WIENER-HOPF FACTORIZATION IN THE INVERSE SCATTERING THEORY FOR THE n-D SCHRÖDINGER EQUATION

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We study the n-dimensional Schrödinger equation, $n \geq 2$, with a nonspherically symmetric potential in the class of Agmon's short range potentials without any positive energy bound states. We give sufficient conditions that guarantee the existence of a Wiener-Hopf factorization of the corresponding scattering operator. We show that the potential can be recovered from the scattering operator by solving a related Riemann-Hilbert problem utilizing the Wiener-Hopf factors of the scattering operator. We also study the properties of the scattering operator and show that it is a trace class perturbation of the identity when the potential is also integrable.

1. INTRODUCTION

In this article we study the inverse scattering problem for the n-D Schrödinger equation

$$\nabla_x^2 \psi(k,x,\theta) + k^2 \psi(k,x,\theta) = V(x)\psi(k,x,\theta),$$

where $n \geq 2$, ∇_x^2 is the Laplacian, $x \in \mathbf{R}^n$ is the spatial coordinate, $\theta \in S^{n-1}$ is a unit vector in $\mathbf{R}^n$, and $k^2 \in \mathbf{R}$ is energy. The potential $V(x)$ is assumed to decrease to zero sufficiently fast as $|x| \to \infty$, but need not be spherically symmetric. Then, as $|x| \to \infty$, the wave function $\psi(k,x,\theta)$ behaves as

$$\psi(k,x,\theta) = e^{ik\theta\cdot x} + ie^{-\frac{\pi}{4}i(n-1)} \frac{e^{ik|x|}}{|x|^{\frac{n-1}{2}}} A(k, \frac{x}{|x|}, \theta) + o\left(\frac{1}{|x|^{\frac{n-1}{2}}}\right)$$

where $A(k,\theta,\theta')$ is the scattering amplitude. The scattering operator $S(k,\theta,\theta')$ is then defined as

$$S(k,\theta,\theta') = \delta(\theta-\theta') + i\left(\frac{k}{2\pi}\right)^{\frac{n-1}{2}} A(k,\theta,\theta'),$$

where δ is the Dirac delta distribution on S^{n-1}. The scattering operator acts on $L^2(S^{n-1})$, the Hilbert space of square-integrable complex-valued functions with respect to the surface Lebesgue measure on S^{n-1}. Then, in operator notation, the above equation becomes

$$S(k) = \mathbf{I} + i\left(\frac{k}{2\pi}\right)^{\frac{n-1}{2}} A(k).$$

Here **I** denotes the identity operator. $S(k)$ is known to be unitary and to satisfy

$$S(-k) = QS(k)^{-1}Q, \tag{1.1}$$

where Q is the signature operator on $L^2(S^{n-1})$ defined by $(Qf)(\theta) = f(-\theta)$.

The inverse scattering problem consists of retrieving the potential $V(x)$ from the scattering matrix $S(k)$. For one-dimensional and radial Schrödinger equations the inverse scattering problem is fairly well understood [**CS89**]. In higher dimensions the methods available to solve the inverse scattering problem have not yet led to a complete and satisfactory solution. These methods include the Newton-Marchenko method [**Ne80, Ne81, Ne82**], the generalized Gel'fand-Levitan method [**Ne74, Ne80, Ne81, Ne82**], the $\bar{\partial}$-method [**NA84, BC85, BC86, NH87**], the generalized Jost-Kohn method [**Pr69, Pr76, Pr80, Pr82**], a method based on the Green's function of Faddeev [**Fa65, Fa74, Ne85**], and the generalized Muskhelishvili-Vekua method [**AV89b**]. A comprehensive review of the methods and related open problems in 3-D inverse scattering prior to 1989 can be found in Newton's forthcoming book [**Ne89b**] and in Chapter XIV of [**CS89**].

The basic idea behind the Newton-Marchenko, Gel'fand-Levitan, and Muskhelishvili-Vekua methods is to formulate the inverse scattering problem as a Riemann-Hilbert boundary value problem and to use the Fourier transform to obtain a vector-valued integral equation on the half-line (the Newton-Marchenko method), or to use the solution of the Riemann-Hilbert problem in the kernel of an integral equation (the generalized Gel'fand-Levitan method), or to transform the Riemann-Hilbert problem into a Fredholm integral equation with a weakly singular kernel (the generalized Muskhelishvili-Vekua method). The key Riemann-Hilbert problem in n-D inverse scattering theory is given by (3.3), where the operator $G(k)$ is the x-dependent unitary transform of the scattering operator defined by

$$G(k) = U_x(k)QS(k)QU_x(k)^{-1}, \tag{1.2}$$

where $(U_x(k)f)(\theta) = e^{-ik\theta\cdot x}f(\theta)$. Note that we suppress the x-dependence of $G(k)$. The spectra of the three integral operators mentioned above are closely related to the partial indices of $G(k)$. Hence, the study of the Wiener-Hopf factorization of $G(k)$ not only leads to a direct solution of the Riemann-Hilbert problem (3.3) but also helps us to study the solvability of the integral equations in these three inversion methods.

This paper is organized as follows. In Section 2 we establish the Hölder continuity of the scattering operator by using the limiting absorption principle for the free Hamiltonian [**Ag75, Ku80**] and using the estimates given by Weder [**We90**]. In Section 3 using the Hölder continuity of the scattering operator and the results by Gohberg and Leiterer [**GL73**], we prove the existence of the Wiener-Hopf factorization for $G(k)$. In this section we also study the properties of the partial indices of $G(k)$, solve the Riemann-Hilbert problem (3.3) in terms of the Wiener-Hopf factors of $G(k)$, and show that the potential of the n-dimensional Schrödinger equation can be recovered from the scattering operator. Hence, the results in this paper generalize those in [**AV89a**] from 3-D to n-D. Note also that the generalized Muskhelishvili-Vekua method in 3-D given in [**AV89b**] is now seen to be valid also for n-D because the Hölder continuity of $G(k)$ is basically all that is needed in that method. In Section 4 we prove that the scattering operator $S(k)$ is a trace class perturbation of the identity and evaluate the trace of $S(k)-\mathbf{I}$ as $k\to\pm\infty$.

Throughout we will use the following notation. $\mathbf{C}$ is the complex plane, $\mathbf{C}^\pm=\{z\in\mathbf{C}:\pm\mathrm{Im}z>0\}$, $\mathbf{R}^+=(0,\infty)$, $\mathbf{R}_\infty=\mathbf{R}\cup\{\pm\infty\}$, $\mathbf{T}=\{z\in\mathbf{C}:|z|=1\}$, $\mathbf{T}^+=\{z\in\mathbf{C}:|z|<1\}$, $\mathbf{T}^-=\{z\in\mathbf{C}:|z|>1\}\cup\{\infty\}$ and $\mathbf{D}^\pm=\mathbf{C}^\pm\cup\mathbf{R}^+$. The closure of a set $\mathbf{P}$ in the Riemann sphere $\mathbf{C}_\infty=\mathbf{C}\cup\{\infty\}$ will be denoted by $\overline{\mathbf{P}}$.

The domain, kernel, range, and spectrum of a linear operator T will be denoted by $\mathcal{D}(T)$, $\mathcal{N}(T)$, $\mathcal{R}(T)$, and $\sigma(T)$, respectively. $\mathcal{L}(X;Y)$ will denote the set of bounded linear operators from the Banach space X into the Banach space Y, while $\mathcal{L}(X)$ will stand for $\mathcal{L}(X;X)$. The adjoint of T on a Hilbert space will be denoted by $T^\dagger$.

$\hat{u}(\xi)$ will denote the Fourier transform of $u\in L^2(\mathbf{R}^n)$; i.e.,

$$\hat{u}(\xi)=\lim_{N\to\infty}\frac{1}{(2\pi)^{n/2}}\int_{|x|\le N}e^{-i\xi\cdot x}u(x)dx,$$

and hence $\|u\|_2=\|\hat{u}\|_2$, where $\|\cdot\|_2$ is the norm in $L^2(\mathbf{R}^n)$. We will use $L^2_s(\mathbf{R}^n)$ to denote the Hilbert space of all complex measurable functions $u(x)$ on $\mathbf{R}^n$ such that $(1+|x|^2)^{s/2}u(x)\in L^2(\mathbf{R}^n)$, endowed with the norm

$$\|u\|_s=\|(1+|x|^2)^{s/2}u(x)\|_2. \tag{1.3}$$

By $C_0^\infty(\mathbf{R}^n)$ we will denote the linear space of all C^∞-functions on $\mathbf{R}^n$ of compact support. Then $H^\alpha(\mathbf{R}^n)$ will denote the Sobolev space of order α, which is the completion of $C_0^\infty(\mathbf{R}^n)$ in the norm

$$\|u\|_{H^\alpha(\mathbf{R}^n)}=\|(1+|\xi|^2)^{\alpha/2}\hat{u}(\xi)\|_2.$$

By $H^\alpha_s(\mathbf{R}^n)$ we denote the weighted Sobolev space of order (α,s), which is defined as

$$H^\alpha_s(\mathbf{R}^n)=\{u(x):(1+|x|^2)^{s/2}u(x)\in H^\alpha(\mathbf{R}^n)\}$$

with norm $\|u\|_{H^\alpha_s(\mathbf{R}^n)}=\|(1+|x|^2)^{s/2}u(x)\|_{H^\alpha(\mathbf{R}^n)}$. Note that we have $H^0_s(\mathbf{R}^n)=L^2_s(\mathbf{R}^n)$, $H^\alpha_0(\mathbf{R}^n)=H^\alpha(\mathbf{R}^n)$, and $H^0_0(\mathbf{R}^n)=L^2(\mathbf{R}^n)$.

