Multidimensional Inverse Quantum Scattering Problem and Wiener-Hopf Factorization*

T. Aktosun[1] *and C. van der Mee*[2]

[1]Department of Mathematical Sciences, University of Texas at Dallas, Richardson, TX75083, USA
[2]Department of Mathematical Sciences, University of Delaware, Newark, DE19716, USA
*Research supported in part by the NSF grant DMS-8823102

Abstract: We consider the direct and inverse scattering for the n-dimensional Schrödinger equation, $n \geq 2$, with a potential having no spherical symmetry. Sufficient conditions are given for the existence of a Wiener-Hopf factorization of the corresponding scattering operator. This factorization leads to the solution of a related Riemann-Hilbert problem, which plays a key role in inverse scattering.

1. INTRODUCTION

Consider the n-dimensional ($n \geq 2$) Schrödinger equation

$$\Delta_x \psi + k^2 \psi = V(x)\psi \tag{1}$$

where $x \in \mathbf{R}^n$, Δ_x is the Laplacian, and $V(x)$ is the potential. In quantum mechanics the behavior of a particle in the force field of $V(x)$ is governed by (1). We assume that $V(x) \to 0$ as $|x| \to \infty$, but we do not assume any spherical symmetry for $V(x)$. All of our results hold for real and locally square-integrable potentials $V(x) \in L^2_{\text{loc}}(\mathbf{R}^n)$ belonging to the class $\mathbf{B}_\alpha$ with $0 \leq \alpha < 2$. Here $\mathbf{B}_\alpha$ denotes the class of potentials such that for some $s > 1/2$, $(1+|x|^2)^s V(x)$ is a bounded linear operator from $H^\alpha(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$, where $H^\alpha(\mathbf{R}^n)$ denotes the Sobolev space of order α. For the reader who is solely interested in the case $n = 3$, the following conditions on the potential will suffice:

1. $\int_{\mathbf{R}^3} dx\, |V(x)| \left(\frac{|x|+|y|+1}{|x-y|}\right)^2 < \infty, \quad \forall y \in \mathbf{R}^3.$

2. There exist some $c > 0$ and some $s > 1/2$ such that $|V(x)| \leq \frac{c}{(1+|x|^2)^s}, \quad \forall x \in \mathbf{R}^3.$

3. $\int_{\mathbf{R}^3} dx\, |x|^\beta\, |V(x)| < \infty$ for some $\beta \in (0,1]$.

4. $k = 0$ is not an exceptional point. This condition is satisfied if there are neither bound states nor half-bound states at zero energy.

As $|x| \to \infty$, the wavefunction ψ behaves as

$$\psi(k, x, \theta) = e^{ik\theta \cdot x} + ie^{-\frac{\pi}{4}i(n-1)} \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} A(k, \frac{x}{|x|}, \theta) + o(|x|^{\frac{1-n}{2}}) \tag{2}$$

where $\theta \in S^{n-1}$ is a unit vector in $\mathbf{R}^n$ and $A(k, \theta, \theta')$ is the scattering amplitude. The scattering operator $S(k)$ is defined as

$$S(k, \theta, \theta') = \delta(\theta - \theta') + i(\frac{k}{2\pi})^{\frac{n-1}{2}} A(k, \theta, \theta'), \tag{3}$$

where δ is the Dirac delta distribution.

The inverse quantum scattering problem consists of recovering $V(x)$ for all x from the knowledge of $S(k)$ for all k. Most information about molecular, atomic, and subatomic particles comes from scattering experiments, and one of the most important problems in physics is to understand the forces between these particles. Solving the inverse scattering problem accomplishes this task, and it can be described as the determination of the force from the scattering data. A comprehensive review of the methods and open problems for the 3-D inverse scattering prior to 1989 can be found in [**Ne89**] and in [**CS89**]. None of the methods developed to solve the multidimensional inverse problem have yet led to a complete and satisfactory solution, but there has been a lot of progress made in this research area. The methods to solve the multidimensional inverse scattering problem include the Newton-Marchenko method [**Ne80, Ne81, Ne82**], the generalized Gel'fand-Levitan method [**Ne74, Ne80, Ne81, Ne82**], the $\bar{\partial}$ method [**NA84, BC85, BC86, NH87**], the generalized Jost-Kohn method [**Pr69, Pr76, Pr80, Pr82**], and a method that uses the Green's function of Faddeev [**Fa65, Fa74, Ne85**].

Here we discuss the Wiener-Hopf factorization of the scattering operator and give sufficient conditions for the existence of such a factorization. The Wiener-Hopf factorization of a unitary transform of the scattering operator, which we also establish, leads to the solution of a Riemann-Hilbert problem which plays a major role in inverse quantum scattering. We only state our results here, and for proofs we refer the reader to [**AV89a,** AV89b]. To keep the discussion short, we only consider the case where the potentials do not have any bound states.

