Well-posedness of stationary and time-dependent Spencer–Lewis equations modeling electron slowing down

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(Received 9 September 1987; accepted for publication 7 September 1988)

The unique solvability of the time-dependent and stationary Spencer–Lewis equations is established under natural assumptions on the solution and the data of the problem. The strategy used is the method of characteristics followed by perturbation and monotone approximation arguments. The evolution operator in the time-dependent Spencer–Lewis equation is proved to generate a strongly continuous contraction semigroup.

I. INTRODUCTION

In this article we prove that the Spencer–Lewis equation, originally derived in the 1950's by Spencer[1] and Lewis[2] to describe the continuous slowing down of electrons of intermediate energy in a semiconductor or metallic slab medium when the distribution function of the electrons at the upper end of the considered energy range is known, is uniquely solvable. In its present form the equation was formulated by Bartine *et al.*[3] (see also Arkuszewski *et al.*[4]), who replaced the original term $\beta\, \partial u/\partial E$ by the mathematically more convenient term $\partial(\beta u)/\partial E$. Here $\beta = \beta(x,E)$ represents the stopping power. Both the time-dependent and the stationary problem will be considered under natural initial and boundary conditions (see below).

The solution $u = u(x,\mu,E,t)$ of the Spencer–Lewis equation describes the electron distribution as a function of position $x\in[0,a]$, direction cosine of propagation $\mu\in[-1,1]$, energy $E\in[E_m,E_M]\subseteq(0,\infty)$ and, when the problem is time dependent, time $t\in[0,\infty)$. The equation takes account of the fact that incoming electrons may undergo elastic scattering of electrons by atomic nuclei, inelastic scattering by atomic electrons, and bremsstrahlung producing collisions with atomic nuclei and atomic electrons. Inelastic scattering between an incident electron and an atomic electron may cause ionization and thus add to the free electron population. However, the relatively small contribution to the electron distribution by the electrons stemming from the interaction of photons with matter, through the photoelectric effect, Compton scattering, and pair production, is neglected when deriving the Spencer–Lewis equation. The contribution of the so-called "soft" electron–electron and electron–atomic nuclei collisions leading to an energy transfer of the order of or less than the binding energy of the target electrons is described as a continuous slowing down so that the energy loss per unit distance due to such collisions rather than their cross sections appears in the equation.

In the time-dependent case the equation has the form

$$\frac{\partial u}{\partial t}(x,\mu,E,t) + \mu\frac{\partial u}{\partial x}(x,\mu,E,t) - \frac{\partial(\beta u)}{\partial E}(x,\mu,E,t) + \sigma(x,\mu,E)u(x,\mu,E,t)$$
$$= \int_{-1}^{1}\sigma_s(x,\mu,\mu',E)u(x,\mu',E,t)\,d\mu' + f(x,\mu,E,t), \tag{1}$$

where $\sigma = \sigma(x,\mu,E)$ is the total scattering cross section that usually does not depend on μ, $\sigma_s = \sigma_s(x,\mu,\mu',E)$ the (azimuthally integrated) scattering cross section, and $f(x,\mu,E,t)$ the distribution function for the internal electron sources of intermediate energy. Equation (1) is endowed with the boundary conditions

$$u(x=0,\mu,E,t) = g_0(\mu,E,t),\quad \mu>0,\quad E\in[E_m,E_M], \tag{2a}$$

$$u(x=a,\mu,E,t) = g_a(\mu,E,t),\quad \mu<0,\quad E\in[E_m,E_M], \tag{2b}$$

specifying the distribution of the incident electrons of intermediate energy, the boundary condition

$$u(x,\mu,E=E_M,t) = g_i(x,\mu,t),\quad x\in[0,a],\quad \mu\in[-1,1], \tag{2c}$$

specifying the distribution of the electrons incident at the higher end of the energy range, and the initial condition

$$u(x,\mu,E,t=0) = h_0(x,\mu,E),\quad x\in[0,a],\quad \mu\in[-1,1],\quad E\in[E_m,E_M]. \tag{3}$$

In the stationary case we have the boundary-value problem

$$\mu\frac{\partial u}{\partial x}(x,\mu,E) - \frac{\partial(\beta u)}{\partial E}(x,\mu,E) + \sigma(x,\mu,E)u(x,\mu,E) = \int_{-1}^{1}\sigma_s(x,\mu,\mu',E)u(x,\mu',E)\,d\mu' + f(x,\mu,E), \tag{4}$$

$$u(0,\mu,E) = g_0(\mu,E),\quad \mu>0,\quad E\in[E_m,E_M], \tag{5a}$$

$$u(a,\mu,E) = g_a(\mu,E),\quad \mu<0,\quad E\in[E_m,E_M], \tag{5b}$$

$$u(x,\mu,E_M) = g_i(x,\mu),\quad x\in[0,a]\quad \mu\in[-1,1]. \tag{5c}$$

Contrary to the situation of neutron transport theory, the integral term describing the gain of electrons due to collisions with the host medium does not involve an integration over energy but only over the direction cosine of propagation. Another difference with neutron transport theory is the presence of a term in the equation involving partial differentiation with respect to energy. Natural assumptions on the model are to require u, f, g_0, g_a, g_i, and h_0 as well as β, σ, and σ_s to be non-negative Borel functions and to adopt the hypothesis

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 0022-2488/89/010158-08$02.50

$$\sigma(x,\mu',E)\geqslant\int_{-1}^{1}\sigma_s(x,\mu,\mu',E)d\mu,$$
$$(x,\mu',E)\in[0,a]\times[-1,1]\times[E_m,E_M]\,, \tag{6}$$

where the equality sign holds true if (and only if) there is no electron absorption at intermediate energies.

