6.25) and using Eqs. (6.19), (2.28), and (2.29), one finds that $n(k,x)=n(k,x;b)$. Hence the potential $U(x)$ has the stated properties. ■

If $c\in(-1/(ac_r),0)$, then $w(x)$ in Eq. (5.55) has exactly one zero x_0, where

$$\int_{-\infty}^{x_0}\check{\chi}^{-2}(y;a)dy+c=0.$$

A similar situation occurs in the exceptional case; since $\check{\chi}(x)=\check{m}_l(0,x)$ approaches nonzero constant limits as $x\to\pm\infty$, we replace Eq. (5.55) by

$$w(x)=\check{m}_l(0,x)^2\left[\int_0^x \check{m}_l(0,y)^{-2}\,dy+c\right].$$

Thus $w(x)$ has exactly one zero for any c. A similar situation arises if we consider Eqs. (5.38) and (5.39) in the exceptional case and do not restrict the solutions by imposing Eq. (5.29). We will omit a detailed analysis of the case when $w(x)$ has a zero; a special case has been worked out in Ref. 14.

We will end this section with a simple observation that is easily obtained from Eqs. (6.2), (6.4), and (6.6). If we require that a potential $V\in L_1^1$ has support contained in a half line; i.e., if $V(x)=0$ for $x>a_1$ or $x<a_2$, then, as already mentioned in the Introduction, the bound state norming constants for $V(x)$ are determined by the scattering matrix corresponding to this potential and cannot be chosen arbitrarily. Consequently, in solving the inverse scattering problem for such potentials, contrary to the case of potentials whose support is not contained in a half line, the norming constants need not be specified and in fact cannot be specified arbitrarily. To see this, note that if $V(x)=0$ for $x>a_1$, we have $m_l(k,x)=1$ and $m_r(k,x)=1/T(k)+R(k)e^{2ikx}/T(k)$ for $x\geqslant a_1$, and hence $R(k)$ has a meromorphic extension to $\mathbf{C}^+$ such that the poles of $R(k)$ and those of $T(k)$ are the same; furthermore $R(k)e^{2ikx}\to 0$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$ for $x\geqslant a_1$. Thus at the bound state $k=i\beta_s$, from Eq. (6.2) it follows that

$$c_s=(-1)^{\mathcal{N}}\frac{\check{R}(i\beta_s)}{\check{T}(i\beta_s)}. \tag{6.26}$$

Using Eqs. (6.4) and (6.6) we then see that the norming constant for the bound state at $k=i\beta_s$ is uniquely determined by the scattering matrix. If the potential vanishes for $x<a_2$, a similar computation gives

$$c_s=(-1)^{\mathcal{N}}\frac{\check{L}(i\beta_s)}{\check{T}(i\beta_s)}, \tag{6.27}$$

and in this case $L(k)$ has a meromorphic extension to $\mathbf{C}^+$ with its poles identical to the poles of $T(k)$, and $L(k)e^{-2ikx}\to 0$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$ for $x\leqslant a_2$. Thus, potentials whose support is not contained in a half line cannot have bound state norming constants chosen arbitrarily. Let

$$B_l(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[m_l(k,x)-1]e^{-iky}\,dk,$$

$$B_r(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[m_r(k,x)-1]e^{-iky}\,dk.$$

In the Marchenko theory[2,10,13] of inverse scattering, the potential $V(x)$ is obtained as

$$V(x)=-2\frac{dB_l(x,0+)}{dx}=2\frac{dB_r(x,0+)}{dx}.$$

If a nontrivial reflection coefficient $R(k)$ has a meromorphic extension to $\mathbf{C}^+$ with poles identical to the poles of $T(k)$ and $R(k)e^{2ikx}\to 0$ as $k\to\infty$ for $x\geqslant a_1$, then from Eq. (3.13) of Ref. 15 it follows that

$$B_l(x,y)=\sum_{s=1}^{\mathcal{N}}2\beta_s\check{T}(i\beta_s)e^{-2i\beta_s x}m_l(i\beta_s,x)\left[c_s-(-1)^{\mathcal{N}}\frac{\check{R}(i\beta_s)}{\check{T}(i\beta_s)}\right], \tag{6.28}$$

and hence for such a reflection coefficient $R(k)$, the potential $V(x)$ vanishes for $x>a_1$ if and only if the norming constants are chosen as in Eq. (6.26). In a similar manner, we obtain that if a nontrivial reflection coefficient $L(k)$ has a meromorphic extension to $\mathbf{C}^+$ with poles identical to the poles of $T(k)$ and $L(k)e^{-2ikx}\to 0$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$ for $x\leqslant a_2$, then the potential $V(x)$ vanishes for $x<a_2$ if and only if the norming constants are chosen as in Eq. (6.27).

