Contemporary Mathematics
Volume **122**, 1991

WIENER-HOPF FACTORIZATION IN MULTIDIMENSIONAL INVERSE SCHRÖDINGER SCATTERING[1]

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ABSTRACT. We consider a Riemann-Hilbert problem arising in the study of the inverse scattering for the multidimensional Schrödinger equation with a potential having no spherical symmetry. It is shown that under certain conditions on the potential, the corresponding scattering operator admits a Wiener-Hopf factorization. The solution of the Riemann-Hilbert problem can be obtained using a similar factorization for the unitarily dilated scattering operator. We also study the connection between the Wiener-Hopf factorization and the Newton-Marchenko integral operator.

1. RIEMANN-HILBERT PROBLEM IN QUANTUM SCATTERING. Consider the n-dimensional Schrödinger equation ($n \geq 2$)

$$\Delta\psi + k^2\psi = V(x)\psi \tag{1.1}$$

where $x \in \mathbf{R}^n$, Δ is the Laplacian, k^2 is energy, and $V(x)$ is the potential. In nonrelativistic quantum mechanics the behavior of a particle in the force field of $V(x)$ is governed by (1.1).

Mathematics Subject Classification (1991). 35Q15, 81U05, 35R30, 47A40.

[1]This paper is in final form and no version of it will be submitted for publication elsewhere. The authors are indebted to Roger Newton for his help.

[2]The author is supported by the National Science Foundation under grant DMS 9001903.

[3]The author is supported by the National Science Foundation under grant DMS 8823102.

We assume that $V(x) \to 0$ as $|x| \to \infty$ in some sense which will be made precise in the next paragraph, but we do not assume any spherical symmetry for $V(x)$. As $|x| \to \infty$, the wavefunction ψ behaves as

$$\psi(k,x,\theta) = e^{ik\theta\cdot x} + ie^{-\frac{\pi}{4}i(n-1)}e^{ik|x|}|x|^{\frac{1-n}{2}}A(k,\frac{x}{|x|},\theta) + o(|x|^{\frac{1-n}{2}})$$

where $\theta \in S^{n-1}$ is a unit vector in $\mathbf{R}^n$ and $A(k,\theta,\theta')$ is the scattering amplitude. The scattering operator $S(k)$ is defined as

$$S(k,\theta,\theta') = \delta(\theta-\theta') + i(\frac{k}{2\pi})^{\frac{n-1}{2}}A(k,\theta,\theta'),$$

where δ is the Dirac delta distribution. In operator notation the above equation becomes

$$S(k) = \mathbf{I} + i(\frac{k}{2\pi})^{\frac{n-1}{2}}A(k).$$

All our results presented in this paper hold for real and locally square-integrable potentials $V(x) \in L^2_{\text{loc}}(\mathbf{R}^n)$ belonging to the class $\mathbf{B}_\alpha$ with $0 \le \alpha < 2$. Here $\mathbf{B}_\alpha$, $\alpha \in [0,2)$, denotes the class of potentials such that for some $s > \frac{3}{2} - \frac{1}{n}$, $(1+|x|^2)^s V(x)$ is a bounded linear operator from $H^\alpha(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$, where $H^\alpha(\mathbf{R}^n)$ denotes the Sobolev space of order α. For the reader whose interest is restricted to the case $n=3$, the following conditions on the potential will be sufficient:

1. There exist positive constants a and b such that for all $y \in \mathbf{R}^3$ we have
$$\int_{\mathbf{R}^3} dx\, |V(x)| \left(\frac{|x|+|y|+a}{|x-y|}\right)^2 \le b.$$

2. There exist constants $c > 0$ and $s > 1/2$ such that $|V(x)| \le c(1+|x|^2)^{-s}$ for all $x \in \mathbf{R}^3$.

3. There exist constants $\gamma > 0$ and $\beta \in (0,1]$ such that $\int_{\mathbf{R}^3} dx\, |x|^\beta\, |V(x)| < \gamma$.

4. $k=0$ is not an exceptional point. This condition is satisfied if there are neither bound states nor half-bound states at zero energy.

