*Applicable Analysis*, Vol. 41, pp. 89–110
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53

# Trace Theorems and Kinetic Equations for Non Divergence Free External Forces

Communicated by M. Z. Nashed

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AMS(MOS): 70G35

Abstract For a general class of time dependent linear Boltzmann type equations with (i) an external, non divergence free force term $a \cdot \partial u/\partial \xi$, (ii) a collision term which can be written as the difference of a gain term involving a general nonnegative "collision frequency" $h(x,\xi,t)$ and a loss term involving an arbitrary bounded linear operator J, and (iii) a general boundary operator K which is a (strict) contraction, the method of characteristics and perturbation techniques are used to obtain the well-posedness of the initial-boundary value problem, provided the divergence b of a is bounded above. The functional setting is $L_p$, $1<p<+\infty$. For time independent data the transport operator is shown to generate a $C_0$-semigroup on $L_p(\Sigma,d\mu)$. The results are proven by generalizing a recently established theory of time dependent kinetic equations where the external force is divergence free with respect to velocity. Solutions on spaces of measures are discussed briefly.

KEY WORDS: Boltzman equation, Fokker–Planck equation.

(*Received for Publication 27 February 1990*)

## 1. INTRODUCTION

Since the article of Lehner and Wing [1] on the time dependent monoenergetic neutron transport equation in homogeneous slab media, much effort has been spent on proving the well-posedness of time dependent kinetic equations. The usual strategy has been to use semigroup theory, but in recent years general results have been obtained with the help of the method of characteristics. Continuing the work first expounded by Bardos [2], Beals and Protopopescu [3] have developed a general existence and uniqueness theory of time dependent kinetic equations of the type

[1]A large part of the research was conducted as a visiting professor supported by C.N.R. (Consiglio Nazionale per le Ricerche), Gruppo Nazionale per la Fisica Matematica. The author has greatly benefited from the hospitality rendered by the Dipartimento di Matematica dell' Università di Pavia.

$$\frac{\partial u}{\partial t} + \xi\cdot\frac{\partial u}{\partial x} + a\cdot\frac{\partial u}{\partial \xi} + hu - Ju = f \tag{1}$$

where x, $\xi$ and t denote position, velocity and time, the solution u is the particle density or radiative intensity, h is the cross section, extinction coefficient or collision frequency, J is a bounded operator describing the loss contribution of the collision or single scattering processes, f describes internal particle or radiative sources and a accounts for the self-consistent or external forces. This equation is endowed with the initial condition

$$u(x,\xi,t=0) = g_0(x,\xi), \tag{2}$$

which describes the situation at time t=0, and the boundary condition

$$u_-(x,\xi,t) = (Ku_+)(x,\xi,t) + g_-(x,\xi,t), \tag{3}$$

which describes boundary reflection, absorption and fission processes through a contractive boundary operator K. In [3] it was assumed that the acceleration a is divergence free with respect to the velocity $\xi$. A more elaborate treatment of these results may be found in Chapters XI and XII of [4]. On the other hand, Arlotti [5] has applied the framework of semigroup theory to prove the well-posedness of an extensive class of time dependent kinetic equations with $\Re^3$ as the spatial domain. In her study the acceleration, though non divergence free, is time independent, as is the collision frequency h and the bounded operator J. Recently Cosner et al. [6] have presented a theory of time dependent kinetic equations in an $L_p$-setting, $1<p<\infty$, where the collision operator $(h-J)$ is replaced by a second order differential operator modeling Fokker-Planck type equations. In the present article we shall generalize the results of [3] and [5] and remove the divergence free condition. At this point we remark that Bardos [2] treated the above initial-boundary value problem for a quite general class of non divergence free accelerations a, bounded regions and K=0. His work contains the important clue of decomposing the phase space measure along the integral curves using a weight function which contains the divergence of a, contrary to [3] and [4] where no such weight function was needed. For reviews of the theory of time dependent kinetic equations we refer the reader to [7], [8] and [4].

The crux of the proof of the well-posedness results, whether one adopts semigroup theory or the method of characteristics, is the derivation of a trace theorem, which justifies restricting $L_p$-functions to (parts of) the boundary. In fact, since the equation itself is studied on an $L_p$-space of functions of position, velocity and time and the initial and boundary conditions are equations in an $L_p$-space of functions on the submanifold $\{t=0\}$ and in an $L_p$-space of functions on the incoming and outgoing parts of the boundary, one has to restrict $L_p$-functions on the position-velocity-time phase space to certain submanifolds of lower dimension using a trace theorem. While Bardos [2] treated a quite general class of kinetic equations, his article does not contain a proof of the appropriate trace theorem. For the case $a\equiv 0$, $h=0$ and $J=0$ (and mostly in $L_1$-spaces) this was done by Voigt [9] within the framework of positive semigroups. He unvealed, in particular, the complications that arise when the boundary operator K has unit norm. In this case the notion of a solution has to be weakened, since the traces of $L_p$-solutions are in general local rather than global $L_p$-functions. (If K is a strict contraction, $L_p$-solutions have global $L_p$-traces.) However, the traces are still global $L_p$-functions in a weighted $L_p$-space (see [10] for $a\equiv 0$; [11] for quite general divergence free a), though in one without apparent physical meaning. We shall generalize the trace theorems in [3] and [4] to the non divergence free case. As far as possible, we shall follow the presentation of [4]. Once the trace theorems are established, we will apply perturbation techniques to pass to the general case.

In Section 2 we shall prove the trace theorem. Once it is proved, the well-posedness results can be established almost precisely as in the divergence free case. For this reason we shall indicate, in Section 3, how the results of [3] (but using the presentation of [4]) carry over to the non divergence free case, emphasizing the few differences that occur. Section 4 is devoted to the proof of the semigroup property in the case of time independent acceleration a, collision operator $(h-J)$ and boundary operator K.

