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Scattering of polarized light: properties of the elements of the phase matrix

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Summary. A structural analysis of the phase matrix relevant to the scattering of light by a small volume-element in a plane-parallel atmosphere is presented. This 4×4 matrix transforms the Stokes parameters of an incident beam into those of a scattered beam when planes of reference through the direction of propagation and the normal to the atmosphere are used for the definition of the Stokes parameters. First, relations between elements of the phase matrix and those of the scattering matrix are derived under fairly general assumptions. It is further shown that there are 8 basic equations which only involve elements of the phase matrix and are valid for arbitrary directions. A number of inequalities for the elements of the phase matrix is presented. Finally, relations involving elements of the azimuth decomposed phase matrices are given.

Key words: radiative transfer – atmospheres – light scattering – polarization

1. Introduction

Virtually all theoretical studies of the transfer of polarized light in a scattering and absorbing atmosphere utilize a set of four parameters to describe the intensity and state of polarization of the radiation. The most commonly used parameters are the so-called Stokes parameters (e.g. Chandrasekhar, 1950; Van de Hulst, 1957, 1980; Hansen and Travis, 1974; Hovenier and Van der Mee, 1983). In such a description a plane of reference is needed for each beam of light. If only single scattering is considered an obvious choice is the plane of scattering, both for the incident and scattered beam. The scattering process may then be described by means of a 4×4 matrix which transforms the Stokes parameters of the incident beam into those of the scattered beam. We call this matrix the scattering matrix. In general it consists of 16 different functions of the directions of incidence and scattering. A comprehensive study of general conditions for these 16 functions was made by Hovenier et al. (1986).

In astrophysics (including planetary physics) we are mostly interested in multiple scattering, since a single scattering treatment seldom suffices to provide a satisfactory interpretation of the observations. The scattering planes of successive scatterings will, in general, not coincide. Considering plane-parallel atmospheres we may then obtain useful concepts by adopting the plane through the direction of propagation and the normal to the atmosphere as the plane of reference for the Stokes parameters of a beam. Scattering of light by a small volume-element in such a medium is then described by a 4×4 matrix which we call the phase matrix. It is obtained from the scattering matrix by pre- and postmultiplication by rotation matrices.

The phase matrix plays a fundamental role in theoretical studies and computational methods concerning the transfer of polarized radiation. It occurs, for instance, as the kernel of the equation of transfer for polarized light. A number of symmetry relations exists for the elements of the phase matrix (Hovenier, 1969) which have many useful applications. These symmetry relations provide equations involving elements with different values of the arguments. A deeper analysis of the structure of the phase matrix is presented in this paper. A variety of properties of the elements is derived by elementary means without even using expansions in generalized spherical functions (e.g. Kuščer and Ribarič, 1959; Herman, 1968; Domke, 1974; Siewert, 1981; Hovenier and Van der Mee, 1983). These properties may be used at the formulation of analytical and numerical solution methods for radiative transfer problems, for checking purposes and extensions to reflection and transmission matrices and other matrices relevant to multiple scattering theory.

2. The relationship between the scattering matrix and the phase matrix

The phase matrix may be constructed from amplitude and rotation matrices. We shall first give a short discussion of this construction. For details we refer to Van de Hulst (1957), Hovenier and Van der Mee (1983), and Hovenier et al. (1986).

The scattering of a simple wave by an arbitrary particle may be described by means of a 2×2 amplitude matrix satisfying

$$\begin{bmatrix} E_l \\ E_r \end{bmatrix} = \begin{bmatrix} A_2 & A_3 \\ A_4 & A_1 \end{bmatrix} \begin{bmatrix} E_{l0} \\ E_{r0} \end{bmatrix}. \tag{1}$$

Here E_l and E_r represent the electric field components of the scattered wave parallel and perpendicular to the scattering plane, respectively; in a similar way E_{l0} and E_{r0} relate to the ingoing wave. The elements of the amplitude matrix generally are complex functions of the directions of incidence and scattering. On expressing the Stokes parameters I, Q, U and V of the scattered beam and the Stokes parameters I_0, Q_0, U_0 and V_0 of the incident beam in the electric field components and their complex

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conjugates and making the Stokes parameters elements of column vectors, we obtain from Eq. (1)

$$\boldsymbol{I}=\mathbf{F}(\theta)\boldsymbol{I}_0 \tag{2}$$

where $\mathbf{F}(\theta)$ is the so-called scattering matrix transforming the Stokes vector of the incident beam, $\boldsymbol{I}_0$, into the Stokes vector of the scattered beam, $\boldsymbol{I}$, with the scattering plane as the plane of reference of the Stokes parameters. When dealing with an assembly of independently scattering particles, we add the corresponding elements of the scattering matrices of the single particles to obtain the scattering matrix of the assembly. Symmetries will then lead to a simplification of the scattering matrix for an extensive class of assemblies. We consider a scattering matrix of the following block diagonal form

$$\mathbf{F}(\theta)=\begin{bmatrix} a_1(\theta) & b_1(\theta) & 0 & 0 \\ b_1(\theta) & a_2(\theta) & 0 & 0 \\ 0 & 0 & a_3(\theta) & b_2(\theta) \\ 0 & 0 & -b_2(\theta) & a_4(\theta) \end{bmatrix}, \tag{3}$$

where $0\leq\theta\leq\pi$ is the scattering angle, i.e. the angle between the directions of the incident and scattered beams. This matrix contains 6 real functions and is valid in various situations, such as

(i) scattering by an assembly of randomly oriented particles each of which has a plane of symmetry, like optically inactive ellipsoids, spheres and cylinders;

(ii) scattering by an assembly having particles and their mirror particles in equal numbers and with random orientation;

(iii) Rayleigh scattering (with or without depolarization effects) by an assembly of randomly oriented optically inactive particles.

Let us now consider a plane-parallel layer of scattering particles illuminated at the top. We specify directions by means of $-1\leq u\leq 1$ (cosine of the angle with the downward normal) and $0\leq\varphi\leq 2\pi$ (azimuth measured clockwise when viewing upward). Using the meridian plane as the plane of reference for the Stokes parameters of a beam, the light scattered by a volume-element from a direction (u',φ') into a direction (u,φ) is described by the phase matrix $\mathbf{Z}(u,u',\varphi-\varphi')$. This matrix follows from the scattering matrix by pre- and postmultiplication by rotation matrices. In fact, we have

$$\mathbf{Z}(u,u',\varphi-\varphi')=\mathbf{L}(\pi-\sigma_2)\mathbf{F}(\theta)\mathbf{L}(-\sigma_1), \tag{4}$$

where the rotation matrix $\mathbf{L}(\alpha)$ has the form

$$\mathbf{L}(\alpha)=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

and the variables $u,u',(\varphi-\varphi')$, θ, σ_1 and σ_2 are related by the equalities

$$\cos\theta=uu'+(1-u^2)^{1/2}(1-u'^2)^{1/2}\cos(\varphi-\varphi') \tag{6}$$

$$\cos\sigma_1=(-u+u'\cos\theta)/\{(1-u'^2)(1-\cos^2\theta)\}^{1/2} \tag{7}$$

$$\sin\sigma_1=(1-u^2)^{1/2}\sin(\varphi'-\varphi)/(1-\cos^2\theta)^{1/2} \tag{8}$$

$$\cos\sigma_2=(-u'+u\cos\theta)/\{(1-u^2)(1-\cos^2\theta)\}^{1/2} \tag{9}$$

$$\sin\sigma_2=(1-u'^2)^{1/2}\sin(\varphi'-\varphi)/(1-\cos^2\theta)^{1/2}. \tag{10}$$

The phase matrix can also be written as

$$\mathbf{Z}(u,u',\varphi-\varphi')=\begin{bmatrix} a_1(\theta) & b_1(\theta)C_1 & -b_1(\theta)S_1 & 0 \\ b_1(\theta)C_2 & C_2a_2(\theta)C_1-S_2a_3(\theta)S_1 & -C_2a_2(\theta)S_1-S_2a_3(\theta)C_1 & -b_2(\theta)S_2 \\ b_1(\theta)S_2 & S_2a_2(\theta)C_1+C_2a_3(\theta)S_1 & -S_2a_2(\theta)S_1+C_2a_3(\theta)C_1 & b_2(\theta)C_2 \\ 0 & -b_2(\theta)S_1 & -b_2(\theta)C_1 & a_4(\theta) \end{bmatrix}, \tag{11}$$

where

$$\left.\begin{aligned} C_1&=\cos 2\sigma_1,\ C_2=\cos 2\sigma_2 \\ S_1&=\sin 2\sigma_1,\ S_2=\sin 2\sigma_2 \end{aligned}\right\}. \tag{12}$$

The elements of the phase matrix will be denoted by $Z_{ij}(u,u',\varphi-\varphi')$ with $i,j=1,2,3,4$. From hereon we shall not always write the dependence on u,u' and $(\varphi-\varphi')$ explicitly.