For any Banach space $\mathcal{X}$, Borel set $\mathcal{J}$ in $\mathbf{C}_\infty$, and $\gamma\in(0,1]$, we will denote by $\mathcal{C}(\mathcal{J};\mathcal{X})$ the Banach space of all bounded and continuous functions $\psi:\mathcal{J}\to\mathcal{X}$ endowed with the norm

$$\|\psi\|_{\mathcal{C}(\mathcal{J};\mathcal{X})}=\sup_{t\in\mathcal{J}}\|\psi(t)\|_{\mathcal{X}},$$

and by $\mathcal{H}_\gamma(\mathcal{J};\mathcal{X})$ the Banach space of all uniformly Hölder continuous functions $\psi:\mathcal{J}\to\mathcal{X}$ endowed with the norm

$$\|\psi\|_{\mathcal{H}_\gamma(\mathcal{J};\mathcal{X})}=\sup_{t\in\mathcal{J}}\|\psi(t)\|_{\mathcal{X}}+\sup_{t\neq s\in\mathcal{J}}\frac{\|\psi(t)-\psi(s)\|_{\mathcal{X}}}{|t-s|^\gamma}.$$

Here the continuity pertains to the strong topology of $\mathcal{X}$. If $\mathcal{X}$ is a Banach algebra, so are $\mathcal{C}(\mathcal{J};\mathcal{X})$ and $\mathcal{H}_\gamma(\mathcal{J};\mathcal{X})$. For $\gamma\in(0,1)$, the closed subspace of $\mathcal{H}_\gamma(\mathcal{J};\mathcal{X})$ consisting of those $\psi : \mathcal{J} \to \mathcal{X}$ such that $\|\psi(t) - \psi(s)\|_{\mathcal{X}} = o(|t-s|^\gamma)$ as $t \to s$, will be denoted by $\mathcal{H}^0_\gamma(\mathcal{J};\mathcal{X})$. Note also that we will use the norm $\|\cdot\|$ without a subscript in order to denote the operator norm in $\mathcal{L}(L^2(S^{n-1}))$.

Acknowledgement. The research leading to this article was supported in part by the National Science Foundation under grant No. DMS 8823102.

2. ESTIMATES ON THE SCATTERING OPERATOR

In this section, starting with the representation given in (2.3), we prove that the scattering operator $S(k)$ and its unitary transform $G(k)$ defined in (1.2) are both Hölder continuous.

For $\alpha \geq 0$, a real function $V(x) \in L^2_{loc}(\mathbf{R}^n)$ is said to belong to the class $\mathbf{B}_\alpha$ if, for some $s > \frac{1}{2}$, the multiplication by $(1+|x|^2)^s V(x)$ represents a bounded linear operator from $H^\alpha(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$. (Our definition of $\mathbf{B}_\alpha$ differs from the one used in [**We90**] in that we require $V(x)$ to be locally L^2). For such potentials, the multiplication by $V(x)$ represents a bounded operator from $H^\alpha_{-t}(\mathbf{R}^n)$ into $L^2_t(\mathbf{R}^n)$ for all $t \in (\frac{1}{2}, s]$ with s as in the definition of $\mathbf{B}_\alpha$. It follows from results in Chapter 6 of [**Sc71**] that $V(x) \in \mathbf{B}_\alpha$ if $\exists \epsilon > 0, \beta \in (0, 2\alpha)$ such that

$$\sup_{x\in\mathbf{R}^n} (1+|x|^2)^{1+\epsilon} \int_{|x-y|\leq 1} dy \, \frac{|V(y)|^2}{|x-y|^{n-\beta}} < +\infty.$$

If $V(x) \in \mathbf{B}_0$; i.e., if $\exists c > 0, s > \frac{1}{2}$ such that

$$|V(x)| \leq \frac{c}{(1+|x|^2)^s}, \quad x \in \mathbf{R}^n,$$

then $V(x) \in \mathbf{B}_\alpha$ for every $\alpha \geq 0$.

A real function $V(x) \in L^2_{loc}(\mathbf{R}^n)$ is said to be **short range** or to belong to the class **SR** [**Ag75**] if for some $s > \frac{1}{2}$ the multiplication by $(1+|x|^2)^s V(x)$ represents a compact operator from $H^2(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$. Since $H^2(\mathbf{R}^n)$ is compactly imbedded in any of the spaces $H^\alpha(\mathbf{R}^n)$ for $0 \leq \alpha < 2$, we have

$$\mathbf{B}_\alpha \subset \mathbf{SR}, \quad 0 \leq \alpha < 2.$$

Now let $\mathbf{K}_2$ denote the set of all real functions $V(x) \in L^2_{loc}(\mathbf{R}^n)$ such that the multiplication by $V(x)$ represents a compact operator from $H^2(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$ [**We90**]. Then $\mathbf{SR} \subset \mathbf{K}_2$ and hence

$$\mathbf{B}_\alpha \subset \mathbf{SR} \subset \mathbf{K}_2, \quad 0 \leq \alpha < 2.$$

Let $H_0 = -\nabla^2_x$ be the free Hamiltonian with domain $\mathcal{D}(H_0) = H^2(\mathbf{R}^n)$. Then H_0 is a selfadjoint operator on $L^2(\mathbf{R}^n)$ with absolutely continuous spectrum $[0,\infty]$. Define

$$R_0^\pm(\lambda) = \lim_{z\to\lambda}(H_0 - z)^{-1}, \quad \lambda \in \mathbf{R}^+,\ z \in \mathbf{C}^\pm. \tag{2.1}$$

According to the limiting absorption principle, the limit in (2.1) exists in the uniform operator topology of $\mathcal{L}(L^2_s(\mathbf{R}^n); H^2_{-s}(\mathbf{R}^n))$ for $s > \frac{1}{2}$ **[Ag75, Ku80]**. $R_0^\pm(\lambda)$ can be extended to $\mathbf{C}^\pm$ by defining

$$R_0^\pm(z) = (H_0 - z)^{-1}, \quad z \in \mathbf{C}^\pm. \tag{2.2}$$

Then from (2.1) and (2.2) it follows that $R_0^\pm(z)$ is a bounded linear operator from $L^2_s(\mathbf{R}^n)$ into $H^2_{-s}(\mathbf{R}^n)$, which is continuous (in operator norm) in $z \in \mathbf{D}^\pm$, bounded and analytic for $z \in \mathbf{C}^\pm$.

If $V(x) \in \mathbf{SR}$, then the Hamiltonian $H = H_0 + V(x)$, whose domain is $\mathcal{D}(H) = \mathcal{D}(H_0) = H^2(\mathbf{R}^n)$, is selfadjoint and bounded below with essential spectrum $[0, \infty)$, while its negative eigenvalues have finite multiplicity and can only accumulate at zero. Moreover **[Ag75, Ku80]**, the set, $\sigma_+(H)$, of positive eigenvalues of H consists only of eigenvalues of finite multiplicity, which can only accumulate at zero and at infinity. Apart from that, H may have a bound state or half-bound state at zero energy. If $V(x) = O(\frac{1}{|x|})$ as $x \to \infty$, the set $\sigma_+(H)$ is empty **[Ka59]**. As in (2.1) we define

$$R^\pm(\lambda) = \lim_{z \to \lambda} (H - z)^{-1}, \quad \lambda \in \mathbf{R}^+ \setminus \sigma_+(H),\ z \in \mathbf{C}^\pm.$$

The operator $R^\pm(\lambda)$ exists in the norm topology of $\mathcal{L}(L^2_s(\mathbf{R}^n); H^2_{-s}(\mathbf{R}^n))$ for all $s > \frac{1}{2}$ and it can be extended to $\mathbf{C}^\pm$, by defining $R^\pm(z) = (H - z)^{-1}$ for $z \in \mathbf{C}^\pm$. Then $R^\pm(z)$ becomes a continuous (in operator norm) function on $\mathbf{D}^\pm \setminus \sigma_+(H)$ and analytic for $z \in \mathbf{C}^\pm$.

If $V(x) \in \mathbf{K}_2$, then $V(x)R_0^\pm(z)$ is compact on $L^2_s(\mathbf{R}^n)$ for all $z \in \mathbf{D}^\pm$ and $s > \frac{1}{2}$, while $[\mathbf{I} + V(x)R_0^\pm(z)]$ and $[\mathbf{I} + R_0^\pm(z)V(x)]$ are invertible on $L^2_s(\mathbf{R}^n)$ and $H^2_{-s}(\mathbf{R}^n)$, respectively, for all $z \in \mathbf{D}^\pm \setminus \sigma_+(H)$ and satisfy the identity **[Ag75, Ku80]**

$$R^\pm(z) = R_0^\pm(z)[\mathbf{I} + V(x)R_0^\pm(z)]^{-1} = [\mathbf{I} + R_0^\pm(z)V(x)]^{-1}R_0^\pm(z), \quad z \in \mathbf{D}^\pm \setminus \sigma_+(H).$$

The scattering matrix may then be represented in the form **[Ku80]**

$$S(k) = \mathbf{I} - \frac{ik^{n-2}}{2(2\pi)^{n-1}}\sigma(k)[\mathbf{I} + V(x)R_0^+(k^2)]^{-1}V(x)\sigma^\dagger(k), \quad k^2 \in \mathbf{R}^+ \setminus \sigma_+(H), \tag{2.3}$$

where

$$(\sigma(k)f)(\theta) = \int_{\mathbf{R}^n} dx\ e^{-ik\theta\cdot x} f(x), \quad \theta \in S^{n-1},$$

$$(\sigma^\dagger(k)g)(x) = \int_{S^{n-1}} d\theta\ e^{ik\theta\cdot x} g(\theta), \quad x \in \mathbf{R}^n. \tag{2.4}$$

We will use the multiplicative representation (2.3) in order to prove the Hölder continuity of $S(k)$, but we first need a few propositions concerning the Hölder continuity of the individual factors appearing in (2.3).