## 2. VECTOR FIELDS, FUNCTION SPACES AND TRACES

Let us consider a general setting which contains the vector field of interest. We shall assume that $\Sigma$ is a $C^\infty$ manifold with boundary (denoted by $\partial\Sigma$), imbedded in $\Re^d$ with $\overline{\Sigma} = \Sigma\cup\partial\Sigma$ a closed subset of $\Re^d$, and such that the interior of $\overline{\Sigma}$ does not intersect $\partial\Sigma$. The set $\Sigma$ is assumed to be equipped with a positive Borel measure $\mu$ such that all bounded Lebesgue measurable subsets of $\Sigma$ have finite $\mu$-measure. Let Y be a Lipschitz continuous vector field defined on $\Sigma$ which extends Lipschitz continuously to $\bar{\Sigma}$ and does not vanish at any point of the closure $\bar{\Sigma}$. The vector field Y is assumed to be real and to have a divergence b with respect to $\mu$, in the sense that

$$\int_{\Sigma} \{Yv+bv\}d\mu = 0, \quad v \in C_c^1(\Sigma). \tag{4}$$

Here $C_c^1(\Sigma)$ is the space of continuously differentiable functions defined on $\Sigma$ having compact support in $\Sigma$, and b is a (real) Lebesgue measurable function on $\Sigma$ which is bounded on every bounded Borel set of $\Sigma$ and satisfies the boundedness condition specified below. Given a point $y_0 \in \Sigma$, there is a unique maximal integral curve of Y, passing through $y_0$, specified by

$$\frac{dy}{ds}(s) = Yy(s), \quad y(0) = y_0.$$

Since Y is Lipschitz continuous on $\bar{\Sigma}$, the integral curves of Y do not intersect, neither within $\Sigma$ nor when extended to the boundary $\partial\Sigma$. On defining the length of an integral curve as the length of the corresponding s-interval, we shall assume that every maximal integral curve of Y has a length bounded above by a fixed finite constant T and has left and right limits. Since Y does not vanish on $\bar{\Sigma}$, each boundary point is the left (resp. right) limit of at most one integral curve. We shall assume that the trajectories of Y do not reach infinity in finite time, which is satisfied, for instance, if $|a(x,\xi,t)| \leq C(1+|x|+|\xi|)$ (see [3], [4]).

Let $D^-$ (resp. $D^+$) be the subset of $\partial\Sigma$ consisting of all left (resp. right) endpoints of maximal integral curves of Y in $\Sigma$. If $\partial\Sigma$ is piecewise continuously differentiable, then the union of all integral curves which intersect $D^+ \cap D^-$ has $\mu$-measure zero, as derived by Bardos [2] as a consequence of Sard's theorem (cf. [12],[13]). However, even if $\partial\Sigma$ is not piecewise $C^1$, $D^- \cap D^+$ is a Borel set. In fact, one may prove in general that $D^\pm$ are Borel sets and the length, $\ell(x)$, of the integral curves beginning at $x \in D^-$ is a Borel function of x. We thus obtain the identification

$$\Sigma \sim \{(x,s)\colon x \in D^-,\ 0<s<\ell(x)\}.$$

In this identification Borel sets correspond to Borel sets, points of $D^- \cap D^+$ are counted twice, $Y=\frac{d}{ds}$, and

$$D^- \sim \{(x,0)\colon x\in D^-\},\quad D^+ \sim \{(x,\ell(x)\colon x\in D^-\}.$$

Throughout the paper we make the following

Assumptions: 1. The function

$$B(x,s) = \int_0^s b(x,t)dt$$

is $\mu$-essentially bounded on $\Sigma$.
2. The function h is a Borel function on $\Sigma$ which is $\mu$-integrable on every bounded Borel set of $\Sigma$ and satisfies $h \geq b$ $\mu$-almost everywhere.

In Sections 3 and 4 we will strengthen the assumption on b. We now define the test function space $\Phi$ as the space of all Borel functions v on $\Sigma$ with the following properties:

(i) v is continuously differentiable along each integral curve.

(ii) v and Yv are bounded.

(iii) The support of v is bounded and there is a positive lower bound to the lengths of the integral curves that meet the support of v. It is clear from the boundedness of Yv and the continuous differentiability of v along integral curves that every $v\in\Phi$ can be extended to be continuous at the endpoints of each trajectory.

We shall then define the test function space $\Phi_0$ as the subspace of $\Phi$ for which the functions have limit zero at the endpoints of each integral curve.

LEMMA 1. There exist unique positive Borel measures $\nu^{\pm}$ on $D^{\pm}$ such that

$$\int_\Sigma \{Yu+bu\}d\mu = \int_{D^+} ud\nu^+ - \int_{D^-} ud\nu^-, \quad u\in\Phi. \tag{5}$$

The measures are related by

$$d\mu = \exp\{B(x,s)\}d\nu^-(x)ds = \exp\{-B(x,\ell(x))+B(x,s)\}d\nu^+(x)ds. \tag{6}$$

Proof: Let us consider the auxiliary Borel measure

$$d\tilde{\mu} = \exp\{-B(x,s)\}d\mu,$$

and write

$$\int_\Sigma \{Yu+bu\}d\mu = \int_\Sigma Y(\exp\{B(x,s)\}u)d\tilde{\mu}.$$

Let us suppose that $E_0$ is a bounded subset of $D^-$ on which $\ell(x)$ is bounded away from zero. Suppose that w is a bounded Borel function defined on the set

$$E = \{(x,0)\colon x\in E_0\} \cup \{(x,\ell(x))\colon x\in E_0\}.$$

We extend w by zero on the rest of $D^- \cup D^+$, and for $x\in D^-$ we set

$$\hat{w}(x,s) = \exp\{-B(x,s)\}([1-\ell(x)^{-1}s]w(x,0) + \ell(x)^{-1}s\exp\{B(x,\ell(x))\}w(x,\ell(x))).$$

Then

$$\int_\Sigma \{Y\hat{w}+b\hat{w}\}d\mu = \int_\Sigma \frac{d}{ds}([1-\ell(x)^{-1}s]w(x,0) +$$
$$+ \ell(x)^{-1}s\exp\{B(x,\ell(x))\}w(x,\ell(x)))d\tilde{\mu} = \int_{D^+} wd\nu^+ - \int_{D^-} wd\nu^-$$

for certain unique positive Borel measures $\nu^\pm$ on $D^\pm$, which are related by

$$d\nu^+(x) = \exp\{B(x,\ell(x))\}d\nu^-(x).$$

In order to derive the Green's identity (5) it suffices to prove that $\int_\Sigma(Yv+bv)d\mu=0$ for all $v\in\Phi_0$, since every $v\in\Phi$ can be written as the sum of a function in $\Phi_0$ and a function $w\in\Phi$ that is the product of $\exp\{-B(x,s)\}$ and a function linear along trajectories. Let us first observe that (4) holds true for all $v\in C_c^1(\Sigma)$, by assumption. This implies the validity of (4) for all $v\in\Phi$ of the form

$$v(x,s) = \tau(x,s)w(x), \;\; x\in D^{-}, \;\; 0<s<\ell(x), \tag{7}$$

where $\tau\in C_c^1(\Sigma)$ and w is a bounded Borel function on $D^-$. Then (5) is true on the class $\Psi_0$ of finite linear combinations of functions of the type (7) and therefore on the class $\Psi$ of sums of a function in $\Psi_0$ and a function in $\Phi$ that is linear along trajectories apart from the factor $\exp\{-B(x,s)\}$.