VII. WIENER–HOPF FACTORIZATION

By a Wiener–Hopf factorization of the matrix function $\mathbf{G}(k,x)$ we mean a factorization of the form

$$\mathbf{G}(k,x)=\mathbf{G}_-(k,x)\mathbf{D}(k)\mathbf{G}_+(k,x), \quad k\in\mathbf{R}, \tag{7.1}$$

where

$$\mathbf{D}(k)=\left(\frac{k-i}{k+i}\right)^{\rho_1}\mathbf{Q}_+ + \left(\frac{k-i}{k+i}\right)^{\rho_2}\mathbf{Q}_- .$$

The matrix functions $\mathbf{G}_\pm(k,x)$ are continuous in k for $k\in\mathbf{R}$ with continuous inverses and have analytic extensions into $\mathbf{C}^\pm$. Moreover, $\mathbf{G}_\pm(k,x)\to\mathbf{I}$ as $k\to\infty$ in $\overline{\mathbf{C}^+}$. The matrices $\mathbf{Q}_\pm$ are complementary rank-one projections. The numbers ρ_1 and ρ_2 are integers called "partial indices," and they are uniquely determined by $\mathbf{G}(k,x)$. If $\rho_1=\rho_2=0$ the factorization (7.1) is called canonical, otherwise noncanonical. For more information on Wiener–Hopf factorization of matrix functions, we refer the reader to Refs. 16 and 17. The following results are merely stated here since they can be easily verified using the results of the previous sections. They can be obtained by the method of Ref. 15. In Eqs. (5.1) and (5.18) we used a factorization that is not of the above form since the first factor there is not continuous at $k=0$ in the generic case. Instead of the factorizations used in Eqs. (5.1) and (5.18), one can also use the Wiener–Hopf factorization given below.

In the exceptional case, when there are no bound states, $\mathbf{G}(k,x)$ has the canonical factorization with

$$\mathbf{G}_-(k,x)=\mathbf{M}(-k,x), \quad \mathbf{G}_+(k,x)=\mathbf{q}\mathbf{M}(k,x)^{-1}\mathbf{q}, \quad \mathbf{D}(k)=\mathbf{I},$$

where $\mathbf{M}(k,x)$ is the solution of Eq. (1.18) constructed in Sec. III. In the exceptional case with bound states, we have a noncanonical factorization with factors

$$\mathbf{G}_-(k,x)=\left(\prod_{j=1}^{\mathcal{N}}\frac{k-i}{k-i\beta_j}\right)\mathbf{N}(-k,x),$$

$$\mathbf{G}_+(k,x)=\left(\prod_{j=1}^{\mathcal{N}}\frac{k+i\beta_j}{k+i}\right)\mathbf{q}\mathbf{N}(k,x)^{-1}\mathbf{q},$$

$$\mathbf{D}(k)=\left(\frac{k+i}{k-i}\right)^{\mathcal{N}}\mathbf{I}.$$

Here the matrix $\mathbf{N}(k,x)$ is given by Eq. (5.20). The partial indices are $\rho_1=\rho_2=-\mathcal{N}$.

In the generic case, with or without bound states, we have a noncanonical factorization. We state the factors separately for $\mathcal{N}$ even and $\mathcal{N}$ odd. For $\mathcal{N}$ even, we have

$$\mathbf{G}_{-}(k,x;a)=\left(\prod_{j=1}^{\mathcal{N}}\frac{k-i}{k-i\beta_j}\right)\mathbf{N}(-k,x;a)\left[\mathbf{Q}_{+}+\left(\frac{k}{k-i}\right)\mathbf{Q}_{-}\right], \tag{7.2}$$

$$\mathbf{G}_{+}(k,x;a)=\left(\prod_{j=1}^{\mathcal{N}}\frac{k+i\beta_j}{k+i}\right)\left[\mathbf{Q}_{+}+\left(\frac{k+i}{k}\right)\mathbf{Q}_{-}\right]\mathbf{q}\mathbf{N}(k,x;a)^{-1}\mathbf{q}, \tag{7.3}$$

