Structure of matrices transforming Stokes parameters

C. V. M. van der Mee and J. W. Hovenier
Department of Physics and Astronomy, Free University, De Boelelaan 1081, 1081 HV Amsterdam, The Netherlands

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The structure of matrices that represent a linear transformation of the Stokes parameters of a beam of light into the Stokes parameters of another beam of light is investigated by means of the so-called Stokes criterion. This holds that the degree of polarization of a beam of light can never be changed into a number larger than unity. Several general properties are derived for matrices satisfying the Stokes criterion. These are used to establish conditions for the elements of such matrices. Conditions that are either necessary, or sufficient or both are presented. General 4×4 matrices are treated and a number of special cases is worked out analytically. Several applications are pointed out.

I. INTRODUCTION

The intensity and the state of polarization of a (partially) polarized beam of light are completely determined by the four Stokes parameters I, Q, U, and V. The Stokes parameters form the components of a real four-vector $\mathbf{I}=\{I,Q,U,V\}$, called the Stokes vector,[1] which satisfies the Stokes criterion, i.e., we have

$$I\geqslant(Q^2+U^2+V^2)^{1/2}, \tag{1}$$

so that the degree of polarization

$$p=(Q^2+U^2+V^2)^{1/2}/I\leqslant1. \tag{2}$$

For an introduction of Stokes parameters we refer to Refs. 2–4.

In physics one encounters a multitude of real 4×4 matrices $\mathbf{M}$ that represent a linear transformation of (the Stokes parameters of) a beam of light into (the Stokes parameters of) another beam of light. In order that such a real 4×4 matrix $\mathbf{M}$ be physically meaningful, it must therefore at least have the following property: For every vector $\mathbf{I}_0=\{I_0,Q_0,U_0,V_0\}$ satisfying the inequality

$$I_0\geqslant({Q_0}^2+{U_0}^2+{V_0}^2)^{1/2}, \tag{3}$$

the product vector

$$\mathbf{I}=\mathbf{M}\mathbf{I}_0=\begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14}\\ M_{21} & M_{22} & M_{23} & M_{24}\\ M_{31} & M_{32} & M_{33} & M_{34}\\ M_{41} & M_{42} & M_{43} & M_{44}\end{bmatrix}\begin{bmatrix} I_0\\ Q_0\\ U_0\\ V_0\end{bmatrix} \tag{4}$$

satisfies Eq. (1). If this holds we will say that $\mathbf{M}$ satisfies the Stokes criterion. Examples of physically meaningful 4×4 matrices $\mathbf{M}$ are (i) the Mueller matrices describing optical devices such as polarizers and retarders,[5] (ii) matrices describing the reflection of radiation by smooth and rough surfaces,[6,7] and (iii) several matrices that are relevant to the single and multiple scattering of polarized radiation such as the scattering matrix, the phase matrix, the reflection and transmission matrices, and the matrices describing the internal radiation field in a plane-parallel atmosphere.[2,3,8–10]

The main purpose of this article is to derive conditions for a 4×4 matrix to satisfy the Stokes criterion. Conditions which are both necessary and sufficient, as well as separate necessary conditions and sufficient conditions, will be derived, for general 4×4 matrices and also for so-called block-diagonal matrices. While the research leading to the present article was in progress, the authors obtained in 1991 a preprint by Konovalov.[11] In his work, considering cones in four-dimensional space, necessary and sufficient conditions are derived for general 4×4 matrices in terms of the nonnegativity of a function of two angular variables. Konovalov[11] also gave necessary and sufficient conditions for certain special block-diagonal matrices. In another publication,[12] he only supplied conditions for such block-diagonal matrices. In the present article we seek a more transparent presentation and derivation of conditions for general 4×4 matrices transforming Stokes parameters. In addition, we systematically discuss the algebraic properties of general Stokes vectors and matrices satisfying the Stokes criterion and we work out a number of special cases. We also address the problem of the minimal structure of a matrix transforming Stokes parameters. Some conditions for general as well as special block-diagonal 4×4 scattering matrices to satisfy the Stokes criterion were obtained independently by Nagirner.[13] His approach is based on relativistic notations.

The present article may be viewed as part of a set of studies on the structure of matrices relevant to polarized light scattering for arbitrary scattering angles. For the most common scattering matrices which pertain to certain types of assemblies of particles,[3] some conditions (inequalities, in fact) were given by Kuščer and Ribarič.[14] These as well as some other conditions returned in subsequent articles,[4,15] as well as conditions (equalities and inequalities) for the scattering matrix of a single particle.[15–18] All these conditions were studied[19] by following the itinerary leading from the amplitude matrix through the scattering matrix of a single particle to the scattering matrix of an assembly of particles, and systemizing the relations valid at each stop. This resulted in a comprehensive survey of conditions for a variety of scattering angles. The analysis was extended to the phase matrix occurring, for instance, as the integral kernel of the equation of transfer of polarized light.[20] One surprising result of the work of Refs. 11 and 19 was the discovery of 4×4 matrices that, although satisfying the Stokes criterion, cannot possibly arise as scattering matrices of assemblies of particles. Clearly, the necessary conditions for an arbitrary real 4×4 matrix to satisfy the Stokes criterion, which will be considered in this paper, only constitute the minimal structure of a matrix transforming Stokes parameters into Stokes parameters.

II. TWO BY TWO MATRICES

Before treating the general 4×4 matrix $\mathbf{M}$ occurring in Eq. (4) we first consider conditions for a real 2×2 matrix to satisfy (the analog of) the Stokes criterion. Such matrices occur when averaging the phase matrix, the reflection and transmission matrices or the matrices describing the internal radiation field in a plane-parallel atmosphere over azimuth,[4,10] since in that case the corresponding transformation of the Stokes vector $\{I_0,Q_0,U_0,V_0\}$ is decoupled into separate transformations of the vectors $\{I_0,Q_0\}$ and $\{U_0,V_0\}$. First note that every two-vector $\mathbf{I}=\{I,Q\}$ satisfying $I\geqslant|Q|$ is a linear combination of the vectors $\{1,1\}$ and $\{1,-1\}$ with nonnegative coefficients. More precisely, $\{I,Q\}=\frac{1}{2}(I+Q)\{1,1\}+\frac{1}{2}(I-Q)\{1,-1\}$. Hence, a real matrix $\mathbf{M}$ with entries M_{11}, M_{12}, M_{21}, and M_{22} satisfies the 2×2 version of the Stokes criterion if and only if both $\mathbf{M}\{1,1\}$ and $\mathbf{M}\{1,-1\}$ are two-vectors $\mathbf{I}=\{I,Q\}$ satisfying $I\geqslant|Q|$. To see this, note that for a linear combination of two beams with nonnegative coefficients we have $I\geqslant|Q|$ if the corresponding inequality holds for each one of the beams. As a result, $\mathbf{M}$ satisfies the 2×2 version of the Stokes criterion if and only if the inequalities

$$M_{11}+M_{12}\geqslant|M_{21}+M_{22}|, \tag{5}$$

$$M_{11}-M_{12}\geqslant|M_{21}-M_{22}| \tag{6}$$

hold true.[21] These inequalities imply that M_{11} is nonnegative and it is not exceeded by the absolute value of any other element of $\mathbf{M}$.