The natural functional setting for the above two problems is suggested by the fact that $u=u(x,\mu,E,t)$ is the electron distribution function for given incident electron fluxes $|\mu|g_0(\mu,E,t)$ and $|\mu|g_a(\mu,E,t)$. This means in particular that one should analyze the above problems in the Banach space $\mathscr{N}=L_1(\Lambda)$, where $\Lambda=(0,a)\times(-1,1)\times(E_m,E_M)$, consisting of the functions $u=u(x,\mu,E)$, which are finite with respect to the norm

$$\|u\|_1=\int_0^a\int_{-1}^{1}\int_{E_m}^{E_M}|u(x,\mu,E)|dE\,d\mu\,dx,$$

while f, $h_0\in\mathscr{N}$, $g_0\in\mathscr{N}_+$, $g_a\in\mathscr{N}_-$, and $g_i\in\mathscr{N}_i$ are given functions. Here $\mathscr{N}_\pm$ is the Banach space of all functions $g=g(\mu,E)$ finite with respect to the norm

$$\|g\|_1=\pm\int_0^{\pm1}\int_{E_m}^{E_M}|\mu|\,|g(\mu,E)|dE\,d\mu\,,$$

while $\mathscr{N}_i$ is the Banach space of all functions $g=g(x,\mu)$ finite with respect to the norm

$$\mathscr{N}_i=\int_0^a\int_{-1}^{1}\beta(x,E_M)|g(x,\mu)|d\mu\,dx\,.$$

Note that the stopping power at the higher end of the energy range appears as a weight in the L_1 norm of $\mathscr{N}_i$.

In recent years there has been renewed activity on the Spencer–Lewis equation, in part because of the necessity of proving the convergence of the existing finite difference methods for solving Eqs. (4) and (5). Nelson and Seth[5] proved the convergence of certain finite difference schemes under the assumption that Eqs. (4) and (5) have a unique solution. For a simple rod model the well-posedness of the original problem was proved by Nelson.[6] After the emergence of the abstract time-dependent kinetic theory of Beals and Protopopescu,[7] which can also be found in the monograph of Greenberg *et al.*,[8] these results have been extended for the time-dependent and the stationary problem to the case where (i) $\beta=\beta(x,E)$ is piecewise constant in energy and Lipschitz continuous in position, (ii) $\sigma=\sigma(x,E)$ is bounded and independent of μ, and (iii) when treating the stationary problem, condition (6) is replaced by

$$\sigma(x,\mu',E)\geqslant\delta\int_{-1}^{1}\sigma_s(x,\mu,\mu',E)d\mu,$$
$$\text{for all }(x,\mu',E)\in[0,a]\times[-1,1]\times[E_m,E_M]\,, \tag{7}$$

where $\delta\in(0,1)$. Condition (i) was imposed to make the vector fields appearing in Eqs. (1) and (4) divergence-free so that the theory of Refs. 7 and 8 goes through. Condition (ii) implies the boundedness of the integral operator at the right-hand side of Eqs. (1) and (4) on $\mathscr{N}$, which is another prerequisite of the theory of Refs. 7 and 8. Condition (iii) implies that the evolution semigroup of Eq. (1) is exponentially decreasing in time, which makes the corresponding stationary problem uniquely solvable. For these results we refer to Sec. XIII.3 of Ref. 8.

Recently a number of new developments in abstract kinetic theory have taken place that will enable us to drop the above rather artificial conditions (i)–(iii) from the theory of Spencer–Lewis equations and to prove the unique solvability of both the time-dependent and the stationary problem under more natural assumptions. It has become clear[9,10] how to treat non-divergence-free force fields and thus how to drop condition (i). Furthermore, abstract kinetic theory has been extended to the case where the integral term of the collision operator is a (positive) contraction from $\mathscr{N}_\sigma=L_1(\Lambda;\sigma(x,\mu,E)dx\,d\mu dE)$ into $\mathscr{N}=L_1(\Lambda;dx\,d\mu\,dE)$ (see Refs. 9 and 10 for treatments of similar situations). These novel developments will guide us in the construction of an existence and uniqueness theory for the solution of the Spencer–Lewis equation under the following assumptions.

(A) There exists a partition $E_m=E_0<E_1<\cdots<E_r=E_M$, possibly with $r=1$, of the intermediate energy range such that β is non-negative and Lipschitz continuous on the closure of each set

$$\Lambda_i=(0,a)\times(-1,1)\times(E_{i-1},E_i)\,, \tag{8}$$

where $i=1,\dots,r$.

(B) The stopping power is Lipschitz continuous on the *disjoint* union $\cup_{i=1}^{r}\overline{\Lambda}_i$ and has only finitely many zeros, all of them in the interior of $(0,a)\times(E_m,E_M)$. Thus in defining the stopping power one should distinguish between E_i^- and E_i^+, for $i=1,\dots,r-1$ if $r\geqslant2$.

(C) σ and σ_s satisfy condition (6) and

$$\int_0^a\int_{-1}^{1}\int_{E_m}^{E_M}\sigma(x,\mu,E)dE\,d\mu\,dx<\infty.$$

Apart from this integrability condition, σ may be unbounded. When $r\geqslant2$, we will also require the solutions u of Eqs. (1)–(3) and Eqs. (4) and (5) to be continuous at the energy jumps, i.e., to satisfy $u(E_i^-)=u(E_i^+)$, for $i=1,\dots,r-1$. The physical meaning of this requirement is that discontinuous jumps in the stopping power do not bring about (positive or negative) electron sources. We will apply the method of characteristics in such a way that this continuity requirement is incorporated in the mathematical formulation in such a way that it does not show up as a boundary condition any more. In this fashion we will accomplish a major simplification of the method of characteristics used in Sec. XIII 3 of Ref. 8.

In Sec. II we will solve the time-dependent problem using the method of characteristics. The stationary problem will be the topic of Sec. III. However, we will solve this problem by reformulating it as an *initial*-value problem and applying the method of characteristics in the usual way. Section IV is devoted to semigroup properties and Sec. V to a discussion of the results.

