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KINETIC EQUATIONS WITH COLLISION OPERATORS OF SPECTRAL RADIUS LESS THAN ONE

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ABSTRACT

The unique solvability of a class of abstract kinetic equations with applications to multigroup neutron transport and polarized light transfer is established for the case when the collision operator differs from the identity by an operator of spectral radius <1 for a semi-infinite medium and ≤ 1 for a finite slab medium. Some applications to the critical slab problem are discussed.

1. INTRODUCTION

In recent years much progress has been reported on abstract kinetic equations of the form

$$(T\psi)'(x) = -A\psi(x), \quad 0 < x < 1, \tag{1.1}$$

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with nonreflective boundary conditions of the type

$$Q_+\psi(0) = \varphi_+, \tag{1.2}$$

$$\left.\begin{array}{r} Q_-\psi(\tau) = \varphi_- \quad \text{if } \tau \text{ is finite} \\ \|\psi(x)\|_H = O(1) \quad (x\to\infty) \quad \text{if } \tau \text{ is infinite} \end{array}\right\} \tag{1.3}$$

where T is an injective self-adjoint operator, $Q_\pm$ is the maximal T-invariant projection on whose range $\pm\langle T\cdot,\cdot\rangle$ is positive, and $A=\mathbb{I}-B$ is a compact perturbation of the identity, all three of them defined on the Hilbert space H. These abstract equations model a variety of stationary transport equations in homogeneous plane-parallel media of (optical) thickness τ in radiative transfer, neutron transport, rarefied gas dynamics and other fields[1,2]. In order to impose positivity assumptions on T and B we observe that Eqs. (1.1)-(1.3) with $\tau=\infty$ are uniquely solvable if $\|B\|<1$ (cf. Ref. [1], Th. VII 3.4). The proofs given in Refs. 1 and 2 and many of their predecessor references involve the minor regularity assumption

$$\exists\, \alpha > 0: \;\; \text{Ran } B \subseteq \text{Ran } |T|^\alpha \cap D(|T|^{1+\alpha})$$

and special cases where this condition obviously holds true, though recently[3] it has been shown to be entirely redundant. Thus we will not make such an assumption. However, in a number of concrete applications in neutron transport theory[4-6] and polarized light transfer[7] it appeared possible to prove the unique solvability of the half-space problem if the spectral radius $\text{spr}(B)<1$. The purpose of this article is to prove this result within the context of the abstract theory and thus to unify and extend the existing material. As an ancillary result we shall prove the unique solvability of the finite slab problem (1.1)-(1.3) with $\tau<\infty$ for $\text{spr}(B)\leq 1$. We will also obtain new applications[8] to multigroup neutron transport with anisotropic scattering. Throughout we will assume positivity of the operators involved so that the criticality problem makes sense.

2. MAIN THEOREMS AND PROOFS

It is well-known that T is unitarily equivalent to a direct integral of operators $[\mu]_\xi$ of multiplication by the independent variable on $L_2(\rho_\xi)$ with ρ_ξ positive Borel measures supported by the real line[9]:

$$H = \mathfrak{U} \int_{\oplus_\xi} L_2(\rho_\xi), \quad T = \mathfrak{U} \int_{\oplus_\xi} [\mu]_\xi \, \mathfrak{U}^*. \tag{2.1}$$

In applications we almost always have a direct sum or integral of multiplication operators $[\mu]_\xi$ so that in this case $\mathfrak{U}=\mathbb{I}$. In any case, the decomposition (2.1) can be used to turn H into a complex Banach lattice on which f(T), for all bounded nonnegative continuous f, is a positive operator. With respect to this lattice structure we will assume B positive.