The above treatment clarifies the complexity of our general problem. Giving necessary and sufficient conditions for a 3×3 or 4×4 matrix $\mathbf{M}$ to satisfy the Stokes criterion is not as straightforward as for a 2×2 matrix. The main reason is that the curvature of the boundary of the set of Stokes vectors in the three-vector and four-vector cases generally prevents one from reducing the problem to checking if each vector of a finite set of vectors satisfies the Stokes criterion. Instead, for a 4×4 matrix the problem is reduced to checking the images of the real vectors $\{1,q,u,v\}$ satisfying $q^2+u^2+v^2=1$ under the matrix $\mathbf{M}$. This leads to a minimization problem for a quadratic polynomial defined on the unit spherical surface (see Sec. IV).

III. GENERAL THEOREMS

In this section a number of general theorems will be discussed for general Stokes vectors and matrices satisfying the Stokes criterion.

Let $\mathscr{S}$ be the set of Stokes vectors satisfying Eq. (1). Then $\mathscr{S}$ has the following algebraic properties:

1. If $\mathbf{I}$ belongs to $\mathscr{S}$ and the constant $c\geqslant0$, then the vector $c\mathbf{I}=\{cI,cQ,cU,cV\}$ belongs to $\mathscr{S}$.
2. If $\mathbf{I}_1=\{I_1,Q_1,U_1,V_1\}$ and $\mathbf{I}_2=\{I_2,Q_2,U_2,V_2\}$ belong to $\mathscr{S}$, then their sum vector $\mathbf{I}_1+\mathbf{I}_2=\{I_1+I_2,Q_1+Q_2,U_1+U_2,V_1+V_2\}$ belongs to $\mathscr{S}$.
3. If both $\mathbf{I}$ and $-\mathbf{I}$ belong to $\mathscr{S}$, then $I=Q=U=V=0$.
4. A vector $\mathbf{I}$ belongs to $\mathscr{S}$ if and only if the inner product $\mathbf{I}\cdot\mathbf{I}_0=II_0+QQ_0+UU_0+VV_0$ is nonnegative for all $\mathbf{I}_0\in\mathscr{S}$.

The first property is obvious and the second property follows directly from Schwartz's inequality

$$|Q_1Q_2+U_1U_2+V_1V_2|\leqslant(Q_1^{\,2}+U_1^{\,2}+V_1^{\,2})^{1/2}\times(Q_2^{\,2}+U_2^{\,2}+V_2^{\,2})^{1/2}. \tag{7}$$

The third property is immediate from Eq. (1). Indeed, if $\mathbf{I}$ and $-\mathbf{I}$ belong to $\mathscr{S}$, then $I=0$ and hence, via Eq. (1), $I=Q=U=V=0$. To prove the fourth property, note that if $\mathbf{I}$, $\mathbf{I}_0\in\mathscr{S}$, then Schwartz's inequality (7) implies

$$\mathbf{I}\cdot\mathbf{I}_0\geqslant II_0-(Q^2+U^2+V^2)^{1/2}(Q_0^{\,2}+U_0^{\,2}+V_0^{\,2})^{1/2}\geqslant0. \tag{8}$$

Conversely, suppose $\mathbf{I}=\{I,Q,U,V\}$ and $\mathbf{I}\cdot\mathbf{I}_0\geqslant0$ for every vector $\mathbf{I}_0\in\mathscr{S}$, and put $W=(Q^2+U^2+V^2)^{1/2}$. Then, if $W=0$ (i.e., if $Q=U=V=0$), $II_0\geqslant0$ for any $I_0\geqslant0$ and hence $I\geqslant0$, which implies that $\mathbf{I}\in\mathscr{S}$. On the other hand, if $W>0$, then on choosing $\mathbf{I}_0=(1,-Q/W,-U/W,-V/W)$ we find $I\geqslant(Q^2+U^2+V^2)^{1/2}$, whence $\mathbf{I}\in\mathscr{S}$.

Matrices that satisfy the Stokes criterion have the following properties:

1. If $\mathbf{M}$ satisfies the Stokes criterion and the constant $c\geqslant0$, then the scalar product $c\mathbf{M}$ satisfies the Stokes criterion.

2. If $\mathbf{M}_1$ and $\mathbf{M}_2$ satisfy the Stokes criterion, then their sum $\mathbf{M}_1+\mathbf{M}_2$ satisfies the Stokes criterion.

3. If $\mathbf{M}_1$ and $\mathbf{M}_2$ satisfy the Stokes criterion, then their product $\mathbf{M}_1\mathbf{M}_2$ satisfies the Stokes criterion.

4. If $\mathbf{M}$ satisfies the Stokes criterion, then its transpose $\tilde{\mathbf{M}}$ satisfies the Stokes criterion.

The first two properties follow directly from the first two properties given for vectors in $\mathscr{I}$. The third property is immediate also. To prove the fourth property, let $\mathbf{M}$ satisfy the Stokes criterion and take $\mathbf{I}_1, \mathbf{I}_2 \in \mathscr{I}$. Then $\mathbf{MI}_2$ and $\mathbf{I}_1$ belong to $\mathscr{I}$ and hence the inner product $(\mathbf{MI}_2)\cdot\mathbf{I}_1 \geqslant 0$. This inner product can also be written as $\mathbf{I}_2\cdot(\tilde{\mathbf{M}}\mathbf{I}_1)$ and hence the nonnegativity of the inner product $\mathbf{I}_2\cdot(\tilde{\mathbf{M}}\mathbf{I}_1)$ for every $\mathbf{I}_2 \in \mathscr{I}$ implies that $\tilde{\mathbf{M}}\mathbf{I}_1 \in \mathscr{I}$, as a result of the fourth property of vectors in $\mathscr{I}$. But then the arbitrary choice of $\mathbf{I}_1$ as a vector in $\mathscr{I}$ entails that $\tilde{\mathbf{M}}$ satisfies the Stokes criterion, as claimed.

Let us derive a few consequences of the Stokes criterion or, in other words, necessary conditions. Suppose $\mathbf{M}$ satisfies the Stokes criterion. Then, for $j=2,3,4$, this is also the case for the three 3×3 submatrices of $\mathbf{M}$ obtained from $\mathbf{M}$ by removing its jth row and jth column, i.e., the matrices

$$\begin{bmatrix} M_{11} & M_{13} & M_{14} \\ M_{31} & M_{33} & M_{34} \\ M_{41} & M_{43} & M_{44} \end{bmatrix}, \begin{bmatrix} M_{11} & M_{12} & M_{14} \\ M_{21} & M_{22} & M_{24} \\ M_{41} & M_{42} & M_{44} \end{bmatrix}, \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \tag{9}$$

satisfy the corresponding 3×3 version of the Stokes criterion. To see this, one applies $\mathbf{M}$ to Stokes vectors $\mathbf{I}_0 = \{I_0, Q_0, U_0, V_0\}$ with $Q_0=0$, $U_0=0$, and $V_0=0$, respectively. By the same token, the three submatrices

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \begin{bmatrix} M_{11} & M_{13} \\ M_{31} & M_{33} \end{bmatrix}, \begin{bmatrix} M_{11} & M_{14} \\ M_{41} & M_{44} \end{bmatrix} \tag{10}$$

satisfy the corresponding 2×2 version of the Stokes criterion.