Remark: Recently E. Ringeisen (Centre de Mathématiques Appliquées, École Normale Supérieure, Paris) proved the unique solvability of the stationary Spencer–Lewis equation under the assumptions that (i) the stopping power $\beta=\beta(x,E)$ is continuously differentiable on

$[0,a]\times[E_m,E_M]$, (ii) the cross sections σ and σ_s are L_∞ functions, and

$$\text{(iii)}\quad \int_{-1}^{1}\sigma_s(x,E,\mu,\mu')\times\exp\left[-\frac{\delta(x,\mu',E)(\|\sigma\|_\infty+\lambda_1)}{|\mu|}\right]d\mu'$$

is bounded away from zero for some positive constant λ_1. Here $\delta(x,\mu,E)$ is the length of the maximal integral curve of the vector field X (defined below) passing through (x,μ,E).

II. THE TIME-DEPENDENT PROBLEM

On the set $\Lambda_t = \cup_{i=1}^r \Lambda_i$ with the union thought of as disjoint and endowed with the Lebesgue measure we introduce the vector field

$$X=\mu\frac{\partial}{\partial x}-\beta(x,E)\frac{\partial}{\partial E},$$

which is clearly Lipschitz continuous on the closure Λ_t (when distinguishing between E_i^- and E_i^+, for $i=1,...,r$). Using time t as a parameter there is a unique integral curve of X through each point of Λ_t satisfying the characteristic equations

$$\frac{dx}{dt}=\mu,\quad \frac{d\mu}{dt}=0,\quad \frac{dE}{dt}=-\beta(x,E).$$

In contrast to the practice of Refs. 7–9, we will identify all points of the type (x,μ,E_i^-) with the corresponding points (x,μ,E_i^+), thus obtaining the original manifold Λ, and continue the integral curves of X across the energy interfaces $E=E_i$ $(i=1,2,...,r-1)$. The sets $D_\pm$ of left and right end points of the integral curves of X passing through an interior point of Λ_t are then given by

$$\begin{aligned}D_- &= [\{0\}\times(0,1)\times(E_m,E_M)]\\ &\quad\cup[\{a\}\times(-1,0)\times(E_m,E_M)]\\ &\quad\cup[(0,a)\times(-1,1)\times\{E_M\}],\\ D_+ &= [\{0\}\times(-1,0)\times(E_m,E_M)]\\ &\quad\cup[\{a\}\times(0,1)\times(E_m,E_M)]\\ &\quad\cup[(0,a)\times(-1,1)\times\{E_m\}].\end{aligned}$$

Along the integral curves the energy E is steadily decreasing. We now parametrize $\overline{\Lambda}$ as

$$\overline{\Lambda}=\{(z,t):\ z\in D_-,\ 0\leqslant t<l(z)\},$$

where $l(z)$ is the travel time along the entire trajectory of X starting from $z\in D_-$. From nonzero μ there is a maximal travel time along a trajectory having μ as a constant of motion which is bounded above by $a/|\mu|$. However, on the trajectory with $\mu=0$ and $x\in(0,a)$ as constants of motion the total travel time

$$l(z)=\int_{E_m}^{E_M}\frac{1}{\beta(x,E)}\,dE=+\infty,$$

because $\beta(x,E_M)\neq 0$, β has only finitely many zeros on Λ, and β is Lipschitz continuous. [Note that $l(z)=+\infty$ if $\beta(x,E_0)=0$ for some $E_0\in(E_m,E_M)$, because the Lipschitz condition on β implies $|\beta(x,E)|\leqslant L|E-E_0|$ with $L>0$.] Similarly, one may parametrize $\overline{\Lambda}\times[0,T]$ as

$$\overline{\Lambda}\times[0,T]=\{(z,s):\ z\in[\Lambda\times\{0\}]\cup[D_-\times(0,T)],\ 0\leqslant s\leqslant\max(l(z),T)\},$$

where s is the travel time parameter along the trajectory of $Y=\partial/\partial t+X$ from its left end point on either $\Lambda\times\{0\}$ or $D_-\times(0,T)$. To avoid confusion between t as a variable appearing in the vector field Y and the parameter in the characteristic equations of Y, we will use t as the time variable and s as the travel time parameter.

For every $u\in L_1(\Sigma)$ with $\Sigma=\Lambda\times(0,T)$ one may define

$$Yu=\frac{\partial u}{\partial t}+Xu,\quad Xu=\mu\frac{\partial u}{\partial x}-\beta\frac{\partial u}{\partial E},$$

as distributional directional derivates by

$$\int_\Sigma\left\{(Yu)v+u(Yv)+\frac{\partial\beta}{\partial E}uv\right\}dx\,d\mu\,dE\,dt=0$$

and

$$\int_\Lambda\left\{(Xu)v+u(Xv)+\frac{\partial\beta}{\partial E}uv\right\}dx\,d\mu\,dE=0,$$

where v belongs to the test function space Φ_0 of all real Borel functions on Σ (resp. Λ) that are bounded, are continuously differentiable along the trajectories of Y (resp. X) with bounded directional derivative Yv (resp. Xv), vanish at the end points of each trajectory and have the property that the lengths of the trajectories meeting the support of v are bounded away from zero. The latter means in particular that $|\mu|$ is bounded away from zero on the support of each $v\in\Phi_0$. Note that $\partial\beta/\partial E$ exists almost everywhere as a result of the absolute continuity of β.

Below we will employ the spaces $\mathscr{M}$, $\mathscr{M}_\pm$, and $\mathscr{M}_i$. These spaces are defined in the same way as the $\mathscr{N}$ spaces, i.e., again as L_1 spaces but with F replaced by $F\times(0,T)$ and the underlying measure replaced by its product with the Lebesgue measure on $(0,T)$. These spaces may also be represented as the spaces of all Bochner integrable functions from $(0,T)$ into the corresponding $\mathscr{N}$ space endowed with the L_1 norm (cf. Ref. 11).

Lemma 2.1: Suppose $h_0\in\mathscr{N}$, $g_0\in\mathscr{M}_+$, $g_a\in\mathscr{M}_-$, $g_i\in\mathscr{M}_i$, and $f\in\mathscr{M}$. Then there exists a unique solution u of the initial-boundary-value problem

$$Yu+\left\{\sigma(x,\mu,E)-\frac{\partial\beta}{\partial E}\right\}u=f, \tag{9}$$

$$u(x,\mu,E,t=0)=h_0(x,\mu,E), \tag{10}$$

$$u(x=0,\mu,E,t)=g_0(\mu,E,t),\quad \mu>0, \tag{11}$$

$$u(x=a,\mu,E,t)=g_a(\mu,E,t)\quad \mu<0, \tag{12}$$

$$u(x,\mu,E=E_M,t)=g_i(x,\mu,t). \tag{13}$$

The solution u and the left-hand sides of Eqs. (10)–(13) have the following properties.