PROPOSITION 2.1. *Let $s > \frac{1}{2}$. Then, for every compact subset $\mathcal{J}$ of $\mathbf{R}_\infty \setminus \{0\}$, $\sigma^\dagger(k)$ is a uniformly Hölder continuous function from $\mathcal{J}$ into $\mathcal{L}(L^2(S^{n-1}); L^2_{-s}(\mathbf{R}^n))$ of exponent γ where $0 < \gamma < s - \frac{1}{2}$.*

Proof: It is known ([**Ag75**], Lemma 5.2; [**We90**], Lemma 2.4) that $\sigma^\dagger(k)$ belongs to $\mathcal{L}(L^2(S^{n-1}); L^2_{-s}(\mathbf{R}^n))$ for all $s > \frac{1}{2}$ and $k \in \mathbf{R} \setminus \{0\}$; i.e., for all $g \in L^2(S^{n-1})$

$$\|\sigma^\dagger(k)g\|_{-s} \leq C_{k,s}\|g\|_{L^2(S^{n-1})}. \tag{2.5}$$

More details on $C_{k,s}$ are given in Appendix A. For $s > \frac{3}{2}$ we have

$$\|\frac{d}{dk}(\sigma^\dagger(k)g)\|^2_{-s} \leq \sum_{j=1}^n \|\sigma^\dagger(k)(\theta_j g)\|^2_{-(s-1)} \leq nC^2_{k,s-1}\|g\|^2_{L^2(S^{n-1})},$$

so that for $k_1 \neq k_2 \in \mathbf{R} \setminus \{0\}$

$$\|[\sigma^\dagger(k_1) - \sigma^\dagger(k_2)]g\|_{-s} \leq \sqrt{n}\ C_{\mathcal{J},s-1}|k_1 - k_2|\ \|g\|_{L^2(S^{n-1})},$$

where $C_{\mathcal{J},s} = \sup_{k\in\mathcal{J}} C_{k,s}$. So, if $s > \frac{3}{2}$, $\sigma^\dagger(k)$ belongs to $\mathcal{H}_1(\mathcal{J}; \mathcal{L}(L^2(S^{n-1}); L^2_{-s}(\mathbf{R}^n)))$ with its norm bounded above by $(1 + \sqrt{n})C_{\mathcal{J},s-1}$. Now note that, by Hölder's inequality, for $s_1, s_2 > 0$ and $\epsilon \in (0,1)$

$$\|f\|_{-s} \leq \|f\|^\epsilon_{-s_1}\|f\|^{1-\epsilon}_{-s_2} \tag{2.6}$$

whenever $s = \epsilon s_1 + (1-\epsilon)s_2$. Thus, for $s_1 > \frac{3}{2}$, $s_2 > \frac{1}{2}$, $\epsilon \in (0,1)$ and $s = \epsilon s_1 + (1-\epsilon)s_2$, using (2.6) we obtain

$$\begin{aligned}\|[\sigma^\dagger(k_1) - \sigma^\dagger(k_2)]g\|_{-s} \leq \\ \leq \left(2C_{\mathcal{J},s_2}\|g\|_{L^2(S^{n-1})}\right)^{1-\epsilon}\left((1+\sqrt{n})C_{\mathcal{J},s_1-1}\|g\|_{L^2(S^{n-1})}|k_1-k_2|\right)^\epsilon = \\ = 2^{1-\epsilon}(1+\sqrt{n})^\epsilon(C_{\mathcal{J},s_1})^{1-\epsilon}(C_{\mathcal{J},s_2-1})^\epsilon|k_1-k_2|^\epsilon\|g\|_{L^2(S^{n-1})},\end{aligned}$$

so that $\sigma^\dagger(k)$ belongs to $\mathcal{H}^0_\epsilon(\mathcal{J}; \mathcal{L}(L^2(S^{n-1}); L^2_{-s}(\mathbf{R}^n)))$. In fact, its Hölder norm of exponent ϵ is bounded above by $C_{\mathcal{J},s} + 2^{1-\epsilon}(1+\sqrt{n})^\epsilon(C_{\mathcal{J},s_1})^{1-\epsilon}(C_{\mathcal{J},s_2-1})^\epsilon$, which is in turn bounded above by

$$(2+\sqrt{n})(C_{\mathcal{J},s_1})^{1-\epsilon}(C_{\mathcal{J},s_2-1})^\epsilon. \tag{2.7}$$

Here we have used (2.6) and the inequality $1 + 2^{1-\epsilon}(1+\sqrt{n})^\epsilon \leq 2 + \sqrt{n}$. ∎

PROPOSITION 2.2. *Let $s > \frac{1}{2}$ and $\alpha \geq 0$. Then, for every compact subset $\mathcal{J}$ of $\mathbf{R}_\infty \setminus \{0\}$, $\sigma^\dagger(k)$ is a uniformly Hölder continuous function from $\mathcal{J}$ into $\mathcal{L}(L^2(S^{n-1}); H^\alpha_{-s}(\mathbf{R}^n))$ of exponent γ where $0 < \gamma < s - \frac{1}{2}$.*

Proof: Clearly,

$$(\nabla^2_x\sigma^\dagger(k)g)(x) = -k^2\int_{S^{n-1}} d\theta\ e^{-ik\theta\cdot x}g(\theta), \tag{2.8}$$

so that for $s > \frac{1}{2}$

$$\|\sigma^\dagger(k)g\|_{2,-s} \le \left(\|\sigma^\dagger(k)g\|_{-s}^2 + \|\nabla_x^2\sigma^\dagger(k)g\|_{-s}^2\right)^{\frac{1}{2}} \le$$

$$\le (1+k^2)^{\frac{1}{2}}\|\sigma^\dagger(k)g\|_{-s} \le (1+k^2)^{\frac{1}{2}}C_{k,s}\|g\|_{L^2(S^{n-1})}.$$

Repeated application of the above process leads to the estimate

$$\|\sigma^\dagger(k)g\|_{\alpha,-s} \le (1+k^2)^{\alpha/4}C_{k,s}\|g\|_{L^2(S^{n-1})}$$

for $\alpha = 0,2,4,6,\cdots$ and hence, by interpolation, this inequality remains valid for all $\alpha \ge 0$.

We may now repeat the proof of Proposition 2.1 using (2.8) instead of (2.5). The result is the Hölder continuity of $\sigma^\dagger(k)$ from $\mathcal{J}$ into $\mathcal{L}(L^2(S^{n-1}); H^\alpha_{-s}(\mathbf{R}^n))$ with its Hölder norm bounded above by

$$(2+\sqrt{n})\left(D_{\mathcal{J},s_1}\right)^{1-\epsilon}\left(D_{\mathcal{J},s_2-1}\right)^{\epsilon},$$

where $D_{\mathcal{J},s} = \sup_{k\in\mathcal{J}}\,(1+k^2)^{\alpha/4}C_{k,s}$ and $s = \epsilon s_1 + (1-\epsilon)s_2$ for some $s_1 > \frac{1}{2}$, $s_2 > \frac{3}{2}$ and $\epsilon \in (0,1)$. ∎

PROPOSITION 2.3. *Let $s > \frac{1}{2}$. Then, for every compact subset $\mathcal{J}$ of $\mathbf{R}_\infty \setminus \{0\}$, $\sigma(k)$ is a uniformly Hölder continuous function from $\mathcal{J}$ into $\mathcal{L}(L^2_s(\mathbf{R}^n); L^2(S^{n-1}))$ of exponent γ where $\gamma \in (0, s-\frac{1}{2})$.*

Proof: By duality from Proposition 2.1. The Hölder norm again is bounded by the quantity in (2.7). ∎

PROPOSITION 2.4. *Let $s > \frac{1}{2}$. Then, for every compact subset $\mathcal{J}$ of $\mathbf{R}_\infty \setminus \{0\}$, $R_0^+(k^2)$ is a uniformly Hölder continuous function from $\mathcal{J}$ into $\mathcal{L}(L^2_s(\mathbf{R}^n); H^2_{-s}(\mathbf{R}^n))$ of exponent γ where $\gamma \in (0, s-\frac{1}{2})$.*

Proof: According to [**Ag75**], Eq. (4.7),

$$(2.9) \qquad < R_0^\pm(k^2)f, g >= \pm\pi i\Phi_{f,g}(k) + CPV\int_0^\infty dt\ \frac{t^{(n-2)/2}}{t-k^2}\Phi_{f,g}(t),$$

where CPV stands for Cauchy's principal value, $\Phi_{f,g}$ is defined as

$$\Phi_{f,g}(k) = \frac{1}{2|k|}\int_{|k|S^{n-1}} d\theta\ \hat{f}(\theta)\overline{\hat{g}(\theta)} = \frac{1}{2} < \sigma(k)f, \sigma(k)g >_{L^2(S^{n-1})}$$

and $f, g \in L^2_s(\mathbf{R}^n)$. According to Proposition 2.3, $\Phi_{f,g}$ is a Hölder continuous complex function on $\mathcal{J}$ of exponent $\gamma \in (0, s-\frac{1}{2})$ and the Hölder norm is bounded by

$$\|\Phi_{f,g}\|_{\mathcal{H}_\gamma} \le M_\gamma\|f\|_s\|g\|_s,$$

where M_γ is a constant independent of f and g. Then (2.9) implies that $< R_0^\pm(k^2)f, g >$ is a Hölder continuous complex function on $\mathcal{J}$ of exponent $\gamma \in (0, s-\frac{1}{2})$ with its Hölder norm bounded above by $N_{\gamma,\mathcal{J}}\|f\|_s\|g\|_s$ for some constant $N_{\gamma,\mathcal{J}}$ not depending on f and g. Thus we obtain

$$|<[R_0^{\pm}(k_1^2)-R_0^{\pm}(k_2^2)]f,g>|\leq N_{\gamma,\mathcal{J}}|k_1-k_2|^{\gamma}\|f\|_s\|g\|_s,$$

which completes the proof. ∎.