So far we have summarized some well-known material. We will now investigate relations between elements of the phase matrix and elements of the scattering matrix. We will group them into relations for the corner elements, relations for the noncorner elements along the boundary and relations involving the middle block. First of all, for the corner elements of the scattering matrix of Eq. (3) we have the trivial identities

$$Z_{11}=a_1 \tag{13}$$

$$Z_{44}=a_4 \tag{14}$$

$$Z_{14}=0 \tag{15}$$

$$Z_{41}=0. \tag{16}$$

Secondly, for the noncorner elements of the first and fourth row of the scattering and phase matrices we find the 4 equations

$$Z_{12}^2+Z_{13}^2=b_1^2 \tag{17}$$

$$Z_{42}^2+Z_{43}^2=b_2^2 \tag{18}$$

$$Z_{12}Z_{42}+Z_{13}Z_{43}=0 \tag{19}$$

$$Z_{12}Z_{43}-Z_{13}Z_{42}=-b_1b_2. \tag{20}$$

Similarly, for the noncorner elements of the first and fourth column we have the 4 equations

$$Z_{21}^2+Z_{31}^2=b_1^2 \tag{21}$$

$$Z_{24}^2+Z_{34}^2=b_2^2 \tag{22}$$

$$Z_{21}Z_{24}+Z_{31}Z_{34}=0 \tag{23}$$

$$Z_{21}Z_{34}-Z_{31}Z_{24}=b_1b_2. \tag{24}$$

Finally, we obtain the 2 equations

$$Z_{22}^2+Z_{23}^2+Z_{32}^2+Z_{33}^2=a_2^2+a_3^2 \tag{25}$$

$$Z_{22}Z_{33}-Z_{23}Z_{32}=a_2a_3. \tag{26}$$

Equations (13)–(26) may be derived by using Eq. (11) and algebraic manipulations. They hold for arbitrary values of u,u' and $(\varphi-\varphi')$ in the allowed ranges. Other equations may be derived also, e.g.

$$(Z_{22}+Z_{33})^2+(Z_{23}-Z_{32})^2=(a_2+a_3)^2, \tag{27}$$

which follows from Eqs. (25) and (26). We may also write down equations expressing the invariance of the determinant and the sum of the squared elements of the entire scattering matrix under pre- and postmultiplication by rotation matrices. In this way we find

$$\det \mathbf{Z} = \det \mathbf{F} \tag{28}$$

$$\sum_{i=1}^{4} \sum_{j=1}^{4} Z_{ij}^2 = \sum_{i=1}^{4} \sum_{j=1}^{4} F_{ij}^2. \tag{29}$$

Special situations occur for forward ($\theta=0$) and backward scattering ($\theta=\pi$). The scattering matrix then has the additional properties (cf. Van de Hulst, 1957)

$$b_1(0)=b_2(0)=0 \tag{30}$$

$$a_2(0)=a_3(0) \tag{31}$$

$$b_1(\pi)=b_2(\pi)=0 \tag{32}$$

$$a_2(\pi)=-a_3(\pi). \tag{33}$$

For $\theta=0$ we have $u=u'$, $\varphi-\varphi'=0$ and the rotation matrices reduce to unit matrices [cf. Eqs. (6)–(10)] yielding

$$\mathbf{Z}(u,u,0)=\mathrm{diag}\,\{a_1(0), a_2(0), a_2(0), a_4(0)\}. \tag{34}$$

Similarly for $\theta=\pi$ we find $u=-u'$, $\varphi-\varphi'=\pi$ and the rotation matrices to reduce to unit matrices so that

$$\mathbf{Z}(u,-u,\pi)=\mathrm{diag}\,\{a_1(\pi), a_2(\pi), -a_2(\pi), a_4(\pi)\}. \tag{35}$$

3. Relations for the elements of the phase matrix

In this section we shall consider relations for the elements of the phase matrix and study their independence and completeness properties. We have met some relations of this type already in the preceding section. Since the 16 elements of the phase matrix depend on 8 quantities, namely a_1, a_2, a_3, a_4, b_1, b_2, σ_1 and σ_2, we expect the existence of 8 basic relations involving elements of the phase matrix only.

3.1. The boundary elements

For the 12 elements on the boundary of the phase matrix we have first of all the trivial equalities (15) and (16) as well as Eqs. (19) and (23). From Eqs. (17) and (21), and from Eqs. (18) and (22) [or from Eqs. (11) and (12)] we easily obtain

$$Z_{12}^2+Z_{13}^2-Z_{21}^2-Z_{31}^2=0 \tag{36}$$

$$Z_{42}^2+Z_{43}^2-Z_{24}^2-Z_{34}^2=0. \tag{37}$$

From Eq. (11) we have directly

$$Z_{13}Z_{24}+Z_{31}Z_{42}=0 \tag{38}$$

$$Z_{12}Z_{34}+Z_{21}Z_{43}=0 \tag{39}$$

$$Z_{12}Z_{24}-Z_{31}Z_{43}=0 \tag{40}$$

$$Z_{13}Z_{34}-Z_{21}Z_{42}=0. \tag{41}$$

Finally, from Eqs. (20) and (24) we have immediately

$$Z_{12}Z_{43}-Z_{13}Z_{42}+Z_{21}Z_{34}-Z_{31}Z_{24}=0. \tag{42}$$

From these simple equations we may easily derive more complicated equations. For example, from Eqs. (36), (37) and (42) we find

$$(Z_{12}\mp Z_{43})^2+(Z_{13}\pm Z_{42})^2-(Z_{21}\pm Z_{34})^2-(Z_{31}\mp Z_{24})^2=0. \tag{43}$$

Here and in the sequel, a relation containing $\pm$ or $\mp$ will be viewed as two relations, where one carries the upper signs and the other one the lower signs. Other identities resembling Eq. (42) may be found by adding or subtracting two of the Eqs. (19), (23) and (38)–(41).