(i) $u\in\mathscr{M}$ while the left-hand sides of Eqs. (10)–(13) belong to $\mathscr{M}$, $\mathscr{M}_+$, $\mathscr{M}_-$, and $\mathscr{M}_i$, respectively.

(ii) Together with $u(x,\mu,E=E_m,t)$, which belongs to the space $L_1(M;\beta(x,E_m)dx\,d\mu\,dt)$ with $M=(0,a)$

$\times(-1,1)\times(0,T)$, and $u(x,\mu,E,t=T)$ in $\mathcal{N}$, these functions are related by the Green identity

$$\int_\Sigma \left(\frac{\partial u}{\partial t}+\mu\frac{\partial u}{\partial x}-\frac{\partial}{\partial E}(\beta u)\right)dx\,d\mu\,dE\,dt$$
$$=\int_M (\beta(x,E_m)u(x,\mu,E_m,t)$$
$$-\beta(x,E_M)u(x,\mu,E_M,t))dx\,d\mu\,dt$$
$$+\int_0^T\int_{E_m}^{E_M}\int_{-1}^{1}\mu\{u(a,\mu,E,t)$$
$$-u(0,\mu,E,t)\}d\mu\,dE\,dt$$
$$+\int_\Lambda \{u(x,\mu,E,T)-u(x,\mu,E,0)\}dx\,d\mu\,dE. \quad (14)$$

Here the functions $u(x=0,\mu,E,t)$ and $u(x=a,\mu,E,t)$ belong to the L_1 space $L_1((-1,1)\times(E_m,E_M)\times(0,T);|\mu|d\mu\,dE\,dt)$.

(iii) $\sigma u\in\mathcal{M}$ and

$$\|\sigma u\|_{\mathcal{M}}\leqslant\|f\|_{\mathcal{M}}+\|h_0\|_{\mathcal{N}}+\|g_0\|_{\mathcal{M}}$$
$$+\|g_a\|_{\mathcal{M}_+}+\|g_i\|_{\mathcal{M}_i}. \quad (15)$$

Proof: Writing

$$h(x,\mu,E)=\sigma(x,\mu,E)-\frac{\partial\beta}{\partial E}$$

and using the above parametrization of Y, we reduce Eqs. (9)–(13) to the initial-value problem

$$\frac{du}{ds}=h(z,s)u(z,s)=f(z,s), \quad (16)$$
$$u(z,s=0)=g(z), \quad (17)$$

where, modulo the parametrization, $g(z)$ coincides with $h_0(x,\mu,E)$ on $\Lambda\times\{0\}$, $g_0(\mu,E,t)$ on $\{0\}\times(0,1)\times(E_m,E_M)\times(0,T)$, $g_a(\mu,E,t)$ on $\{a\}\times(-1,0)\times(E_m,E_M)\times(0,T)$, and $g_i(x,\mu,t)$ on $(0,a)\times(-1,1)\times(0,T)$. Since β is (piecewise) absolutely continuous on $\overline{\Lambda}$, it has an almost everywhere defined derivative $\partial\beta/\partial E$, which belongs to $L_1(\Lambda)$; hence, by assumption (C), $h\in L_1(\Lambda)$. As the unique solution we find

$$u(z,s)=\exp\left\{-\int_0^s h(z,\sigma)d\sigma\right\}g(z)$$
$$+\int_0^s\exp\left\{-\int_\tau^s h(z,\sigma)d\sigma\right\}f(z,\tau)d\tau.$$

Here the uniqueness, with the derivatives in Eq. (16) taken in distributional sense, follows from the Green's identity applied to Eqs. (16) and (17) with $g=0$ and $f=0$. Hence $u(x,\mu,E,t=0)$, $u(x=0\,\mu,E,t)$, $u(x=a,\mu,E,t)$, and $u(x,\mu,E=E_M,t)$ have the appropriate properties and satisfy Eqs. (10)–(13). Further, $u(x,\mu,E=E_m,t)$, $u(x,\mu,E,t=T)$, $u(x=0,\mu,E,t)$, and $u(x=a,\mu,E,t)$ also have the appropriate properties and Eq. (14) is satisfied. In fact, Eq. (14) can be written as

$$\int_\Sigma\left(Y-\frac{\partial B}{\partial E}\right)u\,dx\,d\mu\,dE\,dt$$
$$=\int_{D^+}u^+\,dv^+(z)-\int_{D^-}u^-\,dv^-(z), \quad (18)$$

where $D^-=[\Lambda\times\{0\}]\cup[D_-\times(0,T)]$ and $D^+=[\Lambda\times\{T\}]\cup[D_+\times(0,T)]$ for appropriate positive Borel measures $dv^\pm(z)$ which are weighted Lebesgue measures with $|\mu|$ as the weight on $\{x=0\}$ and $\{x=a\}$, 1 as the weight on $\{t=0\}$ and $\{t=T\}$, $\beta(x,E_M)$ as the weight on $\{E=E_M\}$, and $\beta(x,E_m)$ as the weight on $\{x,E_m\}$. Finally, to prove (15) it suffices to restrict onself to non-negative g_0, g_0, g_a, g_i, and f. For non-negative data we have

$$\|\sigma u\|=\left\|\left(h+\frac{\partial\beta}{\partial E}\right)u\right\|$$
$$=\|f\|-\left\|\left(Y-\frac{\partial\beta}{\partial E}\right)u\right\|$$
$$=\|f\|+\|u^-\|-\|u^+\|\leqslant\|f\|+\|u^-\|$$
$$=\|f\|+\|h_0\|+\|g_0\|+\|g_a\|+\|g_i\|,$$

which proves the lemma. □

We now define the positive operator

$$(Ju)(x,\mu,E)=\int_\Lambda \sigma_s(x,\mu,\mu',E)u(x,\mu',E)d\mu', \quad (19)$$

which satisfies

$$\|Ju\|_{\mathcal{N}}\leqslant\|\sigma u\|_{\mathcal{N}},\quad u\in\mathcal{N}_\sigma=L_1(\Lambda;\sigma\,dx\,d\mu\,dE). \quad (20)$$

We will denote the norm of J as a contraction from $\mathcal{N}_\sigma$ into $\mathcal{N}$ by $\|J\|_\#$.

