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Representations of Internal Field Solutions for Generalized Kinetic Models

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For a class of generalized kinetic equations on the half line with nonreflective half range boundary conditions and modeling various equations from radiative transfer, neutron transport, and gas kinetics, we express the internal field solution in generalizations of Chandrasekhar's H-functions. Contrary to the case of computing the solution at the boundary, an analysis has to be made of the behavior of the dispersion function near the right half plane spectrum of the evolution operator appearing in the equation. Both submultiplying and conservative media are considered. Three instructive examples are worked out and various generalizations are discussed briefly.

1. INTRODUCTION

In this article we study the problem of finding explicit representations for the solutions of half space boundary value problems of the type

$$T\psi'(x) = -A\psi(x), \qquad 0 < x < \infty, \tag{1.1}$$

$$Q_+\psi(0) = \varphi_+, \tag{1.2}$$

$$\|\psi(x)\|_H = O(1) \qquad (\text{as } x \to \infty), \tag{1.3}$$

where T is an injective self adjoint operator and A is a compact perturbation of the identity, both defined on a complex Hilbert space H. Further, Q_+ is the projection of H onto the maximal T-positive T-invariant subspace and $\|\cdot\|_H$ denotes the norm in H. Under the assumption that Eqs. (1.1)–(1.3) are uniquely solvable, we shall obtain a closed form expression of the solution $\psi(x)$ in terms of the solution of a coupled set of nonlinear integral equations.

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The main problem of this article is one of the traditional topics in the theory of neutron transport, radiative transfer, rarefied gas dynamics, and related areas. To illustrate the problem under consideration, we discuss the elementary example

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \psi(x, \mu) = \frac{c}{2} \int_{-1}^{1} \psi(x, \mu')\, d\mu', \qquad 0 < x < \infty, \tag{1.4}$$

$$\psi(0, \mu) = \varphi_{+}(\mu), \qquad \mu > 0, \tag{1.5}$$

$$\left[\int_{-1}^{1} |\psi(x, \mu)|^2\, d\mu \right]^{1/2} = O(1) \qquad (\text{as } x \to \infty), \tag{1.6}$$

where $c \in (0, 1)$. This problem describes steady neutron transport or radiative transfer in an absorbing medium with isotropic scattering and has been studied in a multitude of articles ranging from engineering studies to rigorous mathematics. Among the methods for computing explicit solutions we mention in particular the invariant imbedding approach of Ambarzumian [2] and Chandrasekhar [9] and the singular eigenfunction expansion method of Case [5] and Van Kampen [13]. Both of them lead to explicit formulae for $\psi(x, \mu)$. For $x = 0$ we have, in particular,

$$\psi(0, \mu) = \frac{c}{2} \int_{-1}^{1} \frac{\nu}{\nu - \mu} H(-\mu) H(\nu) \varphi_{+}(\nu)\, d\nu, \qquad \mu < 0, \tag{1.7}$$

where $H(\cdot)$ is the unique solution of the so-called H-equation

$$\frac{1}{H(\mu)} = 1 - \frac{c}{2} \int_{0}^{1} \frac{\mu}{\mu + \nu} H(\nu)\, d\nu, \qquad \mu > 0, \tag{1.8}$$

which is analytic on the right half plane. Equation (1.8) is the well-known H-equation introduced by Ambarzumian [1] and studied systematically by Chandrasekhar [9]. Equations (1.4)–(1.6), which are known to be uniquely solvable (cf. [4] and several later works), fit into the framework of Eqs. (1.1)–(1.3) if we take $H = L_2[-1, 1]$, $(Th)(\mu) = \mu h(\mu)$, $(Ah)(\mu) = h(\mu) - (c/2) \int_{-1}^{1} h(\mu')\, d\mu'$, $(Q_{+} h)(\mu) = h(\mu)$ for $\mu > 0$, and $(Q_{+} h)(\mu) = 0$ for $\mu < 0$.

Let us return to the abstract boundary value problem (1.1)–(1.3). In a previous article (cf. [20]; also [12, Sect. 8.1]) we have derived an explicit formula for the solution of this boundary value problem under the sole assumption that it is uniquely solvable. The solution was written in terms of the spectral function of the self adjoint operator T (something easy to get in most applications) as well as the solutions of a coupled set of non-linear integral equations which generalize the H-equation (1.8). From this expression one may, in principle, derive the solution $\psi(x)$ as follows. First we observe that $\psi(0)$ belongs to a subspace which is $T^{-1}A$-invariant and on which the semigroup $\exp\{-xT^{-1}A\}$ makes sense and is bounded and analytic. We may then use the spectral representation $\sigma_1(\cdot)$ of the operator $T^{-1}A$ to find $\psi(x)$ from $\psi(0)$ with the help of the representation

$$\psi(x) = e^{-xT^{-1}A}\psi(0) = \int e^{-x/t}\sigma_1(dt)\,\psi(0),$$

where the vector integral is taken over the spectrum of $T^{-1}A$ in the right half plane. Such a method, however, requires the explicit computation of the spectral representation of $T^{-1}A$ and the computation of a double integral. In order to do so rigorously, we would have to go through a detailed spectral analysis of $T^{-1}A$ and to simplify a complicated expression involving repeated integrations. It is this type of analysis we seek to avoid in this article as much as possible.

A basic assumption underlying the representation of the solution is the unique solvability of Eqs. (1.1)–(1.3). At present, there is a compehensive existence and uniqueness theory of such problems, covering the cases when A is strictly positive self adjoint, $\mathbb{1} - A$ is a strict contraction or A has a strictly positive real part. The theory is more complicated if A is positive self adjoint (or has a positive real part) as well as an isolated eigenvalue at zero, but for this case we also have a comprehensive theory. The theory at date covers one-speed and symmetric (and certain types of nonsymmetric) neutron transport in nonmultiplying media, radiative transfer with and without polarization, and linearized Boltzmann equations under BGK conditions or for hard and Maxwellian interactions with angular cut-off. For these results, which were gradually developed by various authors, and their history, we refer to [12, Chaps. 2, 3, 4, and 9].