LEMMA 1. Let C_1 and C_2 be two bounded lattice positive operators on H. Then

$$\|C_1\lambda(\lambda-T)^{-1}C_2\|_{L(H)} \leq \|C_1C_2\|_{L(H)}, \quad \mathrm{Re}\ \lambda=0.$$

Proof: Considering the inequality

$$|\langle C_1\lambda(\lambda-T)^{-1}C_2h,h\rangle| \leq \int_{\sigma(T)} \frac{|\lambda|}{[|\lambda|^2+t^2]^{1/2}}\langle\sigma(dt)C_2|h|,C_1^*|h|\rangle \leq \langle C_1C_2|h|,|h|\rangle,$$

where $|h|$ is the lattice absolute value of h, we obtain the lemma, since $\||h|\|=\|h\|$. □

THEOREM 2. If spr(B)<1, the half-space problem is uniquely solvable.

Proof: Recall that B is lattice positive. Using an obvious generalization of the lemma to k positive operators $C_1,\cdots,C_k$ we find

$$\|[\lambda(\lambda-T)^{-1}B]^k\| \leq \|B\lambda(\lambda-T)^{-1}B\cdots B\lambda(\lambda-T)^{-1}B\| \leq \|B^k\|, \quad \mathrm{Re}\ \lambda = 0,$$

so that

$$\text{spr}\ (\lambda(\lambda-T)^{-1}B) \leq \text{spr}\ (B) < 1, \quad \text{Re}\ \lambda = 0.$$

Choosing a constant $r \in (1, \text{spr}(B)^{-1})$ and constructing the equivalent Hilbert space norm[10]

$$\|h\|_{\bullet} = \left(\sum_{k=0}^{\infty} r^{2k} \|B^k\|^2\right)^{1/2} \tag{2.2}$$

on H, we obtain immediately

$$\|\lambda(\lambda-T)^{-1}B\|_{\bullet} \leq \tfrac{1}{r} \|h\|_{\bullet}, \quad \text{Re}\ \lambda = 0.$$

Now recall that the half-space problem is equivalent to the Wiener-Hopf operator integral equation

$$\psi(x) - \int_0^{\infty} \mathcal{H}(x-y)B\psi(y)dy = e^{-xT^{-1}}\varphi_+, \quad 0 < x < \infty, \tag{2.3}$$

where $\mathcal{H}(\pm|x|) = \pm\int_0^{\pm\infty} |t|^{-1}e^{-x/t}\sigma(dt)$ [Cf. Ref. 1, Ch. 6]. The symbol of this equation, which is given by

$$W(\lambda) = \mathbb{1} - \int_{-\infty}^{+\infty} e^{x/\lambda}\mathcal{H}(x)Bdx = \mathbb{1} - \lambda(\lambda-T)^{-1}B, \quad \text{Re}\ \lambda = 0,$$

satisfies

$$\sup_{\text{Re}\lambda=0} \|W(\lambda)-\mathbb{1}\|_{\bullet} < 1$$

with respect to the operator norm of the Hilbert space $(H, \|\cdot\|_{\bullet})$. Using a factorization result of Gohberg and Leiterer[11], we obtain the unique solvability of Eq. (2.3) and hence the well-posedness of the half-space problem. □

We may now exploit Theorem 2 plus an analyticity argument to derive the following result.

THEOREM 3. If spr(B)$\leq$1, the finite slab problem is uniquely solvable.

Proof: Observe that the convolution equation

$$\psi(x) - \int_o^\tau \mathcal{H}(x-y)B\psi(y)dy = e^{-xT^{-1}}\varphi_+ + e^{(\tau-x)T^{-1}}\varphi_-, \quad 0 \leq x \leq \tau,$$

is equivalent to Eqs. (1.1)-(1.3) with $\tau<\infty$. When rescaled to the interval $0\leq x\leq 1$, the convolution operator

$$(\hat{\mathcal{L}}_\tau\psi)(x) = \int_0^1 \tau\mathcal{H}(\tau(x-y))B\psi(y)dy$$

on the Banach space X of strongly continuous functions ψ:[0,1]$\to$H with supremum norm depends analytically on the parameter τ, while the positivity of its kernel implies that its spectral radius is monotonically nondecreasing from 0 (if $\tau\downarrow 0$) to spr(B) (if $\tau\uparrow\infty$). Since this convolution operator is compact for $\tau<\infty$ and lattice positive, its spectral radius when nonzero is an eigenvalue which is monotonically nondecreasing[12] and analytic[13] in τ, except for algebraic branch points. This prevents the spectral radius from matching or exceeding spr(B) if τ is finite. Hence, the finite-slab problem is uniquely solvable if spr(B)$\leq$1. □