In order to prove (5) for arbitrary $u\in\Phi$, we use the already established Green's identity on $\Psi$ to derive (6), from which the general Green's identity (5) is immediate. Indeed, consider a general $w\in\Psi$ and set

$$v_1(x,s) = -\int_s^{\ell(x)} \exp\{B(x,t)-B(x,s)\}w(x,t)dt.$$

Then $(Y+b)v_1=w$, and $v_1=0$ on $D^+$. Thus (5), proven above for all $w\in\Psi$, together with Fubini's theorem, implies

$$\int_\Sigma wd\mu = \int_{D^+} v_1(x,\ell(x))d\nu^+(x) - \int_{D^-} v_1(x,0)d\nu^-(x) =$$
$$= \int_\Sigma \exp\{B(x,s)\}w(x,s)d\nu^-(x)ds.$$

On the other hand, putting

$$v_2(x,s) = \int_0^s \exp\{B(x,t)-B(x,s)\}w(x,t)dt,$$

we get $(Y+b)v_2=w$ and $v_2=0$ on $D^-$, and therefore

$$\int_\Sigma wd\mu = \int_{D^+} v_2(x,\ell(x))d\nu^+(x) - \int_{D^-} v_2(x,0)d\nu^-(x) =$$
$$= \int_\Sigma \exp\{-B(x,\ell(x))+B(x,s)\}w(x,s)d\nu^+(x)ds.$$

Observe that $\Psi$ is dense in $L_p(\Sigma,\exp\{B(x,s)\}d\nu^-ds)$ for $p\in[1,\infty)$ and that both $\mu$ and the above weighted product measure are Borel measures. Then every function $w\in L_p(\Sigma,\exp\{B(x,s)\}d\nu^-ds)$ has the same integral over $\Sigma$ with respect to this product measure as with respect to $\mu$, which completes the proof of (6). From (6) we immediately have (5) for arbitrary $u\in\Phi$ by direct integration with respect to s. □

The decomposition (6) implies that $\exp\{B(x,s)\}$ is the Jacobian of the transformation of $\Sigma$ consisting of a translation along trajectories over a distance s in the direction of increasing s, as found by Bardos (cf. [2], Proposition 2.1).

We are now in a position to derive an important proposition on which the proofs of the existence and uniqueness theorems of the next section will be based. For $1\leq p<\infty$ we denote by $L_{p,loc}(\Sigma,d\mu)$ the linear space of all $\mu$-measurable functions u on $\Sigma$ with the property that $|u|^p$ is $\mu$-integrable on every bounded $\mu$-measurable subset of $\Sigma$ on which $\ell(x)$ is bounded away from zero. $L_{p,loc}(D^{\pm},d\nu^{\pm})$ is defined analogously. We define the distributional derivative by the formula

$$<Yu,v> = -<u,Yv> - <bu,v>, \quad v\in\Phi_0.$$

Here we have to keep in mind that heuristically

$$<Yu,v> + <u,Yv> + <bu,v> = \int_{D^-}\int_0^{\ell(x)} \Big(\frac{d}{ds}\, e^{B(x,s)}uv\Big)ds\,d\nu^-(x) = 0,$$

since $v\in\Phi_0$.

<u>Definition</u>: If u and Yu belong to $L_p(\Sigma,d\mu)$, we shall define a <u>trace</u> of u as a pair of functions $u^{\pm}$ in $L_{p,loc}(D^{\pm},d\nu^{\pm})$ such that the extended Green's identity is valid:

$$<Yu,v> + <u,Yv> + <bu,v> = \int_{D^+} u^+v\,d\nu^+ - \int_{D^-} u^-v\,d\nu^-, \quad v\in\Phi. \tag{8}$$

PROPOSITION 2. Suppose that u and $(Y+h)u$ belong to $L_p(\Sigma,d\mu)$, $1\leq p<\infty$. Assume that b is $\mu$-essentially bounded below if $p>1$. Then the following statements hold true:

(i) u has a unique trace $u^{\pm}$.

(ii) If $u^-$ belongs to $L_p(D^-,d\nu^-)$, then $u^+$ belongs to $L_p(D^+,d\nu^+)$, in which case $h|u|^p$ and $|u|^{p-1}(Y+\frac{b}{p})u$ are $\mu$-integrable and

$$\int_{D^+} |u^+|^p d\nu^+(x) + (p-1)\int_{\Sigma} b|u|^p d\mu + p\int_{\Sigma} \sigma|u|^p d\mu =$$

$$= \int_{D^-} |u^-|^p d\nu^-(x) + p\int_{\Sigma} (\mathrm{sgn}\ u)|u|^{p-1}(Y+h)u d\mu, \tag{9}$$

where $\sigma = h - b \geq 0$.

Proof: Suppose first u and (Y+h)u belong to $L_{p,loc}(\Sigma, d\mu)$. Writing $u=u_0+u_1$, where $u_0=[1-\ell(x)^{-1}s]u(x,s)$, we deduce that

$$v_0 \equiv Yu_0 = [1-\ell(x)^{-1}s]Yu - \ell(x)^{-1}u$$

belongs to $L_{p,loc}(\Sigma, d\mu)$. It follows from the above lemma that, for almost all $x \in D^-$, $v_0(x,.) \in L_p((0,\ell(x)), \exp\{B(x,s)\}ds)$. Indeed, if $v_0 \in L_{p,loc}(\Sigma, d\mu)$ and $\chi_S$ denotes the characteristic function of a set S, then $v_0\chi_S \in L_p(\Sigma, d\mu)$ for every S of the form $S=\{(x,s): x \in E_0, 0<s<\ell(x)\}$, where $E_0$ is bounded and $\ell(x)$ is bounded away from zero on $E_0$. Hence, for such x,

$$\int_{\Sigma} |v_0\chi_S|^p d\mu = \int_{E_0}\int_0^{\ell(x)} |v_0(x,s)|^p \exp\{B(x,s)\} ds d\nu^-(x) < \infty,$$

whence $v_0(x,.) \in L_p((0,\ell(x)), \exp\{B(x,s)\}ds)$ almost everywhere on $D^-$, which proves our assertion. For such x set