Our main lead in deriving representations of solutions is the integral form

$$\psi(x) - \int_0^\infty \mathscr{H}(x-y)B\psi(y)\,dy = e^{-xT^{-1}}\varphi_+, \qquad 0 < x < \infty, \qquad (1.9)$$

of the boundary value problem, which may be proven equivalent to Eqs. (1.1)–(1.3) under the minor regularity condition

$$\operatorname{Ran} B \subset \operatorname{Ran} |T|^{\alpha} \cap D(|T|^{\beta}), \qquad (1.10)$$

where $\alpha > 0$, $\beta > 1$, and $B = \mathbb{1} - A$ is compact on H. For the proof we refer to [17, 18, 20] (also [12, Chap. 6]). Since in typical applications B has finite rank, it appears opportune to consider a closed subspace $\mathscr{M}$ of H, the orthogonal projection $\pi\colon H \to \mathcal{M}$, and the natural imbedding $j\colon \mathcal{M} \to H$ such that $B = Bj\pi$. We then find

$$\psi(x) = e^{-xT^{-1}}\varphi_+ + \int_0^\infty H(x-y)\, Bj\chi(y)\, dy, \qquad 0 < x < \infty, \quad (1.11)$$

where

$$\chi(x) - \int_0^\infty \pi\mathcal{H}(x-y)\, Bj\chi(y)\, dy = \pi e^{-xT^{-1}}\varphi_+, \qquad 0 < x < \infty. \quad (1.12)$$

Solving the latter in the "classical" way by Wiener–Hopf factorization of the "symbol"

$$\Lambda(z) = \mathbb{1} - \int_{-\infty}^{\infty} e^{x/z}\pi\mathcal{H}(x) Bj\, dx = \mathbb{1} - \int_{\sigma(T)} \frac{z}{z-t}\, \pi\sigma(dt) Bj, \qquad \operatorname{Re} z = 0, \quad (1.13)$$

where $\sigma(\cdot)$ is the spectral function of T, we find a closed form solution in terms of the Wiener–Hopf factors of the inverse of the dispersion function $\Lambda(z)^{-1}$. The factors satisfy generalized H-equations. So far we have been describing a method introduced by Burniston, Mullikin, and Siewert [3] for two-group neutron transport, applied by Mullikin [21] to radiative transfer with anisotropic scattering and Kelley [16] to a class of multigroup neutron transport type equations, and made abstract by Van der Mee [20] (also [12, Sect. 8.1]). In this article, however, we shall go beyond these results for $x = 0$ by also computing the solution for $x > 0$. If $T^{-1}A$ has zero or purely imaginary eigenvalues, no Wiener–Hopf factorization is possible. Nevertheless, we will indicate how to obtain similar solution formulae in this so-called singular case.

For $x > 0$ the solution is much more involved than for $x = 0$, since it contains a contour integral around the spectrum of $A^{-1}T$ in the right half-plane. If this spectrum consists of a real interval plus finitely many isolated eigenvalues, the contour may be contracted to this interval leading to an integral plus finitely many terms arising from the residues at the poles of the contour integrand, provided no eigenvalues are imbedded in the interval. Such a procedure is well known from the singular eigenfunction approach (cf. [6]). We shall perform this contraction and obtain the explicit formula for the solution at $x > 0$. Hereafter we will adapt our method to the case when $T^{-1}A$ has an eigenvalue at zero but no purely imaginary eigenvalues. Here we will utilize stability properties of solutions under perturbations of the collision operator A.

In Section 2 we shall derive the solution of Eqs. (1.1)–(1.3) in terms of contour integrals involving generalized H-functions for the case when

Eqs. (1.1)–(1.3) are uniquely solvable and $T^{-1}A$ does not have zero or purely imaginary eigenvalues. In Section 3 we will make the contraction of the contour to the spectrum of $T^{-1}A$ in the right half plane, while in Section 4 we will discuss the necessary modifications for the case when $T^{-1}A$ has an eigenvalue at zero but no purely imaginary eigenvalues. In Sections 5 and 6 we will discuss some illustrative examples and a variety of extensions of our approach.

2. Representations of Solutions via Contour Integrals

Throughout Sections 2 to 4 we assume that T is an injective self adjoint operator and B is a compact operator satisfying (1.10), both defined on a complex Hilbert space H, and put $A = \mathbb{1} - B$. Suppose Q_+ and Q_- are the orthogonal projections of H onto the maximal T-positive and T-negative T-invariant subspaces, respectively, and that Eqs. (1.1)–(1.3) are uniquely solvable. In the present section we also assume that $T^{-1}A$ does not have zero or purely imaginary eigenvalues. Under these hypotheses we will derive the solution of Eqs. (1.1)–(1.3) in terms of generalized H-functions. The expression for this solution will extend the formula

$$\psi(0) = \varphi_+ + \int_{-\infty}^{0} \int_{0}^{\infty} \frac{\nu}{\nu - \mu}\, \sigma(d\nu)\, BjH_l(-\mu)\, H_r(\nu)\, \pi\sigma(d\nu)\, \varphi_+ \tag{2.1}$$

derived in [20] (also [12, Sect. 8.1]). In this formula $H_l(z)$ and $H_r(z)$ are the unique solutions of the generalized H-equations

$$H_l(z)^{-1} = \mathbb{1} - z \int_0^\infty (z+t)^{-1} H_r(t)\, \pi\sigma(dt)\, Bj, \tag{2.2}$$

$$H_r(z)^{-1} = \mathbb{1} - z \int_0^\infty (z+t)^{-1}\, \pi\sigma(-dt)\, BjH_l(t), \tag{2.3}$$

which are analytic on the right half plane and continuous up to the extended imaginary line. These solutions are exactly the unique functions which are analytic on the right half plane, are continuous up to the imaginary line, and satisfy the factorization formula

$$A(z)^{-1} = H_l(-z)\, H_r(z), \qquad \operatorname{Re} z = 0, \tag{2.4}$$

as well as the equalities $H_l(0^+) = H_r(0^+) = \mathbb{1}$.

In order to find the generalization of Eq. (2.1) to the case $x > 0$, we consider the vector-valued Wiener–Hopf equation (1.9), write its unique solution as (1.11) with $\chi(x)$ satisfying the uniquely solvable Eq. (1.12), and observe that

$$\chi(x) = \pi e^{-xT^{-1}} \varphi_+ + \int_0^\infty \gamma(x, y) \pi e^{-yT^{-1}} \varphi_+ \, dy, \tag{2.5}$$

where $\gamma(x, y)$ is the resolvent kernel of the integral equation (1.12). Here $\pi \exp\{-xT^{-1}\} \varphi_+$ may be replaced by any right hand side $\zeta(x)$ belonging to the Banach space $L_p(\mathcal{M})_0^\infty$ of strongly measurable L_p-functions $\chi: (0, \infty) \to \mathcal{M}$; the solution of Eq. (1.12) belongs to the same space (see, for instance, [12, Sect. 7.1], plus references therein). We then have (cf. [20]; also [12, Sect. 8.1])

$$\int_0^\infty dy \int_0^\infty dz \, e^{y/\mu} e^{-z/\nu} \{\delta(y-z) + \gamma(y, z)\} = \frac{\mu\nu}{\mu - \nu} H_l(-\mu) \, H_r(\nu). \tag{2.6}$$

We first derive two propositions.