Lemma 2.2: Suppose $\|J\|_\#<1$, i.e., suppose

$$\sigma(x,\mu',E)\geqslant\delta\int_{-1}^{1}\sigma_{s(x,\mu,\mu',E)}\,d\mu,$$

for some $\delta\in(0,1)$. Then there exists a unique solution u of the initial-boundary-value problem

$$Yu+\left\{\sigma(x,\mu,E)-\frac{\partial\beta}{\partial E}\right\}u=Ju+f, \quad (21)$$
$$u(x,\mu,E,t=0)=h_0(x,\mu,E), \quad (22)$$
$$u(x=0,\mu,E,t)=g_0(\mu,E,t),\quad \mu>0, \quad (23)$$
$$u(x=a,\mu,E,t)=g_a(\mu,E,t),\quad \mu<0, \quad (24)$$
$$u(x,\mu,E=E_M,t)=g_i(x,\mu,t). \quad (25)$$

The solution u and the left-hand sides of Eqs. (22)–(25) have the properties (i) and (ii) in the statement of Lemma 2.1, while (iii) is replaced by (iii') σu belongs to $\mathcal{M}$ and

$$\|\sigma u\|\leqslant(1-\|J\|_\#)^{-1}(\|f\|_{\mathcal{M}}+\|h_0\|_{\mathcal{N}}+\|g_0\|_{\mathcal{M}}$$
$$+\|g_a\|_{\mathcal{M}_+}+\|g_i\|_{\mathcal{M}_i}). \quad (26)$$

Proof: Let us write Eqs. (21)–(25) as

$$(Y+h)u=Ju+f, \quad (27)$$
$$u^-=g, \quad (28)$$

where g is defined as in the proof of Lemma 2.1. Denoting the solution of Eqs. (16) and (17) as $u=S(f,g)$, we represent the solution of Eqs. (27) and (28) as $u=S(f^*,g)$. Then $f^*\in\mathcal{M}$ satisfies the equation

$$(\mathbb{1}+L)f^*=f+JS(0,g), \quad (29)$$

where

$$Lf^*=-JS(f^*,0).$$

Then

$$\|Lf^*\|_{\mathscr{M}} \leqslant \|J\|_\# \|\sigma S(f^*,0)\|_{\mathscr{M}} \leqslant \|J\|_\# \|f^*\|_{\mathscr{M}},$$

so that Eq. (29) has a unique solution f^* satisfying

$$\|f^*\| \leqslant \sum_{n=0}^{\infty} \|J\|_\#^n \|f + JS(0,g)\|$$
$$\leqslant (1 - \|J\|_\#)^{-1}(\|f\| + \|J\|_\# \|g\|) .$$

Hence

$$\|\sigma u\| = \|\sigma S(f^*,g)\| \leqslant \|f^*\| + \|g\|$$
$$\leqslant (1 - \|J\|_\#)^{-1}(\|f\| + \|g\|),$$

which proves the lemma. □

For non-negative data we directly obtain from Eq. (27)

$$\|(Y+b)u\| + \|(h-b)u\| + \|g\| = \|Ju\| + \|f\| + \|g\|,$$

where $b = -(\partial\beta/\partial E)$ and all norms are L_1 norms. Using the Green's identity (18) we have

$$\|u^+\| + \|\sigma u\| = \|Ju\| + \|f\| + \|g\|,$$

whence

$$\|u^+\| + (1 - \|J\|_\#)\|\sigma u\| \leqslant \|f\| + \|g\|. \tag{30}$$

Here

$$u^+ = (u(t=T), u(E=E_m),$$
$$u(x=0,\mu<0), u(x=a,\mu \geqslant 0))$$

on a direct sum of L_1 spaces with certain weights.

When $\|J\|_\# = 1$, we cannot apply the same perturbation arguments as in the proof of Lemma 2.2. Instead we approximate u monotonically by the unique solutions u_n of the initial-boundary-value problems

$$(Y+h)u_n = \beta_n J u_n + f, \tag{31}$$

$$(u_n)^- = g, \tag{32}$$

where $\{\beta_n\}_{n=1}^{\infty} \uparrow 1$. These solutions are non-negative, are nondecreasing with n, and satisfy

$$\|u_n^+\| + (1-\beta_n)\|\sigma u_n\| \leqslant \|f\| + \|g\|.$$

Hence there exists u^+ such that

$$\lim_{n\to\infty} \|u^+ - u_n^+\| = 0 \tag{33}$$

in the norm of $\mathscr{N}_+$ and Eq. (28) is satisfied. On the other hand,

$$f - \left(Y - \frac{\partial\beta}{\partial E}\right) u_n = \sigma u_n - \beta_n J u_n,$$

while

$$\int_\Lambda \left(Y - \frac{\partial\beta}{\partial E}\right)[u_k - u_n]\,dx\,d\mu\,dE\,dt = \|u_k^+ - u_n^+\| \to 0,$$

as $k,n\to\infty$. Hence there exists $w\in\mathscr{M}$ such that

$$\lim_{n\to\infty} \|\sigma u_n - \beta_n J u_n - w\| = 0. \tag{34}$$

Then we also have

$$\lim_{n\to\infty} \left\| \left(Y - \frac{\partial\beta}{\partial E}\right) u_n - [f - w] \right\| = 0. \tag{35}$$

We now solve the initial-boundary-value problem

$$\left(Y - \frac{\partial\beta}{\partial E}\right) u = f - w, \quad u^- = g,$$

and find a solution $u\in\mathscr{M}$ having the properties (i) and (ii) in the statement of Lemma 2.1. We will show that u is a solution of Eqs. (22)–(25), but in a rather weak sense.