The inverse quantum scattering problem consists of recovering the potential $V(x)$ for all x when $S(k)$ is known for all k. Information about molecular, atomic, and subatomic particles is usually obtained from scattering experiments. An important problem in physics is to understand the forces between these particles. Solving the inverse scattering problem is equivalent to the determination of the force from the scattering data. For a review of the methods and open problems for 3-D inverse scattering prior to 1989 we refer the reader to **[Ne89]** and **[CS89]**. None of the methods developed to solve the multidimensional inverse problem have led to a complete and satisfactory solution yet, but there has been a lot of progress made in this research area especially during the last ten years. The methods to solve the multidimensional inverse scattering problem include the Newton-Marchenko method **[Ne80, Ne81, Ne82]**, the generalized Gel'fand-Levitan method **[Ne74, Ne80, Ne81, Ne82]**, the $\bar{\partial}$ method **[NA84, BC85, BC86, NH87]**, the generalized Jost-Kohn method **[Pr69, Pr76, Pr80, Pr82]**, a method that uses the Green's function of Faddeev **[Fa65, Fa74, Ne85]**, and the generalized Muskhelishvili-Vekua method **[AV91b]**. The principal idea behind the methods of Newton-Marchenko, generalized Gel'fand-Levitan, and generalized Muskhelishvili-Vekua is to formulate the inverse scattering problem as a Riemann-Hilbert problem and to transform this latter problem into an integral equation that uses the scattering data in its kernel and its inhomogeneous term. Then, the potential is recovered from the solution of the integral equation. Here we will solve the same Riemann-Hilbert problem by using a Wiener-Hopf factorization for operator functions utilizing some results of Gohberg and Leiterer **[GL73]**.

In the Schrödinger equation k appears as k^2, and as a result $\psi(-k,x,\theta)$ is also a solution whenever $\psi(k,x,\theta)$ is a solution. These two solutions are related to each other by the functional equation **[Ne80]**

$$\psi(k,x,\theta) = \int_{S^{n-1}} d\theta'\, S(k,-\theta,\theta')\, \psi(-k,x,\theta')$$

or equivalently

$$f_+(k,x,\theta) = \int_{S^{n-1}} d\theta'\, G(k,x,\theta,\theta')\, f_-(k,x,\theta'), \qquad k \in \mathbf{R} \tag{1.2}$$

where

$$f_\pm(k,x,\theta) = e^{\mp ik\theta\cdot x}\psi(\pm k,x,\pm\theta)$$

and

$$G(k,x,\theta,\theta') = e^{-ik(\theta-\theta')\cdot x} S(k,-\theta,-\theta'). \tag{1.3}$$

For potentials specified in the beginning of this section, in the absence of bound states, $f_\pm$ has an analytic extension in $k \in \mathbf{C}^\pm$. If there are bound states, these can be removed by the reduction technique **[Ne89]** before the analysis is carried out. Let us suppress the x-dependence and write (1.2) in vector form as

$$f_+(k) = G(k) f_-(k), \qquad k \in \mathbf{R},$$

or equivalently as

$$X_+(k) = G(k)\,X_-(k) + [G(k) - \mathbf{I}]\hat{1}, \qquad k \in \mathbf{R}, \tag{1.4}$$

where

$$X_\pm(k) = f_\pm(k) - \hat{1}.$$

For potentials considered in this paper $X_\pm \in L^2(S^{n-1})$, the Hilbert space of square integrable functions on S^{n-1}, and the strong limit of $f_\pm$ is $\hat{1}$ as $k \to \infty$ in $\mathbf{C}^\pm$. Note that in our notation $\mathbf{I}$ denotes the identity operator on $L^2(S^{n-1})$ and $\hat{1}$ denotes the vector in $L^2(S^{n-1})$ such that $\hat{1}(\theta) = 1$ for $\theta \in S^{n-1}$. Hence, (1.4) constitutes a Riemann-Hilbert problem: Given $G(k)$, determine $X_\pm(k)$. Note also that from (1.3) it is seen that $G(k)$ is the unitarily dilated scattering operator.

2. SOLUTION OF THE RIEMANN-HILBERT PROBLEM. We have the following result concerning the Wiener-Hopf factorization of the operator $G(k)$ that appears in the Riemann-Hilbert problem (1.4). In order to keep the discussion short, we assume that there are no bound states. If there are bound states, these can be removed by a reduction technique **[Ne80, Ne89]** before the factorization is accomplished. For details we refer the reader to **[AV90]**.