If we apply $\mathbf{M}$ to the Stokes vector $\{1,0,0,0\}$, we get

$$M_{11} \geqslant (M_{21}{}^2+M_{31}{}^2+M_{41}{}^2)^{1/2}. \tag{11}$$

On the other hand, on applying $\tilde{\mathbf{M}}$, which also satisfies the Stokes criterion, we find

$$M_{11} \geqslant (M_{12}{}^2+M_{13}{}^2+M_{14}{}^2)^{1/2}. \tag{12}$$

From Eqs. (11) and (12) we obtain

$$|M_{1j}| \leqslant M_{11}, \quad |M_{j1}| \leqslant M_{11}, \quad j=2,3,4. \tag{13}$$

If we apply $\mathbf{M}$ to the Stokes vectors $\{1,\pm 1,0,0\}$, $\{1,0,\pm 1,0\}$, and $\{1,0,0,\pm 1\}$, respectively, we get for $j=2,3,4$

$$M_{11} \pm M_{1j} \geqslant [(M_{21} \pm M_{2j})^2 + (M_{31} \pm M_{3j})^2 + (M_{41} \pm M_{4j})^2]^{1/2}, \tag{14}$$

and therefore

$$|M_{k1}+M_{kj}| \leqslant M_{11}+M_{1j}, \tag{15}$$

$$|M_{k1}-M_{kj}| \leqslant M_{11}-M_{1j}, \tag{16}$$

where $j,k=2,3,4$. Adding Eqs. (15) and (16) we find

$$|M_{kj}| \leqslant M_{11}, \quad j,k=2,3,4. \tag{17}$$

From Eqs. (13) and (17) it is now clear that M_{11} is nonnegative and is not exceeded by the absolute value of any other element of $\mathbf{M}$ if $\mathbf{M}$ satisfies the Stokes criterion. If we write down Eqs. (14)–(16) for the transpose $\tilde{\mathbf{M}}$ of $\mathbf{M}$, which also satisfies the Stokes criterion, we have

$$M_{11} \pm M_{j1} \geqslant [(M_{12} \pm M_{j2})^2 + (M_{13} \pm M_{j3})^2 + (M_{14} \pm M_{j4})^2]^{1/2}, \tag{18}$$

and for $j,k=2,3,4$

$$|M_{1k}+M_{jk}| \leqslant M_{11}+M_{j1}, \tag{19}$$

$$|M_{1k}-M_{jk}| \leqslant M_{11}-M_{j1}. \tag{20}$$

The last two equations can also be written as

$$|M_{j1}+M_{1k}| \leqslant M_{11}+M_{jk}, \tag{21}$$

$$|M_{j1}-M_{1k}| \leqslant M_{11}-M_{jk}. \tag{22}$$

Squaring the six inequalities (14) (three $\pm$ pairs) and adding the results yield

$$3M_{11}{}^2 + \sum_{j=2}^{4} M_{1j}{}^2 \geqslant 3\sum_{j=2}^{4} M_{j1}{}^2 + \sum_{j=2}^{4}\sum_{k=2}^{4} M_{jk}{}^2. \tag{23}$$

Starting from Eq. (18) we get in a similar way

$$3M_{11}{}^2 + \sum_{j=2}^{4} M_{j1}{}^2 \geqslant 3\sum_{j=2}^{4} M_{1j}{}^2 + \sum_{j=2}^{4}\sum_{k=2}^{4} M_{jk}{}^2. \tag{24}$$

Hence, adding Eqs. (23) and (24) we find after some algebra

$$\sum_{j=1}^{4}\sum_{k=1}^{4} M_{jk}{}^2 \leqslant 4M_{11}{}^2, \tag{25}$$

which generalizes a well-known relation for scattering matrices [cf. Ref. 15, Eq. (11); Ref. 20, Eq. (104)].

Finally, in view of the symmetry properties of many matrices transforming Stokes parameters,[22,23] we note that Properties 3 and 4 imply that if **M** satisfies the Stokes criterion so do $\mathbf{P}\tilde{\mathbf{M}}\mathbf{P}$ and $\mathbf{\Delta M \Delta}$, where $\mathbf{P}=\mathrm{diag}(1,1,-1,1)$ and $\mathbf{\Delta}=\mathrm{diag}(1,1,-1,-1)$.

IV. CONDITIONS FOR GENERAL MATRICES

In order to find necessary and sufficient conditions for a general real 4×4 matrix to satisfy the Stokes criterion, we first reduce the set of Stokes vectors $\mathbf{I}_0$ to which the matrix is to be applied, to those pertaining to fully polarized light. Note that for an arbitrary vector $\mathbf{I}_0 = \{I_0,Q_0,U_0,V_0\}$ in $\mathscr{I}$ and arbitrary real numbers q, u, and v such that $q^2+u^2+v^2=1$ the identity

$$\begin{bmatrix} I_0 \\ Q_0 \\ U_0 \\ V_0 \end{bmatrix} = \begin{bmatrix} (Q_0{}^2+U_0{}^2+V_0{}^2)^{1/2} \\ Q_0 \\ U_0 \\ V_0 \end{bmatrix} + \frac{\Delta}{2}\begin{bmatrix} 1 \\ q \\ u \\ v \end{bmatrix} + \frac{\Delta}{2}\begin{bmatrix} 1 \\ -q \\ -u \\ -v \end{bmatrix} \tag{26}$$

holds true, where

$$\Delta = I_0 - [Q_0{}^2+U_0{}^2+V_0{}^2]^{1/2} \tag{27}$$

and the vectors on the right-hand side of Eq. (26) are vectors $\{I,Q,U,V\}$ satisfying $I=(Q^2+U^2+V^2)^{1/2}$, i.e., they are Stokes vectors with degree of polarization 1. Equation (26) expresses the well-known fact[2] that an arbitrary beam of light can be decomposed in a fully polarized beam and an unpolarized beam, while the latter can be decomposed in two fully polarized beams. Hence, using Properties 1 and 2 of **I** [see Sec. III] we find that a 4×4 matrix **M** satisfies the Stokes criterion if and only if the image of any real vector $\{1,q,u,v\}$ with $q^2+u^2+v^2=1$ under **M** is a vector in $\mathscr{I}$. We may thus confine ourselves to incident beams that are fully polarized. Using Eq. (4) we now have

$$M\begin{bmatrix} 1 \\ q \\ u \\ v \end{bmatrix} = \begin{bmatrix} M_{11}+qM_{12}+uM_{13}+vM_{14} \\ M_{21}+qM_{22}+uM_{23}+vM_{24} \\ M_{31}+qM_{32}+uM_{33}+vM_{34} \\ M_{41}+qM_{42}+uM_{43}+vM_{44} \end{bmatrix}. \tag{28}$$