PROPOSITION 2.5. *Let $V(x)\in\mathbf{B}_\alpha$ for some $\alpha\in[0,2)$, and let $s>\frac{1}{2}$ be the constant such that multiplication by $(1+|x|^2)^sV(x)$ is a bounded linear operator from $H^\alpha(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$. Then the scattering operator $S(k)$ is a uniformly Hölder continuous function from $\mathbf{R}\setminus[\sigma_+(H)^{1/2}\cup\{0\}]$ into $\mathcal{L}(L^2(S^{n-1}))$ satisfying*

$$\|S(k)-\mathbf{I}\|\leq\frac{C_\delta}{(1+|k|^2)^{1-\delta}} \tag{2.10}$$

for all $0<\delta<\frac{1}{2}$. Here $\sigma_+(H)^{1/2}=\{z\in\mathbf{R}:z^2\in\sigma_+(H)\}$ and C_δ is a constant.

Proof: According to (2.3) and Propositions 2.1-2.4, we have the following commutative diagram of bounded linear operators for all nonzero k such that $k^2\notin\sigma_+(H)$:

$$\begin{array}{ccccc}
L^2(S^{n-1}) & \xrightarrow{S(k)-\mathbf{I}} & L^2(S^{n-1}) & \xleftarrow{\frac{-ik^{n-2}}{2(2\pi)^{n-1}}\mathbf{I}} & L^2(S^{n-1}) \\
\downarrow{\scriptstyle \sigma^{\dagger}(k)} & & & & \uparrow{\scriptstyle \sigma(k)} \\
H^{\alpha}_{-s}(\mathbf{R}^n) & \xrightarrow{(\mathbf{I}+R_0^+(k^2)V)^{-1}} & H^{\alpha}_{-s}(\mathbf{R}^n) & \xrightarrow{V} & L^2_s(\mathbf{R}^n)
\end{array}$$

Then, for each compact subset $\mathcal{J}$ of $\mathbf{R}\setminus[\sigma_+(H)^{1/2}\cup\{0\}]$, every operator in the diagram is uniformly Hölder continuous in k as a function from $\mathcal{J}$ into $\mathcal{L}(\mathcal{X};\mathcal{Y})$ where $\mathcal{X}$ and $\mathcal{Y}$ are the spaces at the beginning and the end of the corresponding arrow, respectively. Hence, $S(k)$ is a uniformly Hölder continuous function from any such $\mathcal{J}$ into $\mathcal{L}(L^2(S^{n-1}))$. Moreover, the Hölder index γ may be chosen to satisfy $0<\gamma<s-\frac{1}{2}$.

Using the estimate (A.4) in Appendix A we obtain, for some constant C_1,

$$\|S(k)-\mathbf{I}\|\leq C_1|k|^{n-2}\left(|k|^{s-\frac{1}{2}n}\right)^2,$$

which implies (2.10) on all compact subsets $\mathcal{J}$ of $\mathbf{R}_\infty\setminus[\sigma_+(H)^{1/2}\cup\{0\}]$. ∎

Suppose $V(x)\in\mathbf{K}_2$ and $[\mathbf{I}+V(x)R_0^+(k^2)]$ has a limit in the operator norm of $L^2_s(\mathbf{R}^n)$ as $k\to 0$. Then this limit is a compact perturbation of the identity. We denote it by $\mathbf{I}+V(x)R_0^+(0)$. We call $k=0$ an **exceptional point [Ne89b, We90]** if $[\mathbf{I}+V(x)R_0^+(0)]$ is not boundedly invertible on $L^2_s(\mathbf{R}^n)$. In that case there exists $0\neq\varphi\in L^2_s(\mathbf{R}^n)$ such that $\varphi=-VR_0^+(0)\varphi$.

The estimates obtained so far must be refined in order to deal with the case $k \to 0$. This will lead us to the additional assumption that $s > \frac{3}{2} - \frac{1}{n}$ in the definition of $\mathbf{B}_\alpha$. First of all, for all $s > \frac{3}{2} - \frac{1}{n}$, $k \in \mathbf{R} \setminus \{0\}$ and $g \in L^2(S^{n-1})$, we have

$$\|k^{\frac{1}{2}(n-2)}\sigma^\dagger(k)g\|_{-s} \leq D_{n,s}|k|^{\frac{(2s-3)n+2}{2(n-1)}}\|g\|_{L^2(S^{n-1})},$$

as a result of (A.4). As in the proof of Proposition 2.1, we obtain for $s > \frac{5}{2} - \frac{1}{n}$

$$\|\frac{d}{dk}(k^{\frac{1}{2}(n-2)}\sigma^\dagger(k)g)\|_{-s} \leq D'_{n,s}|k|^{\frac{(2s-5)n+2}{2(n-1)}}\|g\|_{L^2(S^{n-1})}.$$

Here $D_{n,s}$ and $D'_{n,s}$ are constants which do not depend on k. Then through interpolation it follows that, for $s > \frac{3}{2} - \frac{1}{n}$, the operator $k^{\frac{1}{2}(n-2)}\sigma^\dagger(k)$ is a uniformly Hölder continuous function from $[-1,1]$ into $\mathcal{L}(L^2(S^{n-1}); L^2_{-s}(\mathbf{R}^n))$ of exponent γ where $0 < \gamma < s - \frac{3}{2} + \frac{1}{n}$. Next, converting the integral on the right-hand side of (2.9) to a CPV-integral on all of $\mathbf{R}$ we obtain the Hölder continuity of $R_0^+(k^2)$ as a function from $[-1,1]$ into $\mathcal{L}(L^2_{-s}(\mathbf{R}^n); H^2_{-s}(\mathbf{R}^n))$ of exponent γ where $0 < \gamma < s - \frac{3}{2} + \frac{1}{n}$. Thus, in the absence of an exceptional point at $k = 0$, using (2.3) we conclude that $S(k)$ is uniformly Hölder continuous from any compact subset of $[-1,1] \setminus \sigma_+(H)^{1/2}$ into $\mathcal{L}(L^2(S^{n-1}))$ of exponent γ where $0 < \gamma < s - \frac{3}{2} + \frac{1}{n}$. We then readily obtain the following result.

THEOREM 2.6. *Let $V(x) \in \mathbf{B}_\alpha$ for some $\alpha \in (0,2)$, and let $s > \frac{3}{2} - \frac{1}{n}$ be the constant such that multiplication by $(1+|x|^2)^s V(x)$ is a bounded linear operator from $H^\alpha(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$. Suppose $\sigma_+(H) = \emptyset$ while $k = 0$ is not an exceptional point. Then $S(k)$ is a uniformly Hölder continuous function from $\mathbf{R}$ into $\mathcal{L}(L^2(S^{n-1}))$ satisfying (2.10) for all $0 < \delta < \frac{1}{2}$.*

In order to prove the existence of a Wiener-Hopf factorization of $S(k)$, we transform Theorem 2.6 to the unit circle $\mathbf{T}$. Let us define

$$\widetilde{S}(\xi) = S(i\frac{1+\xi}{1-\xi}), \quad \xi \in \mathbf{T}. \tag{2.11}$$

Throughout $\widetilde{\ }$ will denote the Möbius transform of a function on the real line to the unit circle, according to the rule (2.11). The next theorem shows that $\widetilde{S}(\xi)$ is also Hölder continuous.

THEOREM 2.7. *Let $V(x) \in \mathbf{B}_\alpha$ for some $\alpha \in (0,2)$, and let $s > \frac{3}{2} - \frac{1}{n}$ be the constant such that multiplication by $(1+|x|^2)^s V(x)$ is a bounded linear operator from $H^\alpha(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$. Suppose $\sigma_+(H) = \emptyset$ while $k = 0$ is not an exceptional point. Then $\widetilde{S}(\xi)$ is a uniformly Hölder continuous function from $\mathbf{T}$ into $\mathcal{L}(L^2(S^{n-1}))$ satisfying $\widetilde{S}(1) = \mathbf{I}$. The Hölder exponent can be any ζ satisfying $0 < \zeta < \min\{\frac{1}{2}, 1 - (s - \frac{1}{2} + \frac{1}{n})^{-1}\}$.*

Proof: From Proposition 2.5 we have

$$\|S(k_1) - S(k_2)\| \leq M_1|k_1 - k_2|^{\epsilon_1}$$

for every $\epsilon_1 \in (0, \min\{1, s - \frac{3}{2} + \frac{1}{n}\})$, and

$$\|S(k) - \mathbf{I}\| \leq \frac{M_2}{(1+|k|)^{2(1-\epsilon_2)}}$$

for every $\epsilon_2 \in (0,1)$. Here M_1 and M_2 are constants independent of k. Now put

$$\lambda(k,\delta) = (k^2+1)^{\zeta/2}[(k+\delta)^2+1]^{\zeta/2}\delta^{-\zeta}\|S(k+\delta) - S(k)\|,$$

where $\zeta \in (0,1)$ is to be determined later. Then for $\xi = \frac{k-i}{k+i}$ and $\eta = \frac{k+\delta+i}{k+\delta-i}$ we have

$$\frac{\|\tilde{S}(\xi) - \tilde{S}(\eta)\|}{|\xi - \eta|^{\zeta}} = 2^{-\zeta}\lambda(k,\delta),$$

so that the theorem follows if we can prove the boundedness of $\lambda(k,\delta)$.