2. RIEMANN-HILBERT PROBLEM

It is possible to pose the inverse scattering problem as a vector-valued Riemann-Hilbert problem (Hilbert's twenty-first problem in a generalized sense). The solution of the Schrödinger equation satisfies the functional equation

$$\psi(k,x,\theta) = \int_{S^{n-1}} S(k,-\theta,\theta')\,\psi(-k,x,\theta')\,d\theta' \tag{4}$$

or equivalently

$$f_+(k,x,\theta) = \int_{S^{n-1}} G(k,x,\theta,\theta')\,f_-(k,x,\theta')\,d\theta', \quad k \in \mathbf{R} \tag{5}$$

where

$$f_\pm(k,x,\theta) = e^{\mp ik\theta\cdot x}\psi(\pm k,x,\pm\theta) \tag{6}$$

and

$$G(k,x,\theta,\theta') = e^{-ik(\theta-\theta')\cdot x}S(k,-\theta,-\theta'). \tag{7}$$

For potentials as specified in Section 1, $f_\pm$ has an analytic extension in $k \in \mathbf{C}^\pm$. Here $\mathbf{C}^+$ denotes the upper half complex plane and $\mathbf{C}^-$ denotes the lower half complex plane. Let us write (5) in vector form and suppress the x-dependence as

$$f_+(k) = G(k)f_-(k), \quad k \in \mathbf{R}, \tag{8}$$

or equivalently as

$$X_+(k) = G(k)\,X_-(k) + [G(k) - \mathbf{I}]\hat{1}, \tag{9}$$

where

$$X_\pm(k) = f_\pm(k) - \hat{1}. \tag{10}$$

In $L^2(S^{n-1})$, the Hilbert space of square integrable functions on S^{n-1}, the strong limit of $f_\pm$ is $\hat{1}$ as $k \to \infty$ in $\mathbf{C}^\pm$. Note that in our notation $\mathbf{I}$ denotes the identity operator on $L^2(S^{n-1})$ and $\hat{1}$ denotes the vector in $L^2(S^{n-1})$ such that $\hat{1}(\theta) = 1$, $\forall\theta \in S^{n-1}$. Hence, (9) constitutes a Riemann-Hilbert problem: From the knowledge of $G(k)$, determine $X_\pm(k)$.

3. WIENER-HOPF FACTORIZATION

We have the following result.

Theorem 1 For potentials as specified in Section 1, $G(k)$ defined in (7) has a (left) Wiener-Hopf factorization; i.e., there exist operators $G_+(k)$, $G_-(k)$, and $D(k)$ such that $G(k) = G_+(k)\, D(k)\, G_-(k)$ where

1. $G_+(k)$ is continuous in $\mathbf{C}^+$ in the operator norm of $\mathcal{L}(L^2(S^{n-1}))$ and is boundedly invertible there. Here $\mathcal{L}(L^2(S^{n-1}))$ denotes the Banach space of linear operators acting on $L^2(S^{n-1})$. Similarly, $G_-(k)$ is continuous in $\mathbf{C}^-$ in the operator norm of $\mathcal{L}(L^2(S^{n-1}))$ and is boundedly invertible there.

2. $G_+(k)$ is analytic in $\mathbf{C}^+$ and $G_-(k)$ is analytic in $\mathbf{C}^-$.

3. $G_+(\pm\infty) = G_-(\pm\infty) = \mathbf{I}$.

4. $D(k) = P_0 + \sum_{j=1}^m \left(\frac{k-i}{k+i}\right)^{\rho_j} P_j$, where $P_1, \ldots, P_m$ are mutually disjoint, rank-one projections, and $P_0 = \mathbf{I} - \sum_{j=1}^m P_j$. The (left) partial indices $\rho_1, \ldots, \rho_m$ are nonzero integers. In case there are no partial indices; i.e., when $D(k) = \mathbf{I}$, the resulting Wiener-Hopf factorization is called canonical.

Note that, as seen from (7), $G(k)$ is a unitary transform of the scattering operator $S(k)$. In particular, when $x = 0$, $G(k)$ reduces to $S(k)$. The proof of Theorem 1, using some results of Gohberg and Leiterer regarding factorization of operator functions on contours in the complex plane [**GL73**], is essentially based on the Hölder continuity of $S(k)$ and can be found in [**AV89a, AV89b**].

The solution of the Riemann-Hilbert problem (9) is obtained in terms of the Wiener-Hopf factors of $G(k)$ and can be written as

$$X_+(k) = [G_+(k) - \mathbf{I}]\hat{1} + G_+(k) \sum_{\rho_j > 0} \frac{\phi_j(k)}{(k+i)^{\rho_j}} \pi_j \tag{11}$$

and

$$X_-(k) = [G_-(k)^{-1} - \mathbf{I}]\hat{1} + G_-(k)^{-1} \sum_{\rho_j > 0} \frac{\phi_j(k)\pi_j + [(k+i)^{\rho_j} - (k-i)^{\rho_j}]P_j\hat{1}}{(k-i)^{\rho_j}}, \tag{12}$$

provided $P_j\hat{1} = 0$ whenever $\rho_j < 0$. Here π_j is a fixed nonzero vector in the range of P_j, and $\phi_j(k)$ is an arbitrary polynomial of degree less than ρ_j associated with each $\rho_j > 0$. We can state our result as follows.