$$u_0^*(x,s) = -\int_s^{\ell(x)} v_0(x,t)dt. \tag{10}$$

Assume $\varphi \in \Phi$ and set

$$w_0(x,s) = \int_0^s \varphi(x,t)\exp\{B(x,t)\}dt.$$

Then $w_0 \in \Phi$ and $w_0=0$ on $D^-$, so $[1-\ell(x)^{-1}s]w_0 \in \Phi_0$ and we have

$$<u_0^*,\varphi> = \int_{D^-}\int_0^{\ell(x)} u_0^*\varphi \exp\{B(x,s)\}ds d\nu^-(x) = \int_{D^-} \{[u_0^* w_0]_0^{\ell(x)} -$$
$$- \int_0^{\ell(x)} v_0 w_0 ds\} d\nu^-(x) = -\int_{D^-}\int_0^{\ell(x)} \{(1-\ell^{-1}s)Yu - \ell^{-1}u\} w_0 ds d\nu^-(x) =$$

$$= \int_{D^-} \{-[(1-\ell^{-1}s)uw_0]_0^{\ell(x)} + \int_0^{\ell(x)} [\ell^{-1}uw_0 + (1-\ell^{-1}s)u \exp\{B(x,s)\}\varphi]ds\} d\nu^-(x) -$$

$$- \int_{D^-}\int_0^{\ell(x)} \ell^{-1}uw_0 ds d\nu^-(x) = <u_0,\varphi>.$$

Thus we may identify $u_0$ with $u_0^*$. For any $z\in\Phi$ an integration by parts and the decomposition (6) give

$$<u_0,Yz> + <Yu_0,z> + <bu,v> = <u_0^*,Yz> + <v_0,z> + <bu_0,v> =$$

$$= \int_{D^-} [u_0^* z]_{s=0}^{\ell(x)} d\nu^-(x) = -\int_{D^-} u_0(x,0)z(x,0)d\nu^-(x).$$

This shows that $u_0$ has traces $u_0^- = u_0^*(.,0)$ and $u_0^+ \equiv 0$. Hölder's inequality applied to (10) shows that $u^- = u_0^-$ is in $L_{p,loc}(D^-, d\nu^-)$.

Next, put $u_1 = \ell(x)^{-1}su$, $v_1 \equiv Yu_1 = \ell^{-1}sYu + \ell^{-1}u$ and

$$u_1^*(x,s) = \int_0^s v_1(x,t)dt. \tag{11}$$

Taking $\varphi\in\Phi$ and putting $w_1(x,s) = -\int_s^{\ell(x)} \exp\{B(x,t)\}\varphi(x,t)dt$, we get $w_1\in\Phi$, and $w_1=0$ on $D^+$; thus $(sw_1/\ell(x))\in\Phi_0$. As before we obtain

$$<u_1^*,\varphi> = -\int_{D^-}\int_0^{\ell(x)} v_1 w_1 ds d\nu^-(x) = -\int_{D^-}\int_0^{\ell(x)} \ell^{-1}s(Yu)w_1 d\nu^-(x) -$$

$$- \int_{D^-}\int_0^{\ell(x)} \ell^{-1}uw_1 ds d\nu^-(x) = <u_1,\varphi>,$$

whence we may identify $u_1$ and $u_1^*$. Thus for any $z\in\Phi$ we obtain

$$<u_1,Yz> + <Yu_1,z> + <bu_1,z> = \int_{D^+} u_1(x,\ell(x))z(x,\ell(x))d\nu^+(x).$$

Hence, the trace of $u_1$ equals $u_1^- = 0$, $u_1^+ = u_1^*(.,\ell(x))$, while $u^+ = u_1^+ \in L_{p,loc}(D^+, d\nu^+)$ as a consequence of Hölder's inequality applied to (11). Now observe that $u = u_0 + u_1$ with $u_0$ vanishing on $D^+$ and $u_1$ vanishing on $D^-$. Since both $u_0$ and $u_1$ have a trace, so does u by linearity.

To prove the uniqueness of the trace we start with a function w on $D^-\cup D^+$ as in the proof of the above lemma. Then we must have

$$\int_{D^+} wu^+ d\nu^+ - \int_{D^-} wu^- d\nu^- = <u,Y\hat{w}> + <Yu,\hat{w}> + <bu,\hat{w}>. \tag{12}$$

The right hand side of (12) is uniquely determined by w and u, and therefore $u^\pm$ are unique.

Now suppose that u and (Y+h)u belong to $L_p(\Sigma,d\mu)$, and set

$$\hat{u}(x,s) = u(x,s)\exp\{\int_0^s h(x,t)dt\} \equiv u(x,s)H(x,s).$$

Then $Y\hat{u} = H(Y+h)u$, so that $\hat{u}$ and $Y\hat{u}$ belong to $L_{p,loc}(\Sigma,d\mu)$. From the first part of the proof it follows that $\hat{u}$ has a trace. To complete the proof, we first note that $|u|^p$ has the distributional derivative

$$Y(|u|^p) = p(\text{sgn } u)|u|^{p-1}Yu,$$

which, together with $|u|^p$, belongs to $L_{1,loc}(\Sigma,d\mu)$. Using the trace formula (8) with u replaced by $|u|^p$ and v=1 we obtain Eq. (9) in part (ii) of the theorem, taking into account that the trace of $|u|^p$ is $|u^\pm|^p$. Finally, if we apply the Green's identity to $w\in\Phi$ defined as the characteristic function of the set $\Sigma_0=\{(x,s): x\in C^-\subseteq D^-, 0<s<\ell(x)\}$, then the identity (5) holds with $D^\pm$ replaced by Borel sets $C^\pm$ on which $\ell^{-1}$ is bounded, and with $\Sigma$ replaced by $\Sigma_0$. Furthermore, since $u\in L_p(\Sigma,d\mu)$ and $(Y+h)u\in L_p(\Sigma,d\mu)$, one readily concludes that $(\text{sgn } u)|u|^{p-1}(Y+h)u\in L_1(\Sigma,d\mu)$, and therefore its integral over $\Sigma$ converges. Hence, if $u^-\in L_p(D^-,d\nu^-)$, then the above equation implies that $u^+\in L_p(D^+,d\nu^+)$ and $\sigma|u|^p\in L_1(\Sigma,d\mu)$, provided b is $\mu$-essentially bounded below if $p>1$. This extra assumption is not necessary if $p=1$, because in this case the second term in Eq. (9) vanishes. Finally, we take $\Sigma$ as the union of such sets $\Sigma_0$ and pass to the limit to obtain the identity (9). □