Let us point out a set of four basic equations which are independent and complete for the 8 elements Z_{12}, Z_{13}, Z_{21}, Z_{31}, Z_{42}, Z_{43}, Z_{24} and Z_{34}. By independent we mean that none of these equations can be derived from the others, and by complete that all other equations for the same elements which stem from Eqs. (11) and (12) can be derived from them. Assuming $Z_{12}\neq 0$ we first note that it is straightforward to derive Eqs. (23), (37), (38), (41) and (42) from Eqs. (19), (36), (39) and (40). The last four equations are independent, since in the consecutive sequence consisting of Eq. (36) and the three equations

$$Z_{42}=-Z_{13}Z_{43}/Z_{12} \tag{44}$$

$$Z_{34}=-Z_{21}Z_{43}/Z_{12} \tag{45}$$

$$Z_{24}=Z_{31}Z_{43}/Z_{12}, \tag{46}$$

which are modifications of Eqs. (19), (39) and (40), there always appear quantities that did not appear in any of the preceding equations in the sequence. A rigorous proof of the completeness of Eqs. (19), (36), (39) and (40) runs as follows. Suppose we have 8 quantities satisfying Eqs. (19), (36), (39) and (40). We first assume $Z_{12}\neq 0$ and define

$$b_1=\pm(Z_{12}^2+Z_{13}^2)^{1/2} \tag{47}$$

$$C_1=Z_{12}/b_1 \tag{48}$$

$$C_2=Z_{21}/b_1 \tag{49}$$

$$S_1=-Z_{13}/b_1 \tag{50}$$

$$S_2=Z_{31}/b_1 \tag{51}$$

$$b_2=-Z_{43}/C_1, \tag{52}$$

therewith taking for granted a common sign indeterminacy in b_1, b_2, C_1, C_2, S_1 and S_2. Equations (47), (48) and (50) imply that an angle ($2\sigma_1$) can be chosen so that C_1 and S_1 are its cosine and sine, respectively. Similarly C_2 and S_2 can be made the cosine and sine of the angle $2\sigma_2$, because of Eqs. (36), (47), (49) and (51). We now use Eqs. (19), (39) and (40) and find our 8 quantities Z_{12}, Z_{13}, Z_{21}, Z_{31}, Z_{42}, Z_{43}, Z_{24} and Z_{34} to be exactly of the type given by Eqs. (11) and (12). This reconstruction evidently implies the completeness of Eqs. (19), (36), (39) and (40). As a result all equations for the above 8 elements of the phase matrix stemming from Eqs. (11) and (12) can be derived from these four equations if $Z_{12}\neq 0$. If $Z_{12}=0$ but one of the other 7 elements is nonzero, the unique set consisting of the 4 equations among Eqs. (19), (23), and (36)–(41) that contain the selected nonzero quantity may be chosen as the basic set. We note that in this case C_1 may vanish while $b_1\neq 0$. We should then replace Eq. (52) by

$$b_2=-Z_{42}/S_1 \tag{53}$$

when reconstructing b_2. On the other hand, if $Z_{12}=Z_{13}=Z_{21}=Z_{31}=0$ and thus b_1 vanishes, one should replace Eqs. (47)–(52) by equations where the roles of b_1 and b_2 have been interchanged. If all 8 elements vanish, Eqs. (19), (23) and (36)–(42) reduce to tautologies of the type $0=0$ and the choice of a basic set of equations is immaterial. This situation occurs, for instance, for

forward ($\theta=0$) and backward ($\theta=\pi$) scattering [cf. Eqs. (34) and (35)], or if polarization is neglected, i.e. if all elements of the scattering matrix except a_1 vanish.

3.2. Relations involving elements of the middle block

To obtain equations involving the elements Z_{22}, Z_{23}, Z_{32} and Z_{33} of the middle block of the phase matrix, we first express $a_3S_1C_1$ and $a_2S_1C_1$ in these elements [cf. Eq. (11)]. To illustrate how $a_3S_1C_1$ is obtained, we use Eq. (11), and (i) multiply Z_{22} by C_1S_2 and Z_{32} by $-C_1C_2$ and add the resulting expressions, or (ii) multiply Z_{23} by S_1S_2 and Z_{33} by $-S_1C_2$ and add the resulting equations. It is then readily verified that

$$\begin{bmatrix} t_1 & t_2 & t_3 & t_4 \\ t_4 & -t_3 & -t_2 & t_1 \end{bmatrix} \begin{bmatrix} Z_{22} \\ Z_{23} \\ Z_{32} \\ Z_{33} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{54}$$

where t_i, $i=1, 2, 3, 4$, are given by

$$\left.\begin{aligned} t_1&=cC_1S_2 \\ t_2&=-cS_1S_2 \\ t_3&=-cC_1C_2 \\ t_4&=cS_1C_2 \end{aligned}\right\} \tag{55}$$

for $c=1$. If we multiply both equations of this system of equations by b_1^2 we obtain from Eq. (11)

$$(Z_{12}Z_{31})Z_{22}+(Z_{13}Z_{31})Z_{23}-(Z_{12}Z_{21})Z_{32} -(Z_{13}Z_{21})Z_{33}=0 \tag{56}$$

$$-(Z_{13}Z_{21})Z_{22}+(Z_{12}Z_{21})Z_{23}-(Z_{13}Z_{31})Z_{32} +(Z_{12}Z_{31})Z_{33}=0. \tag{57}$$

In an analogous fashion we may multiply both equations of the system by b_2^2 to obtain the pair of equations

$$(Z_{43}Z_{24})Z_{22}-(Z_{42}Z_{24})Z_{23}+(Z_{43}Z_{34})Z_{32}-(Z_{42}Z_{34})Z_{33}=0 \tag{58}$$

$$-(Z_{42}Z_{34})Z_{22}-(Z_{43}Z_{34})Z_{23}+(Z_{42}Z_{24})Z_{32} +(Z_{43}Z_{24})Z_{33}=0. \tag{59}$$

A third pair of equations is obtained by applying the above procedure with b_1^2 or b_2^2 replaced by b_1b_2. As a result we find

$$(Z_{31}Z_{43})Z_{22}-(Z_{31}Z_{42})Z_{23}-(Z_{21}Z_{43})Z_{32} +(Z_{21}Z_{42})Z_{33}=0 \tag{60}$$

$$(Z_{21}Z_{42})Z_{22}+(Z_{21}Z_{43})Z_{23}+(Z_{31}Z_{42})Z_{32}+(Z_{31}Z_{43})Z_{33}=0. \tag{61}$$

However, since a product of the type $b_1b_2C_1S_2$ can be decomposed alternatively as $(b_1C_1)(b_2S_2)$ or $(b_1S_2)(b_2C_1)$, each one of the equations (60) and (61) can be written in 16 equivalent forms.

Using the identities

$$C_1^2S_2^2-C_2^2S_1^2=S_2^2-S_1^2=C_1^2-C_2^2 \tag{62}$$

$$C_1^2C_2^2-S_1^2S_2^2=C_1^2-S_2^2=C_2^2-S_1^2, \tag{63}$$

we may eliminate one of the variables Z_{22}, Z_{23}, Z_{32} and Z_{33} from Eqs. (54) and (55) and obtain the following four equations

$$C_2S_2Z_{23}+C_1S_1Z_{32}+(C_1^2-C_2^2)Z_{33}=0 \tag{64}$$

$$C_2S_2Z_{22}+(S_1^2-C_2^2)Z_{32}+S_1C_1Z_{33}=0 \tag{65}$$

$$-C_1S_1Z_{22}+(S_2^2-C_1^2)Z_{23}-S_2C_2Z_{33}=0 \tag{66}$$

$$(S_2^2-S_1^2)Z_{22}-S_1C_1Z_{23}-S_2C_2Z_{32}=0. \tag{67}$$

Premultiplication by b_1^2 and using Eqs. (11), (62) and (63) yield

$$Z_{21}Z_{31}Z_{23}-Z_{12}Z_{13}Z_{32}+(Z_{12}^2-Z_{21}^2)Z_{33}=0 \tag{68}$$

$$Z_{21}Z_{31}Z_{22}+(Z_{31}^2-Z_{12}^2)Z_{32}-Z_{12}Z_{13}Z_{33}=0 \tag{69}$$

$$-Z_{12}Z_{13}Z_{22}+(Z_{12}^2-Z_{31}^2)Z_{23}+Z_{21}Z_{31}Z_{33}=0 \tag{70}$$

$$(Z_{21}^2-Z_{12}^2)Z_{22}-Z_{12}Z_{13}Z_{23}+Z_{21}Z_{31}Z_{32}=0. \tag{71}$$

Each one of these equations may be written in two different forms by using Eq. (36). In the same way one obtains 8 equations by premultiplication by b_2^2 and 32 equations by premultiplication by b_1b_2.