Indeed, for $|k| \leq 1 \leq \delta$ we have $\lambda(k,\delta) \leq 2^{\zeta/2}(5\delta^2)^{\zeta/2}\delta^{-\zeta}\|S(k+\delta) - S(k)\| \leq 2\cdot 10^{\zeta/2}$. For $\max(|k|,\delta) \leq 1$ we have $\lambda(k,\delta) \leq 2^{\zeta/2}5^{\zeta/2}\delta^{\epsilon_1-\zeta}M_1$. For $1 \leq |k| \leq \delta$ we get $\lambda(k,\delta) \leq (2k^2)^{\zeta/2}(5k^2)^{\zeta/2}2M_2(1+|k|)^{-2(1-\epsilon_2)}$, since $\delta^{-\zeta} \leq 1$. Finally, for $\delta \leq 1 \leq |k|$ we have for $0 < \zeta \leq \epsilon_1$

$$\lambda(k,\delta) \leq (2k^2)^{\zeta/2}(5k^2)^{\zeta/2}M_1^{\zeta/\epsilon_1}\left[\frac{2M_2}{(1+|k|)^{2(1-\epsilon_2)}}\right]^{1-\frac{\zeta}{\epsilon_1}}.$$

Hence, to have $\lambda(k,\delta)$ bounded we must choose $0 < \zeta \leq \epsilon_1(1-\epsilon_2)/(1+\epsilon_1-\epsilon_2)$. Taking the maximum over $\epsilon_2 \in (0,1)$ with $0 < \epsilon_1 < \min\{1, s - \frac{3}{2} + \frac{1}{n}\}$, we get $\zeta \in (0, \min\{\frac{1}{2}, 1 - (s - \frac{1}{2} + \frac{1}{n})^{-1}\})$. ∎

3. WIENER-HOPF FACTORIZATION OF THE SCATTERING OPERATOR

The incoming and outgoing scattering solutions $\psi(k,x,\theta)$ and $\psi(-k,x,\theta)$ of the n-D Schrödinger equation are related to each other, as in the 3-D case [**Ne80**], as

$$\psi(k,x,\theta) = \int_{S^{n-1}} d\theta'\; S(k,-\theta,\theta')\psi(-k,x,\theta'),$$

where $x \in \mathbf{R}^n$, $k \in \mathbf{R}$ and $\theta \in S^{n-1}$. Defining

$$f(k,x,\theta) = e^{-ik\theta\cdot x}\psi(k,x,\theta), \tag{3.1}$$

we obtain the Riemann-Hilbert problem

$$f(k,x,\theta) = \int_{S^{n-1}} d\theta'\; e^{-ik\theta\cdot x}S(k,-\theta,\theta')e^{-ik\theta'\cdot x}f(-k,x,\theta'),$$

where $f(k,x,\theta) = 1 + O(\frac{1}{|k|})$ as $|k| \to \infty$ from $\mathbf{C}^+ \cup \mathbf{R}$. In the absence of bound states, $f(k,x,\theta)$ has an analytic continuation in k to $\mathbf{C}^+$. Using $G(k)$ given in (1.2) and defining the sectionally analytic functions

$$X_{\pm}(k,x,\theta) = f(\pm k,x,\pm\theta) - 1, \tag{3.2}$$

we obtain the vector Riemann-Hilbert problem

$$X_+(k) = G(k)X_-(k) + [G(k) - \mathbf{I}]\hat{1} \tag{3.3}$$

on $L^2(S^{n-1})$, where $\hat{1}(\theta) \equiv 1$ and the x-dependence has been suppressed. In general, $f(k, x, \theta)$ is meromorphic on $\mathbf{C}^+$ with simple poles at $k = i\gamma$ where $-\gamma^2$ are the bound state energies. It is possible to remove these simple poles from the Riemann-Hilbert problem by a reduction method [**Ne82**] and to obtain a Riemann-Hilbert problem of the form (3.3) where $X_\pm(k)$ are continuous on $\mathbf{C}^\pm \cup \mathbf{R}$, are analytic on $\mathbf{C}^\pm$, and vanish as $k \to \infty$ from $\mathbf{C}^\pm \cup \mathbf{R}$. For $n = 3$, we refer the reader to [**Ne89b, AV89a**] for details. Note that once (3.3) is solved, the solution of the Schrödinger equation can be obtained using (3.2) and (3.1).

We will solve the Riemann-Hilbert problem (3.3) by using the Wiener-Hopf factors [**Go64**] of the operator function of $G(k)$. By a (**left**) **Wiener-Hopf factorization** of an operator function $G : \mathbf{R}_\infty \to \mathcal{L}(L^2(S^{n-1}))$, we mean a representation of $G(k)$ in the form

$$G(k) = G_+(k)D(k)G_-(k), \quad k \in \mathbf{R}_\infty, \tag{3.4}$$

with

$$D(k) = P_0 + \sum_{j=1}^{m} \left(\frac{k-i}{k+i}\right)^{\rho_j} P_j,$$

where
1. $G_+(k)$ is continuous on $\mathbf{C}^+ \cup \mathbf{R}$ in the operator norm of $\mathcal{L}(L^2(S^{n-1}))$ and is boundedly invertible there. Similarly, $G_-(k)$ is continuous on $\mathbf{C}^- \cup \mathbf{R}$ in the operator norm and is boundedly invertible there.
2. $G_+(k)$ is analytic on $\mathbf{C}^+$ and $G_-(k)$ is analytic on $\mathbf{C}^-$.
3. $G_+(\pm\infty) = G_-(\pm\infty) = \mathbf{I}$.

The projections $P_1, \cdots, P_m$ are finite in number, are mutually disjoint, have rank one, and $P_0 = \mathbf{I} - \sum_{j=1}^m P_j$. The (**left**) **partial indices** $\rho_1, \cdots, \rho_m$ are nonzero integers. In the absence of partial indices, we have $D(k) = \mathbf{I}$, in which case the Wiener-Hopf factorization is called (**left**) **canonical**. The partial indices of $G(k)$ depend neither on the choice of the factors $G_+(k)$ and $G_-(k)$ nor on the choice of the projections $P_1, \cdots, P_m$. If the factorization is (left) canonical, the factors $G_+(k)$ and $G_-(k)$ are unique as a result of Liouville's theorem.

In the same way we define a right Wiener-Hopf factorization, right partial indices, and a right canonical factorization by interchanging the roles of $G_+(k)$ and $G_-(k)$ in (3.4). The right indices may be different, both in number and in value, from the left indices, but the sum of the left indices coincides with the sum of the right indices. This sum is called the **sum index** of $G(k)$.

By using the Möbius transformation defined in (2.11), we can define the left and right Wiener-Hopf factorizations of operator functions on $\mathbf{T}$. The left and right partial indices are invariant under this Möbius transformation. For details, we refer the reader to [**AV89a**].

REMARK 3.1. The scattering operator $S(k)$ is unitary, and hence from (1.2) it follows that $G(k)$ is also unitary. As a consequence [**AV89a**], the sets of left and right partial indices of $G(k)$ coincide. Moreover, the projections and factors appearing in the right and left Wiener-Hopf factorizations of $G(k)$ are related by

$$\hat{P}_j = (P_j)^\dagger \quad \text{for} \quad j = 1, \cdots, m; \quad \text{and} \quad \hat{G}_\pm(k)^{-1} = G_\mp(\bar{k})^\dagger.$$

Here the quantities with ^ pertain to the right Wiener-Hopf factorization of $G(k)$.

THEOREM 3.2. *Let $V(x) \in \mathbf{B}_\alpha$ for some $\alpha \in (0,2)$, and let $s > \frac{3}{2} - \frac{1}{n}$ be the constant such that multiplication by $(1+|x|^2)^s V(x)$ is a bounded linear operator from $H^\alpha(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$. Suppose $\sigma_+(H) = \emptyset$ while $k = 0$ is not an exceptional point. Then the operator function $G(k)$ defined by (1.2) has right and left Wiener-Hopf factorizations. The sets of left and right indices coincide, and the sum index is independent of the choice of $x \in \mathbf{R}^n$.*
Proof: According to Theorem 6.1 (or 6.2) of [**GL73**], it is sufficient to show the following:
1. $G(k)$ is boundedly invertible for every $k \in \mathbf{R}_\infty$.
2. $G(k)$ is a compact perturbation of the identity for every $k \in \mathbf{R}_\infty$.
3. $\widetilde{G}(\xi) \in \mathcal{H}_\alpha(\mathbf{T}; \mathcal{L}(L^2(S^{n-1})))$ for some $\alpha \in (0,1)$, where $\widetilde{G}(\xi)$ is the Möbius transform of $G(k)$, as defined in (2.11).