Indeed, from (30) and $\|\beta_n J\|_\# = \beta_n$ it is clear that

$$(1-\beta_n)\|Ju_n\| \leqslant (1-\beta_n)\|\sigma u_n\| \leqslant \|f\| + \|g\|,$$

so that $\{(\sigma - J)u_n\}_{n=1}^{\infty}$ is a bounded sequence in $\mathscr{M}$. Since

$$\lim_{n\to\infty} \|[f - (\sigma - \beta_n J)u_n] - [f - w]\| = 0,$$

we have

$$\lim_{n\to\infty} \|u - u_n\| = 0.$$

To establish the uniqueness of the solution, we assume that u is a (real) solution of the homogeneous time-dependent problem

$$(Y+b)u + (\sigma - J)u = 0, \quad u^- = 0,$$

where $b = -\partial\beta/\partial E$. Then $u = \operatorname{sgn}(u)|u|$ and hence

$$(Y+b)u + (\sigma - J)|u| + \{J|u| - \operatorname{sgn}(u)Ju\} = 0,$$
$$|u|^- = 0.$$

Integrating over position–velocity–energy–time phase space and using $u^- = 0$ we get

$$\|u^+\| + \iiiint (\sigma - J)|u|\,dx\,d\mu\,dE\,dt$$
$$+ \|\{J|u| - \operatorname{sgn}(u)Ju\}\| = 0.$$

The second term on the left-hand side is non-negative [cf. Eq. (6)], so that $u^+ = 0$, $(\sigma - J)|u|$ has a zero integral over phase space, and $\{J|u| - \operatorname{sgn}(u)Ju\} = 0$, so that $|u|$ is a solution of Eqs. (27) with $f = 0$. Thus without loss of generality we may assume $u \geqslant 0$. We then find

$$u(z,s) = \int_0^s \exp\left\{-\int_\tau^s (\sigma + b)(z,\tau')d\tau\right\}(Ju)(z,\tau)d\tau.$$

For $s = l(z)$ we get u^+, which vanishes. Moreover, $Ju \geqslant 0$. Hence $Ju \equiv 0$ and $u \equiv 0$, which settles the uniqueness issue.

We have therefore established the following.

Theorem 2.3: There exists a unique solution u of the initial-boundary-value problem (21)–(25). The solution u and the left-hand sides of Eqs. (22)–(25) have the properties (i) and (ii) in the statement of Lemma 2.1, $(\sigma u - Ju)\in\mathscr{M}$, but σu and Ju themselves need not belong to $\mathscr{M}$.

Corollary 2.4: If $g_0 \equiv 0$, $g_a \equiv 0$, $g_i \equiv 0$, and $f \equiv 0$, the unique solution of Eqs. (22)–(25) can be represented as $u(t) = S(t)h_0$, $t \geqslant 0$, where $\{S(t)\}_{t\geqslant 0}$ is a positive contraction semigroup on $\mathscr{N}$. This semigroup satisfies $\|S(t)h_0\| = \|h_0\|$, for all non-negative $h_0\in\mathscr{N}$ and all $t\geqslant 0$, if and only if $\|Ju\| = \|\sigma u\|$ for all non-negative $u\in\mathscr{N}_\sigma$.

Proof: The first part is clear from the estimates

$$\|u(t=T)\| \leqslant \|u^+\| \leqslant \|f\| + \|g\| = \|h_0\|.$$

The second part is a simple consequence of the Green's identity for non-negative data. □

III. THE STATIONARY PROBLEM

The stationary problem is given by Eqs. (4) and (5) and can thus be written as

$$Xu + \left(\sigma - \frac{\partial\beta}{\partial E}\right) u = Ju + f, \tag{36}$$

$$u_+ = g, \tag{37}$$

where $g = (g_0, g_a, g_i)$. We have to find a function $u \in \mathcal{N}$ such that $u_- \in \mathcal{N}_- \oplus \mathcal{N}_+ \oplus \mathcal{N}_i$ and Eqs. (36) and (37) are satisfied. Here we assume that $f \in \mathcal{N}$ and $g \in \mathcal{N}_- \oplus \mathcal{N}_+ \oplus \mathcal{N}_i$. As in the previous section we distinguish between the cases $\|J\|_\# < 1$ and $\|J\|_\# = 1$.

Lemma 3.1: Suppose $\|J\|_\# < 1$. Then Eqs. (36) and (37) have a unique solution u such that $\sigma u \in \mathcal{N}$ and

$$\|\sigma u\| \leqslant (1 - \|J\|_\#)^{-1}\{\|f\| + \|g\|\}. \tag{38}$$

The solution is non-negative for non-negative data f and g.

Proof: Writing $h = \sigma - (\partial\beta/\partial E)$, we solve Eqs. (36) and (37) for $J = 0$ and obtain

$$u(z,s) = \exp\left\{ - \int_0^s h(z,\sigma) d\sigma \right\} g(z) + \int_0^s \exp\left\{ - \int_\tau^s h(z,\sigma) d\sigma \right\} f(z,\tau) d\tau,$$

which has the desired properties. The Green's identity for X gives as before

$$\|\sigma u\| \leqslant \|f\| + \|g\|.$$

We write $u = S(f,g)$.