PROPOSITION 2.1. *Let $\Phi(x)$ belong to $L_1(\mathcal{M})_0^\infty \cap L_2(\mathcal{M})_0^\infty$, and suppose*

$$G(\mu) = -\frac{1}{\mu} \int_0^\infty e^{y/\mu} \Phi(y) \, dy, \qquad \operatorname{Re} \mu \leqslant 0.$$

Then

$$\Phi(y) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{-y/\mu} G(\mu) \frac{d\mu}{\mu}. \tag{2.7}$$

Proof. Since $\Phi(y)$ is an L_1-function, $G(\mu)$ is continuous in μ on the closed left half plane. Its being L_2 allows for the substitution $1/\mu = -i\zeta$ and the application of inverse Fourier transformation to arrive at (2.7). ∎

PROPOSITION 2.2. *Suppose $\sigma(\cdot)$ is the spectral function of T. Then the forward and backward contributions to the solution $\psi(x)$ of Eqs.* (1.1)–(1.3) *are given by*

$$Q_+ \psi(x) - e^{-xT^{-1}} \varphi_+ = \int_0^\infty \int_0^\infty \sigma(d\mu) BjF(x, \mu, \nu) \pi \sigma(d\nu) \varphi_+, \tag{2.8}$$

$$Q_- \psi(x) = \int_{-\infty}^0 \int_0^\infty \sigma(d\mu) BjF(x, \mu, \nu) \pi \sigma(d\nu) \varphi_+, \tag{2.9}$$

where

$$F(x,\mu,\nu)=\begin{cases} +\dfrac{1}{\mu}\displaystyle\int_0^x e^{-(x-y)/\mu}\Phi(y,\nu)\,dy, & \mu>0,\\ -\dfrac{1}{\mu}\displaystyle\int_x^\infty e^{-(x-y)/\mu}\Phi(y,\nu)\,dy, & \mu<0\end{cases}$$

and

$$F(0,\mu,\nu)=\frac{\nu}{\nu-\mu}H_l(-\mu)H_r(\nu) \tag{2.10}$$

whenever $\mu<0$ *and* $\nu>0$.

Proof. The solution $\psi(x)$ may be written as (1.11) where $\chi(x)$ is the unique solution of the Wiener–Hopf equation (1.12). Writing

$$\mathscr{H}(x-y)=\int\frac{1}{|t|}e^{-(x-y)/t}\sigma(dt),$$

with the integration over $(0,\infty)$ if $x>y$ and $(-\infty,0)$ if $x<y$, as well as

$$e^{-yT^{-1}}Q_+=\int_0^\infty e^{-y/t}\sigma(dt),$$

we obtain

$$Q_+\psi(x)-e^{-xT^{-1}}\varphi_+=\int_0^x\int_0^\infty\frac{1}{\mu}e^{-(x-y)/\mu}\sigma(d\mu)\,Bj\chi(y)\,dy,$$

$$Q_-\psi(x)=-\int_x^\infty\int_{-\infty}^0\frac{1}{\mu}e^{-(x-y)/\mu}\sigma(d\mu)Bj\chi(y)\,dy.$$

At this point we remark that $\Phi\colon(0,\infty)\to L(H)$ belongs to $L_1(L(H))_0^\infty\cap L_2(L(H))_0^\infty$ for Re $\nu>0$, since

$$\Phi(y,\nu)=e^{-y/\nu}\mathbb{1}+\int_0^\infty\gamma(y,z)e^{-z/\nu}\,dz$$

(cf . [12, Prop. VIII 1.1]) and the integral operator with kernel $\gamma(y,z)$ maps the space of $L(H)$-valued L_2-functions into itself. Here $L(H)$ denotes the Banach algebra of bounded linear opeators on H . Noting that

$$\chi(x)=\int_0^\infty\Phi(y,\nu)\pi\sigma(d\nu)\varphi_+,$$

substituting (2.5), and changing the order of integration we find (2.8) and (2.9), which completes the proof. ▮

From the above propositions we obtain the following theorem.

THEOREM 2.3. *Suppose $\sigma(\cdot)$ is the spectral function of T. Then the forward and backward contributions to the solution $\psi(x)$ of Eqs.* (1.1)–(1.3) *are given by*

$$Q_+\psi(x) - e^{-xT^{-1}}\varphi_+ = \int_0^\infty \int_0^\infty \sigma(d\mu)\, BjF(x,\mu,\nu)\pi\sigma(d\nu)\varphi_+, \tag{2.11}$$

$$Q_-\psi(x) = \int_{-\infty}^0 \int_0^\infty \sigma(d\mu) BjF(x,\mu,\nu)\pi\sigma(d\nu)\varphi_+, \tag{2.12}$$

where

$$F(x,\mu,\nu) = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} \frac{\nu}{\nu-p}\,\frac{e^{-x/p}-e^{-x/\mu}}{p-\mu} H_l(-p)H_r(\nu)\,dp, & \mu>0, \\[2ex] \dfrac{1}{2\pi i}\displaystyle\int_{-i\infty}^{i\infty} \frac{\nu}{(\nu-p)(p-\mu)} e^{-x/p} H_l(-p)H_r(\nu)\,dp, & \mu<0. \end{cases}$$

Proof. Applying Proposition 2.1 to the above expression for $F(x,\mu,\nu)$, we find the present theorem, provided we evaluate one integral involving exponentials for $\mu>0$ and one integral involving exponentials for $\mu<0$. For $\mu>0$ we compute

$$\frac{1}{p\mu}\int_0^x e^{-(x-y)/\mu} e^{-y/p}\,dy = \frac{e^{-x/\mu}-e^{-x/p}}{\mu-p},$$

which yields the desired expression for $\mu>0$. For $\mu<0$ we calculate

$$-\frac{1}{p\mu}\int_x^N e^{-(x-y)/\mu} e^{-y/p}\,dy = \frac{1}{p-\mu}\left(e^{-x/p} - e^{-N/p} e^{(N-x)/\mu}\right),$$

which tends to $(p-\mu)^{-1}\exp\{-x/p\}$ as $N\to\infty$ and yields the desired expression for $\mu<0$. ▮

3. DEFORMATION OF THE CONTOUR OF INTEGRATION

For $x=0$ it is particularly simple to reduce the representation formulae (2.11) and (2.12) to their final fom. For $\mu>0$ the first formula yields zero, as to be expected from the boundary condition (1.2). For $\mu<0$ the exponential factor in the integrand disappears. One may then deform the imaginary line to a closed contour in the *left* half plane and use the analyticity of $H_l(-p)$ in this half plane to derive (2.10). For $x>0$, however, the presence of exponential factors forces us to deform the imaginary line to a closed contour in the *right* half plane, which is the wrong half plane for a complex analysis calculation which does not rely on specific information about the position of the singularities of $H_l(-p)$. In order to still implement such a deformation of the integration curve, we have to make additional hypotheses on the operators T and B.