Under these conditions there exists a left Wiener-Hopf factorization of $\widetilde{G}(\xi)$ with respect to $\mathbf{T}$ which is given by

$$\widetilde{G}(\xi) = \widetilde{G}_+(\xi)\widetilde{D}(\xi)\widetilde{G}_-(\xi),$$

where

$$\widetilde{D}(\xi) = P_0 + \sum_{j=1}^{m} \xi^{\rho_j} P_j,$$

$\widetilde{G}_+(\xi) \in \mathcal{H}_\alpha(\mathbf{T}^+; \mathcal{L}(L^2(S^{n-1})))$ and is invertible there, $\widetilde{G}_-(\xi) \in \mathcal{H}_\alpha(\mathbf{T}^-; \mathcal{L}(L^2(S^{n-1})))$ and is invertible there, $\widetilde{G}_+(\xi)$ and $\widetilde{G}_-(\xi)$ are analytic on $\mathbf{T}^+$ and on $\mathbf{T}^-$, respectively. The inverse of the Möbius transformation given by (2.11) then yields a left Wiener-Hopf factorization for $G(k)$ of the type (3.4) where the Möbius transformed factors $\widetilde{G}_+(\xi)$ and $\widetilde{G}_-(\xi)$ as well as their inverses are Hölder continuous of exponent α in operator norm on $\mathbf{T}^+$ and $\mathbf{T}^-$, respectively. These properties are easily seen to extend to $G(k)$ for all $x \in \mathbf{R}^n$, since the operator function $U_x(k)$ appearing in (1.2) is continuously differentiable with respect to k and the norm of $G(k) - \mathbf{I}$ does not depend on x. As mentioned in Remark 3.1, the coincidence of the sets of left and right indices of $G(k)$ for every $x \in \mathbf{R}^n$ is a direct consequence of the unitarity of this operator function. The x-independence of the sum index is due to the fact that $\widetilde{G}(\xi)$ depends continuously on x in $\mathcal{H}_\alpha(\mathbf{T}; \mathcal{L}(L^2(S^{n-1})))$ [cf. [**GL73**], Section 7]. ∎

The Riemann-Hilbert problem (3.3) can be solved in terms of the Wiener-Hopf factors of $G(k)$ as in the case of 3-D [**AV89a**] to obtain

$$X_+(k) = [G_+(k)^{-1} - \mathbf{I}]\hat{1} + G_+(k) \sum_{\rho_j > 0} \frac{\phi_j(k)}{(k+i)^{\rho_j}} \pi_j$$

and

$$X_-(k) = [G_-(k)^{-1} - \mathbf{I}]\hat{1} + G_-(k)^{-1} \sum_{\rho_j > 0} \frac{\phi_j(k)\pi_j + [(k+i)^{\rho_j} - (k-i)^{\rho_j}]P_j\hat{1}}{(k-i)^{\rho_j}}$$

provided $P_j\hat{1} = 0$ whenever $\rho_j < 0$. Here π_j is a fixed nonzero vector in the range of P_j and $\phi_j(k)$ is an arbitrary polynomial of degree less than ρ_j. Using the Schrödinger equation the potential is obtained as

$$V(x) = \frac{[\nabla_x^2 + 2ik\theta \cdot \nabla_x]X_+(k)}{1 + X_+(k)}$$

provided the right-hand side is independent of θ and k; as in the case of 3-D [**AV89a**], this θ- and k-independence is equivalent to the "miracle" condition of Newton [**Ne89b**].

From (1.1) and (1.2), we obtain

$$G(-k) = QG(k)^{-1}Q.$$

It can be shown that the factors $G_+(k)$ and $G_-(k)$ in (3.4) can be chosen in such a way that

$$G_\pm(-k) = QG_\mp(k)^{-1}Q.$$

For details we refer the reader to Remark 4.3 of [**AV89a**].

In the special case where $S(k)$ is a meromorphic function on $\mathbf{C}$ with only finitely many poles and zeros, we can obtain additional information on the partial indices of $G(k)$. If $\sigma_-(H) = \emptyset$ and $k = 0$ is not an exceptional point, these poles and zeros are nonreal. Using that $\lim_{k\to\pm\infty} \|S(k) - \mathbf{I}\| = 0$, we may represent $G(k)$ in the form [**BGK79**]

$$G(k) = E_x(k)\left[\mathbf{I} + C_x(k\mathbf{I} - A_x)^{-1}B_x\right]F_x(k), \tag{3.5}$$

where $E_x(k)$, $E_x(k)^{-1}$, $F_x(k)$ and $F_x(k)^{-1}$ are entire operator functions satisfying the identities $\lim_{k\to\pm\infty} \|E_x(k) - \mathbf{I}\| = 0$ and $\lim_{k\to\pm\infty} \|F_x(k) - \mathbf{I}\| = 0$ and A_x, B_x and C_x are bounded linear operators. In that case

$$G(k)^{-1} = F_x(k)^{-1}\left[\mathbf{I} - C_x(k\mathbf{I} - A_x^\times)^{-1}B_x\right]E_x(k)^{-1},$$

where $A_x^\times = A_x - B_xC_x$. One may choose A_x, B_x, and C_x in such a way that A_x acts on a finite-dimensional space $\mathcal{X}$, and B_x and C_x act between $L^2(S^{n-1})$ and $\mathcal{X}$, and that the spectra of A_x and $A_x^\times$ coincide with the sets of poles and zeros of $S(k)$, respectively. Moreover, the representation (3.5) may be chosen in such a way that it is minimal [i.e., $\cap_{i=0}^\infty \mathcal{N}(C_xA_x^i) = \{0\}$ and $\sum_{i=0}^\infty \mathcal{R}(A_x^iB_x) = \mathcal{X}$] and there exists a signature operator J [i.e., $J = J^\dagger = J^{-1}$] such that $A^\times$ coincides with $(-A)$ and iA is J-selfadjoint. Since a zero of $S(k)$ is a pole of $S(k)^{-1}$, the poles of $G(k)$ in $\mathbf{C}^+$ correspond exactly with the (negative) bound states and hence these poles are simple and located on the imaginary axis.

THEOREM 3.3. *Let $V(x) \in \mathbf{B}_\alpha$ for some $\alpha \in (0,2)$, and let $s > \frac{3}{2} - \frac{1}{n}$ be the constant such that multiplication by $(1+|x|^2)^s V(x)$ is a bounded linear operator from $H^\alpha(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$. Suppose $\sigma_+(H) = \emptyset$ while $k = 0$ is not an exceptional point. Finally, assume $S(k)$ extends to a meromorphic operator function on the entire complex plane with only finitely many poles and zeros. Then the partial indices of $G(k)$ are nonnegative.*

Proof: Let us construct the representation (3.5) with the properties mentioned above and let us suppress the x-dependence. If T is a bounded linear operator without real spectrum and $\Gamma_\pm$ is a simple, positively oriented rectifiable Jordan contour in $\mathbf{C}^\pm$ enclosing the spectrum of T in $\mathbf{C}^\pm$, we write $\mathcal{M}_\pm(T)$ for the ranges of the complementary bounded projections $\frac{1}{2\pi i}\int_{\Gamma_\pm} dz\, (z\mathbf{I}-T)^{-1}$. Then, according to the main result of [**BGK86**], the sum of the positive (resp. negative) indices ρ_j of $G(k)$ satisfies

$$\sum_{\pm\rho_j>0} |\rho_j| = \dim[\mathcal{M}_\mp(A) \cap \mathcal{M}_\pm(A^\times)].$$

Here we have used the fact that the sets of left and right partial indices of $G(k)$ coincide. If $^\perp$ denotes the orthogonal complement with respect to the indefinite scalar product $[\cdot,\cdot] = < J\cdot,\cdot >$, then

$$\sum_{\pm\rho_j>0} |\rho_j| = \dim\{\mathcal{M}_\mp(A) \cap \mathcal{M}_\mp(A)^\perp\},$$

which is the J-neutral part of $\mathcal{M}_\mp(A)$; i.e., the subspace of those vectors u of $\mathcal{M}_\mp(A)$ such that $[u,u] = 0$. Now let $i\kappa_1, \cdots, i\kappa_q$ be the *different* positive imaginary eigenvalues of A. Then in terms of a J-orthogonal direct sum we have

$$\mathcal{M}_+(A) = \bigoplus_{j=1}^{q} \bigoplus_{l=1}^{\infty} \mathcal{N}(A - i\kappa_j)^l.$$

Since there are no generalized eigenvectors associated with these eigenvalues, we obtain

$$\mathcal{M}_+(A) = \bigoplus_{j=1}^{q} \mathcal{N}(A - i\kappa_j).$$

Hence,

$$\mathcal{M}_+(A) \cap \mathcal{M}_+(A)^\perp = \bigoplus_{j=1}^{q} \{\mathcal{N}(A - i\kappa_j) \cap \mathcal{R}(-A + i\kappa_j)\} = \{0\},$$

the last equality being clear from the absence of generalized eigenvectors. Thus, the sum of the negative indices is zero, and hence there are no negative indices, as claimed. ∎

Even if $S(k)$ does not have the analyticity properties of Theorem 3.3, it is possible to obtain expressions for the sum of the positive indices and the sum of the negative indices of $G(k)$. Let $\mathcal{X}$ be one of the Banach spaces $\mathcal{H}_\gamma(\mathbf{T}; L^2(S^{n-1}))$ where $0 < \gamma < 1$, and let $\mathcal{X}^\pm$ be the subspaces of $\mathcal{X}$ consisting of those $F \in \mathcal{X}$ which extend to a function in $\mathcal{H}_\gamma(\mathbf{T}^\pm \cup \mathbf{T}; L^2(S^{n-1}))$ that are analytic on $\mathbf{T}^\pm$ and, in the case of $\mathcal{X}^-$, satisfy $F(\infty) = 0$. Then $\mathcal{X}^+ \oplus \mathcal{X}^- = \mathcal{X}$ ([**Mu46**], extended to the vector-valued case). Then (cf. [**AV89b**])

$$\text{(3.6)} \qquad \sum_{\rho_j > 0} \rho_j = \dim\left\{\widetilde{G}[\mathcal{X}^-] \cap \mathcal{X}^+\right\}, \quad -\sum_{\rho_j < 0} \rho_j = \dim\left\{\mathcal{X}^- \cap \widetilde{G}[\mathcal{X}^+]\right\}.$$

Indeed, if $f_+ \in \widetilde{G}[\mathcal{X}^-] \cap \mathcal{X}^+$, then $f_+ = \widetilde{G} f_-$ for some $f_- \in \mathcal{X}^-$. Representing $G(k)$ as in (3.4), we obtain

$$\left[P_0 + \sum_{j=1}^{m} \xi^{\rho_j} P_j\right] \widetilde{G}_-(\xi) f_-(\xi) = \widetilde{G}_+(\xi)^{-1} f_+(\xi).$$

Premultiplication by P_0 gives $P_0 \widetilde{G}_-(\xi) f_-(\xi) = P_0 \widetilde{G}_+(\xi)^{-1} f_+(\xi)$; both sides vanish, because they belong to $\mathcal{X}^+ \cap \mathcal{X}^-$. Similarly, we get $\xi^{\rho_j} P_j \widetilde{G}_-(\xi) f_-(\xi) = P_j \widetilde{G}_+(\xi)^{-1} f_+(\xi)$. For $\rho_j < 0$ both sides vanish. For $\rho_j > 0$, however, Liouville's theorem implies that the two sides equal a scalar polynomial $\varphi_j(\xi)$ of degree at most $(\rho_j - 1)$ multiplied by P_j. But then, by adding the contributions of P_0 and the various P_j,

$$f_+(\xi) = \widetilde{G}_+(\xi) \sum_{\rho_j > 0} \varphi_j(\xi) P_j$$

and hence the first identity in (3.6) follows. The second identity in (3.6) can be proven by employing a right Wiener-Hopf factorization of $G(k)$ and using the fact that the left and right indices of $G(k)$ coincide.