This vector belongs to $\mathscr{I}$ if and only if

$$M_{11}+qM_{12}+uM_{13}+vM_{14} \geqslant 0 \tag{29}$$

and

$$D(q,u,v) \geqslant 0 \tag{30}$$

for all real vectors $\{q,u,v\}$ with $q^2+u^2+v^2=1$, where

$$\begin{aligned} D(q,u,v) = {} & (M_{11}+qM_{12}+uM_{13}+vM_{14})^2 \\ & -(M_{21}+qM_{22}+uM_{23}+vM_{24})^2 \\ & -(M_{31}+qM_{32}+uM_{33}+vM_{34})^2 \\ & -(M_{41}+qM_{42}+uM_{43}+vM_{44})^2. \end{aligned} \tag{31}$$

In terms of the lightbeam with Stokes vector (28), $D(q,u,v)=I^2(1-p^2)$, where p is the degree of polarization. Now note that Eq. (29) is equivalent to the requirement that the inner product $\mathbf{m}\cdot\mathbf{I}\geqslant 0$, where $\mathbf{m} = \{M_{11},M_{12},M_{13},M_{14}\}$, for every real vector $\mathbf{I} = \{I,Q,U,V\}$ with $I=(Q^2+U^2+V^2)^{1/2}$. Using Eqs. (26) and (27), the latter inequality is equivalent to $\mathbf{m}\cdot\mathbf{I}\geqslant 0$ for every vector $\mathbf{I}\in\mathscr{I}$ and hence, according to the above Property 4 of Stokes vectors, to $\mathbf{m}\in\mathscr{I}$. In view of these considerations, the matrix **M** satisfies the Stokes criterion if and only if

$$M_{11} \geqslant (M_{12}{}^2+M_{13}{}^2+M_{14}{}^2)^{1/2} \tag{32}$$

and

$$\min_{q^2+u^2+v^2=1} D(q,u,v) \geqslant 0 \tag{33}$$

are simultaneously fulfilled. Hence, to prove that **M** satisfies the Stokes criterion we may check Eq. (32), determine the minimum of $D(q,u,v)$ under the constraint

$$q^2+u^2+v^2=1, \tag{34}$$

and prove it to be nonnegative. By applying Schwartz's inequality to Eq. (31) we find that a sufficient condition for **M** to satisfy the Stokes criterion is

$$[M_{11}-(M_{12}{}^2+M_{13}{}^2+M_{14}{}^2)^{1/2}]^2 \geqslant 2\sum_{i=2}^{4}\sum_{j=1}^{4} M_{ij}^2. \tag{35}$$

Replacing **M** by $\tilde{\mathbf{M}}$ it turns out that

$$[M_{11}-(M_{21}{}^2+M_{31}{}^2+M_{41}{}^2)^{1/2}]^2 \geqslant 2\sum_{i=1}^{4}\sum_{j=2}^{4} M_{ij}^2 \tag{36}$$

is another sufficient condition for **M** to satisfy the Stokes criterion.

Let us simplify Eq. (31). Defining

$$N_{ij}=M_{1i}M_{1j}-M_{2i}M_{2j}-M_{3i}M_{3j}-M_{4i}M_{4j} \tag{37}$$

for $1\leqslant i,j\leqslant 4$, we have

$$N_{ij}=N_{ji}=\text{real}, \quad 1\leqslant i,j\leqslant 4, \tag{38}$$

and

$$D(q,u,v)=N_{11}+q^2N_{22}+u^2N_{33}+v^2N_{44}+2qN_{12}+2uN_{13}+2vN_{14}+2quN_{23}+2qvN_{24}+2uvN_{34}. \quad (39)$$

Writing (q,u,v) in spherical coordinates,

$$q=\sin\theta\cos\varphi, \quad u=\sin\theta\sin\varphi, \quad v=\cos\theta, \quad (40)$$

where $0\leqslant\theta\leqslant\pi$ and $0\leqslant\varphi\leqslant 2\pi$, we may express Eq. (33) in the form

$$\min_{0\leqslant\theta\leqslant\pi,0\leqslant\varphi\leqslant 2\pi} D(\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta)\geqslant 0. \quad (41)$$

Using $\sin\theta=(1-v^2)^{1/2}$, we find

$$D((1-v^2)^{1/2}\cos\varphi,(1-v^2)^{1/2}\sin\varphi,v)$$
$$=[N_{11}+\tfrac{1}{2}(N_{22}+N_{33})+\tfrac{1}{2}(N_{22}-N_{33})\cos 2\varphi+N_{23}\sin 2\varphi]+v^2[-\tfrac{1}{2}(N_{22}+N_{33})-\tfrac{1}{2}(N_{22}-N_{33})\cos 2\varphi$$
$$-N_{23}\sin 2\varphi+N_{44}]+2vN_{14}+2(1-v^2)^{1/2}[N_{12}\cos\varphi+N_{13}\sin\varphi]+2v(1-v^2)^{1/2}[N_{24}\cos\varphi+N_{34}\sin\varphi]. \quad (42)$$

Consequently, the matrix **M** satisfies the Stokes criterion if and only if Eq. (32) holds and simultaneously the expression in Eq. (42) is nonnegative for all $-1\leqslant v\leqslant 1$ and $0\leqslant\varphi\leqslant 2\pi$.

For an arbitrary matrix **M** one may check numerically whether Eq. (42) is nonnegative for all $-1\leqslant v\leqslant 1$ and $0\leqslant\varphi\leqslant 2\pi$ by means of a small computer program. We have done so by computing $D((1-v^2)^{1/2}\cos\varphi, (1-v^2)^{1/2}\sin\varphi,v)$ in a number of cases for a sequence of (v,φ) grids with decreasing mesh. In this way we readily found the minimum in six significant figures. In particular, we checked the example of **M** given in the appendix of Ref. 11 and we corroborated its conclusion that his **M** does not satisfy the Stokes criterion.

V. SPECIAL TYPES OF MATRICES

In this section we derive necessary and sufficient conditions for certain special classes of matrices to satisfy the Stokes criterion.

A. Scattering matrices of single particles

The scattering matrix of a single particle satisfies the Stokes criterion in the form (33) in a rather obvious way. Indeed, for such scattering matrices we have the identities [cf. Ref. 19, Eqs. (147), (150), (152), (168), (155), (171), and (178)]

$$N_{11}=-N_{22}=-N_{33}=-N_{44}, \quad (43)$$

$$N_{12}=N_{13}=N_{14}=N_{23}=N_{24}=N_{34}=0, \quad (44)$$

so that in this case

$$D(q,u,v)=N_{11}(1-q^2-u^2-v^2)\equiv 0, \quad (45)$$

where N_{11} equals the squared absolute value of the determinant of the amplitude matrix.[19] Equation (45) expresses the fact that a beam of fully polarized light always remains fully polarized after scattering by a single particle. Equations (43)–(45) also hold for an assembly of particles that are either spherical or are maintaining the same orientation in space.[19] It is clear from the derivations of Eqs. (43)–(45) that they hold for all interactions between an electromagnetic wave and a body that can be described by means of a 2×2 matrix transforming the electric field components. Particularly, optical devices such as polarizers and retarders can be described by 4×4 Mueller matrices or 2×2 Jones matrices,[5] so that Eqs. (43)–(45) hold for such Mueller matrices and their products.