We shall now prove that either Eqs. (56) and (57), Eqs. (58) and (59) or Eqs. (60) and (61) are independent, provided four independent and complete equations for the 8 quantities Z_{12}, Z_{13}, Z_{21}, Z_{31}, Z_{42}, Z_{43}, Z_{24} and Z_{34} are given. Indeed, the three pairs of equations (56) and (57), (58) and (59), and (60) and (61) can be written in the concise form of Eq. (54) where $c=b_1^2$, b_2^2 and b_1b_2, respectively. This makes each t_i a $\pm$ product of two non-corner elements along the boundary of the phase matrix. We further have

$$t_1^2+t_2^2+t_3^2+t_4^2=c^2 \tag{72}$$

$$t_1t_4=t_2t_3. \tag{73}$$

It now follows from Eq. (11) that, if at least one of the elements Z_{12}, Z_{13}, Z_{21}, Z_{31}, Z_{42}, Z_{43}, Z_{24} and Z_{34} does not vanish, b_1 and b_2 do not vanish simultaneously so that we can select a pair of equations for which $c\neq 0$. Owing to the conditions (72) and (73) with $c\neq 0$, it will then be impossible to simultaneously violate the three conditions

$$t_2^2\neq t_3^2 \tag{74}$$

$$t_1^2\neq t_4^2 \tag{75}$$

$$t_1t_2+t_3t_4\neq 0. \tag{76}$$

Since one of these conditions is sufficient to guarantee the 2×4-matrix of Eq. (54) to have rank 2, it follows that Eqs. (56) and (57) for $b_1\neq 0$, Eqs. (58)–(59) for $b_2\neq 0$ and Eqs. (60)–(61) for $b_1b_2\neq 0$ are independent.

3.3. Independent and complete equations for all elements of the phase matrix

Let us assume that at least one of the non-corner elements along the boundary of the phase matrix does not vanish. Then one may indicate 6 equations for the non-corner elements which are independent and complete. Here independence means that no equation from such a set of 6 can be derived from the other five. Completeness means that every equation for the non-corner elements of the phase matrix which is based on Eq. (11) may be derived from such a set of 6 equations. If either Z_{12}, Z_{13}, Z_{21} or Z_{31} does not vanish, then the four equations among Eqs. (19), (23) and (36)–(41) containing a particular nonzero quantity plus Eqs. (56) and (57) will be proven to be a basic set. On the other hand, if either Z_{42}, Z_{43}, Z_{24} or Z_{43} does not vanish, a basic set will be shown to consist of the four equations among Eqs. (19), (23) and (36)–(41) containing a particular nonzero quantity plus Eqs. (58) and (59).

Indeed, to prove independence we assume $Z_{12} \neq 0$ and consider the set of 6 equations (19), (36), (39), (40), (56) and (57). If either Eq. (56) or Eq. (57) could be derived from the remaining five, then the completeness of Eqs. (19), (36), (39) and (40) as a set of four basic equations for the non-corner elements along the boundary would imply the linear dependence of Eqs. (56) and (57) for given values of the non-corner elements along the boundary. The latter contradicts with the 2×4 matrix on the left-hand side of Eq. (54) having rank two for $c = b_1^2 \neq 0$. On the other hand, each one of Eqs. (19), (39) or (40) is independent of the remaining five from the above set of 6 equations, because each one of these three equations contains a quantity (Z_{42}, Z_{34} and Z_{24}, respectively) not present in any of the other five equations. Finally, Eq. (36) cannot be derived from Eqs. (19), (39), (40), (56) and (57), because multiplication of Z_{21} and Z_{31} by, say, 2 and simultaneous division of Z_{42} and Z_{43} by 2 leaves those five equations invariant but affects Eq. (36).

In Sect 3.1 we pointed out a set of four independent and complete equations for the non-corner elements along the boundary. If these elements and equations are given we can find unique quantities b_1, b_2, C_1, C_2, S_1 and S_2, apart from a common sign indeterminacy. Suppose now we have four quantities Z_{22}, Z_{23}, Z_{32} and Z_{33} obeying at least one of the pairs of Eqs. (56) and (57), or (58) and (59). On dividing by b_1^2 or b_2^2 one then arrives at the system (54) where t_1, t_2, t_3 and t_4 are given by Eq. (55) with $c = 1$. In addition, one finds Eqs. (72) and (73) with $c = 1$. Two linearly independent solutions $\{Z_{22}, Z_{23}, Z_{32}, Z_{33}\}$ of the system (54) are then given by the column vectors $\{C_1C_2, -S_1C_2, C_1S_2, -S_1S_2\}$ and $\{-S_1S_2, -C_1S_2, S_1C_2, C_1C_2\}$, as is readily verified. Hence, we can find unique scalars a_2 and a_3 such that

$$\begin{bmatrix} Z_{22} \\ Z_{23} \\ Z_{32} \\ Z_{33} \end{bmatrix} = a_2 \begin{bmatrix} C_1C_2 \\ -S_1C_2 \\ C_1S_2 \\ -S_1S_2 \end{bmatrix} + a_3 \begin{bmatrix} -S_1S_2 \\ -C_1S_2 \\ S_1C_2 \\ C_1C_2 \end{bmatrix} \tag{77}$$

Consequently, Z_{22}, Z_{23}, Z_{32} and Z_{33} are exactly of the type given by Eqs. (11) and (12). This completes the reconstruction. Summarizing, if at least one of the non-corner elements along the boundary of the phase matrix is nonzero, there exists a set of 6 independent and complete equations for the 12 non-corner elements of the phase matrix. Together with Eqs. (15) and (16) this results in a set of 8 independent and complete equations for the 16 elements of the phase matrix valid for arbitrary values of u, u' and $\varphi - \varphi'$. For instance, if $Z_{13} \neq 0$, one takes Eqs. (19), (36), (38), (41), (56) and (57), i.e. the 4 equations among Eqs. (19), (23), (36)–(41) containing Z_{13} plus a pair of additional equations of the type (54) for which $c = b_1^2 \neq 0$.

If $Z_{12} = Z_{13} = Z_{21} = Z_{31} = Z_{42} = Z_{43} = Z_{24} = Z_{34} = 0$, no equations for Z_{22}, Z_{23}, Z_{32} and Z_{33} can be found but tautologies of the type $0 = 0$. This is not surprising since in this case 10 of the 16 elements vanish and the remaining 6 are expressed in 6 quantities, namely a_1, a_2, a_3, a_4, σ_1 and σ_2.

3.4. Symmetry relations

Symmetry relations provide simple equations involving elements of the phase matrix with different values of the arguments. A basic set of symmetry relations is (cf. Hovenier, 1969)

$$\mathbf{Z}(u, u', \varphi' - \varphi) = \mathbf{D}\mathbf{Z}(u, u', \varphi - \varphi')\mathbf{D} \tag{78}$$

$$\mathbf{Z}(u', u, \varphi - \varphi') = \mathbf{P}\tilde{\mathbf{Z}}(u, u', \varphi - \varphi')\mathbf{P} \tag{79}$$

$$\mathbf{Z}(-u, -u', \varphi - \varphi') = \mathbf{D}\mathbf{Z}(u, u', \varphi - \varphi')\mathbf{D} \tag{80}$$

where a tilde above a matrix signifies transposition and $\mathbf{D}$ and $\mathbf{P}$ are diagonal matrices, namely

$$\mathbf{D} = \mathrm{diag}\{1, 1, -1, -1\} \tag{81}$$

and

$$\mathbf{P} = \mathrm{diag}\{1, 1, -1, 1\}. \tag{82}$$

The symmetry relations may be combined with the relations for elements having the same values of the arguments which were considered earlier. We mention the following applications.