THEOREM 1. *For potentials as specified in Section 1, $G(k)$ defined in (1.3) has a (left) Wiener-Hopf factorization; i.e., there exist operators $G_+(k)$, $G_-(k)$, and $D(k)$ such that $G(k) = G_+(k)\,D(k)\,G_-(k)$ where*

1. *$G_+(k)$ is continuous in $\mathbf{C}^+$ in the operator norm of $\mathcal{L}(L^2(S^{n-1}))$ and is boundedly invertible there. Here $\mathcal{L}(L^2(S^{n-1}))$ denotes the Banach space of linear operators acting on*

$L^2(S^{n-1})$. Similarly, $G_-(k)$ is continuous in $\mathbf{C}^-$ in the operator norm of $\mathcal{L}(L^2(S^{n-1}))$ and is boundedly invertible there.

2. *$G_+(k)$ is analytic in $\mathbf{C}^+$ and $G_-(k)$ is analytic in $\mathbf{C}^-$.*

3. *$G_+(\pm\infty) = G_-(\pm\infty) = \mathbf{I}$.*

4. *$D(k) = P_0 + \sum_{j=1}^m \left(\frac{k-i}{k+i}\right)^{\rho_j} P_j$, where $P_1, \ldots, P_m$ are mutually disjoint, rank-one projections, and $P_0 = \mathbf{I} - \sum_{j=1}^m P_j$. The (left) partial indices $\rho_1, \ldots, \rho_m$ are nonzero integers. In case there are no partial indices; i.e., when $D(k) = \mathbf{I}$, the resulting Wiener-Hopf factorization becomes canonical.*

Note that, as seen from (1.3), $G(k)$ is a unitary transform of the scattering operator $S(k)$. In particular, when $x = 0$, $G(k)$ reduces to $S(k)$. The proof of Theorem 1 uses some results of Gohberg and Leiterer regarding factorization of operator functions on contours in the complex plane **[GL73]**. When $S(k)$ is boundedly invertible, is a compact perturbation of the identity, and $\tilde{S}(\xi) = S(i\frac{1+\xi}{1-\xi})$ is uniformly Hölder continuous on the unit circle $\mathbf{T}$ in the complex plane, its unitary transform $G(k)$ also satisfies these three conditions and admits a Wiener-Hopf factorization. The Hölder-continuity of $\tilde{S}(\xi)$ and $\tilde{G}(\xi) = G(i\frac{1+\xi}{1-\xi})$ can be established using either an additive representation of the scattering amplitude or a multiplicative representation. We refer the reader to **[AV90]** for the proof that uses an additive representation of the scattering amplitude and to **[AV91a]** for the proof that uses a multiplicative representation of the scattering amplitude. The conditions on the potential in 3-D specified in Section 1 were used in the additive representation, and the conditions specified in that section in n-D were used in the multiplicative representation. We also refer the reader to **[Ne90]** for various results related to the Wiener-Hopf factorization of the scattering operator; in this reference Professor Newton introduced a related factorization called the Jost function factorization and studied the relationship between these two factorizations; in this reference Theorem 5.1 gives a characterization of the scattering operator for the existence of a potential.

The solution of the Riemann-Hilbert problem (1.4) is obtained in terms of the Wiener-Hopf factors of $G(k)$ **[AV90]** and is given as

$$X_+(k) = [G_+(k) - \mathbf{I}]\hat{1} + G_+(k) \sum_{\rho_j > 0} \frac{\phi_j(k)}{(k+i)^{\rho_j}} \pi_j \tag{2.1}$$

$$X_-(k) = [G_-(k)^{-1} - \mathbf{I}]\hat{1} + G_-(k)^{-1} \sum_{\rho_j>0} \frac{\phi_j(k)\,\pi_j + [(k+i)^{\rho_j} - (k-i)^{\rho_j}]P_j\hat{1}}{(k-i)^{\rho_j}}, \tag{2.2}$$

provided $P_j\hat{1} = 0$ whenever $\rho_j < 0$. Here π_j is a fixed nonzero vector in the range of P_j, and $\phi_j(k)$ is an arbitrary polynomial of degree less than ρ_j associated with each $\rho_j > 0$. We can state our result as follows.

THEOREM 2. *For potentials as specified in Section 1, the Riemann-Hilbert problem* (1.4) *has a solution if and only if* $P_j\hat{1} = 0$ it *whenever* $\rho_j < 0$. *When this happens, the solution is given by* (2.1) *and* (2.2). *The solution, if it exists, is unique when the operator* $G(k)$ *has no positive partial indices.*

A simple condition that assures the unique solvability of the Riemann-Hilbert problem (1.4) is given by $\sup_{k\in\mathbf{R}} ||S(k) - \mathbf{I}|| < 1$, where the norm is the operator norm in $\mathcal{L}(L^2(S^{n-1}))$. If this holds, neither the scattering operator $S(k)$ nor its unitary transform $G(k)$ has any partial indices. As a result, in this case, (1.4) is uniquely solvable.