The results of Proposition 2 may be strengthened if one replaces $L_p(D^\pm,d\nu^\pm)$ by $L_p(D^\pm,\ell d\nu^\pm)$, where the length $\ell(x)$ of the characteristic appears as an additional weight. Here we must use the assumption that B(x,s) is bounded, i.e. that $0<C\leq\exp\{B(x,s)\}\leq D$ for certain constants C and D, and also assume that h and b are $\mu$-essentially bounded. In order to prove that the trace $u^\pm$ of an arbitrary $u\in L_p(\Sigma,d\mu)$ belongs to $L_p(D^\pm,\ell d\nu^\pm)$, one

follows an argument of Ukai ([11]; also [4], Section XI.3). Starting from an arbitrary $u \in L_p(\Sigma, d\mu)$, one derives the estimate

$$\ell(x)|u^-(x)|^p \leq \frac{2^{p-1}}{C}\int_0^{\ell(x)} \{|u(x,s)|^p + \frac{1}{p}\left(\frac{DT}{C}\right)^p |(Yu)(x,s)|^p\} e^{B(x,s)} ds\}$$

with minor modifications of Ukai's argument. One then integrates both sides with respect to $d\nu^-(x)$ and completes the proof as in the divergence free case, yielding

$$\|u^\pm\|_\ell \leq M(\|u\| + \|(Y+h+\tfrac{b}{p})u\|),$$

where M is the maximum of the constants $K_p$, $K_pL_p$ and $K_pL_p(\|h\|_\infty + \frac{1}{p}\mu\text{-esssup } b)$, with $K_p = (2^{p-1}/C)^{1/p}$ and $L_p = p^{-1/p}\frac{DT}{C}$. Moreover, $\|.\|_\ell$ denotes the norm in $L_p(D^\pm, \ell d\nu^\pm)$. Therefore $u^\pm \in L_p(D^\pm, \ell d\nu^\pm)$ whenever $u \in L_p(\Sigma, d\mu)$ and $(Y+h)u \in L_p(\Sigma, d\mu)$, using that $h, B \in L_\infty(\Sigma, d\mu)$ and b is $\mu$-essentially bounded.

PROPOSITION 3. Given $f \in L_p(\Sigma, d\mu)$ and $g \in L_p(D^-, d\nu^-)$, $1 \leq p < \infty$, there is a unique function $u \in L_p(\Sigma, d\mu)$ such that $(Y+h)u = f$ on $\Sigma$, $u^- = g$ on $D^-$ and $u^+ \in L_p(D^+, d\nu^+)$. Moreover, if f and g are nonnegative, then u is nonnegative.

Proof: Given $f \in L_p(\Sigma, d\mu)$ and $g \in L_p(D^-, d\nu^-)$, set

$$u(x,s) = \exp\{-\int_0^s h(x,\sigma)d\sigma\}g(x) + \int_0^s \exp\{-\int_t^s h(x,\sigma)d\sigma\}f(x,t)dt. \tag{13}$$

Clearly, u is nonnegative if f and g are, and an integration by parts gives

$$<u,(Y-h)u> + <f,v> = \int_{D^+} u^+ v d\nu^+(x) - \int_{D^-} g v d\nu^-(x) - <bu,v>$$

for all $v \in \Phi$, yielding $(Y+h)u = f$ and $u^- = g$ in the distributional sense as a result of (8). Using $0 < C \leq \exp\{B(x,s)\} \leq D$ we now estimate

$$|u(x,s)| \leq \frac{1}{C} e^{B(x,s)}|u(x,s)| \leq \frac{1}{C}\left\{|g(x)| + \int_0^s |f(x,t)|e^{B(x,t)}dt\right\} \leq$$

$$\leq \frac{1}{C}\left\{|g(x)| + D^{\frac{p-1}{p}} s^{\frac{p-1}{p}} \left(\int_0^s |f(x,t)|^p e^{B(x,t)} dt\right)^{1/p}\right\} \leq$$

$$\leq 2^{\frac{p-1}{p}} \frac{1}{C}\Big(|g(x)|^p + (Ds)^{p-1}\int_0^s |f(x,t)|^p e^{B(x,t)} dt\Big)^{1/p},$$

and therefore

$$\int_0^{\ell(x)} |u(x,s)|^p e^{B(x,s)} ds \leq \frac{2^{p-1}}{C^p}\Big(DT|g(x)|^p + \frac{1}{p}(DT)^p \int_0^{\ell(x)} |f(x,t)|^p e^{B(x,t)} dt\Big),$$

which implies

$$\|u\|_p \leq 2^{\frac{p-1}{p}} \frac{1}{C}\Big(DT\|g\|_p^p + \frac{1}{p}(DT)^p \|f\|_p^p\Big)^{1/p}.$$

Hence, $u \in L_p(\Sigma, d\mu)$. Similarly,

$$|u^+(x,\ell(x))| \leq |g(x)| + \frac{1}{C}\int_0^{\ell(x)} |f(x,t)| e^{B(x,t)} dt \leq$$

$$\leq 2^{\frac{p-1}{p}}\Big(|g(x)|^p + \frac{D^{p-1}}{C^p}\ell(x)^{p-1}\int_0^{\ell(x)} |f(x,t)|^p e^{B(x,t)} dt\Big)^{1/p},$$

whence, using $d\nu^+(x) = \exp\{B(x,\ell(x))\} d\nu^-(x)$,

$$\|u^+\|_p \leq 2^{\frac{p-1}{p}}\Big(DT\|g\|_p^p + \frac{1}{p}\Big(\frac{DT}{C}\Big)^p \|f\|_p^p\Big)^{1/p}.$$

Hence, $u^+ \in L_p(\Sigma, d\mu)$.