(i) Checks may be obtained on equations in Sect. 2 and subsections 3.1–3.3. For example, since according to Eq. (79)

$$Z_{21}(u', u, \varphi - \varphi') = Z_{12}(u, u', \varphi - \varphi') \tag{83}$$

and

$$Z_{31}(u', u, \varphi - \varphi') = -Z_{13}(u, u', \varphi - \varphi'), \tag{84}$$

the validity of Eq. (17) for all values of the arguments implies that Eq. (21) must hold for all values of the arguments. Similarly Eqs. (23) and (41) may be derived from Eqs. (19) and (40), respectively. In fact, it is readily verified that all equations for elements having the same values of the arguments can be checked by using the symmetry relations.

(ii) Equations of a new type may be obtained, such as

$$[Z_{21}(u, u', \varphi - \varphi')]^2 + [Z_{31}(u, u', \varphi - \varphi')]^2 = [Z_{21}(u', u, \varphi - \varphi')]^2 + [Z_{31}(u', u, \varphi - \varphi')]^2. \tag{85}$$

This equation follows from Eqs. (36), (83) and (84) and corresponds to the physically well-known fact that for incident unpolarized light the degree of polarization of singly scattered light only depends on the scattering angle.

4. Inequalities for the elements of the phase matrix

In this section we shall describe several methods for obtaining inequalities for elements of the phase matrix and discuss the relationships between the different methods. In doing so we shall apply some of the methods and results of Hovenier et al. (1986) relevant to the elements of the scattering matrix and employ them to find inequalities for the elements of the phase matrix. It should be noted that a certain inequality may sometimes be obtained by different methods, but the amount of algebra may depend considerably on the method used. We do not aim at a comprehensive set of inequalities but only give some illustrative examples.

4.1. Omitting terms in equalities

Equalities for the elements of the phase matrix may be turned into inequalities by omitting terms that are never negative. Thus Eqs. (36), (37) and (43) yield for example

$$Z_{12}^2 + Z_{13}^2 \geq Z_{21}^2 \tag{86}$$

$$Z_{21}^2 + Z_{31}^2 \geq Z_{13}^2 \tag{87}$$

$$Z_{42}^2 + Z_{43}^2 \geq Z_{34}^2 \tag{88}$$

$$(Z_{12} \mp Z_{43})^2 + (Z_{13} \pm Z_{42})^2 \geq (Z_{21} \pm Z_{34})^2. \tag{89}$$

4.2. Rotation of axes for the amplitude matrix

In Eq. (1) E_l and E_r represent the electric field components of the scattered wave parallel and perpendicular to the scattering plane. If we put the particle in a plane-parallel atmosphere we can write for fixed directions of the incident and scattered beams

$$\begin{bmatrix} \bar{E}_l \\ \bar{E}_r \end{bmatrix} = \mathbf{L}_2(\tfrac{1}{2}(\pi-\sigma_2)) \begin{bmatrix} A_2 & A_3 \\ A_4 & A_1 \end{bmatrix} \mathbf{L}_2(-\tfrac{1}{2}\sigma_1) \begin{bmatrix} \bar{E}_{l0} \\ \bar{E}_{r0} \end{bmatrix}. \tag{90}$$

Here $\mathbf{L}_2(\alpha)$ is the 2×2 rotation matrix occurring as the middle block of the matrix on the right-hand side of Eq. (5) and a bar above a symbol means that instead of the scattering plane the local meridian plane is used as the plane of reference. Performing the matrix multiplications we can rewrite Eq. (90) in the form

$$\begin{bmatrix} \bar{E}_l \\ \bar{E}_r \end{bmatrix} = \bar{\mathbf{A}} \begin{bmatrix} \bar{E}_{l0} \\ \bar{E}_{r0} \end{bmatrix} \tag{91}$$

where $\bar{\mathbf{A}}$ is the amplitude matrix pre- and postmultiplied by suitable rotation matrices. If we now use the definition of the Stokes parameters we obtain

$$\bar{I} = \bar{\mathbf{F}}\bar{I}_0. \tag{92}$$

Since in general the elements of $\bar{\mathbf{F}}$ are obtained from 7 quantities (the complex elements of $\bar{\mathbf{A}}$ minus an irrelevent phase; cf. Van de Hulst, 1957) there are 9 basic relations between the 16 elements of $\bar{\mathbf{F}}$. These are quadratic and exactly the same as those found and discussed by Hovenier et al. (1986) for the elements of $\mathbf{F}$, since for the derivation of the equations it is immaterial whether Eqs. (1) and (2) are used or Eqs. (91) and (92). In fact, $\bar{\mathbf{F}}$ is the phase matrix of a single particle. For an assembly of many independently scattering particles we must add the Stokes parameters and therefore the phase matrices of the individual particles to obtain the phase matrix $\mathbf{Z}$ of the assembly. The quadratic interrelations between the elements of the phase matrix of an individual particle are, generally, lost in the summation process. As a result (cf. Hovenier et al, 1986 for a completely analogous treatment of the mathematics involved) we find

$$(Z_{33}+Z_{44})^2+(Z_{34}-Z_{43})^2 \le (Z_{11}+Z_{22})^2-(Z_{12}+Z_{21})^2 \tag{93}$$

$$(Z_{31}-Z_{32})^2+(Z_{41}-Z_{42})^2 \le (Z_{11}-Z_{12})^2-(Z_{21}-Z_{22})^2 \tag{94}$$

$$(Z_{13}-Z_{23})^2+(Z_{14}-Z_{24})^2 \le (Z_{11}-Z_{21})^2-(Z_{12}-Z_{22})^2 \tag{95}$$

$$(Z_{13}+Z_{23})^2+(Z_{14}+Z_{24})^2 \le (Z_{11}+Z_{21})^2-(Z_{12}+Z_{22})^2 \tag{96}$$

$$(Z_{31}+Z_{32})^2+(Z_{41}+Z_{42})^2 \le (Z_{11}+Z_{12})^2-(Z_{21}+Z_{22})^2 \tag{97}$$

$$(Z_{33}-Z_{44})^2+(Z_{34}+Z_{43})^2 \le (Z_{11}-Z_{22})^2-(Z_{12}-Z_{21})^2, \tag{98}$$

where the equality signs hold for an assembly of identical particles with the same orientation or for a cloud of identical spheres. By adding Eqs. (93) and (98), (94) and (97), and (95) and (96), we find the three inequalities

$$Z_{33}^2+Z_{44}^2+Z_{34}^2+Z_{43}^2 \le Z_{11}^2+Z_{22}^2-Z_{12}^2-Z_{21}^2 \tag{99}$$

$$Z_{31}^2+Z_{32}^2+Z_{41}^2+Z_{42}^2 \le Z_{11}^2+Z_{12}^2-Z_{21}^2-Z_{22}^2 \tag{100}$$

$$Z_{13}^2+Z_{23}^2+Z_{14}^2+Z_{24}^2 \le Z_{11}^2+Z_{21}^2-Z_{12}^2-Z_{22}^2. \tag{101}$$

Using Eqs. (15), (16) and (36) we may greatly simplify Eqs. (100) and (101) and obtain

$$Z_{22}^2+Z_{32}^2+Z_{13}^2+Z_{42}^2 \le Z_{11}^2 \tag{102}$$

$$Z_{22}^2+Z_{23}^2+Z_{31}^2+Z_{24}^2 \le Z_{11}^2. \tag{103}$$

By adding Eqs. (99)–(101) we obtain the interesting inequality

$$\sum_{i=1}^{4}\sum_{j=1}^{4} Z_{ij}^2 \le 4Z_{11}^2, \tag{104}$$

which is the analog of an inequality for the scattering matrix.