As in the previous section we represent the solution of Eqs. (36) and (37) as $u = S(f^*,g)$, where $(\mathbb{1} + L) f^* = f + JS(0,g)$ and $\|L\| \leqslant \|J\|_\# < 1$. We then find a unique $f^* \in \mathcal{N}$, which is non-negative for non-negative f and g because $(-L) \geqslant 0$. A simple estimation then gives (38). □

To pass to the case $\|J\|_\# = 1$, we use monotone approximation by the solutions u_n of the stationary problem

$$Xu_n + \left(\sigma - \frac{\partial\beta}{\partial E}\right) u_n = \beta_n J u_n + f, \tag{39}$$

$$u_{n,-} = g, \tag{40}$$

where $(\beta_n)_{n=1}^\infty \uparrow 1$. Using the Green's identity for X we find

$$\|u_{n,-}\| + (1 - \beta_n)\|\sigma u_n\| \leqslant \|f\| + \|g\|,$$

so that $\{u_{n,-}\}_{n=1}^\infty$ converges monotonically to some $u_- \in \mathcal{N}_- \oplus \mathcal{N}_+ \oplus \mathcal{N}_i$ in the strong sense. Then

$$\left\| \left(X - \frac{\partial\beta}{\partial E}\right)[u_n - u_k] \right\| \leqslant \|u_{n,-} - u_{k,-}\|$$

implies that $\{(X - \partial\beta/\partial E)u_n\}_{n=1}^\infty$ converges in $\mathcal{N}$ to some limit w. We then have

$$\lim_{n\to\infty} \|(\sigma u_n - \beta_n J u_n) - (f - w)\| = 0. \tag{41}$$

Letting u be the unique solution in $\mathcal{N}$ of the trivial stationary problem

$$\left(X - \frac{\partial\beta}{\partial E}\right) u = f - w,$$

$$u_- = g,$$

which depends continuously on $(f - w)$ and g, we find from (41) that

$$\lim_{n\to\infty} \|u_n - u\| = 0. \tag{42}$$

The uniqueness issue is settled in the same way as for the time-dependent problem with $\|J\|_\# = 1$.

We have the following theorem.

Theorem 3.2: Suppose $\|J\|_\# \leqslant 1$. Then Eqs. (36) and (37) have a unique solution u such that $(X - \partial\beta/\partial E)u \in \mathcal{N}$ and $(\sigma u - Ju) \in \mathcal{N}$. This solution is non-negative for non-negative data f and g.

IV. SEMIGROUP FORMULATION OF THE TIME-DEPENDENT SOLUTION

We have proved the unique solvability of Eqs. (1)–(3) in an L_1 space of functions u on $\Lambda \times [0,T]$. We have proved these solutions to have L_1 traces on each hyperplane $t = t_0$ with $t_0 \in [0,T]$. This follows from the inclusion $(t = T) \subseteq D^+$, the finiteness of $\|u^+\|$, and the arbitrariness of T (so that we may replace T by t_0). Nevertheless, we have not studied their continuity properties as a function of t. In this section we intend to do so. In order to apply the Hille–Yosida theorem (cf. Ref. 12) we will first study the stationary equation

$$Xu_\lambda + \left(\sigma + \lambda - \frac{\partial\beta}{\partial E}\right) u_\lambda = Ju_\lambda + f, \tag{43}$$

with boundary conditions

$$u_{\lambda_-} = g, \tag{44}$$

i.e., Eqs. (36) and (37) with σ replaced by $\sigma + \lambda$, where $\lambda > 0$. According to Lemma 3.1, Eqs. (43) and (44) have a unique solution $u_\lambda \in \mathcal{N}$ such that $(\sigma + \lambda)u_\lambda \in \mathcal{N}$ and

$$\|(\sigma + \lambda)u_\lambda\| \leqslant (1 - \|J\|_\#)^{-1}\{\|f\| + \|g\|\}$$

whenever $\|J\|_\# < 1$, and this solution u_λ is non-negative for non-negative data f and g. Thus if f and g are non-negative, then u_λ satisfies Eq. (38) as well as the estimate

$$\|u_\lambda\| \leqslant \lambda^{-1}(1 - \|J\|_\#)^{-1}\{\|f\| + \|g\|\}.$$

Hence if we define the operator $\mathcal{J}$ by

$$\mathcal{J} = -X - \left(\sigma(x,\mu,E) - \frac{\partial\beta}{\partial E}\right),$$

$$D(\mathcal{J}) = \left\{ u \in \mathcal{N}: \left(X - \frac{\partial\beta}{\partial E}\right) u \in \mathcal{N},\ \sigma u \in \mathcal{N},\ u|_{D_-} \equiv 0 \right\},$$

then for $g \equiv 0$ there exists a unique solution $u_\lambda \in D(\mathcal{J})$ such that

$$(\lambda - \mathcal{J} - J)u_\lambda = f,$$

which satisfies

$$\|u_\lambda\| \leqslant \lambda^{-1}(1 - \|J\|_\#)^{-1}\|f\|.$$

Thus for $\|J\|_\# < 1$ the operator $\mathcal{J} + J$ with domain

$$D(\mathcal{J} + J) = D(\mathcal{J})$$

generates a bounded strongly continuous semigroup on $\mathscr{N}$, which we will denote as $\{S(t)\}_{t\geqslant 0}$.

Let us apply Lemma 2.1 to Eqs. (10)–(13) for nonnegative data. We obtain immediately

$$\|f\| = \int_\Sigma \left(\frac{\partial u}{\partial t} + Xu + \left\{\sigma - \frac{\partial \beta}{\partial E} - J\right\}u\right) dx\, d\mu\, dE\, dt$$
$$= \|u^+\| - \|u^-\| + \|\sigma u\| - \|Ju\| \geqslant \|u^+\| - \|u^-\|,$$

so that

$$\|u(t=T)\| \leqslant \|u(t=0)\|$$

whenever $f\equiv 0$, $g_0\equiv 0$, $g_a\equiv 0$, and $g_i\equiv 0$. Hence the semigroup $\{S(t)\}_{t\geqslant 0}$ is a contraction semigroup.

We have the following theorem.

Theorem 4.1: Suppose $\|J\|_\# < 1$. Then the operator $-X - (\sigma - \partial\beta/\partial E + J)$ generates a strongly continuous contraction semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathscr{N}$.