3. PARTIAL INDICES. In this section we relate the partial indices of the unitarily dilated scattering operator given in (1.3) to the Newton-Marchenko integral operator **[Ne89]**. We also discuss the relationship between solutions of the Riemann-Hilbert problem and the Newton-Marchenko integral equation. The proofs of the results stated in this section will be published elsewhere.

We let Q be the operator on $L^2(S^{n-1})$ such that $(Qf)(\theta) = f(-\theta)$. As in **[Ne89]** we define

$$\mathbf{G}(\alpha) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ik\alpha}[G(k) - \mathbf{I}]Q \tag{3.1}$$

and we also define the operators $\mathcal{G}$, $\mathcal{G}^*$, and $\mathcal{H}^*$ on $L^2(\mathbf{R}^+)$

$$(\mathcal{G}\eta)(\alpha) = \int_0^{\infty} d\beta\, \mathbf{G}(\alpha+\beta)\,\eta(\beta), \qquad \alpha > 0 \tag{3.2}$$

$$(\mathcal{G}^*\eta)(\alpha) = \int_0^{\infty} d\beta\, \mathbf{G}(-\alpha-\beta)\,\eta(\beta), \qquad \alpha > 0 \tag{3.3}$$

$$(\mathcal{H}^*\eta)(\alpha) = \int_0^{\infty} d\beta\, \mathbf{G}(-\alpha+\beta)\,\eta(\beta), \qquad \alpha > 0.$$

The Fourier transform maps $L^2(\mathbf{R}^+)$ onto the Hardy space of analytic operator functions $X_+(k)$ on $\mathbf{C}^+$ such that

$$\sup_{\epsilon>0} \int_{-\infty}^{\infty} dk \, \|X_+(k+i\epsilon)\|^2_{L^2(S^{n-1})} < +\infty.$$

We will denote this Hardy space by $\mathbf{H}_2^+$.

Defining

$$\eta(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-ik\alpha} X_+(k)$$

$$f(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, e^{-ik\alpha} [G(k) - \mathbf{I}]\hat{1},$$

from (1.4) we obtain

$$\eta(\alpha) = \int_{-\infty}^{\infty} d\beta \, \mathbf{G}(\alpha+\beta) \, \eta(\beta) \, + Q\eta(-\alpha) + f(\alpha), \qquad \alpha \in \mathbf{R}. \tag{3.4}$$

Since $X_+ \in \mathbf{H}_2^+$, we have $\eta(\alpha) = 0$ for $\alpha < 0$. Hence, we see that (3.4) is equivalent to

$$\begin{cases} \eta(\alpha) = \int_0^{\infty} d\beta \, \mathbf{G}(\alpha+\beta) \, \eta(\beta) + f(\alpha), & \alpha > 0 \\ 0 = \int_0^{\infty} d\beta \, \mathbf{G}(-\alpha+\beta) \, \eta(\beta) + Q\eta(\alpha) + f(-\alpha), & \alpha > 0. \end{cases} \tag{3.5}$$

We can write (3.5) in the form

$$\begin{cases} \eta = \mathcal{G}\eta + f \\ (Q + \mathcal{H}^*)\eta = -f^*, \end{cases} \tag{3.6}$$

where $f^*(\alpha) = f(-\alpha)$. Since (1.4) and (3.6) are equivalent, it follows that every solution $X_+ \in \mathbf{H}_2^+$ of the Riemann-Hilbert problem (1.4) leads to a solution $\eta \in L^2(\mathbf{R}^+)$ of (3.6), and conversely. The first equation in (3.6) is the Newton-Marchenko integral equation and $\mathcal{G}$ is the Newton-Marchenko integral operator.

Since $\tilde{G}(\xi)$ is Hölder continuous on $\mathbf{T}$, $\tilde{G}(\xi) - \mathbf{I}$ is a compact operator, and $\tilde{G}(\xi)$ is boundedly invertible for all $\xi \in \mathbf{T}$, it follows that $G(k)$ has a (left) Wiener-Hopf factorization **[GL73, AV90, AV91a]**. In that case, we can solve the Riemann-Hilbert problem (1.4) in terms of the Wiener-Hopf factors of $G(k)$ and obtain the following **[AV90, AV91a]**.