4.3. Using the inequalities valid for the elements of the scattering matrix

For a scattering matrix of the type given by Eq. (3) we have (cf. Hovenier et al., 1986)

$$a_1 \ge 0 \tag{105}$$

$$(a_3+a_4)^2+4b_2^2 \le (a_1+a_2)^2-4b_1^2 \tag{106}$$

$$|a_3-a_4| \le |a_1-a_2| \tag{107}$$

$$|a_2 \pm b_1| \le |a_1 \pm b_1|, \tag{108}$$

which implies, for example,

$$a_1 \ge \max\{|a_2|, |a_3|, |a_4|, |b_1|, |b_2|\} \tag{109}$$

$$a_1^2 \ge b_1^2+b_2^2+a_4^2 \tag{110}$$

$$\sum_{i=1}^{4}\sum_{j=1}^{4} F_{ij}^2 \le 4a_1^2. \tag{111}$$

On the other hand, Eq. (11) provides

$$a_1 = Z_{11} \tag{112}$$

$$b_1 = \pm(Z_{12}^2+Z_{13}^2)^{1/2} = \pm(Z_{21}^2+Z_{31}^2)^{1/2} \tag{113}$$

$$b_2 = \pm(Z_{42}^2+Z_{43}^2)^{1/2} = \pm(Z_{24}^2+Z_{34}^2)^{1/2} \tag{114}$$

$$a_4 = Z_{44} \tag{115}$$

$$a_2 = \frac{Z_{21}}{Z_{12}} Z_{22} + \frac{Z_{31}}{Z_{12}} Z_{32} \tag{116}$$

$$a_3 = \frac{Z_{31}}{Z_{13}} Z_{22} - \frac{Z_{21}}{Z_{13}} Z_{32} \tag{117}$$

where various alternatives are possible for the last two equations. In these we must, of course, have $Z_{12} \neq 0$ and $Z_{13} \neq 0$, respectively. In addition, we had already found Eq. (29). Thus we may use Eqs. (105)–(111) to reestablish Eq. (104) and to provide, for instance,

$$Z_{11} \ge 0 \tag{118}$$

$$Z_{11} \ge \left|\frac{Z_{21}Z_{22}+Z_{31}Z_{32}}{Z_{12}}\right| \tag{119}$$

$$Z_{11} \ge \left|\frac{Z_{31}Z_{22}-Z_{21}Z_{32}}{Z_{13}}\right| \tag{120}$$

$$Z_{11}^2-Z_{21}^2-Z_{31}^2 \ge 0 \tag{121}$$

$$Z_{11}^2-Z_{42}^2-Z_{43}^2 \ge 0 \tag{122}$$

$$Z_{11}^2-Z_{24}^2-Z_{34}^2 \ge 0 \tag{123}$$

$$\left|\frac{Z_{31}}{Z_{13}} Z_{22} - \frac{Z_{21}}{Z_{13}} Z_{32} - Z_{44}\right| \le \left|Z_{11} - \frac{Z_{21}}{Z_{12}} Z_{22} - \frac{Z_{31}}{Z_{12}} Z_{32}\right| \tag{124}$$

$$Z_{11}^2 \ge (Z_{21} \pm Z_{24})^2 + (Z_{31} \pm Z_{34})^2 + Z_{44}^2. \tag{125}$$

The last pair of inequalities follows from Eq. (110) if we substitute each element of the phase matrix according to Eq. (11) and use

that $C_k^2+S_k^2=1$ for $k=1, 2$. For the special cases $\theta=0$ and $\theta=\pi$ Eqs. (34), (35) and (107) yield

$$|Z_{33}-Z_{44}|\leq|Z_{11}-Z_{22}|. \tag{126}$$

4.4. The Stokes vector criterion

Another method for obtaining inequalities for elements of the phase matrix is based on the observation that the phase matrix transforms four-vectors $\boldsymbol{I}=\{I, Q, U, V\}$ satisfying

$$I\geq(Q^2+U^2+V^2)^{1/2}\geq 0 \tag{127}$$

into vectors of the same type. Mathematically, this property of the phase matrix follows easily from the product representation (4) and the analogous properties of the rotation matrices and the scattering matrix. Physically, Eq. (127) follows from the fact that the degree of polarization, p, of a beam of light always satisfies $0\leq p\leq 1$. On applying the phase matrix to the vector $\{1, 0, 0, 0\}$ we obtain $\{Z_{11}, Z_{21}, Z_{31}, Z_{41}\}$. In combination with Eqs. (13) and (16) we then find the positivity condition (118) and the quadratic inequality (121). A similar application of the phase matrix to the vectors $\{1, \pm 1, 0, 0\}$, $\{1, 0, \pm 1, 0\}$ and $\{1, 0, 0, \pm 1\}$ yields the respective inequalities

$$(Z_{11}\pm Z_{12})^2\geq(Z_{21}\pm Z_{22})^2+(Z_{31}\pm Z_{32})^2+(Z_{41}\pm Z_{42})^2 \tag{128}$$

$$(Z_{11}\pm Z_{13})^2\geq(Z_{21}\pm Z_{23})^2+(Z_{31}\pm Z_{33})^2+(Z_{41}\pm Z_{43})^2 \tag{129}$$

$$(Z_{11}\pm Z_{14})^2\geq(Z_{21}\pm Z_{24})^2+(Z_{31}\pm Z_{34})^2+(Z_{41}\pm Z_{44})^2. \tag{130}$$

Equation (128) coincides with the pair of Eqs. (94) and (97), which imply Eqs. (100) and (102). Adding the $\pm$ inequalities (129)–(130) we obtain

$$Z_{11}^2+Z_{13}^2\geq Z_{21}^2+Z_{23}^2+Z_{31}^2+Z_{33}^2+Z_{41}^2+Z_{43}^2 \tag{131}$$

$$Z_{11}^2+Z_{14}^2\geq Z_{21}^2+Z_{24}^2+Z_{31}^2+Z_{34}^2+Z_{41}^2+Z_{44}^2. \tag{132}$$

With the help of Eqs. (15), (16) and (36) one may simplify Eqs. (131)–(132) to obtain

$$Z_{11}^2\geq Z_{12}^2+Z_{23}^2+Z_{33}^2+Z_{43}^2 \tag{133}$$

$$Z_{11}^2\geq Z_{21}^2+Z_{24}^2+Z_{31}^2+Z_{34}^2+Z_{44}^2. \tag{134}$$

In Eq. (134) we may simultaneously replace Z_{21} and Z_{31} by Z_{12} and Z_{13}, and Z_{24} and Z_{34} by Z_{42} and Z_{43} [cf. Eqs. (36)–(37)]. From Eqs. (15), (16), (102), (103), (118), (133) and (134) we obtain

$$Z_{11}\geq|Z_{ij}| \tag{135}$$

where $i, j=1, 2, 3, 4$.

4.5. Miscellaneous methods

First of all, we obtain a class of inequalities by omitting terms in Eqs. (93)–(98). This yields

$$\max\{|Z_{12}\pm Z_{21}|, |Z_{33}\pm Z_{44}|, |Z_{34}\mp Z_{43}|\}\leq Z_{11}\pm Z_{22} \tag{136}$$

$$\max\{|Z_{21}\pm Z_{22}|, |Z_{31}\pm Z_{32}|, |Z_{42}|\}\leq Z_{11}\pm Z_{12} \tag{137}$$

$$\max\{|Z_{12}\pm Z_{22}|, |Z_{13}\pm Z_{23}|, |Z_{24}|\}\leq Z_{11}\pm Z_{21}, \tag{138}$$

where Eqs. (15), (16) and (135) have been used also. Following the reasoning of sub-section 4.2 we may also derive Eqs. (136)–(138) from inequalities valid for the elements of the scattering matrix given by Hovenier et al. (1986) (cf. their Eqs. (206)–(212) and (236)]. By omitting only one term from Eqs. (93)–(98) we may obtain still other inequalities, such as

$$(Z_{33}\pm Z_{44})^2+(Z_{34}\mp Z_{43})^2\leq(Z_{11}\pm Z_{22})^2, \tag{139}$$

in analogy with Eq. (213) of Hovenier et al. (1986).