In this section we make the following assumptions on T and B:

(i) T has purely absolutely continuous spectrum consisting of a finite number of mutually disjoint intervals. We write $\mathscr{S}$ for the interior of $\sigma(T)$ (in $\mathbb{R}$) and $\mathscr{S}_\pm = \mathscr{S} \cap (\pm[0, \infty))$. We also write $\Delta(\cdot)$ for the Radon–Nikodym derivative of $\sigma(\cdot)$.

(ii) B and π have finite rank. We write $\rho(t) = \pi\Delta(t)j$ and assume ρ to be Hölder continuous.

(iii) $T^{-1}A$ has finitely many discrete eigenvalues.

(iv) $T^{-1}A$ does not have zero or purely imaginary eigenvalues.

We now deform the union of the imaginary intervals $[-iR, -i\varepsilon]$ and $[i\varepsilon, iR]$, ordered from the left to the right endpoint, to the union $\mathscr{C}(\varepsilon, R)$ of finitely many nonintersecting curves in the right half plane consisting of the circular arcs $\{z \in \mathbb{C}: |z| = R,\ \operatorname{Re} z \geqslant 0,\ |\operatorname{Im} z| \geqslant \varepsilon\}$ with counterclockwise orientation, the intervals $[i\varepsilon, i\varepsilon + (R^2 - \varepsilon^2)^{1/2}]$ ordered from the left to the right endpoint and $[-i\varepsilon, -i\varepsilon + (R^2 - \varepsilon^2)^{1/2}]$ ordered from the right to the left endpoint, and negatively oriented circles around each isolated eigenvalue of $A^{-1}T$ in the open right half plane. Here ε and R are chosen in such a way that all isolated eigenvalues λ of $A^{-1}T$ satisfy $|\operatorname{Im} \lambda| > \varepsilon$ and $|\lambda| < R$. As $\varepsilon \downarrow 0$ and $R \to \infty$ we obtain the following five types of contributions to the integral defining $F(x, \mu, \nu)$ for $\operatorname{Re} \nu > 0$ and either of $\pm \operatorname{Re} \mu > 0$:

(I) the contribution of $p = \infty$ obtained by letting the radius R of the circular arcs tend to infinity,

(II) the cut $-\mathscr{S}_-$ of the meromorphic function $H_l(-p)$, without the imbedded eigenvalues of $A^{-1}T$,

(III) the discrete eigenvalues of $A^{-1}T$ in the open left half plane, whose opposites appear as the poles of $H_l(-p)$ in the open right half plane,

(IV) the imbedded eigenvalues of $-A^{-1}T$ in the interior of $-\mathscr{S}_-$.

As a consequence of the Hölder condition on $\rho(\cdot)$ we have

$$\Lambda^\pm(t) = \lim_{\varepsilon \downarrow 0} \Lambda(t \pm i\varepsilon) = \lambda(t) \pm i\pi t \rho(t), \qquad t \in \mathscr{S}, \tag{3.1}$$

where

$$\lambda(t)=\frac{1}{2}\{\Lambda^+(t)+\Lambda^-(t)\}=\mathbb{1}-\mathscr{P}\int_{\sigma(T)}\frac{t}{t-s}\rho(s)\,ds. \tag{3.2}$$

As a result, $\Lambda^{\pm}(t)$ is invertible (or, by the same token, $\det \Lambda^{\pm}(t)\neq 0$; here we use that $\mathscr{M}$ has finite dimension) if and only if t is not an imbedded eigenvalue of $A^{-1}T$. Furthermore,

$$H_l(-p)H_r(v)=\Lambda(p)^{-1}H_r(p)^{-1}H_r(v). \tag{3.3}$$

On the circular arc $|p|=R$, $\operatorname{Re} p\geqslant 0$, we have uniformly in v on the closed right half plane

$$\|H_l(-p)H_r(v)\|_{L(\mathscr{M})}\leqslant C_R,$$

where C_R is a bounded function of R on $[R_0,\infty)$ with R_0 exceeding all discrete and imbedded eigenvalues of $A^{-1}T$ in absolute value and $L(\mathscr{M})$ denotes the algebra of linear operators on $\mathscr{M}$. Here (3.3) plays a role, as well as the invertibility of $\Lambda^{\pm}(p)$ for $p\notin\mathscr{I}_+\cup\mathscr{I}_-$. The contribution of this circular arc to the integrals defining $F(x,\mu,v)$ can then be majorized in absolute value by expressions of order $O(R^{-1})$, which vanish as $R\to\infty$.

Using (3.1) and the identity

$$\lim_{\varepsilon\downarrow 0}\frac{1}{2\pi i}\int_{-\mathscr{S}_-}\frac{f(p)}{p\pm i\varepsilon-v}\,dp=\frac{1}{2\pi i}\mathscr{P}\int_{-\mathscr{S}_-}\frac{f(p)}{p-v}\,dp\mp\frac{1}{2}f(v),\qquad v\in-\mathscr{S}_-,$$

which is valid to Hölder continuous functions f, we obtain for the contribution to $F(x,\mu,v)$ of the continuous spectrum the expressions

$$\begin{aligned}F_c(x,\mu,v)&=v\,\frac{e^{-x/v}-e^{-x/\mu}}{v-\mu}\,\Lambda^+(v)^{-1}\lambda(v)\Lambda^-(v)^{-1}\\&\quad-\mathscr{P}\int_{-\mathscr{S}_-}\frac{vp}{v-p}\,\frac{e^{-x/p}-e^{-x/\mu}}{p-\mu}\\&\quad\times\Lambda^+(p)^{-1}\rho(p)\Lambda^-(p)^{-1}H_r(p)^{-1}H_r(v)\,dp,\end{aligned}$$

for $\mu>0$, and

$$\begin{aligned}F_c(x,\mu,v)&=\frac{v}{v-\mu}\,e^{-x/v}\Lambda^+(v)^{-1}\lambda(v)\Lambda^-(v)^{-1}\\&\quad-\mathscr{P}\int_{-\mathscr{S}_-}\frac{vp}{(v-p)(p-\mu)}\\&\quad\times e^{-x/p}\Lambda^+(p)^{-1}\rho(p)\Lambda^-(p)^{-1}H_r(p)^{-1}H_r(v)\,dp,\end{aligned}$$

for $\mu<0$, provided there are no negative imbedded eigenvalues of $T^{-1}A$.