B. Diagonal matrices

If **M** is purely diagonal, which occurs for the Mueller matrices of some optical devices[5] and also for certain scattering and phase matrices,[3,10] we have

$$D(q,u,v)=M_{11}{}^2-q^2M_{22}{}^2-u^2M_{33}{}^2-v^2M_{44}{}^2, \quad (46)$$

so that **M** satisfies the Stokes criterion if and only if

$$M_{11}\geqslant|M_{jj}| \quad (47)$$

for $j=2,3,4$. This simple condition shows that in case **M** is nondiagonal it may be useful to seek a diagonalization.

C. Vanishing rows and columns

If the third and fourth row of **M** vanish, then Eqs. (29) and (31) show that the Stokes criterion holds for **M** if and only if

$$M_{11}+qM_{12}+uM_{13}+vM_{14}\geqslant|M_{21}+qM_{22}+uM_{23}+vM_{24}| \quad (48)$$

for all vectors $\{q,u,v\}$ with $q^2+u^2+v^2=1$. The latter inequality is easily seen to be equivalent to the pair of inequalities

$$M_{11}\pm M_{21}\geqslant[(M_{12}\pm M_{22})^2+(M_{13}\pm M_{23})^2+(M_{14}\pm M_{24})^2]^{1/2}. \tag{49}$$

Hence, $\mathbf{M}$ satisfies the Stokes criterion if and only if Eq. (49) holds true. Similarly, if the third and fourth column of $\mathbf{M}$ vanish, then by considering the transpose of $\mathbf{M}$ one sees that $\mathbf{M}$ satisfies the Stokes criterion if and only if the pair of inequalities

$$M_{11}\pm M_{12}\geqslant[(M_{21}\pm M_{22})^2+(M_{31}\pm M_{32})^2+(M_{41}\pm M_{42})^2]^{1/2} \tag{50}$$

holds true. Analogous necessary and sufficient conditions may be formulated if, for instance, the second and the fourth row or the second and the third column of $\mathbf{M}$ vanish.

D. Block-diagonal matrices

Let us now derive necessary and sufficient conditions in order that the block-diagonal matrix

$$\mathbf{M}=\begin{bmatrix} a_1 & b_1 & 0 & 0 \\ c_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & c_2 & a_4 \end{bmatrix} \tag{51}$$

satisfies the Stokes criterion. Physical examples of matrices of this type are provided by light scattered in certain atmospheres or reflected by certain surfaces if the direction of the incident light is perpendicular to the atmosphere or surface.[10] Another example of Eq. (51) is the reflection matrix of a surface or atmosphere if the mirror principle holds and the planes of incidence and reflection coincide [cf. Ref. 23, Eq. (3)]. We will exploit the block-diagonal structure of the matrix $\mathbf{M}$ in Eq. (51) and diagonalizations. To do so, we define for any pair of real numbers (α,γ) the matrix

$$\mathbf{R}(\alpha,\gamma)=\begin{bmatrix} \cosh\alpha & \sinh\alpha & 0 & 0 \\ \sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & \cos\gamma & -\sin\gamma \\ 0 & 0 & \sin\gamma & \cos\gamma \end{bmatrix}. \tag{52}$$

These matrices have the following algebraic properties:

1. $\mathbf{R}(\alpha_1,\gamma_1)\mathbf{R}(\alpha_2,\gamma_2)=\mathbf{R}(\alpha_1+\alpha_2,\gamma_1+\gamma_2)$.
2. $\mathbf{R}(0,0)$ is the identity matrix.
3. $\mathbf{R}(\alpha,\gamma)$ is an invertible matrix and its inverse is given by $\mathbf{R}(-\alpha,-\gamma)$. Hence, the matrices $\mathbf{R}(\alpha,\gamma)$ form a group with respect to matrix multiplication.
4. $\mathbf{R}(\alpha,\gamma)$ satisfies the Stokes criterion. Indeed, if $\mathbf{I}_0=\{I_0,Q_0,U_0,V_0\}$ belongs to $\mathscr{S}$ and $\mathbf{I}=\{I,Q,U,V\}=\mathbf{R}(\alpha,\gamma)\mathbf{I}_0$, then $I^2-Q^2-U^2-V^2=I_0^2-Q_0^2-U_0^2-V_0^2$ and I and I_0 have the same sign.

Let us try to utilize the matrices $\mathbf{R}(\alpha,\gamma)$ and $\mathbf{R}(\beta,\delta)$ for specific pairs (α,γ) and (β,δ) to diagonalize $\mathbf{M}$. Exploiting the block-diagonal structure of $\mathbf{M}$, we diagonalize the 2×2 matrix

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & a_2 \end{bmatrix}, \tag{53}$$

by pre- and postmultiplying it by the matrices

$$\begin{bmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{bmatrix}, \begin{bmatrix} \cosh\beta & -\sinh\beta \\ -\sinh\beta & \cosh\beta \end{bmatrix}, \tag{54}$$

respectively, and the 2×2 matrix

$$\begin{bmatrix} a_3 & b_2 \\ c_2 & a_4 \end{bmatrix} \tag{55}$$

by pre- and postmultiplying it by the matrices

$$\begin{bmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{bmatrix}, \begin{bmatrix} \cos\delta & \sin\delta \\ -\sin\delta & \cos\delta \end{bmatrix}, \tag{56}$$

respectively. The special case $c_1=b_1$ and $c_2=-b_2$ to be considered in the next subsection showed us that it is necessary to replace $\mathbf{M}$ by the matrix $\mathbf{\Sigma M}$, where $\mathbf{\Sigma}=\mathrm{diag}(1,-1,1,-1)$, before implementing the diagonalization. We easily compute that

$$\mathbf{R}(\alpha,\gamma)\mathbf{\Sigma M R}(-\beta,-\delta)=\begin{bmatrix} L_{11} & L_{12} & 0 & 0 \\ L_{21} & L_{22} & 0 & 0 \\ 0 & 0 & L_{33} & L_{34} \\ 0 & 0 & L_{43} & L_{44} \end{bmatrix}, \tag{57}$$

where L_{11}, L_{12}, L_{21}, L_{22}, L_{33}, L_{34}, L_{43}, and L_{44} are given in the Appendix. If we try to choose (α,β) in such a way that $L_{12}=L_{21}=0$, we find the two equalities

$$(b_1\pm c_1)\cosh(\alpha\pm\beta)=\pm(a_1\pm a_2)\sinh(\alpha\pm\beta). \tag{58}$$

This is only possible if the inequalities

$$|b_1+c_1|\leqslant a_1+a_2, \tag{59}$$

$$|b_1-c_1|\leqslant a_1-a_2 \tag{60}$$

are satisfied with the equality sign occurring only if both sides of Eqs. (59) and (60) vanish, as follows by using the identities $\cosh^2(\alpha\pm\beta)-\sinh^2(\alpha\pm\beta)=1$. Note that if $\mathbf{M}$ satisfies the Stokes criterion, Eqs. (59) and (60) are fulfilled [See Eqs. (5) and (6)]. In fact,

$$\cosh(\alpha-\beta)=\frac{a_1-a_2}{[(a_1-a_2)^2-(b_1-c_1)^2]^{1/2}}, \tag{61}$$

$$\sinh(\alpha-\beta)=\frac{-(b_1-c_1)}{[(a_1-a_2)^2-(b_1-c_1)^2]^{1/2}}, \tag{62}$$

$$\cosh(\alpha+\beta)=\frac{a_1+a_2}{[(a_1+a_2)^2-(b_1+c_1)^2]^{1/2}}, \tag{63}$$

$$\sinh(\alpha+\beta)=\frac{b_1+c_1}{[(a_1+a_2)^2-(b_1+c_1)^2]^{1/2}}. \tag{64}$$