Another method is to use the inequality $a^2+b^2\geq 2|ab|$ to replace the sum of two squares by a double product. From Eq. (121) one then obtains

$$Z_{11}^2\geq 2|Z_{21}Z_{31}|. \tag{140}$$

We can further employ the well-known inequality

$$(x_1y_1+x_2y_2+\cdots+x_ny_n)^2\leq(x_1^2+\cdots+x_n^2)(y_1^2+\ldots+y_n^2) \tag{141}$$

for real numbers x_k, y_k, $k=1, 2, \ldots, n$, where the equality sign holds if and only if x_k/y_k is constant. Taking $|y_1|=|y_2|=\cdots=|y_n|=1$ we find that

$$\sum_{k=1}^{n} x_k^2\leq r \tag{142}$$

implies

$$\sum_{k=1}^{n} |x_k|\leq(rn)^{1/2}. \tag{143}$$

Here the equality sign holds if and only if it holds in Eq. (142) and all $|x_k|$ are the same. Applying this rule to Eqs. (133) and (134) we obtain

$$|Z_{12}|+|Z_{23}|+|Z_{33}|+|Z_{43}|\leq 2Z_{11} \tag{144}$$

$$|Z_{21}|+|Z_{24}|+|Z_{31}|+|Z_{34}|+|Z_{44}|\leq Z_{11}\sqrt{5} \tag{145}$$

where we have used Eq. (118). Equations (15), (16) and (104) provide

$$\sum_{i=1}^{4}\sum_{j=1}^{4}|Z_{ij}|\leq Z_{11}(1+\sqrt{39}). \tag{146}$$

The symmetry relations expressed by Eqs. (78)–(80) can be employed to obtain checks on the preceding inequalities or to obtain new inequalities.

5. Relations for the azimuth components of the phase matrix

In this section we shall derive equalities and inequalities for the Fourier components of the phase matrix. Such relations are of interest, since many calculations are based on Fourier components of the phase matrix rather than on the phase matrix itself. In order to derive such relations, we consider the Fourier decomposition of the phase matrix in azimuthal angle

$$\mathbf{Z}(u, u', \varphi-\varphi')=\mathbf{Z}^{c0}(u, u')+2\sum_{j=1}^{\infty}[\mathbf{Z}^{cj}(u, u')\cos\{j(\varphi-\varphi')\}+\mathbf{Z}^{sj}(u, u')\sin\{j(\varphi-\varphi')\}], \tag{147}$$

whose components satisfy various symmetry relations, such as (cf. Eq. (78))

$$\mathbf{Z}^{c0}(u, u')=\mathbf{D}\mathbf{Z}^{c0}(u, u')\mathbf{D} \tag{148}$$

$$\mathbf{Z}^{cj}(u, u')=\mathbf{D}\mathbf{Z}^{cj}(u, u')\mathbf{D} \tag{149}$$

$$\mathbf{Z}^{sj}(u, u')=-\mathbf{D}\mathbf{Z}^{sj}(u, u')\mathbf{D}. \tag{150}$$

On defining the matrices (cf. Siewert, 1981; Hovenier and Van der Mee, 1983)

$$\mathbf{W}^j(u, u')=\mathbf{Z}^{cj}(u, u')-\mathbf{D}\mathbf{Z}^{sj}(u, u')=\mathbf{Z}^{cj}(u, u')+\mathbf{Z}^{sj}(u, u')\mathbf{D} \tag{151}$$

we find the relations

$$[\mathbf{W}^j(u,u')]_{pq}=\begin{cases}[\mathbf{Z}^{cj}(u,u')]_{pq}, (p,q)\in \mathbf{UL} \text{ or } \mathbf{LR}\\ [\mathbf{Z}^{sj}(u,u')]_{pq}, (p,q)\in \mathbf{LL}\\ -[\mathbf{Z}^{sj}(u,u')]_{pq}, (p,q)\in \mathbf{UR}.\end{cases} \tag{152}$$

Here $\mathbf{UL}=\{(1,1), (1,2), (2,1), (2,2)\}$, $\mathbf{LR}=\{(3,3), (3,4), (4,3), (4,4)\}$, $\mathbf{LL}=\{(3,1), (3,2), (4,1), (4,2)\}$ and $\mathbf{UR}=\{(1,3), (1,4), (2,3), (2,4)\}$ are the index sets of the upper left, lower right, lower left and upper right corners, respectively. We shall derive from Eqs. (19), (23), (36)–(42) and (56)–(61) a set of equations for the elements of $\mathbf{W}^j(u,u')$. Below we shall write W^j_{pq} for the (p, q) element of $\mathbf{W}^j(u,u')$, without displaying its dependence on u and u'.

A major tool in subsequent derivations is the product formula for Fourier series, viz. if one has the product representation

$$c^{c0}+2\sum_{j=1}^{\infty}(c^{cj}\cos jx+c^{sj}\sin jx)=$$
$$=\left[a^{c0}+2\sum_{j=1}^{\infty}(a^{cj}\cos jx+a^{sj}\sin jx)\right]\times$$
$$\left[b^{c0}+2\sum_{j=1}^{\infty}(b^{cj}\cos jx+b^{sj}\sin jx)\right], \tag{153}$$

one has the following relations between the coefficients:

$$c^{c0}=a^{c0}b^{c0}+2\sum_{j=1}^{\infty}(a^{cj}b^{cj}+a^{sj}b^{sj}) \tag{154}$$

$$c^{cj}=a^{cj}b^{c0}+a^{c0}b^{cj}+\sum_{\substack{i+k=j\\ i,k\geq 1}}(a^{ci}b^{ck}-a^{si}b^{sk})$$
$$+\sum_{\substack{|i-k|=j\\ i,k\geq 1}}(a^{ci}b^{ck}+a^{si}b^{sk}) \tag{155}$$

$$c^{sj}=a^{sj}b^{c0}+a^{c0}b^{sj}+\sum_{\substack{i+k=j\\ i,k\geq 1}}(a^{ci}b^{sk}+a^{si}b^{ck})$$
$$+\sum_{\substack{|i-k=j|\\ i,k\geq 1}}\operatorname{sgn}(i-k)(-a^{ci}b^{sk}+a^{si}b^{ck}) \tag{156}$$

where $\operatorname{sgn}(i-k)=1$ if $(i-k)>0$, $\operatorname{sgn}(i-k)=0$ if $(i-k)=0$ and $\operatorname{sgn}(i-k)=-1$ if $(i-k)<0$. From Eqs. (153)–(156) it is immediate that the product of two cosine series or the product of two sine series is a cosine series and the product of a cosine series and a sine series is a sine series. By definition, the series $a^{c0}+2\sum_{j=1}^{\infty}\{a^{cj}\cos jx+a^{sj}\sin jx\}$ is a cosine series if $a^{sj}=0$ for $j\geq 1$, and a sine series if $a^{cj}=0$ for $j\geq 0$.