For every eigenvalue $-\nu_0$ of $A^{-1}T$ in the open left half plane the contribution to $F(x, \mu, \nu)$ can be computed by calculus of residues, which yields

$$F_{\nu 0}(x, \mu, \nu) = -\operatorname{Res}_{p=\nu 0} \frac{\nu}{\nu - p} \frac{e^{-x/p} - e^{-x/\mu}}{p - \mu} A(p)^{-1} H_r(p)^{-1} H_r(\nu), \qquad \mu > 0,$$

and

$$F_{\nu 0}(x, \mu, \nu) = -\operatorname{Res}_{p=\nu 0} \frac{\nu}{(\nu - p)(p - \mu)} e^{-x/p} A(p)^{-1} H_r(p)^{-1} H_r(\nu), \qquad \mu < 0.$$

If $A^{-1}T$ does not have generalized eigenvectors at the eigenvalue $-\nu_0$, one must compute residues at a simple pole yielding the elementary contributions

$$F_{\nu_0}(x, \mu, \nu) = -\frac{\nu}{\nu - \nu_0} \frac{e^{-x/\nu_0} - e^{-x/\mu}}{\nu_0 - \mu} N_{\nu_0} H_r(\nu_0)^{-1} H_r(\nu), \qquad \mu > 0, \tag{3.4}$$

$$F_{\nu_0}(x, \mu, \nu) = -\frac{\nu}{(\nu - \nu_0)(\nu_0 - \mu)} e^{-x/\nu_0} N_{\nu_0} H_r(\nu_0)^{-1} H_r(\nu), \qquad \mu < 0, \tag{3.5}$$

where

$$N_{\nu_0} = \lim_{p \to \nu_0} (p - \nu_0) A(p)^{-1}.$$

4. Representation Formulae for Kinetic Equations in Conservative Media

In the present section we make the same assumptions on T and B as in the previous two sections, except for the fact that we allow A to have an eigenvalue at zero. However, we still exclude $T^{-1}A$ from having purely imaginary eigenvalues. We assume $T^{-1}A$ to have a finite dimensional root manifold Z_0 at the eigenvalue zero which is contained in $D(T)$ and consists of all vectors $h \in H$ satisfying $(T^{-1}A)^n h = 0$ for some $n \in \mathbb{N}$. For positive self adjoint A we always have $n = 2$, but in general n may exceed 2. The existence and uniqueness theory of Eqs. (1.1)–(1.3) is now much more complicated, since there may now exist solutions ψ of Eqs. (1.1)–(1.2) such that $\|\psi(x)\|_H = o(x^n)$ (as $x \to \infty$) for some $n \in \mathbb{N}$ but not for any lower $n \in \mathbb{N} \cup \{0\}$. For a comprehensive theory of such half-space problems as developed through the joint efforts of various authors we refer to [12, Chaps. 3, 4, 7, and 8]. In this section we shall study Eqs. (1.1)–(1.2) with such a boundary condition at infinity as to make the problem uniquely solvable and to force all solutions ψ to satisfy $\|\psi(x)\|_H = O(x^n)$ (as $x \to \infty$) for some $n \in \mathbb{N}$.

Under the above hypotheses there exists a bounded operator E_+ which maps every $\varphi_+ \in Q_+[H]$ into the initial value $\psi(0)$ of the solution ψ and satisfies $E_+ Q_- = 0$. As shown in [20] (also [12, Sect. 8.1]), one may write $\psi(0)$ in the form (2.1) where the H-functions H_l and H_r satisfy Eqs. (2.2)–(2.3) as well as the factorization formula (2.4), are analytic and invertible on the right half plane, have continuous boundary values on the imaginary line but may fail to be continuous at infinity. The behavior of the H-functions at infinity is difficult to describe, since it depends on the boundary condition at ∞. In any case, these functions are $O(z^n)$ (as $z \to \infty$, $\operatorname{Re} z \geqslant 0$) for some $n \in \mathbb{N}$, but this property alone usually does not specify them uniquely. Here we shall consider them as given.

In the present situation we may start our derivation of representations for the solution of the half space problem from Eqs. (1.11) and (1.12). Following the proof of Proposition 2.2 and Theorem 2.3 and using the well-known expression (2.1), we again obtain Eqs. (2.11)–(2.12). The expression for $F(x, \mu, \nu)$, however, cannot always be given as in the statement of Theorem 2.3, since the integral over the imaginary line may not converge and the assumptions of Proposition 2.1 (used in its derivation) may not be satisfied. For this reason it is hazardous to repeat the analysis of Section 3.

There are two principal ways of deriving the desired expression for the solution of the half space problem using the results of Sections 2 and 3. One way is to interpret the operator E_+ mapping φ_+ into $\psi(0)$ as the "albedo" operator of a modified but uniquely solvable half space problem of the type (1.1)–(1.3) where $T^{-1}A$ does not have zero or purely imaginary eigenvalues, using a method detailed in [12, Chap. 3]. This may be attained by replacing A with a finite rank perturbaton A_β so that Eqs. (1.1)–(1.3) with A replaced by A_β are uniquely solvable and $\psi(0)$ remains unchanged. The part of the solution corresponding to initial values belonging to the root manifold Z_0 will change, but the solution of the modified problem can easily be replaced by the solution of the problem of interest, because Eq. (1.1) with $\psi(0) \in Z_0$ is a trivial problem to solve.

The second method consists of the application of stability properties under perturbation of the operator A. In many cases of interest it is possible to view the problem of interest as the limiting case of a sequence of half space problems (1.1)–(1.3) where $T^{-1}A$ does not have zero or purely imaginary eigenvalues. One must then know that the solution of the approximating problem converges to the solution of the problem of interest. One such case is provided by positive self adjoint A where $(Th, k) = 0$ for any pair of vectors $h, k \in \operatorname{Ker} A$ (see [19] if $\dim \operatorname{Ker} A = 1$, [23] in general). When such a stability property holds true, it also holds true in the uniform sense (i.e., in $L(H)$) for the corresponding albedo operators E_+ and the operators mapping φ_+ into $\psi(x)$ as well as for the corresponding H-matrices, provided one chooses the auxiliary space $\mathcal{M}$ in such a way that π is stable under the perturbation.

5. Examples

In this section we discuss some illustrative examples which exhibit different aspects of representing solutions of the half space problem. The first sample problem, radiative transfer with isotropic scattering, can be solved by straightforward analysis (cf. [5, 13, 6], for instance). The second example, the scalar BGK equation, displays an imbedded infinite zero of the dispersion function (cf. [7, 14]), a phenomenon typical of equations in gas dynamics. The last example, another neutron transport model, is a problem where $T^{-1}A$ has an imbedded eigenvalue at the endpoints of the continuous spectrum; for this reason it was studied in [22]. In all these cases the half space problem (1.1)–(1.3) is known to be uniquely solvable.