4. TRACE CLASS PROPERTIES OF THE SCATTERING OPERATOR

In this section we will prove that $S(k)$ is a trace class operator on $L^2(S^{n-1})$ and study its effect on its Wiener-Hopf factorization.

THEOREM 4.1. *Let $V(x) \in \mathbf{B}_\alpha$ for some $\alpha \in (0,1)$, and let $s > 1$ be the constant such that multiplication by $(1+|x|^2)^s V(x)$ is a bounded linear operator from $H^\alpha(\mathbf{R}^n)$ into $L^2(\mathbf{R}^n)$. Suppose $0 \neq k \notin \sigma_+(H)^{1/2}$ and $V(x) \in L^1(\mathbf{R}^n)$. Then $S(k) - \mathbf{I}$ is a trace class operator on $L^2(S^{n-1})$ and*

$$\text{(4.1)} \qquad \lim_{k \to \pm\infty} k^{2-n} \mathrm{tr}[S(k) - \mathbf{I}] = 2i(2\pi)^{n-1} \Sigma_n < V >,$$

where $< V > = \int_{\mathbf{R}^n} dx\, V(x)$ and Σ_n is the surface area of S^{n-1}.

Proof: Observe that $V[H^\alpha_{-t}(\mathbf{R}^n)] \subset L^2_t(\mathbf{R}^n)$ for all $t \in [0,s]$, and put $V^{1/2} = |V|^{1/2}\mathrm{sgn}(V)$. Note that

$$\left|< u, v >_{H^\alpha_t(\mathbf{R}^n)}\right| \leq \|u\|_{H^{\alpha_1}_{t_1}(\mathbf{R}^n)} \|v\|_{H^{\alpha_2}_{t_2}(\mathbf{R}^n)},$$

where $\alpha_1 + \alpha_2 = 2\alpha$ and $t_1 + t_2 = 2t$. Hence, since $V(x) \in \mathbf{B}_\alpha \subset \mathbf{B}_{2\alpha}$, we have

$$\|V^{1/2} u\|^2_{H^\alpha(\mathbf{R}^n)} \leq \|Vu\|_{L^2_s(\mathbf{R}^n)} \|u\|_{H^{2\alpha}_{-s}(\mathbf{R}^n)}.$$

Furthermore,

$$\|V^{1/2}u\|^2_{H^{\frac{1}{2}\alpha}_{\frac{1}{2}s}(\mathbf{R}^n)} \leq \|Vu\|_{L^2_s(\mathbf{R}^n)}\|u\|_{H^\alpha(\mathbf{R}^n)}.$$

Since $\frac{1}{2}s > 1$, the following diagram consists of bounded linear operators:

$$H^\alpha(\mathbf{R}^n) \xrightarrow{|V|^{1/2}} H^{\frac{1}{2}\alpha}_{\frac{1}{2}s}(\mathbf{R}^n) \xrightarrow{\text{imbedding}} L^2_{\frac{1}{2}s}(\mathbf{R}^n) \xrightarrow{R_0^+(k^2)} H^{2\alpha}_{-s}(k^2) \xrightarrow{V^{1/2}} H^\alpha(\mathbf{R}^n).$$

Then $0 \neq k \notin \sigma_+(H)^{\frac{1}{2}}$ implies that $\left[\mathbf{I} + V^{1/2}R_0^+(k^2)|V|^{1/2}\right]$ is boundedly invertible on $H^\alpha(\mathbf{R}^n)$ and the following diagram is commutative:

$$\begin{array}{ccc}
H^{2\alpha}_{-s}(\mathbf{R}^n) & \xrightarrow{[\mathbf{I}+R_0^+(k^2)V]^{-1}} & H^{2\alpha}_{-s}(\mathbf{R}^n) \\
{\scriptstyle V^{1/2}}\downarrow & & \downarrow{\scriptstyle V^{1/2}} \\
H^\alpha(\mathbf{R}^n) & \xrightarrow{[\mathbf{I}+V^{1/2}R_0^+(k^2)|V|^{1/2}]^{-1}} & H^\alpha(\mathbf{R}^n)
\end{array}$$

Here we have used that $\alpha \in [0,1]$ so that $V(x) \in \mathbf{B}_{2\alpha} \in \mathbf{SR}$. Also, defining

$$\mathbf{S}(k) = \mathbf{I} - \frac{ik^{n-2}}{2(2\pi)^{n-1}}[\mathbf{I} + V^{1/2}R_0^+(k^2)|V|^{1/2}]^{-1}V^{1/2}\sigma^\dagger(k)\sigma(k)|V|^{1/2},$$

we have the commutative diagram

$$\begin{array}{ccccc}
H^\alpha(\mathbf{R}^n) & \xrightarrow{\mathbf{S}(k)-\mathbf{I}} & H^\alpha(\mathbf{R}^n) & \xleftarrow{\frac{-ik^{n-2}}{2(2\pi)^{n-1}}N(k)} & H^\alpha(\mathbf{R}^n) \\
{\scriptstyle |V|^{1/2}}\downarrow & & & & \uparrow{\scriptstyle V^{1/2}} \\
L^2_{\frac{1}{2}s}(\mathbf{R}^n) & \xrightarrow{\sigma(k)} & L^2(S^{n-1}) & \xrightarrow{\sigma^\dagger(k)} & H^{2\alpha}_{-s}(\mathbf{R}^n)
\end{array}$$

where

$$N(k) = [\mathbf{I} + V^{1/2}R_0^+(k^2)|V|^{1/2}]^{-1}.$$

A swift comparison with the diagram in the proof of Proposition 2.5 yields that

$$S(k) = \mathbf{I} + \sigma(k)|V|^{1/2}T(k), \quad \mathbf{S}(k) = \mathbf{I} + T(k)\sigma(k)|V|^{1/2},$$

for bounded operators $\sigma(k)|V|^{1/2} : H^\alpha(\mathbf{R}^n) \to L^2(S^{n-1})$ and $T(k) : L^2(S^{n-1}) \to H^\alpha(\mathbf{R}^n)$. As a result, the nonzero spectra of $S(k) - \mathbf{I}$ and $\mathbf{S}(k) - \mathbf{I}$ coincide. Since $S(k) - \mathbf{I}$ is also a normal compact operator, the nonzero spectra of $S(k) - \mathbf{I}$ and $\mathbf{S}(k) - \mathbf{I}$ consist of **the same** discrete set of eigenvalues without associated generalized eigenvectors. Even the multiplicities of the nonzero eigenvalues coincide. Hence, it suffices to prove that

$$\Sigma(k) = |V|^{1/2}\sigma^\dagger(k)\sigma(k)|V|^{1/2}$$

is a trace class operator on $H^{\alpha}(\mathbf{R}^n)$. This will immediately imply that

$$\mathbf{S}(k) - \mathbf{I} = \frac{-ik^{n-2}}{2(2\pi)^{n-1}}[\mathbf{I} + V^{1/2}R_0^+(k^2)|V|^{1/2}]^{-1}\mathrm{sgn}(V)\Sigma(k)$$

is a trace class operator on $H^{\alpha}(\mathbf{R}^n)$. The approximation numbers $\{s_n(\mathbf{S}(k)-\mathbf{I})\}_{n=1}^{\infty}$ [i.e. the non-increasing sequence of eigenvalues of $\{[\mathbf{S}(k)-\mathbf{I}]^{\dagger}[\mathbf{S}(k)-\mathbf{I}]\}^{\frac{1}{2}}$ (cf. [**GK65**])] form a sequence in ℓ^1. As a result, the eigenvalues $\{\lambda_n(k)\}_{n=1}^{\infty}$ of $S(k)-\mathbf{I}$ [or of $\mathbf{S}(k)-\mathbf{I}$] satisfy

$$\sum_{n=1}^{\infty} |\lambda_n(k)| \le \sum_{n=1}^{\infty} s_n(\mathbf{S}(k)-\mathbf{I}) \le \|[\mathbf{I} + V^{1/2}R_0^+(k^2)|V|^{1/2}]^{-1}\| \sum_{n=1}^{\infty} s_n(\Sigma(k)) < +\infty.$$

However, due to the unitarity of $S(k)$ the operator $S(k)-\mathbf{I}$ does not have a Volterra part and hence the trace norm of $S(k)-\mathbf{I}$ satisfies

$$\|S(k)-\mathbf{I}\|_{S_1} = \sum_{n=1}^{\infty} |\lambda_n(k)| \le \|[\mathbf{I} + V^{1/2}R_0^+(k^2)|V|^{1/2}]^{-1}\|\|\Sigma(k)\|_{S_1} < +\infty,$$

which proves $S(k)-\mathbf{I}$ to be a trace class operator on $L^2(S^{n-1})$.