Let us derive equations for the elements of $\mathbf{W}^j(u,u')$ from Eqs. (15), (16), (19), (23) and (36)–(42). Here we use (152), the product formulae (153)–(156) and the fact that the Fourier series of Z_{11}, Z_{12}, Z_{21}, Z_{22}, Z_{33}, Z_{34}, Z_{43} and Z_{44} are cosine series and those of Z_{13}, Z_{14}, Z_{23}, Z_{24}, Z_{31}, Z_{32}, Z_{41} and Z_{42} are sine series [cf. Eq. (78)]. On computing the Fourier components for $j=0$ we use $\mathbf{Z}^{c0}=\mathbf{W}^0$ and obtain (apart from tautologies)

$$W^0_{14}=W^0_{41}=0 \tag{157}$$

$$(W^0_{12})^2+2\sum_{j=1}^{\infty}\{(W^j_{12})^2+(W^j_{13})^2\}=(W^0_{21})^2+2\sum_{j=1}^{\infty}\{(W^j_{21})^2$$
$$+(W^j_{31})^2\} \tag{158}$$

$$(W^0_{43})^2+2\sum_{j=1}^{\infty}\{(W^j_{43})^2+(W^j_{42})^2\}=(W^0_{34})^2$$
$$+2\sum_{j=1}^{\infty}\{(W^j_{24})^2+(W^j_{34})^2\} \tag{159}$$

$$\sum_{j=0}^{\infty}\{W^j_{13}W^j_{24}+W^j_{31}W^j_{42}\}=0 \tag{160}$$

$$\sum_{j=0}^{\infty}(2-\delta_{0j})\{W^j_{12}W^j_{34}+W^j_{21}W^j_{43}\}=0 \tag{161}$$

$$\sum_{j=0}^{\infty}(2-\delta_{0j})\{W^j_{12}W^j_{43}+W^j_{21}W^j_{34}\}$$
$$-2\sum_{j=1}^{\infty}\{W^j_{13}W^j_{42}+W^j_{31}W^j_{24}\}=0. \tag{162}$$

In a similar way equalities stemming from Eqs. (15), (16), (19), (23) and (36)–(42) may be obtained by computing the other Fourier components, but the formulae thus derived are quite complicated.

In order to write down the equations for the Fourier components stemming from Eqs. (56)–(61), which consist of terms that are products of three elements of the phase matrix, one must generalize Eqs. (153)–(156) to a product of three Fourier series. Since in Eqs. (56)–(61) each term consists of a product of three Fourier series which are either two cosine series and one sine series or three sine series, one may restrict oneself to generalizing Eqs. (153)–(156) to such a situation only. As a result of the fact that the resulting Fourier series is a sine series, the $j=0$ component equations stemming from Eqs. (56)–(61) are tautologies of the type $0=0$.

A second way of generating equalities for the Fourier components is to consider the cases $\varphi-\varphi'=0$ and $\varphi-\varphi'=\pi$, using the identities

$$\mathbf{Z}(u,u',0)=\mathbf{Z}^{c0}(u,u')+2\sum_{j=1}^{\infty}\mathbf{Z}^{cj}(u,u')$$
$$=\mathbf{F}(uu'+(1-u^2)^{1/2}(1-u'^2)^{1/2}) \tag{163}$$

$$\mathbf{Z}(u,u',\pi)=\mathbf{Z}^{c0}(u,u')+2\sum_{j=1}^{\infty}(-1)^j\mathbf{Z}^{cj}(u,u')$$
$$=\mathbf{F}(uu'-(1-u^2)^{1/2}(1-u'^2)^{1/2}) \tag{164}$$

and the special form (3) of the scattering matrix. This yields

$$W^0_{12}+2\sum_{j=1}^{\infty}(\pm 1)^jW^j_{12}=W^0_{21}+2\sum_{j=1}^{\infty}(\pm 1)^jW^j_{21} \tag{165}$$

$$W^0_{43}+2\sum_{j=1}^{\infty}(\pm 1)^jW^j_{43}=-\left\{W^0_{34}+2\sum_{j=1}^{\infty}(\pm 1)^jW^j_{34}\right\}. \tag{166}$$

It is also clear that

$$W^0_{pq}+2\sum_{j=1}^{\infty}(\pm 1)^jW^j_{pq}=0 \tag{167}$$

whenever (p, q) belongs to either the upper right or the lower left corner.

Inequalities for the Fourier components of the elements of the phase matrix may be obtained exploiting the linearity of some of the inequalities considered in Sect. 4. From Eqs. (136)–(138) we easily derive

$$\max\{|W^0_{12}\pm W^0_{21}|, |W^0_{33}\pm W^0_{44}|, |W^0_{34}\mp W^0_{43}|\}\leq W^0_{11}\pm W^0_{22} \tag{168}$$

$$|W^0_{21}\pm W^0_{22}|\leq W^0_{11}\pm W^0_{12} \tag{169}$$

$$|W^0_{12} \pm W^0_{22}| \leq W^0_{11} \pm W^0_{21} \tag{170}$$

where the last two relations are equivalent. Their derivation runs as follows. Let us consider just one of the inequalities, e.g. $|Z_{21}+Z_{22}| \leq Z_{11}+Z_{12}$, viewed as the two linear inequalities

$$Z_{11}+Z_{12} \pm (Z_{21}+Z_{22}) \geq 0. \tag{171}$$

Integrating Eq. (171) over azimuth we easily find by linearity

$$W^0_{11}+W^0_{12} \pm (W^0_{21}+W^0_{22}) \geq 0, \tag{172}$$

which implies the desired inequality

$$|W^0_{21}+W^0_{22}| \leq W^0_{11}+W^0_{12}. \tag{173}$$

By a similar derivation we obtain

$$|W^0_{ij}| \leq W^0_{11} \tag{174}$$

for all $i,j=1, 2, 3, 4$ (cf. Eq. (135)) and, for example, (cf. Eqs. (104), (133), (134), (142) and (143))

$$\sum_{i=1}^{4} \sum_{j=1}^{4} |W^0_{ij}| \leq W^0_{11}(1+\sqrt{21}) \tag{175}$$

$$|W^0_{12}|+|W^0_{33}|+|W^0_{43}| \leq W^0_{11}\sqrt{3} \tag{176}$$

$$|W^0_{21}|+|W^0_{34}|+|W^0_{44}| \leq W^0_{11}\sqrt{3}. \tag{177}$$

For the other Fourier components such a derivation is not possible since $\cos j(\varphi-\varphi')$ and $\sin j(\varphi-\varphi')$ change sign if $j \geq 1$.

6. Conclusions

The main conclusions of the present work will be summarized in this section.

The essentials of the relationship between the scattering matrix and the phase matrix were brought to light for many realistic types of light scattering [Sect. 2]. It was further shown that it is not necessary to treat the elements of the phase matrix for the same values of the arguments as if they were 16 unrelated quantities. Indeed, in Sect. 3 eight basic relations have been derived which may be divided into (i) two relations for the corner elements (Eqs. (15) and (16)), (ii) four relations for the non-corner elements along the boundary (Eqs. (19), (36), (39) and (40) if the element $Z_{12} \neq 0$), and (iii) two relations for the elements of the middle block (Eqs. (56) and (57) if $Z_{12} \neq 0$). The degrees of these equations are one, two and three, respectively. In addition to the basic equations a selection of others has been presented. Symmetry relations, which involve elements with different values of the arguments, were used to check the preceding results and to show how relations of a new type may be created. In Sect. 4 a number of ways to obtain inequalities for the elements of the phase matrix has been expounded together with many illustrative examples. Perhaps the most elegant one of these methods is the one based on first performing the rotations on the amplitude matrix of each individual particle and then constructing the phase matrix per particle followed by summation over various particles. Equalities and inequalities for the azimuth components of the phase matrix have been considered in Sect. 5 with emphasis on the azimuth-independent term which usually is the most difficult one to handle in multiple scattering calculations.

In this paper we have restricted ourselves to a representation of polarized light in terms of Stokes parameters. However, since we have not only given results but also explained the ways and means utilized in our treatment the translation into another representation of polarized light (e.g. one based on two oppositely circularly polarized states) may always be made if desired.

The light shed in this paper on the structure of the phase matrix and its relationship to the scattering matrix provides new insight into the nature of transfer of polarized light in plane-parallel atmospheres. In particular, the methods and results may be used to devise or improve analytical and numerical solution strategies for radiative transfer problems as well as for checking purposes and extensions to other matrices involved in scattering theory.

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