$$\mathbf{D}(k)=\left(\frac{k+i}{k-i}\right)^{\mathcal{N}}\left[\mathbf{Q}_{+}+\left(\frac{k-i}{k+i}\right)\mathbf{Q}_{-}\right], \tag{7.4}$$

and for $\mathcal{N}$ odd, we have

$$\mathbf{G}_{-}(k,x;a)=\left(\prod_{j=1}^{\mathcal{N}}\frac{k-i}{k-i\beta_j}\right)\mathbf{N}(-k,x;a)\left[\left(\frac{k}{k-i}\right)\mathbf{Q}_{+}+\mathbf{Q}_{-}\right],$$

$$\mathbf{G}_{+}(k,x;a)=\left(\prod_{j=1}^{\mathcal{N}}\frac{k+i\beta_j}{k+i}\right)\left[\left(\frac{k+i}{k}\right)\mathbf{Q}_{+}+\mathbf{Q}_{-}\right]\mathbf{q}\mathbf{N}(k,x;a)^{-1}\mathbf{q},$$

$$\mathbf{D}(k)=\left(\frac{k+i}{k-i}\right)^{\mathcal{N}}\left[\left(\frac{k-i}{k+i}\right)\mathbf{Q}_{+}+\mathbf{Q}_{-}\right],$$

where the projections $\mathbf{Q}_{\pm}$, are given by

$$\mathbf{Q}_{\pm}=\frac{1}{2}\begin{bmatrix}1 & \pm 1\\ \pm 1 & 1\end{bmatrix}.$$

Hence, in the generic case with bound states the partial indices are

$$\rho_1=\begin{cases}-\mathcal{N}, & \mathcal{N}\text{ even}\\ -\mathcal{N}+1, & \mathcal{N}\text{ odd},\end{cases}$$

$$\rho_2=\begin{cases}-\mathcal{N}+1, & \mathcal{N}\text{ even}\\ -\mathcal{N}, & \mathcal{N}\text{ odd}.\end{cases}$$

In the generic case without bound states, we can use Eqs. (7.2)–(7.4) with $\mathcal{N}=0$.

The sum of the partial indices of $\mathbf{G}(k,x)$ is equal to

$$\rho_1+\rho_2=\text{ind det }\mathbf{S}(k)=\frac{1}{2\pi}\arg\det[\mathbf{S}(k)]\,\Big|_{-\infty}^{\infty}=\frac{1}{2\pi}\left[\arg\frac{T(k)}{T(-k)}\right]_{-\infty}^{\infty},$$

where "ind" stands for the index.[18] Letting $\Theta(k)=\arg T(k)$ and using the argument principle and the continuity of $T(k)/T(-k)$, we have

$$\text{ind det }\mathbf{S}(k)=\frac{2}{\pi}[\Theta(+\infty)-\Theta(0+)]=\begin{cases}-2\mathcal{N}+1, & \text{generic case}\\ -2\mathcal{N}, & \text{exceptional case,}\end{cases}$$

which is equivalent to Levinson's theorem.[1,11]

VIII. EXAMPLES

The following examples have been included in order to allow the reader to check the various statements in the article. In Examples 1 and 3, we have chosen a scattering matrix that does not come from a potential in L_1^1 in order to avoid lengthy formulas.

Example 1: Let $V(x)=2\delta(x)$. This is the generic case without bound states. The scattering matrix is given by

$$T(k)=\frac{k}{k+i}, \quad R(k)=L(k)=\frac{-i}{k+i}.$$

Then

$$m_l(k,x)=\begin{cases}1, & x>0\\ 1+\frac{i}{k}(1-e^{-2ikx}), & x<0,\end{cases} \tag{8.1}$$

$$m_r(k,x)=\begin{cases}1+\frac{i}{k}(1-e^{2ikx}), & x>0\\ 1, & x<0.\end{cases} \tag{8.2}$$

Hence

$$\rho(x;a)=\begin{cases}\frac{2a}{1+a+2ax}, & x>0\\ \frac{-2}{1+a-2x}, & x<0,\end{cases}$$

and thus

$$n_l(k,x)=\begin{cases}1+\frac{i}{k}\frac{2a}{1+a+2ax}, & x>0\\ 1+\frac{i}{k}(1+e^{-2ikx})+\left[-\frac{i}{k}+\frac{1}{k^2}-\frac{e^{-2ikx}}{k^2}\right]\frac{2}{1+a-2x}, & x<0,\end{cases} \tag{8.3}$$

$$n_r(k,x)=\begin{cases}1+\frac{i}{k}(1+e^{2ikx})+\left[-\frac{i}{k}+\frac{1}{k^2}-\frac{1}{k^2}e^{2ikx}\right]\frac{2a}{1+a+2ax}, & x>0\\ 1+\frac{i}{k}\frac{2}{1+a-2x}, & x<0.\end{cases} \tag{8.4}$$