Note that the kernel of the integral operator $\Sigma(k)$ is given by

$$\Sigma(k;x,y) = \int_{S^{n-1}} d\theta\ e^{ik\theta\cdot(x-y)}|V(x)V(y)|^{\frac{1}{2}}.$$

Now let us first consider the case $\alpha = 0$ with $V(x) \in \mathbf{B}_0$. Then $\Sigma(k)$ is a positive self-adjoint operator on $L^2(\mathbf{R}^n)$, the space that takes the place of $H^{\alpha}(\mathbf{R}^n)$. Thus, if $V(x) \in L^1(\mathbf{R}^n)$ and $V(x)$ is continuous on $\mathbf{R}^n$, $\Sigma(k)$ is a trace-class operator and

$$\mathrm{tr}(\mathrm{sgn}(V)\Sigma(k)) = \int_{\mathbf{R}^n} dx\ \mathrm{sgn}(V(x))\Sigma(k;x,x) = \Sigma_n \int_{\mathbf{R}^n} dx\ V(x),$$

where Σ_n is the surface area of S^{n-1}. If $V(x) \in L^1(\mathbf{R}^n)$ and $V(x)$ is not necessarily continuous on $\mathbf{R}^n$, we put for every $h > 0$

$$V_h(x) = \frac{1}{V_n}h^{-n}\int_{|x-y|\le h} dy\ V(y),$$

where V_n is the volume of the unit sphere in $\mathbf{R}^n$. Then $\|V_h\|_1 \le \|V\|_1$, $\lim_{h\to 0} \|V_h - V\|_1 = 0$ and V_h is a bounded continuous function on $\mathbf{R}^n$. If the original $V(x) \in \mathbf{B}_0$ and $s > 1$ is the constant such that multiplication by $(1+|x|^2)^s V(x)$ is a bounded operator from $L^2_{-s}(\mathbf{R}^n)$, the space that stands for $H^{\alpha}_{-s}(\mathbf{R}^n)$ if $\alpha = 0$, into $L^2_s(\mathbf{R}^n)$, then $|V(x)| \le C(1+|x|^2)^{-s}$ for some constant C. Using the estimates $\frac{1+|x|^2}{1+|y|^2} \le 1+|x|^2 \le 1+h^2 \le (h+1)^2$ if $|x| \le h$ and $\frac{1+|x|^2}{1+|y|^2} \le 1+|x|^2 \le \frac{1+(h+1)^2}{2}$ if $|x| \ge h$, we obtain

$$|V_h(x)| \le \frac{C}{(1+|x|^2)^s}\frac{(h+1)^{2s}}{V_n h^n}\int_{|x-y|\le h} dy\ = \frac{C(h+1)^{2s}}{(1+|x|^2)^s},$$

so that $V_h(x)$ belongs to $\mathbf{B}_0$ if $V(x)$ does. However, then $S(k)-\mathbf{I}$ with $V(x)$ replaced by $V_h(x)$ is trace class with a trace norm which is $O(1)$ as $h \to 0$. Hence, $S(k)-\mathbf{I}$ is trace class for the original $V(x)$ and the trace of $S(k)-\mathbf{I}$ with the original $V(x)$ is obtained from the trace of $S(k)-\mathbf{I}$ with $V(x)$ replaced by $V_h(x)$ by taking $h \to 0$. On the other hand, since $\mathrm{tr}(\mathrm{sgn}(V)\Sigma(k)) = \Sigma_n < V >$ for continuous $V(x)$, this must also be the case for discontinuous $V \in L^1(\mathbf{R}^n)$. From the special form of $\mathbf{S}(k)-\mathbf{I}$, the fact that its trace is the sum of its eigenvalues and its eigenvalues coincide with those of $S(k)-\mathbf{I}$, we eventually get (4.1), where we have also used that $[\mathbf{I}+V^{1/2}R_0^+(k^2)|V|^{1/2}]^{-1}$ approaches $\mathbf{I}$ in the norm as $k \to \pm\infty$.

Next, consider arbitrary $\alpha \in [0,1)$, but $V(x) \in \mathbf{B}_0$. Then a simple compactness argument yields that $S(k)-\mathbf{I}$ has the same eigenvalues with the same multiplicities, and hence the same trace, as an operator on either $L^2(\mathbf{R}^n)$ or $H^\alpha(\mathbf{R}^n)$. Thus (4.1) is immediate. More generally, if $V(x) \in \mathbf{B}_\alpha \cap L^1(\mathbf{R}^n)$ for some $\alpha \in [0,1)$, we can always approximate it by potentials in $\mathbf{B}_0 \cap L^1(\mathbf{R}^n)$ in the L^1-norm. Then the expression for the trace of $\mathrm{sgn}(V)\Sigma(k)$ will extend to these more general potentials and hence (4.1) will apply to them. ∎

For $V(x) \in \mathbf{B}_0$ we simply have the diagram of bounded operators

$$L^2_{-s}(\mathbf{R}^n) \xrightarrow{V^{\frac{1}{2}}} L^2(\mathbf{R}^n) \xrightarrow{|V|^{1/2}} L^2_s(\mathbf{R}^n)$$

with $\sigma(k): L^2_s(\mathbf{R}^n) \to L^2(S^{n-1})$, $\sigma^\dagger(k): L^2(S^{n-1}) \to L^2_{-s}(\mathbf{R}^n)$ and $R_0^+(k^2): L^2_s(\mathbf{R}^n) \to L^2_{-s}(\mathbf{R}^n)$ bounded. Hence, in that case Theorem 4.1 is valid if the constant s satisfies $s > \frac{1}{2}$.

If, in addition to the hypotheses of Theorem 4.1, zero is not an exceptional point, the trace norm of $S(k)-\mathbf{I}$ is easily seen to be $O(k^{n-2})$ as $k \to 0$, due to the boundedness of $[\mathbf{I}+R_0^+(k^2)]^{-1}$ on a neighborhood of $k=0$. Hence, in that case $S(0)-\mathbf{I}$ is a trace class operator on $L^2(S^{n-1})$ if $n \geq 3$. For $n=2$ and zero not an exceptional point, $s>1$ implies that $\sigma(0)$ and $\sigma^\dagger(0)$ are bounded operators between suitable spaces [cf. (A.3)]. Hence, we may then repeat the entire proof of Theorem 4.1 and prove that $S(0)-\mathbf{I}$ is trace class on $L^2(S^{n-1})$ if $n=2$ and if zero is not an exceptional point. Summarizing, if zero is not an exceptional point, $S(0)-\mathbf{I}$ is trace class on $L^2(S^{n-1})$ if $V(x) \in \mathbf{B}_\alpha$ for some $\alpha \in [0,1)$ and $s>1$ is the constant such that multiplication by $(1+|x|^2)^s V(x)$ is a bounded operator from $H^\alpha_{-s}(\mathbf{R}^n)$ into $L^2_s(\mathbf{R}^n)$.

APPENDIX A: NORM ESTIMATES FOR $\sigma^\dagger(k)$

From (2.4) we have $(\sigma^\dagger(k)g)(x) = (\sigma^\dagger(1)g)(kx)$. Letting $y = kx$ and using the identity $\frac{1+|y|^2}{k^2+|y|^2} \leq \max(1, \frac{1}{k^2})$, from (2.4) we obtain

$$\|\sigma^\dagger(k)g\|^2_{-s} \leq |k|^{2s-n}\max(1,|k|^{-2s})\|\sigma^\dagger(1)g\|^2_{-s}. \tag{A.1}$$

where the norm $\|\cdot\|_{-s}$ is the norm defined in (1.3). From the paragraph following the proof of Theorem 4.1, we have $\sigma^\dagger(1) \in \mathcal{L}(L^2(S^{n-1}); L^2_s(\mathbf{R}^n))$ for $s > \frac{1}{2}$. Hence, from (A.1) and the definition of $C_{k,s}$ given in (2.5), it follows that

$$C_{k,s} \leq C_{1,s}\max(|k|^{s-\frac{1}{2}n}, |k|^{-\frac{1}{2}n}). \tag{A.2}$$

Note also that from (1.3) and (2.4) we obtain, for $s > \frac{1}{2}n$,

$$\|\sigma^\dagger(k)g\|_{-s}^2 \leq \int_{\mathbf{R}^n} \frac{dx}{(1+|x|^2)^s}\Big(\int_{S^{n-1}} d\theta\ |g(\theta)|\Big)^2 \leq (\Sigma_n)^2 \int_0^\infty dr\ \frac{r^{n-1}}{(1+r^2)^s}\|g\|_{L^2(S^{n-1})}^2,$$

where Σ_n is the surface area of S^{n-1}. Hence, a comparison with (2.5) shows that

$$(A.3) \qquad C_{k,s} \leq \Sigma_n \Big[\int_0^\infty dr\ \frac{r^{n-1}}{(1+r^2)^s}\Big]^{\frac{1}{2}}, \quad s > \frac{1}{2}n.$$

It is possible to improve the estimates in (A.2) and (A.3) as follows. Using (2.6) we obtain

$$C_{k,s} \leq C \max(|k|^{\epsilon(s_1 - \frac{1}{2}n)}, |k|^{-\frac{1}{2}n\epsilon})$$

for some constant C if $s = \epsilon s_1 + (1-\epsilon)s_2$ with $\epsilon \in [0,1]$, $s_1 > \frac{1}{2}$ and $s_2 > \frac{1}{2}n$, which restricts ϵ to $\frac{n-2s}{n-1} < \epsilon \leq 1$. Maximizing $-\frac{1}{2}n\epsilon$ and minimizing $\epsilon(s_1 - \frac{1}{2}n)$ under these constraints we get

$$(A.4) \qquad C_{k,s} = \begin{cases} O\big(|k|^{\min(0, s-\frac{1}{2}n)}\big) & (k \to \pm\infty) \\ O\big(|k|^{-\frac{n(n-2s)}{2(n-1)}+\delta}\big) & (k \to 0, \forall\delta > 0), \end{cases}$$

which for $k \to \pm\infty$ corresponds with [**We90**].

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