Suppose $\|J\|_\# = 1$ and let us approximate J from below by $\beta_n J$, where $\{\beta_n\}_{n=1}^\infty \uparrow 1$. Denoting the corresponding contraction semigroups on $\mathscr{N}$ by $\{S_n(t)\}_{t\geqslant 0}$ we use that $S_n(t) \leqslant S_m(t)$, for $n\leqslant m$, as well as the upper bound $\|S_n(t)\| \leqslant 1$. We thus obtain the family of contraction operators $\{S_n(t)\}_{t\geqslant 0}$ on $\mathscr{N}$ satisfying

$$\lim_{n\to\infty} \|[S(t) - S_n(t)]g\| = 0, \quad t\geqslant 0,$$

as well as the semigroup property. If we now define

$$R(\lambda)g = \int_0^\infty e^{-\lambda t} S(t)g\, dt, \tag{45}$$

for $\operatorname{Re}\lambda \geqslant 0$, we obtain

$$\|R(\lambda)g\| \leqslant [1/(\operatorname{Re}\lambda)]\|g\|, \quad \operatorname{Re}\lambda > 0.$$

On the other hand,

$$(\lambda - [\mathscr{J} + \beta_n J])^{-1} g = \int_0^\infty e^{-\lambda t} S_n(t) g\, dt.$$

Thus by dominated convergence we obtain

$$\lim_{n\to\infty} \|[R(\lambda) - (\lambda - [\mathscr{J} + \beta_n J])^{-1}]g\| = 0, \quad \operatorname{Re}\lambda > 0.$$

We then find the resolvent identity

$$R(\lambda) - R(\mu) = -(\lambda - \mu) R(\lambda) R(\mu). \tag{46}$$

Using (46) we find that Ker $R(\lambda)$ and Ran $R(\lambda)$, the kernel and range of $R(\lambda)$, do not depend on λ. Since every $g\in$Ker $R(\lambda)$ satisfies $S(t)g\equiv 0$ and $u\equiv S(t)g$ is a solution of Eqs. (1)–(3) in $\mathscr{M}$ for $f\equiv 0, g_0\equiv 0, g_a\equiv 0, g_i\equiv 0$, and, given $u(t=0) = g$, we obtain $g\equiv 0$ by the unique solvability of the time-dependent problem so that Ker $R(\lambda) = \{0\}$. By a similar argument on the adjoint semigroup we get the density of Ran $R(\lambda)$ in $\mathscr{N}$. Hence $R(\lambda) = (\lambda - \mathscr{G})^{-1}$ for some closed and densely defined operator $\mathscr{G}$. Thus, by the Hille–Yosida theorem and the uniqueness of the Laplace transform, $\mathscr{G}$ is the generator of a strongly continuous semigroup of $\mathscr{N}$ that must necessarily coincide with $\{S(t)\}_{t\geqslant 0}$. Thus $\{S(t)\}_{t\geqslant 0}$ is a strongly continuous contraction semigroup on $\mathscr{N}$.

We have the following theorem.

Theorem 4.2: Suppose $\|J\|_\# = 1$. Then the closure of operator $-X - (\sigma - \partial\beta/\partial E + J)$ generates a strongly continuous contraction semigroup $\{S(t)\}_{t\geqslant 0}$ on $\mathscr{N}$.

Proof: Clearly the generator $\mathscr{G}$ of $\{S(t)\}_{t\geqslant 0}$ is a closed extension of $-X - (\sigma - \partial\beta/\partial E + J)$. It remains to prove its minimality. Indeed, observe that

$$D(\mathscr{G}) = \{R(\lambda)g : g\in\mathscr{N}\},$$

where $\operatorname{Re}\lambda > 0$. Then for every $h\in D(\mathscr{G})$ we have

$$\lim_{n\to\infty} \|k_n - h\| = 0,$$

where $k_n = (\lambda - \mathscr{J} - \beta_n J)^{-1} g$ and g is the unique vector in $\mathscr{N}$ such that $R(\lambda)g = h$. Note that $k_n \in D(\mathscr{J} + J)$, which is true because $D(\mathscr{J}) \subseteq \mathscr{N}_\sigma$ so that J is well-defined on $D(\mathscr{J})$. Moreover,

$$(\lambda - \mathscr{J} - J)k_n = g + (1 - \beta_n)Jk_n,$$

where $\{\sigma k_n)$ is bounded; hence $(\lambda - \mathscr{J} - J)k_n \to g$. Thus every h belongs to the domain of the closure of $\mathscr{J} + J$ while $(\overline{\mathscr{J} + J})h = g$. But then we must have $\mathscr{G} = \overline{\mathscr{J} + J}$. □

V. DISCUSSION

We have established the unique solvability of the time-dependent and stationary Spencer–Lewis equations under natural assumptions on the stopping power and the cross section and in natural function spaces. These results are far more general than the existence and uniqueness results given by Nelson[6] and Greenberg *et al.*[8] On the other hand, Nelson and Seth[5] have established the convergence of a number of finite difference schemes for solving the stationary Spencer–Lewis equation numerically under the assumption that the corresponding stationary Spencer–Lewis equation is uniquely solvable. If we combine their conditional convergence proof with our well-posedness results, we obtain a convergence proof for the numerical schemes used by Nelson and Seth.[5]

The section on the stationary problem was very concise, because it appeared possible to treat both the time-dependent and the stationary problem by the method of characteristics as introduced in transport theory by Beals and Protopopescu.[7] Certain peculiarities of the Spencer–Lewis equation, however, forced us to go off the path followed by Ref. 7. The one rather artificial assumption left, assumption (B) on the number and position of the zeros of $\beta(x,E)$, may be dropped in the time-dependent case, provided one does not seek a restatement of the time-dependent result within the framework of semigroup theory. When adopting the semigroup framework or sticking to the stationary problem, assumption (B) is a necessary tool to avoid the intricacies of a singular vector field.

ACKNOWLEDGMENTS

The author wishes to express his gratitude to P. Nelson for stimulating discussions on the Spencer–Lewis equation. Also discussions with L. Arlotti and G. Frosali (University of Ancona) and S. L. Paveri-Fontana (University of Rome) who are dealing with similar mathematical questions in other transport problems turned out to be most useful.

This research was conducted while the author was a visiting professor supported by C.N.R. (Consiglio Nazionale per le Ricerche), Gruppo Nazionale per la Fisica Matematica.

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