In a similar fashion, $L_{34}=L_{43}=0$ leads to the pair of equalities

$$(a_3\pm a_4)\sin(\gamma\pm\delta)=\mp(b_2\mp c_2)\cos(\gamma\pm\delta). \tag{65}$$

As a result,

$$\cos(\gamma-\delta)=\frac{a_3-a_4}{[(a_3-a_4)^2+(b_2+c_2)^2]^{1/2}}, \tag{66}$$

$$\sin(\gamma-\delta)=\frac{b_2+c_2}{[(a_3-a_4)^2+(b_2+c_2)^2]^{1/2}}, \tag{67}$$

$$\cos(\gamma+\delta)=\frac{a_3+a_4}{[(a_3+a_4)^2+(b_2-c_2)^2]^{1/2}}, \tag{68}$$

$$\sin(\gamma+\delta)=\frac{-(b_2-c_2)}{[(a_3+a_4)^2+(b_2-c_2)^2]^{1/2}}, \tag{69}$$

where all right-hand sides of Eqs. (66)–(69) may also be multiplied by -1, but this will not affect the final result. Substituting Eqs. (61)–(64) and Eqs. (66)–(69) into Eq. (57) and utilizing Eqs. (A1)–(A8) in the process, we find

$$\mathbf{R}(\alpha,\gamma)\mathbf{\Sigma}\mathbf{M}\mathbf{R}(-\beta,-\delta)=\mathrm{diag}(\Delta_1,\Delta_2,\Delta_3,\Delta_4), \tag{70}$$

where

$$\Delta_1=\tfrac{1}{2}[(a_1-a_2)^2-(b_1-c_1)^2]^{1/2}$$
$$+\tfrac{1}{2}[(a_1+a_2)^2-(b_1+c_1)^2]^{1/2}, \tag{71}$$

$$\Delta_2=\tfrac{1}{2}[(a_1-a_2)^2-(b_1-c_1)^2]^{1/2}$$
$$-\tfrac{1}{2}[(a_1+a_2)^2-(b_1+c_1)^2]^{1/2}, \tag{72}$$

$$\Delta_3=\tfrac{1}{2}[(a_3-a_4)^2+(b_2+c_2)^2]^{1/2}$$
$$+\tfrac{1}{2}[(a_3+a_4)^2+(b_2-c_2)^2]^{1/2}, \tag{73}$$

$$\Delta_4=\tfrac{1}{2}[(a_3-a_4)^2+(b_2+c_2)^2]^{1/2}$$
$$-\tfrac{1}{2}[(a_3+a_4)^2+(b_2-c_2)^2]^{1/2}. \tag{74}$$

Note that

$$\Delta_1\geqslant|\Delta_2|,\quad \Delta_3\geqslant|\Delta_4|. \tag{75}$$

In the special case $c_1=b_1$ and $c_2=-b_2$ the numbers Δ_1, Δ_2, Δ_3, and Δ_4 are the eigenvalues of $\mathbf{\Sigma M}$, since in this case $\alpha=\beta$ and $\gamma=\delta$ [cf. Eqs. (62) and (67)].

Before continuing our discussion, we discuss the degenerate cases of the diagonalization. If both sides of Eq. (59) vanish and the inequality sign holds in Eq. (60), $\alpha+\beta$ is undetermined while $\alpha-\beta$ follows from Eqs. (61) and (62). Further, if both sides of Eq. (60) vanish and the inequality sign holds in Eq. (59), $\alpha-\beta$ is undetermined and $\alpha+\beta$ follows from Eqs. (63) and (64). Finally, if the two sides of both of Eqs. (59) and (60) vanish, α and β are both undetermined and Eq. (70) follows for all pairs (α,β) with $\Delta_1=\Delta_2=0$. In these three cases, Eq. (70) is obtained with Δ_1 and Δ_2 as in Eqs. (71) and (72). If the equality sign occurs in at least one of Eqs. (59) and (60) while the left-hand and right-hand sides of these equations are nonzero, one cannot find a pair (α,β) for which $L_{12}=L_{21}=0$; we will discuss this case shortly. As to the right lower block of $\mathbf{M}$, if $a_3=a_4$ and $b_2=-c_2$, $\gamma-\delta$ is undetermined, while $\gamma+\delta$ is undetermined if $a_3=-a_4$ and $b_2=c_2$. Nevertheless, Eq. (70) is still valid with Δ_3 and Δ_4 being the quantities defined in Eqs. (73) and (74), respectively.

Next, observe that $\mathbf{R}(\alpha,\gamma)\mathbf{\Sigma}\mathbf{M}\mathbf{R}(-\beta,-\delta)$ satisfies the Stokes criterion if and only if $\mathbf{M}$ does, because $\mathbf{R}(\alpha,\gamma)$, $\mathbf{\Sigma}$ and $\mathbf{R}(-\beta,-\delta)$ and their inverse matrices satisfy the Stokes criterion. Since a diagonal 4×4 matrix satisfies the Stokes criterion precisely when its (1,1) element is nonnegative and is not exceeded by the absolute value of any other element [cf. Eq. (47)], we find that the block-diagonal matrix given by Eq. (51) satisfies the Stokes criterion if and only if Eqs. (59) and (60) as well as the following condition are fulfilled [cf. Eq. (75)]:

$$[(a_3-a_4)^2+(b_2+c_2)^2]^{1/2}+[(a_3+a_4)^2+(b_2-c_2)^2]^{1/2}$$
$$\leqslant[(a_1-a_2)^2-(b_1-c_1)^2]^{1/2}$$
$$+[(a_1+a_2)^2-(b_1+c_1)^2]^{1/2}. \tag{76}$$

In giving necessary and sufficient conditions for a block-diagonal matrix to satisfy the Stokes criterion, we have hitherto refrained from discussing the situation in which the equality sign occurs in one of Eqs. (59) and (60) with the two sides being nonzero. If Eq. (59) is true and $a_1-a_2=|b_1-c_1|\neq 0$, Eqs. (59), (60), and (76) remain true if b_1 and c_1 are replaced by σb_1 and σc_1 for some $0<\sigma<1$ sufficiently close to unity. Then the corresponding matrix **M** satisfies the Stokes criterion. Passing to the limit as $\sigma\to 1$, this is also the case for the original matrix **M**. The same approximation argument may be applied if $a_1+a_2=|b_1+c_1|\neq 0$, with as a result that Eqs. (59), (60), and (76) are necessary and sufficient conditions for **M** to satisfy the Stokes criterion.