3. APPLICATIONS AND DISCUSSIONS

In Sec. IX.4 of Ref. 1 we have discussed the existence of critical eigenvalues of the multigroup neutron transport equation

$$\mu\frac{\partial\psi}{\partial x}(x,\omega) + \Sigma\psi(x,\omega) = \frac{1}{4\pi}\int_\Omega C\otimes P(\omega\cdot\omega')d\omega',$$

where $[A\otimes B]_{ij}=[A]_{ij}[B]_{ij}$, Ω is the unit sphere in $\mathbb{R}^3$ and Σ, C and $P(\omega\cdot\omega')$ are nonnegative N$\times$N-matrices, the first one of diagonal type. Writing

$\Sigma=\mathrm{diag}\{\sigma_i\}_{i=1}^N$ we may treat this equation on the Hilbert space of all Lebesgue measurable vector functions $h=\{h_i\}_{i=1}^N:\Omega\to\mathbb{C}^N$ which are bounded with respect to the norm

$$\|h\| = \Big(\sum_{i=1}^N \sigma_i\int_\Omega |h_i(\omega)|^2 d\omega\Big)^{1/2}.$$

Following the reasoning of Section IX.4 of Ref. 1, in particular of the proof of Corollary IX 4.5, we may prove that for (order) irreducible B the finite slab problem with nonnegative $\varphi_\pm$ does not have nonnegative solutions for r(B)>1, unless $\varphi_+\equiv\varphi_-\equiv 0$. This answers in the affirmative a question left open when writing Ref. 1 (cf. its Theorem IX 4.1). For other criticality results we refer to Sec. IX.4 of Ref. 1. Following the reasoning of this section we may easily extend Theorems 4.1-4.3, Corollary 4.4 and the above improvement of Corollary 4.5 to the abstract setting.

The equation of transfer of polarized light reads [Cf. Ref. 1, Sec. IX.2]

$$u\frac{\partial I}{\partial\tau} + I(\tau,u,\varphi) = \frac{a}{4\pi}\int_{-1}^{1}\int_0^{2\pi} Z(u,u',\varphi-\varphi')I(\tau,u',\varphi')d\varphi' du',$$

where $a\in(0,1]$ and $I=\{I,Q,U,V\}$ is a real four-vector satisfying

$$I \geq (Q^2+U^2+V^2)^{1/2} \geq 0. \tag{3.1}$$

This inequality expresses that the intensity of the light, I, be nonnegative and its degree of polarization, $(Q^2+U^2+V^2)^{1/2}/I$, belongs to [0,1]. We now obtain unique solvability of the finite slab problem for $a\in(0,1]$ and the half-space problem for $a\in(0,1)$, because spr(B)=a. The functional space to be used is $H=\oplus_{i=1}^4 L_2(\Omega)$ with Ω the unit sphere in $\mathbb{R}^3$.

Strictly speaking the results of Section 2 do not apply to the equation of transfer of polarized light on H, since the L_2-norm on H is not monotonic with respect to the cone of vectors $I=\{I,Q,U,V\}$ satisfying (3.1). However, one may replace H by a different functional space, namely

$L_1(\Omega)\oplus(\oplus_{i=2}^{4}L_2(\Omega))$, where the norm is monotonic with respect to the partial order induced by the positive cone of vectors I={I,Q,U,V}, and use a compactness argument[14] to prove the spectral radius of the convolution operator $\hat{\mathcal{L}}_T$ to be independent of the functional space. A similar compactness argument will yield the independence of the unique solvability and criticality properties of the multigroup neutron transport operator on the functional space.

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14. See Ref. 1, Sec. VII.4

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