The uniqueness of the solution u given by (13) follows from the fact that, for almost every $x \in D^-$, $u(x,.)$ satisfies an ordinary differential equation on $(0,\ell(x))$ with given initial condition. □

## 3. EXISTENCE, UNIQUENESS AND POSITIVITY OF SOLUTIONS

Let us consider the boundary value problem (1)-(3) on the Banach space $L_p(\Sigma, d\mu)$ of functions of $(x,\xi,t)$, where the initial-boundary value of the solution belongs to $L_p(D^-, d\nu-)$. To this purpose we consider a phase space $\Sigma$ which is an open subset of $\Re^d \times (0,T)$ for some fixed time $T>0$, in order to treat the position-velocity variable $(x,\xi)$ and the time variable t in a symmetric way. We assume that the interior of the closure $\overline{\Sigma}$ of $\Sigma$ in $\Re^{d+1}$ does not intersect $\partial\Sigma$. In addition, we define as our solution space the space $E_p$ of those functions $u \in L_p(\Sigma, d\mu)$ such that $(Y+h)u \in L_p(\Sigma, d\mu)$ and the trace $u^{\pm} \in L_p(D^{\pm}, d\nu^{\pm})$ for $1<p<\infty$, again under the assumption that B is bounded. The initial-boundary value problem (1)-(3) can then be written in the form

$$(Y + h - J)u = f, \tag{14}$$

$$u^{-} = \mathcal{K}u^{+} + g. \tag{15}$$

Here $g=(g_0,g_-)$ accounts for the initial and incoming boundary value of the solution and $\mathcal{K}=(0,K)$ maps $u^{+}=(u_T,u_+)$ into $(0,u_-)$, while we have decomposed $D^{\pm}$ as

$$D^{-} = \{(x,\xi,0) \in D^{-}\} \cup \{(x,\xi,t) \in D^{-}: 0<t<T\},$$

$$D^{+} = \{(x,\xi,T) \in D^{+}\} \cup \{(x,\xi,t) \in D^{+}: 0<t<T\},$$

as to split up the incoming and outgoing parts of the boundary in "temporal" and "spatial-velocity" pieces. Correspondingly one decomposes the measures $\nu^{\pm}$ as

$$\nu^{-} = (\nu_0,\nu_-), \quad \nu^{+} = (\nu_T,\nu_+).$$

One then assumes that $\mu$ is a Borel measure on $\Sigma$ that is bounded on every bounded Lebesgue measurable subset of $\Sigma$, $h \geq b$ and integrable over each bounded Lebesgue measurable subset of $\Sigma$, J is bounded on $L_p(\Sigma,d\mu)$ and K is a bounded operator from $L_p(D_+,d\nu_+)$ into $L_p(D_-,d\nu_-)$ of norm $\leq 1$. The acceleration a has to be chosen in such a way that the vector field $Y=\frac{\partial}{\partial t}+X$ where $X=\xi\cdot\frac{\partial}{\partial x}+a\cdot\frac{\partial}{\partial \xi}$, satisfies the general assumptions of the previous section. In particular, a must be real, Lipschitz continuous on $\overline{\Sigma}$, its integral curves may not run off to infinity in finite time, and the antiderivative B of b must be bounded on $\Sigma$ if $p>1$. Throughout we write $\beta$ for the $\mu$-essential supremum of $|b|$. The operators J and K have to be real and local in time, i.e. $J(ru)=r(Ju)$ and $K(ru)=r(Ku)$ for every bounded continuous function $r=r(t)$ of time alone.

By virtue of the locality in time of b, h, J and K, the unique solvability of Eqs.

(14)-(15) is equivalent to the unique solvability of the system

$$(Y+h+\lambda-J)u_\lambda = f_\lambda, \tag{16}$$

$$u_\lambda^- = \mathcal{K}u_\lambda^+ + g_\lambda, \tag{17}$$

where $\lambda\in\Re$ is arbitrary but fixed. The relationship between the solutions of Eqs. (14)-(15) and those of Eqs. (16)-(17) is given by $u_\lambda=e^{-\lambda t}u$, $f_\lambda=e^{-\lambda t}f$ and $g_\lambda=e^{-\lambda t}g$.

When applying (9) to Eqs. (16)-(17), one must replace h, u, f, $u^\pm$ and g by $h+\lambda$, $u_\lambda$, $f_\lambda$, $u_\lambda^\pm$ and $g_\lambda$, respectively. Substituting $(Y+h+\lambda)u_\lambda=Ju_\lambda+f_\lambda$ we get

$$\|u_\lambda^+\|_p^p + \{\lambda p-(p-1)\beta\}\|u_\lambda\|_p^p \le \|u_\lambda^-\|_p^p + p\int_\Sigma |u_\lambda|^{p-1}|Ju_\lambda+f_\lambda|d\mu. \tag{18}$$

For $p>1$ we first apply the inequality $ab\le\frac{1}{p}a^p+\frac{1}{q}b^q$ for $a,b\ge0$, where $a=\lambda^{-1/q}|Ju_\lambda+f_\lambda|$ and $b=\lambda^{1/q}|u_\lambda|^{p-1}$. We may then estimate the second term at the right-hand side of (18), rearrange the resulting inequality and obtain

$$\|u_\lambda^+\|_p^p + \{\lambda-(p-1)\beta\}\|u_\lambda\|_p^p \le \|u_\lambda^-\|_p^p + \lambda^{1-p}\|Ju_\lambda+f_\lambda\|_p^p. \tag{19}$$

For $p=1$ the latter is immediate from (18).

Let $u_\lambda=S_\lambda(f_\lambda,g_\lambda)$ denote the unique solution of the system of equations

$$(Y+h+\lambda)u_\lambda = f_\lambda,\ \ u_\lambda^- = g_\lambda.$$

From Eq. (19) we immediately have for $\lambda>(p-1)\beta$

$$\|S_\lambda(f_\lambda,0)\| \le (\lambda-(p-1)\beta)^{-1/p}\lambda^{-(1-\frac{1}{p})}\|f_\lambda\|, \tag{20}$$

$$\|S_\lambda(f_\lambda,0)^+\| \le \lambda^{-(1-\frac{1}{p})}\|f_\lambda\|,$$

$$\|S_\lambda(0,g_\lambda)\| \le (\lambda-(p-1)\beta)^{-1/p}\|g_\lambda\|,$$

$$\|S_\lambda(0,g_\lambda)^+\| \le \|g_\lambda\|. \tag{21}$$

Using the results of the previous section, one can follow almost verbatim the existence and uniqueness proof for Eqs. (14)-(15), as presented in [4] (i.e. from Proposition XI 3.4 up to Section XI.5), apart from minor modifications due to the appearance of $\beta$ in (20). We will therefore give a rather sketchy proof of the following theorem.

THEOREM 4. Suppose $1 \leq p < \infty$ and $\|K\| < 1$. Then for every $f \in L_p(\Sigma, d\mu)$ and $g \in L_p(D^-, d\nu -)$ there exists a unique function $u \in E_p$ such that Eqs. (14) and (15) are satisfied. If K and $\lambda_0 + J$ are positive operators (in lattice sense) and f and g are nonnegative functions, then the solution u is nonnegative.