a. *Radiative Transfer with Isotropic Scattering*

Radiative transfer with isotropic scattering is described by Eqs. (1.4)–(1.6) where the corresponding operators T, B, A, Q_+ and Q_- are given in the Introduction. For this example Eqs. (1.1)–(1.3) are uniquely solvable if $c \in (0, 1]$, while the solution for $c = 1$ is the limit (in the L_2-sense) of the solution for $c \in (0, 1)$ as $c \uparrow 1$. Introducing $e(\mu) \equiv 1$, $\mathcal{M} = \operatorname{span}\{e\}$, $j(\xi e) = \xi e$, and $\pi h = \eta e$ with $\eta = \frac{1}{2}\int_{-1}^{1} h(\mu)\, d\mu$, and identifying all vectors and operators on $\mathcal{M}$ with scalars acting on e, we obtain the well-known expression

$$\Lambda(z) = 1 - \frac{cz}{2}\int_{-1}^{1} \frac{dt}{z-t}, \qquad z \notin [-1, 1].$$

This function has two simple real zeros $\pm\nu_0$ if $c \in (0, 1)$ (using the convention $\nu_0 > 1$), a double infinite zero if $c = 1$, and two simple imaginary zeros $\pm\nu_0$ if $c > 1$, while imbedded zeros are absent. For $c \in (0, 1]$ let $H(\mu)$ be the (unique) L_1-function on $(0, 1)$ which satisfies Eq. (1.8) and is analytic on the right half plane. Then the unique solution $\psi(x, \mu)$ in Eqs. (1.4)–(1.6) is given by the formulae

$$\psi(x, \mu) - e^{-x/\mu}\varphi_+(\mu) = \frac{c}{2}\int_0^1 F(x, \mu\, \nu)\varphi_+(\nu)\, d\nu, \qquad \mu > 0,$$

$$\psi(x, \mu) = \frac{c}{2}\int_0^1 F(x, \mu, \nu)\varphi_+(\nu)\, d\nu, \qquad \mu < 0,$$

where

$$F(x,\mu,\nu)=\nu\,\frac{e^{-x/\nu}-e^{-x/\mu}}{\nu-\mu}\,\frac{\lambda(\nu)}{\Lambda^+(\nu)\Lambda^-(\nu)}$$
$$-\frac{c}{2}\mathscr{P}\int_0^1\frac{\nu p}{\nu-p}\,\frac{e^{-x/p}-e^{-x/\mu}}{p-\mu}\,\frac{1}{\Lambda^+(p)\Lambda^-(p)}\,\frac{H(\nu)}{H(p)}\,dp$$
$$-\frac{\nu}{\nu-\nu_0}\,\frac{e^{-x/\nu_0}-e^{-x/\mu}}{\nu_0-\mu}\,N_{\nu_0}\frac{H(\nu)}{H(\nu_0)}$$

for $\mu>0$, and

$$F(x,\mu,\nu)=\frac{\nu}{\nu-\mu}\,e^{-x/\nu}\,\frac{\lambda(\nu)}{\Lambda^+(\nu)\Lambda^-(\nu)}$$
$$-\frac{c}{2}\mathscr{P}\int_0^1\frac{\nu p}{(\nu-p)(p-\mu)}\,e^{-x/p}\,\frac{1}{\Lambda^+(p)\Lambda^-(p)}\,\frac{H(\nu)}{H(p)}\,dp$$
$$-\frac{\nu}{(\nu-\nu_0)(\nu_0-\mu)}\,e^{-x/\nu_0}N_{\nu_0}\frac{H(\nu)}{H(\nu_0)}$$

for $\mu<0$. Here $N_{\nu_0}=1/\Lambda'(\nu_0)$. For $c=1$ one finds the same expressions for $F(x,\mu,\nu)$ where the terms involving ν_0 are to be replaced by

$$\nu\sqrt{3}\,H(\nu)\{1-\varphi(x,\mu)\}$$

with $\varphi(x,\mu)\equiv 0$ for $\mu<0$ and $\varphi(x,\mu)=e^{-x/\mu}$ for $\mu>0$. This is easly seen using the asymptotic formula

$$\nu_0(c)\sim\frac{1}{\sqrt{3}}(1-c)^{-1/2}$$

(see [24, Eq. (I.2.24)]), the formula $\int_0^1 tH(t)\,dt=2/\sqrt{3}$ for $c=1$ (see [24, Sect. 8.3.3]) and the identity

$$\frac{-\nu}{\nu-\nu_0}\,\frac{e^{-x/\nu_0}-\varphi(x,\mu)}{\nu_0-\mu}\,N_{\nu_0}\frac{H(\nu)}{H(\nu_0)}$$
$$=\nu\,\frac{\nu_0^2-1}{(\nu_0-\nu)(\nu_0-\mu)}\,\{e^{-x/\nu_0}-\varphi(x,\mu)\}\,g(\nu_0)H(\nu)$$

with

$$g(\nu_0)=\frac{N_{\nu_0}}{(\nu_0^2-1)N(\nu_0)}=\frac{\nu_0}{1-(1-c)\nu_0^2}\left((1-c)^{1/2}+\frac{c}{2}\int_0^1\frac{tH(t)}{t+\nu_0}\,dt\right).$$

In the latter expression we take the limit as $c \uparrow 1$ and obtain

$$\lim_{c \uparrow 1} g(v_0) = \frac{3}{2}\left(\frac{1}{\sqrt{3}} + \frac{1}{2}\int_0^1 tH(t)\, dt\right) = \frac{3}{2}\left(\frac{1}{\sqrt{3}} + \frac{1}{2}\frac{2}{\sqrt{3}}\right) = \sqrt{3},$$

which settles our claim.

b. *The Scalar BGK Equation*

This equation has the form

$$v \frac{\partial \psi}{\partial x}(x, v) + \psi(x, v) = c\pi^{-1/2} \int_{-\infty}^{\infty} \psi(x, w)\, e^{-w^2}\, dw, \qquad 0 < x < \infty \quad (5.1)$$

with boundary conditions

$$\psi(0, v) = \varphi_+(v), \qquad v > 0, \qquad (5.2)$$

$$\left[\pi^{-1/2} \int_{-\infty}^{\infty} |\psi(x, v)|^2\, e^{-v^2}\, dv\right]^{1/2} = O(1) \qquad (\text{as } x \to \infty), \qquad (5.3)$$

where $c = 1$. We will study the physically relevant $c = 1$ problem as the limit of the physically irrelevant $c < 1$ problem.