The potentials $U(x;a)$ are given by

$$U(x;a)=-2\delta(x)+H(x)\frac{8a^2}{(1+a+2ax)^2}+H(-x)\frac{8}{(1+a-2x)^2}. \tag{8.5}$$

Example 2: Let

$$T(k)=\frac{k+i\beta}{k-i\beta}, \quad R(k)=L(k)=0,$$

where $\beta>0$. Since $T(0)=-1$, this is the exceptional case with one bound state. Then using Eqs. (4.2)–(4.4) we obtain $\check{T}(k)=1$, $\check{R}(k)=\check{L}(k)=0$, and thus $\check{V}(x)=\check{U}(x)=0$ and $\check{m}_l(k,x)=\check{m}_r(k,x)=\check{n}_l(k,x)=\check{n}_r(k,x)=1$. By solving Eqs. (5.36), (5.37), (5.43), (5.44) with $\gamma_2(x)=\theta_2(x)=0$, we get

$$\gamma(x)=\beta\frac{1-ce^{-2\beta x}}{1+ce^{-2\beta x}},\quad \theta(x)=\beta\frac{1-de^{-2\beta x}}{1+de^{-2\beta x}},$$

where $\gamma(x)=\gamma_1(x)$, $\theta(x)=\theta_1(x)$, and $c=c_1$, $d=d_1$ are the constants in Eqs. (6.2) and (6.3), respectively. Then

$$m_l(k,x)=1-\frac{i\beta}{k+i\beta}\frac{2c}{c+e^{2\beta x}},\quad m_r(k,x)=1-\frac{2i\beta}{k+i\beta}\frac{e^{2\beta x}}{c+e^{2\beta x}},$$

$$n_l(k,x)=1-\frac{i\beta}{k+i\beta}\frac{2d}{d+e^{2\beta x}},\quad n_r(k,x)=1-\frac{2i\beta}{k+i\beta}\frac{e^{2\beta x}}{d+e^{2\beta x}},$$

and the potentials are given by

$$V(x)=\frac{-8c\beta^2e^{2\beta x}}{(c+e^{2\beta x})^2},\quad U(x)=\frac{-8d\beta^2e^{2\beta x}}{(d+e^{2\beta x})^2},$$

respectively.

Example 3: Consider the generic case with one bound state with

$$T(k)=\frac{k}{k-i},\quad R(k)=L(k)=\frac{i}{k-i}.$$

Then $\check{T}(k)$, $\check{R}(k)$, and $\check{L}(k)$ agree with the $T(k)$, $R(k)$, and $L(k)$ of Example 1. Thus $\check{m}(k,x)$ is given by Eqs. (8.1) and (8.2), and $\check{n}(k,x)$ is given by Eqs. (8.3) and (8.4). One finds

$$\gamma(x)=\begin{cases}\dfrac{2+(1-c)e^{-2x}}{2-(1-c)e^{-2x}}-\dfrac{2a}{1+a+2ax}, & x\geqslant 0\\[2ex] \dfrac{1-c-2ce^{-2x}}{1-c+2ce^{-2x}}+\dfrac{2}{1+a-2x}, & x\leqslant 0,\end{cases} \tag{8.6}$$

$$\frac{1}{\theta(x)}=\begin{cases}\dfrac{2+(1+d)e^{-2x}}{2-(1+d)e^{-2x}}-\dfrac{2a}{1+a+2ax}, & x\geqslant 0\\[2ex] \dfrac{1+d+2de^{-2x}}{1+d-2de^{-2x}}+\dfrac{2}{1+a-2x}, & x\leqslant 0,\end{cases}$$

with the same notation as in Example 2, and where a is the parameter of the Darboux transform. Having obtained $\gamma(x)$ and $\theta(x)$, one can use them in Eqs. (5.32) and (5.33) and thus obtain $m(k,x)$ and $n(k,x)$. The associated potentials can be found from Eqs. (5.34) and (5.35), where $\check{U}(x;a)$ is given by Eq. (8.5). From Eqs. (5.35), (8.5), and (8.6) it can be seen that the potential $V(x)$ only depends on the constant c. When $c=1$, we obtain $V(x)=-2\delta(x)$.

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