Proof: Above we have considered the case $J = 0$ and $K = 0$ in detail. Let us now consider the case $K = 0$ and J bounded with $\lambda > (p-1)\beta$ and look for a solution of Eqs. (16)-(17) having the form $u_\lambda = S_\lambda(f_\lambda^*, g_\lambda)$ with $f_\lambda^* \in L_p(\Sigma, d\mu)$ to be determined. We then get

$$(I + L_\lambda) f_\lambda^* = f_\lambda + J S_\lambda(0, g_\lambda), \tag{22}$$

where

$$L_\lambda f_\lambda^* = -J S_\lambda(f_\lambda^*, 0).$$

Now (20) implies that $\|L_\lambda\| < 1$, provided $(\lambda - (p-1)\beta)\lambda^{p-1} > \|J\|^p$, a condition which can easily be satisfied by taking $\lambda$ large enough. Thus Eq. (22) has the unique solution

$$f_\lambda^* = \sum_{m=0}^{\infty} (-L_\lambda)^m [f_\lambda + J S_\lambda(0, g_\lambda)],$$

and hence $u_\lambda = T_\lambda(f_\lambda, g_\lambda)$ has been found. Using (21) we have

$$\|T_\lambda(0, g_\lambda)^+\| \leq \left(1 + \left(\lambda\left(\frac{\lambda - (p-1)\beta}{\lambda}\right)^{1/p} - \|J\|\right)^{-1} \|J\|\right) \|g_\lambda\|.$$

Next we consider arbitrary K and J, where $\lambda > (p-1)\beta$ and $(\lambda - (p-1)\beta)\lambda^{p-1} > \|J\|^p$. Writing the solution $u_\lambda$ of Eqs. (16)-(17) as $u_\lambda = T_\lambda(f_\lambda, g_\lambda^*)$ with $g_\lambda^* \in L_p(D^-, d\nu -)$ to be determined, we find

$$(\mathbb{I} - M_\lambda)g_\lambda^* = g_\lambda + \mathcal{K}(T_\lambda(f_\lambda,0)^+),$$

where $M_\lambda g_\lambda^* = \mathcal{K}(T_\lambda(0,g_\lambda^*)^+)$. Thus

$$\|M_\lambda g_\lambda^*\| \leq \|K\|\Big(1 + \Big(\lambda\Big(\frac{\lambda-(p-1)\beta}{\lambda}\Big)^{1/p} - \|J\|\Big)^{-1}\|J\|\Big)\|g_\lambda^*\|,$$

which is strictly less than $\|g_\lambda^*\|$ if $\lambda$ is large enough, provided $\|K\|<1$. We then find $g_\lambda^*$ in the form

$$g_\lambda^* = \sum_{m=0}^{\infty} M_\lambda{}^m[g_\lambda + \mathcal{K}(T_\lambda(f_\lambda,0)^+)],$$

which completes the proof. □

If K is a positive operator of unit norm and $\lambda_0+J$ is positive for some $\lambda_0\in\Re$, one may obtain a "weak" solution of Eqs. (14)-(15), where u and (Y+h)u belong to $L_p(\Sigma,d\mu)$ but $u^\pm$ belongs to $L_{p,loc}(D^\pm,d\nu^\pm)$. For nonnegative data f and g this solution is the monotone strong limit in $L_p(\Sigma,d\mu)$ of the solution of the corresponding problems, where K is replaced by $K_m=\alpha_m K$ for a monotonically increasing sequence $\{\alpha_m\}_{m=1}^{\infty}$ with limit one. The proof is exactly the same as in the non divergence free case, with a slightly different formula for the upper bounds. In the special case when the characteristic length $\ell(x)$ is bounded away from zero on $\Sigma$ and $h\in L_\infty(\Sigma,d\mu)$, one may obtain results as strong as those in Theorem 4, because in this case the solution $u\in E_p$.

By taking $0<t<T$ and replacing $\Sigma$ by $\Sigma'=\{(x,v,t')\in\Sigma\colon 0<t'<t\}$, we obtain a different phase space to which the above theory applies. As a result, every solution $u\in L_p(\Sigma',d\mu')$ has a unique trace $u^\pm$ in $L_p$ if $\|K\|<1$ and in $L_{p,loc}$ if $K\geq 0$ and $\|K\|=1$, whence $u_t$, the solution at time t, can be defined as an $L_p$-function (global $L_p$ if $\|K\|<1$, and local $L_p$ if $K\geq 0$ and $\|K\|=1$).

Following the argument of Section XI.4 of [4], one may now prove that the solution for given g and f=0 is contractive in an $L_p$-space setting, i.e.

$$\|u_T\| \leq \|u_0\|,$$

if the sufficient condition stated in the below theorem is satisfied. We have

THEOREM 5. Suppose that for every $u \in L_p(\Sigma, d\mu)$ the condition

$$\int_\Sigma \mathrm{sgn}(u)|u|^{p-1}(h-\frac{b}{p}-J)u\,d\mu \geq 0 \tag{23}$$

is satisfied. Then the problem (14)-(15) is dissipative in the sense that $\|u_t\| \leq \|u_0\|$ for $t \in [0,T]$ whenever $f \equiv 0$ and $g_- \equiv 0$.

Proof: Suppose $u \in E_p$ satisfies

$$(Y+h-J)u = 0, \quad u_- = Ku_+,$$

while $f=0$ and $g_-=0$. Then the identity (9) gives

$$\int_{D_T} |u_T|^p d\nu_T + \int_{D_+} |u_+|^p d\nu_+ - \int_{D_-} |Ku_+|^p d\nu_- -$$
$$- \int_{D_0} |u_0|^p d\nu_0 + p\int_\Sigma (\mathrm{sgn}\, u)|u|^{p-1}(h-\frac{b}{p}-J)u\,d\mu = 0.$$

Since $\|K\| \leq 1$, the inequality $\|u_T\| \leq \|u_0\|$ is immediate from (23). □

For $p=1$ and positive $K$ and $\lambda_0+J$, with $\lambda_0 \in \Re$ fixed, it is sufficient to require

$$\int_\Sigma (h-b-J)u\,d\mu \geq 0,$$

where $u \geq 0$ in $L_1(\Sigma, d\mu)$.