For $c \in (0, 1]$ the boundary value problem is uniquely solvable when stated in the Hilbert space $H = L_2(\mathbb{R}; \pi^{-1/2} \exp\{-v^2\}\, dv)$ and the solution depends continuously on c. Now define the operators $(Th)(v) = vh(v)$, $(Bh)(v) = c\pi^{-1/2} \int_{-\infty}^{\infty} h(w) \exp\{-w^2\}\, dw$, $(Q_\pm h)(v) = h(v)$ for $\pm v > 0$ and $(Q_\pm h)(v) = 0$ for $\pm v < 0$, the vector $e(v) \equiv 1$, the subspace $\mathcal{M} = \operatorname{span}\{e\}$, and the operators $\pi h = c^{-1} Bh$ and $j(\xi e) = \xi e$, and identify scalars and matrices on $\mathcal{M}$ with scalars acting on e. We then find as the dispersion function

$$\Lambda(z) = 1 - \frac{cz}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{z - w}\, e^{-w^2}\, dw, \qquad z \notin \mathbb{R},$$

where

$$\Lambda^{\pm}(v) = \lambda(v) \pm icv\sqrt{\pi}\, e^{-v^2},$$

$$\lambda(v) = 1 - \frac{cv}{\sqrt{\pi}} \mathscr{P} \int_{-\infty}^{\infty} \frac{1}{v - w}\, e^{-w^2}\, dw,$$

and the H-equation

$$\frac{1}{H(v)} = 1 - \frac{cv}{\sqrt{\pi}} \int_0^{\infty} \frac{H(w)}{v + w}\, e^{-w^2}\, dw, \qquad v > 0,$$

has a unique solution depending continuously on c.

Now observe that $T^{-1}A$ has an eigenvalue at zero and $\Lambda(\cdot)$ has a double zero at infinity, while for $c<1$ there are no discrete eigenvalues and zeros. We may then approximate the solution of Eqs. (5.1)–(5.3) for $c=1$ by the solution of the same problem for $c\in(0,1)$. For $c\in(0,1]$ we obtain

$$\psi(x,v)-e^{-x/v}\varphi_+(v)=c\pi^{-1/2}\int_0^\infty F(x,v,w)e^{-w^2}\varphi_+(w)\,dw$$

for $v>0$, and

$$\psi(x,v)=c\pi^{-1/2}\int_0^\infty F(x,v,w)e^{-w^2}\varphi_+(w)\,dw$$

for $v<0$, where $F(x,v,w)$ depends continuously on c.

In computing $F(x,v,w)$ from Eqs. (2.11)–(2.12) we have to take care at the infinitely long cut along the real line and, for $c=1$, the singularity at infinity. For $c\in(0,1)$ we easily find

$$\begin{aligned}F(x,v,w)={}&w\,\frac{e^{-x/w}-e^{-x/v}}{w-v}\,\frac{\lambda(w)}{\Lambda^+(w)\Lambda^-(w)}\\&-c\pi^{-1/2}\mathscr{P}\int_0^\infty\frac{wp}{w-p}\,\frac{e^{-x/p}-e^{-x/v}}{p-v}\,\frac{1}{\Lambda^+(p)\Lambda^-(p)}\,\frac{H(w)}{H(p)}\,e^{-p^2}\,dp,\end{aligned}$$

where $v>0$, and

$$\begin{aligned}F(x,v,w)={}&\frac{w}{w-v}\,e^{-x/w}\,\frac{\lambda(w)}{\Lambda^+(w)\Lambda^-(w)}\\&-c\pi^{-1/2}\mathscr{P}\int_0^\infty\frac{wp}{(w-p)(p-v)}\,e^{-x/p}\,\frac{1}{\Lambda^+(p)\Lambda^-(p)}\,\frac{H(w)}{H(p)}\,e^{-p^2}\,dp,\end{aligned}$$

where $v<0$. For $c=1$ the dispersion function has a zero and the H-function a pole at infinity, which complicates taking the limit as $c\uparrow 1$. We obtain the same expressions apart from an additional term at each right-hand side of the form

$$v\sqrt{2}\,\{1-\varphi(x,v)\}\,H(w),$$

where $\varphi(x,v)\equiv 0$ for $v<0$ and $\varphi(x,v)=e^{-x/v}$ for $v>0$. Here we use the fact that for $c=1$

$$\lim_{\substack{z\to\infty\\|z|\geqslant\varepsilon}} z^2\Lambda(z)=\frac{-1}{\sqrt{\pi}}\int_{-\infty}^\infty w^2e^{-w^2}\,dw=-\frac{1}{2}$$

so that

$$\lim_{\substack{z \to \infty \\ |z| \geqslant \varepsilon}} \frac{H(z)}{z} = \sqrt{2}.$$

c. *A Neutron Transport Equation with an Imbedded Eigenvalue*

This equation has the form

$$\mu \frac{\partial \psi}{\partial x}(x, \mu) + \psi(x, \mu) = \frac{3}{4} c(1-\mu^2)^{1/2} \times \int_{-1}^{1} (1-\nu^2)^{1/2} \psi(x, \nu)\, d\nu, \qquad 0 < x < \infty, \tag{5.4}$$

where we impose the boundary conditions

$$\psi(0, \mu) = \varphi_+(\mu), \qquad \mu > 0, \tag{5.5}$$

$$\left[\int_{-1}^{1} |\psi(x, \mu)|^2\, d\mu\right]^{1/2} = O(1) \qquad (\text{as } x \to \infty). \tag{5.6}$$

For $c \in (0, 1)$ this problem is uniquely solvable when stated in the Hilbert space $H = L_2[-1, 1]$. Here we define the operators $(Th)(\mu) = \mu h(\mu)$, $(Bh)(\mu) = \frac{3}{4}c(1-\mu^2)^{1/2} \times \int_{-1}^{1} (1-\nu^2)^{1/2} h(\nu)\, d\nu$, $(Q_\pm h)(\mu) = h(\mu)$ for $\pm\mu > 0$ and $(Q_\pm h)(\mu) = 0$ for $\pm\mu < 0$. Moreover, if $e(\mu) = \frac{1}{2}\sqrt{3}\,(1-\mu^2)^{1/2}$, we take $\mathcal{M} = \operatorname{span}\{e\}$, $\pi h = c^{-1} Bh$, and $j(\xi e) = \xi e$. Identifying $\mathcal{M}$ with $\mathbb{C}$ in the natural way we find

$$\Lambda(z) = 1 - \frac{3}{4} cz \int_{-1}^{1} \frac{1-t^2}{z-t}\, dt = 1 - \frac{3}{2} cz^2 + \frac{3}{4} cz(z^2-1) \log \frac{z+1}{z-1},$$

where $z \notin [-1, 1]$ and $\log 1 = 0$, so that $\Lambda(\infty) = 1 - c$ and $\Lambda(\pm 1) = 1 - \frac{3}{2}c$. Thus $\Lambda(\cdot)$ has no discrete or imbedded zeros for $c \in (0, \frac{2}{3})$, one pair of imbedded zeros at $z = \pm 1$ for $c = \frac{2}{3}$, and one pair of simple real zeros $\pm \nu_0$ with $\nu_0 > 1$ for $c \in (\frac{2}{3}, 1)$. For $c \in (\frac{2}{3}, 1)$ the solution is given by

$$\psi(x, \mu) - e^{-x/\mu} \varphi_+(\mu) = \tfrac{3}{4} c(1-\mu^2)^{1/2} \int_0^1 F(x, \mu, \nu)(1-\nu^2)^{1/2} \varphi_+(\nu)\, d\nu, \quad \mu > 0,$$

$$\psi(x, \mu) = \tfrac{3}{4} c(1-\mu^2)^{1/2} \int_0^1 F(x, \mu, \nu)(1-\nu^2)^{1/2} \varphi_+(\nu)\, d\nu, \quad \mu < 0.$$