We can now give an interesting corollary. From the third property of matrices satisfying the Stokes criterion [see Sec. III] and Eq. (47) it is clear that if a general real 4×4 matrix **M** satisfies the Stokes criterion, this is also the case for the matrices $\frac{1}{2}(\mathbf{1}+\boldsymbol{\Delta})\mathbf{M}$, $\frac{1}{2}\mathbf{M}(\mathbf{1}+\boldsymbol{\Delta})$, and $\frac{1}{2}(\mathbf{M}+\boldsymbol{\Delta}\mathbf{M}\boldsymbol{\Delta})$, where $\boldsymbol{\Delta}=\mathrm{diag}\,(1,1,-1,-1)$ and **1** is the identity matrix. Note that these three matrices have zero third and fourth rows, zero third and fourth columns, and the form of Eq. (51), respectively. Consequently, Eqs. (49) and (50) are necessary conditions for a general **M** to satisfy the Stokes criterion and the same thing is true for Eqs. (59), (60), and (76) if we take

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & a_2 \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}, \tag{77}$$

$$\begin{bmatrix} a_3 & b_2 \\ c_2 & a_4 \end{bmatrix} = \begin{bmatrix} M_{33} & M_{34} \\ M_{43} & M_{44} \end{bmatrix}. \tag{78}$$

Similar conditions may be derived by replacing $\boldsymbol{\Delta}$ with either one of the diagonal matrices $\mathrm{diag}(1,-1,-1,1)$ or $\mathrm{diag}(1,-1,1,-1)$.

E. Special block-diagonal matrices

We now consider a matrix of the special block-diagonal form

$$\mathbf{M} = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & 0 & 0 \\ 0 & 0 & a_3 & b_2 \\ 0 & 0 & -b_2 & a_4 \end{bmatrix}. \tag{79}$$

A matrix of this type occurs as the scattering matrix in various physical situations,[3] such as

1. scattering by an assembly of randomly oriented particles each of which has a plane of symmetry (e.g., homogeneous spheres, spheroids, or finite cylinders),

2. scattering by an assembly having particles and their mirror particles in equal numbers and with random orientation, and

3. Rayleigh scattering by optically inactive particles with or without depolarization effects.

The phase matrices of these particles may also be of the type given by Eq. (79) [see Ref. 4, Eq. (88) with $\sigma_1=\sigma_2=\pi/2$]. Another physical example of Eq. (79) is the reflection matrix of a surface or atmosphere obeying the reciprocity principle as well as the mirror principle, if the directions of the incident and reflected beams lie in the same plane as the normal and make the same angle with the normal [see, e.g., Ref. 23, Eqs. (2) and (3)]. Applying the criterion embodied by Eqs. (59), (60), and (76) in the case $c_1=b_1$ and $c_2=-b_2$, we find that the matrix **M** in Eq. (79) satisfies the Stokes criterion if and only if the conditions

$$|b_1|\leqslant\tfrac{1}{2}(a_1+a_2)\leqslant a_1\,, \tag{80}$$

$$|a_3-a_4|+\{(a_3+a_4)^2+4b_2^2\}^{1/2} \leqslant (a_1-a_2)+\{(a_1+a_2)^2-4b_1^2\}^{1/2} \tag{81}$$

are satisfied. In the cases of scattering matrices of assemblies of spherical particles or reflection matrices of one-dimensionally rough surfaces[7] we have $a_1=a_2$ and $a_3=a_4$, so that in this case the matrix **M** satisfies the Stokes criterion if and only if

$$(a_3^2+b_1^2+b_2^2)^{1/2}\leqslant a_1\,. \tag{82}$$

Consequently, we have obtained Eqs. (80) and (81) as a special case of Eqs. (59), (60), and (76) which hold for a general block-diagonal matrix. However, using Eqs. (29) and (31) directly one might try to derive the condition described by Eqs. (80) and (81) from the necessary and sufficient conditions

$$|b_1|\leqslant a_1 \tag{83}$$

and, for all $\{q,u,v\}$ with $q^2+u^2+v^2=1$,

$$D(q,u,v)=(a_1^2-b_1^2)+q^2(b_1^2-a_2^2) -u^2(a_3^2+b_2^2)-v^2(b_2^2+a_4^2) +2qb_1(a_1-a_2)-2uvb_2(a_3-a_4)\geqslant 0. \tag{84}$$

By minimizing the right-hand side of Eq. (84), Konovalov[11] has obtained necessary and sufficient conditions for the matrix in Eq. (79) to satisfy the Stokes criterion, by working out Eqs. (83) and (84). His Theorem 3 is a seemingly different criterion for the matrix in Eq. (79) to satisfy the Stokes criterion. More precisely, putting $e=|b_1(a_1-a_2)|+a_2^2-b_1^2-d^2$ with d denoting the left-hand side of Eq. (81), his necessary and sufficient conditions consist of Eqs. (80) and (81) if $e\leqslant 0$, and only Eq. (80) if $e>0$.[24] However, from Eq. (80) we readily find

$$|b_1(a_1-a_2)| \leqslant \frac{a_1^2-a_2^2}{2} \leqslant \frac{a_1^2-a_2^2}{2} + (a_1-a_2)\left\{\left[\frac{a_1+a_2}{2}\right]^2 - b_1^2\right\}^{1/2}, \tag{85}$$

and hence

$$e+d^2 = |b_1(a_1-a_2)| + a_2^2 - b_1^2 \leqslant \left(\frac{a_1-a_2}{2} + \left\{\left[\frac{a_1+a_2}{2}\right]^2 - b_1^2\right\}^{1/2}\right)^2. \tag{86}$$

As a result, Theorem 3 of Ref. 11 is equivalent to Eqs. (80) and (81).

In the special case of scattering matrices of assemblies of spherical particles, where $a_1=a_2$ and $a_3=a_4$, we must then have $D(q,u,v)=(1-q^2)(a_1^2-a_3^2-b_1^2-b_2^2)\geqslant 0$ for all real vectors $\{q,u,v\}$ with $q^2+u^2+v^2=1$. Thus in this case the matrix **M** in Eq. (79) satisfies the Stokes criterion if and only if Eq. (82) is satisfied.

VI. MINIMAL STRUCTURE

In Sec. V we have considered the structure of matrices transforming Stokes parameters. This structure was expressed in a number of inequalities. It should be realized that in a particular situation more structure may exist. As an example we consider the scattering matrix of an assembly of particles.

On the basis of the amplitude matrix it has been shown that a scattering matrix of the form (79) satisfies the following inequalities [see Ref. 15; Ref. 19, Eqs. (238)–(241)]:

$$|a_2 \pm b_1| \leqslant a_1 \pm b_1, \tag{87}$$

$$|a_3 - a_4| \leqslant a_1 - a_2, \tag{88}$$

$$(a_3+a_4)^2 + 4b_2^2 \leqslant (a_1+a_2)^2 - 4b_1^2. \tag{89}$$

Apparently, Eqs. (80) and (87) are equivalent, while Eq. (81) is immediate from Eqs. (88) and (89). These inequalities imply that not every matrix of the type (79) satisfying the Stokes criterion can occur as the scattering matrix of an assembly of particles. An example is obtained by choosing a matrix of the form (79) which satisfies Eqs. (80) and (81) but violates one of the inequalities (88) and (89). In fact, Hovenier *et al.*[19] [$a_1=8$, $a_2=6$, $b_1=2\sqrt{6}$, $a_3=4$, $a_4=0$ and $b_2=0$] and Konovalov[11] [$a_1=1$, $a_2=0.8$, $b_1=0.82$, $a_3=0.19$, $a_4=0.4$, and $b_2=0$] have given examples of scattering matrices satisfying Eqs. (80), (81), and (89), which violate Eq. (88). Choosing $a_1=8$, $a_2=6$, $b_1=2\sqrt{6}$, $a_3=a_4=5.5$, and $b_2=0$ one gets an example of a scattering matrix which satisfies Eqs. (80), (81), and (88) but violates Eq. (89). Hence, Eqs. (88) and (89) do not follow from the Stokes criterion.