## 4. WELL-POSEDNESS IN A SEMIGROUP SETTING

Let us consider the specific though most common situation of a time dependent kinetic equation with time independent acceleration, collision and boundary operators. Let us assume that $\Sigma = \Lambda \times (0,T)$ for some time independent position-velocity domain $\Lambda$, and that the acceleration a, the cross section h, the loss contribution J to the collision operator and the boundary operator K are independent of time, while the phase space $\Lambda$ is endowed with a Borel measure $\mu$ with respect to which every bounded Lebesgue measurable subset of $\Lambda$ is integrable. We assume that $\Lambda$ is an open subset of $\Re^d$ such that the interior of its closure $\overline{\Lambda}$ does not intersect its boundary $\partial\Lambda$. On $\Lambda$ we consider the vector field $X=\frac{\partial}{\partial x}+a\cdot\frac{\partial}{\partial \xi}$ and its characteristic equations

$$\frac{dx}{dt} = \xi, \ \frac{d\xi}{dt} = a(x,\xi),$$

where t is chosen as the parameter. Under the hypothesis that a is Lipschitz continuous on $\overline{\Lambda}$ and that the characteristics of X do not reach the closure of the set

$$\Lambda_0 = \{(x,\xi)\in\Lambda\colon |\xi|^2 + |a(x,\xi)|^2 = 0\}$$

nor infinity in finite time, one can construct through every point of $\overline{\Lambda}$ where $\xi$ and $a(x,\xi)$ do not vanish simultaneously a unique integral curve, which when maximally extended is either closed, infinite in both directions or reaches the boundary in either or both directions, while no boundary point is the left or right endpoint of more than one integral curve. We shall denote the sets of left and right endpoints by $D_-$ and $D_+$, respectively. The length of the maximal integral curve, i.e. the length of the maximal t-interval of definition, will be denoted by $\ell(x)$, where x is some point on the characteristic. (If the integral curve is closed, the definition of $\ell(x)$ should be modified. However, since closed characteristics do not play any role in the description of boundary processes, we shall not give such a definition.)

In order to formulate the Green's identity and a trace theorem for the vector field X, we have to distinguish between notions related to $\Lambda$ and X and notions related to $Y=\frac{\partial}{\partial t}+X$ and $\Sigma=\Lambda\times(0,T)$, many of which are denoted by the same symbols. For this reason we shall temporarily denote the notions relating to $\Sigma$ and Y by $\nu_{(\pm)}$, $D_{(\pm)}$, etc. First, we observe that we have used essentially the same parameter (s for Y, t for X, with $\frac{dt}{ds}=1$) for the characteristics of X and Y. As a result the X-characteristic is obtained from the Y-characteristic by restricting t to (0,T) and projecting the remainder of the Y-characteristic onto $\Lambda$. Hence, the length of the Y-characteristic is the maximum of T and the length of the X-characteristic. Secondly, since the measures on $\Sigma$ and $\Lambda$ are related by

$$d(\mu) = d\mu dt,$$

we may relate the spatial-velocity parts of the sets $D^{\pm}$ and the corresponding measures $\nu^{\pm}$ by

$$D_{(\pm)} = D_{\pm} \times (0,T),$$

$$d\nu_{(\pm)} = d\nu_{\pm} dt,$$

whence the Green's identity reads

$$\int_{\Lambda} \{Xu+bu\} d\mu = \int_{D_+} u d\nu_+ - \int_{D_-} u d\nu_-$$

for some time independent function b on $\Lambda$ such that $B(x,s)=\int_0^s b(x,t)dt$ is $\mu$-essentially bounded. One may then decompose the measure $\mu$ along the trajectories of X as in the statement of Lemma 1 and prove that for $p\in[1,\infty)$ every $u\in L_p(\Lambda,d\mu)$ has a unique trace $u_{\pm}\in L_{p,loc}(D_{\pm},d\nu_{\pm})$ satisfying

$$<Xu,v> + <u,Xv> + <bu,v> = \int_{D_+} u_+ v d\nu_+ - \int_{D_-} u_- v d\nu_-, \quad v\in\hat{\Phi},$$

where $\hat{\Phi}$ is a suitable test function space. In fact, one defines the space $L_{p,loc}(\Lambda,d\mu)$ as the set of all Lebesgue measurable functions u such that $|u|^p$ is $\mu$-integrable on every bounded Lebesgue measurable subset of $\Lambda$ on which the length of the trajectory is bounded away from zero, and $\hat{\Phi}$ as the set of all Borel functions v on $\Lambda$ such that v and Xv are bounded, v is continuously differentiable along trajectories, and v has bounded support with the trajectory length bounded away from zero for all integral curves of X meeting the support of v. We define $L_{p,loc}(D_{\pm},d\nu_{\pm})$ analogously.

Let $F_p$ denote the set of all functions $u\in L_p(\Lambda,d\mu)$ such that $(X+h-\frac{b}{p})u\in L_p(\Lambda,d\mu)$ and the trace $u_{\pm}\in L_p(D_{\pm},d\nu_{\pm})$. Now let J be a bounded operator on $L_p(\Lambda,d\mu)$ and K a bounded operator from $L_p(D_+,d\nu_+)$ into $L_p(D_-,d\nu_-)$, and let us define $B_{p,K}$ as the restriction of the operator $-(X+h-J)$ to the domain $F_{p,K}=\{u\in F_p\colon u_-=Ku_+\}$. As in Section XII.2 of [4] we may prove the following result under the additional constraint that $|b(x,s)|\le\beta$ $\mu$-almost everywhere on $\Sigma$ for some finite constant $\beta$.

THEOREM 6. Suppose $\|K\|<1$ and $1\leq p<\infty$. Then the following statements hold true:

(i) The transport operator $B_K$ generates a strongly continuous semigroup on $L_p(\Lambda,d\mu)$.

(ii) If $\int_\Lambda (\text{sgn } u)|u|^{p-1}(h-\frac{b}{p}-J)u d\mu \geq 0$ for all $u\in F_p$, then $B_K$ generates a contraction semigroup on $L_p(\Lambda,d\mu)$.

(iii) If $K\geq 0$ and $(\lambda_0+J)\geq 0$ for some real $\lambda_0$, then $B_K$ generates a positive semigroup on $L_p(\Lambda,d\mu)$.

Part (iii) (as well as the combination of parts (ii) and (iii)) of this theorem can also be obtained if $\|K\|=1$ and the length of the integral curves of X is bounded away from zero on $\Lambda$. If one only assumes that K and $\lambda_0+J$ are positive operators, then a closed extension of $B_{p,K}$ (and not necessarily $B_{p,K}$ itself) generates a strongly continuous semigroup, which may be obtained as the monotone strong limit of the semigroups for the problem with K replaced by $\alpha K$ as $\alpha\uparrow 1$.

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