PROPOSITION 3. *There are finitely many, namely $\sum_{\rho_j>0} \rho_j$, linearly independent solutions of the homogeneous problem* (1.4) *where $F(k) \equiv 0$. The inhomogeneous terms $F(k)$*

for which at least one solution of the Riemann-Hilbert problem (1.4) *exists, form a closed subspace of* $L^2(\mathbf{R})$ *of co-dimension equal to* $-\sum_{\rho_j<0}\rho_j$.

Due to the fact that (1.4) and (3.6) are equivalent problems, the above results imply that for all $f, f^* \in L^2(\mathbf{R}^+)$, we have the following.

COROLLARY 4. *There are* $\sum_{\rho_j>0}\rho_j$ *linearly independent solutions* η *of the homogeneous problem* $(Q+\mathcal{H}^*)\eta = 0$. *The right-hand sides* $-f^*$ *for which at least one solution* η *of the equation* $(Q+\mathcal{H}^*)\eta = -f^*$ *exists, form a closed subspace of* $L^2(\mathbf{R}^+)$ *of finite co-dimension equal to* $-\sum_{\rho_j<0}\rho_j$.

The partial indices of the operator $G(k)$ given in (1.3) is related to the Newton-Marchenko operator $\mathcal{G}$ as in the following theorem. Note that $\mathcal{G}$ and $\mathcal{G}^*$ are defined in (3.2) and (3.3).

THEOREM 5. *The partial indices of* $G(k)$ *satisfy*

$$\sum_{\rho_j>0}\rho_j = \dim\,\mathrm{Ker}\,(\mathbf{I}-\mathcal{G}) + \dim\,\mathrm{Ker}\,(\mathbf{I}+\mathcal{G}),$$

$$-\sum_{\rho_j<0}\rho_j = \dim\,\mathrm{Ker}\,(\mathbf{I}-\mathcal{G}^*) + \dim\,\mathrm{Ker}\,(\mathbf{I}+\mathcal{G}^*).$$

Hence, $G(k)$ *has a canonical factorization if and only if* 1 *and* -1 *are not eigenvalues of* $\mathcal{G}$ *and* $\mathcal{G}^*$.

Combining the result of Theorem 5 given above and the results in Lemma 4.3 and Theorem 4.7 in **[Ne90]**, we have the following result. In the absence of bound states, for potentials whose scattering operators belong to the admissible class defined in **[Ne90]**, there are no partial indices. Also using Theorem 5 above and Corollary 4.5 in **[Ne90]**, we see that not only the sum index of $G(k)$ is independent of x **[AV90, AV91a]**, but also the sum of the negative partial indices of $G(k)$ is independent of x and the sum of the positive partial indices of $G(k)$ is independent of x. Since $\sup_{k\in\mathbf{R}}\|G(k)-\mathbf{I}\| = \sup_{k\in\mathbf{R}}\|S(k)-\mathbf{I}\|$, noting that $G(k) = S(k)$ for $x = 0$, it also follows that $\mathcal{G}$ and $\mathcal{G}^*$ do not have eigenvalues ± 1 if $\sup_{k\in\mathbf{R}}\|S(k)-\mathbf{I}\| < 1$. Thus, the Newton-Marchenko integral equation is uniquely solvable if $\sup_{k\in\mathbf{R}}\|S(k)-\mathbf{I}\| < 1$. Here the norms are the operator norm on $L^2(S^{n-1})$.

4. CONCLUSION. If the potential in (1.1) causes bound states, the analysis given in Sections 1, 2, and 3 remains valid, provided we replace $G(k)$ by the reduced operator

$G^{\mathrm{red}}(k)$ obtained after removing the bound states by the reduction technique of Newton **[Ne89, AV90]**. Theorem 5 given in Section 3 remains valid for $G(k)$ even in the presence of bound states.

Combining the result of Theorem 5 given above and the result in Lemma 4.3 in **[Ne90]**, we have the following result. When there are bound states, for potentials whose scattering operators belong to the admissible class defined in **[Ne90]**, the number of bound states $\mathcal{N}$ for the Schrödinger equation (1.1) is related to the sum of the negative partial indices of $G(k)$ as

$$\mathcal{N} = -\frac{1}{2} \sum_{\rho_j < 0} \rho_j .$$

For the same class of potentials, there are still no positive partial indices of $G(k)$.

Using Theorem 5 above and Corollary 4.5 in **[Ne90]**, it follows, even if there are bound states, that both the sum of the negative partial indices of $G(k)$ and that of the positive partial indices of $G(k)$ are independent of x.

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