Generally, the scattering matrix, **M**, of an assembly of particles contains 16 different real elements. Even then six inequalities exist for its elements [see, e.g., Ref. 19, Eqs. (229)–(234); Ref. 15, Eqs. (10a)–(10f)] which can be derived from the amplitude matrices. As shown by the above examples, not all of these six inequalities follow from the Stokes criterion. In fact, Eqs. (229) and (234) of Hovenier *et al.*[19] do not follow from the Stokes criterion, but Eqs. (230)–(233) of that paper do, as is easily verified by applying **M** and $\tilde{\mathbf{M}}$ to the Stokes vectors $\{1, \pm 1,0,0\}$. Consequently, a physically meaningful matrix must satisfy the Stokes criterion, but if it does, one may need additional properties to let this matrix be a scattering matrix.

Summarizing, the minimal structure of a 4×4 matrix transforming Stokes parameters is given by the Stokes criterion. Additional structure is present if the transformation expressed by the 4×4 matrix can also be described by means of a 2×2 matrix [see Eqs. (43)–(45), and Ref. 19] or by a sum of such 4×4 matrices [see, e.g., Eqs. (87)–(89)]. On top of all this, some elements may vanish or equal others in absolute value, as a result of symmetry.[7,10,22–23]

VII. APPLICATIONS

The methods and results of the present study have various applications. One group of applications concerns checks on numerical and experimental results. If a matrix transforming Stokes vectors, which has been obtained by experimental or numerical means, violates the Stokes criterion or one of its spinoffs, there are two possibilities. The first option is that the experimental or numerical results contain a gross error. The second option is that the results are inaccurate, especially if the matrix under consideration narrowly fails to satisfy the Stokes criterion or one of its corollaries. The usefulness of inequalities for checking purposes may be assessed from the work of Stammes[25] and Kuik *et al.*[26] These authors have checked if the elements of theoretically and experimentally determined scattering matrices of several kinds of particles satisfy certain inequalities, as a safeguard against unreliable results in the absence of reliable comparison data. Another group of applications pertains to the use of "artificial" matrices in polarized light scattering studies,[27] and to interpolation and extrapolation of matrices obtained by experimental or numerical methods. When using such matrices one runs the risk of producing negative intensities or degrees of polarization exceeding unity. This may be avoided by making such matrices satisfy the Stokes criterion as well as other conditions whenever available.

The Stokes criterion and its consequences may also be used to obtain interesting theoretical results. The great potential of this approach has already been demonstrated.[28–31] In these papers inequalities for the elements of the scattering matrix were employed to derive inequalities for the coefficients obtained on expanding these elements in a series involving generalized spherical functions. In these four publications the Stokes criterion played an important role in the derivations. Further, the Stokes criterion for the scattering matrix, phase matrix, and reflection matrix of the ground surface has been exploited[32,33] to give a mathematical proof of the unique solvability of the equation of transfer of polarized light in a homogeneous plane-parallel medium. The Stokes criterion was applied[34] to prove the convergence of the iterative processes involving the adding method for homogeneous plane-parallel atmospheres. The Stokes criterion also plays a role in proving the completeness of the eigenfunctions of the equation of transfer of polarized light.[35]

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APPENDIX A: THE QUANTITIES L_{ij}

The elements of the matrix on the right-hand side of Eq. (57) are given by

$$L_{11}=\tfrac{1}{2}(a_1-a_2)\cosh(\alpha-\beta)+\tfrac{1}{2}(a_1+a_2)\cosh(\alpha+\beta)+\tfrac{1}{2}(b_1-c_1)\sinh(\alpha-\beta)-\tfrac{1}{2}(b_1+c_1)\sinh(\alpha+\beta), \tag{A1}$$

$$L_{12}=\tfrac{1}{2}(a_1-a_2)\sinh(\alpha-\beta)-\tfrac{1}{2}(a_1+a_2)\sinh(\alpha+\beta)+\tfrac{1}{2}(b_1-c_1)\cosh(\alpha-\beta)+\tfrac{1}{2}(b_1+c_1)\cosh(\alpha+\beta), \tag{A2}$$

$$L_{21}=\tfrac{1}{2}(a_1-a_2)\sinh(\alpha-\beta)+\tfrac{1}{2}(a_1+a_2)\sinh(\alpha+\beta)+\tfrac{1}{2}(b_1-c_1)\cosh(\alpha-\beta)-\tfrac{1}{2}(b_1+c_1)\cosh(\alpha+\beta), \tag{A3}$$

$$L_{22}=\tfrac{1}{2}(a_1-a_2)\cosh(\alpha-\beta)-\tfrac{1}{2}(a_1+a_2)\cosh(\alpha+\beta)+\tfrac{1}{2}(b_1-c_1)\sinh(\alpha-\beta)+\tfrac{1}{2}(b_1+c_1)\sinh(\alpha+\beta), \tag{A4}$$

$$L_{33}=\tfrac{1}{2}(a_3-a_4)\cos(\gamma-\delta)+\tfrac{1}{2}(a_3+a_4)\cos(\gamma+\delta)+\tfrac{1}{2}(b_2+c_2)\sin(\gamma-\delta)-\tfrac{1}{2}(b_2-c_2)\sin(\gamma+\delta), \tag{A5}$$

$$L_{34}=-\tfrac{1}{2}(a_3-a_4)\sin(\gamma-\delta)+\tfrac{1}{2}(a_3+a_4)\sin(\gamma+\delta)+\tfrac{1}{2}(b_2+c_2)\cos(\gamma-\delta)+\tfrac{1}{2}(b_2-c_2)\cos(\gamma+\delta), \tag{A6}$$

$$L_{43}=\tfrac{1}{2}(a_3-a_4)\sin(\gamma-\delta)+\tfrac{1}{2}(a_3+a_4)\sin(\gamma+\delta)-\tfrac{1}{2}(b_2+c_2)\cos(\gamma-\delta)+\tfrac{1}{2}(b_2-c_2)\cos(\gamma+\delta), \tag{A7}$$

$$L_{44}=\tfrac{1}{2}(a_3-a_4)\cos(\gamma-\delta)-\tfrac{1}{2}(a_3+a_4)\cos(\gamma+\delta)+\tfrac{1}{2}(b_2+c_2)\sin(\gamma-\delta)+\tfrac{1}{2}(b_2-c_2)\sin(\gamma+\delta). \tag{A8}$$

1 In this article we use the term "Stokes vector" for a four-vector $\{I,Q,U,V\}$ as above, where $I(I>0)$ may be an intensity, source function or flux and Q/I, U/I, and V/I describe the state of polarization.

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