Theorem 2 For potentials as specified in Section 1, the Riemann-Hilbert problem (9) has a solution if and only if $P_j\hat{1} = 0$ whenever $\rho_j < 0$. When this happens, the solution is given by (11) and (12).

A simple condition that assures the unique solvability of the Riemann-Hilbert problem (9) is given by $\sup_{k\in\mathbf{R}} ||S(k)-\mathbf{I}|| < 1$, where the norm is the operator norm in $\mathcal{L}(L^2(S^{n-1}))$. If this holds, neither the scattering operator $S(k)$ nor its unitary transform $G(k)$ has any partial indices. As a result, in this case, (9) is uniquely solvable.

REFERENCES

[AV89a] T. Aktosun and C. van der Mee, *Inverse Scattering Problem for the 3-D Schrödinger Equation and Wiener-Hopf Factorization of the Scattering Operator*, J. Math. Phys., to appear.

[AV89b] T. Aktosun and C. van der Mee, *Wiener-Hopf Factorization in the Inverse Scattering Theory for the n-D Schrödinger Equation*, preprint.

[BC85] R. Beals and R. R. Coifman, *Multidimensional Inverse Scattering and Nonlinear P.D.E.'s*, Proc. Symp. Pure Math. **43**, 45-70 (1985).

[BC86] R. Beals and R. R. Coifman, *The D-bar Approach to Inverse Scattering and Nonlinear Evolutions*, Physica D **18**, 242-249 (1986).

[CS89] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed., Springer, New York, 1989.

[Fa65] L. D. Faddeev, *Increasing Solutions of the Schrödinger Equation*, Sov. Phys. Dokl. **10**, 1033-1035 (1965) [Dokl. Akad. Nauk SSSR **165**, 514-517 (1965) (Russian)].

[Fa74] L. D. Faddeev, *Three-dimensional Inverse Problem in the Quantum Theory of Scattering*, J. Sov. Math. **5**, 334-396 (1976) [Itogi Nauki i Tekhniki **3**, 93-180 (1974) (Russian)].

[GL73] I. C. Gohberg and J. Leiterer, *Factorization of Operator Functions with Respect to a Contour. III. Factorization in Algebras*, Math. Nachrichten **55**, 33-61 (1973) (Russian).

[NA84] A. I. Nachman and M. J. Ablowitz, *A Multidimensional Inverse Scattering Method*, Studies in Appl. Math. **71**, 243-250 (1984).

[Ne74] R. G. Newton, *The Gel'fand-Levitan Method in the Inverse Scattering Problem in Quantum Mechanics.* In: J.A. Lavita and J.-P. Marchand (Eds.), *Scattering Theory in Mathematical Physics*, Reidel, Dordrecht, 1974, pp. 193-225.

[Ne80] R. G. Newton, *Inverse Scattering. II. Three Dimensions*, J. Math. Phys. **21**, 1698-1715 (1980); **22**, 631 (1981); **23**, 693 (1982).

[Ne81] R. G. Newton, *Inverse Scattering. III. Three Dimensions, Continued*, J. Math. Phys. **22**, 2191-2200 (1981); **23**, 693 (1982).

[Ne82] R. G. Newton, *Inverse Scattering. IV. Three Dimensions: Generalized Marchenko Construction with Bound States*, J. Math. Phys. **23**, 2257-2265 (1982).

[Ne85] R. G. Newton, *A Faddeev-Marchenko Method for Inverse Scattering in Three Dimensions*, Inverse Problems **1**, 371-380 (1985).

[Ne89] R. G. Newton, *Inverse Schrödinger Scattering in Three Dimensions*, Springer, New York, 1990.

[NH87] R. G. Novikov and G.M. Henkin, *Solution of a Multidimensional Inverse Scattering Problem on the Basis of Generalized Dispersion Relations*, Sov. Math. Dokl. **35**, 153-157 (1987) [Dokl. Akad. Nauk SSSR **292**, 814-818 (1987) (Russian)].

[Pr69] R. T. Prosser, *Formal Solution of Inverse Scattering Problems*, J. Math. Phys. **10**, 1819-1822 (1969).

[**Pr76**] R. T. Prosser, *Formal Solution of Inverse Scattering Problems. II*, J. Math. Phys. **17**, 1775-1779 (1976).

[**Pr80**] R. T. Prosser, *Formal Solution of Inverse Scattering Problems. III*, J. Math. Phys. **21**, 2648-2653 (1980).

[**Pr82**] R. T. Prosser, *Formal Solution of Inverse Scattering Problems. IV*, J. Math. Phys. **23**, 2127-2130 (1982).