Here $F(x, \mu, \nu)$ is given by

$$F(x, \mu, \nu) = \nu \frac{e^{-x/\nu} - e^{-x/\mu}}{\nu - \mu} \frac{\lambda(\nu)}{\Lambda^+(\nu)\Lambda^-(\nu)}$$
$$-\frac{3}{4} c\mathscr{P} \int_0^1 \frac{\nu p}{\nu - p} \frac{e^{-x/p} - e^{-x/\mu}}{p - \mu} \frac{1-p^2}{\Lambda^+(p)\Lambda^-(p)} \frac{H(\nu)}{H(p)} dp$$
$$-\frac{\nu}{\nu - \nu_0} \frac{e^{-x/\nu_0} - e^{-x/\mu}}{\nu_0 - \mu} N_{\nu_0} \frac{H(\nu)}{H(\nu_0)}$$

for $\mu > 0$, and

$$F(x, \mu, \nu) = \frac{\nu}{\nu - \mu} e^{-x/\nu} \frac{\lambda(\nu)}{\Lambda^+(\nu)\Lambda^-(\nu)}$$
$$-\frac{3}{4} c\mathscr{P} \int_0^1 \frac{\nu p}{(\nu - p)(p - \mu)} e^{-x/p} \frac{1-p^2}{\Lambda^+(p)\Lambda^-(p)} \frac{H(\nu)}{H(p)} dp$$
$$-\frac{\nu}{(\nu - \nu_0)(\nu_0 - \mu)} e^{-x/\nu_0} N_{\nu_0} \frac{H(\nu)}{H(\nu_0)}$$

for $\mu < 0$, where $N_{\nu_0} = 1/\Lambda'(\nu_0)$. For $c \in (0, \frac{2}{3})$ we have the same expressions for the solution and its resolvent kernel $F(x, \mu, \nu)$, except for the fact that we must now omit the terms involving ν_0. For $c = \frac{2}{3}$ we have imbedded eigenvalues at $z = \pm 1$. Although in this case the solution may also be obtained by taking the limit of the above expressions as $c \to \frac{2}{3}$, the derivation is so computational that we prefer to obtain it directly from Eqs. (2.11) and (2.12). Indeed, for this value of c we have

$$\Lambda(z) = (1 - z^2)\Lambda_0(z), \qquad z \notin [-1, 1],$$

where

$$\Lambda_0(z) = 1 - \frac{1}{2} z \int_{-1}^1 \frac{dt}{z - t} = 1 - \frac{1}{2} z \log \frac{z+1}{z-1}, \qquad z \notin [-1, 1],$$

is the dispersion function for the $c = 1$ case of isotropic scattering. Starting from Eqs. (2.11)–(2.12) and applying the partial fractions identity

$$\frac{1}{\nu - p} \frac{1}{1 - p} = \frac{1}{\nu - 1} \left(\frac{1}{1 - p} - \frac{1}{\nu - p} \right),$$

we obtain

$$F(x, \mu, \nu) = \frac{\nu}{1 - \nu^2} \frac{e^{-x/\nu} - e^{-x/\mu}}{\nu - \mu} \frac{\lambda_0(\nu)}{\Lambda_0^+(\nu)\Lambda_0^-(\nu)}$$
$$-\frac{1}{2} \mathscr{P} \int_0^1 \frac{\nu p}{(\nu - 1)(p - \nu)} \frac{e^{-x/p} - e^{-x/\mu}}{p - \mu} \frac{1 - p}{\Lambda_0^+(p)\Lambda_0^-(p)} \frac{H(\nu)}{H(p)} dp$$

for $\mu > 0$, and

$$F(x, \mu, \nu) = \frac{\nu}{(1-\nu^2)(\nu-\mu)} e^{-x/\nu} \frac{\lambda_0(\nu)}{\Lambda_0^+(\nu)\Lambda_0^-(\nu)}$$
$$-\frac{1}{2}\mathscr{P}\int_0^1 \frac{\nu p}{(\nu-1)(p-\nu)(p-\mu)} e^{-x/p} \frac{1-p}{\Lambda_0^+(p)\Lambda_0^-(p)} \frac{H(\nu)}{H(p)}\,dp$$

for $\mu < 0$. Here we have disregarded the contribution of the integration around the singularity at $p = 1$, since it vanishes. In order to justify this assertion, we denote by $\mathscr{C}_\varepsilon$ the positively oriented circle about $z = 1$ with radius ε and prove the limit

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathscr{C}_\varepsilon} \frac{\nu}{\nu - 1} \frac{1}{p^2-1} \frac{e^{-x/p} - e^{-x/\mu}}{p-\mu} \frac{1}{\Lambda_0(p)} \frac{H(\nu)}{H(p)}\,dp \tag{5.7}$$

to be zero whenever $\mu > 0$. The limit in (5.7) vanishes, because

$$\lim_{\varepsilon \downarrow 0} \frac{\nu}{\nu-1} \frac{1}{p+1} \frac{e^{-x/p} - e^{-x/\mu}}{p-\mu} \frac{1}{\Lambda_0(p)} \frac{H(\nu)}{H(p)} = 0.$$

For $\mu < 0$ we find the same result if we replace the fraction containing the exponentials by $e^{-x/p}/(p-\mu)$.

It is readily verified that in all these cases the H-function satisfies the equation

$$\frac{1}{H(\mu)} = 1 - \frac{3}{4} c\mu \int_0^1 \frac{1-t^2}{\mu + t} H(t)\,dt.$$

This equation is uniquely solvable, provided we impose one constraint for $c \in [\frac{2}{3}, 1]$ amounting to $H(\mu)$ being analytic at $\mu = \nu_0$.

6. Generalization and Discussion of the Results

We have obtained a closed form expression for the solution of Eqs. (1.1)–(1.3). In deriving this expression we have avoided using the method of singular eigenfunction expansion (as expounded in [6]) and various "rigorizations" of this method. Nevertheless, though the problem was solved in integral form via the classical Wiener–Hopf method, certain "full-range" characteristics of Eq. (1.1) appear in the solution. These characteristics necessitate a study of both the discrete and the imbedded eigenvalues. The advantage of our method, however, is that it provides a way of finding the internal field solution of Eqs. (1.1)–(1.3) without first obtaining the full-range orthogonality and completeness of the corresponding eigenvalue problem. In fact, an analysis of the eigenvalue spectrum suffices.

Various generalizations of our approach can be made rather easily. In the first place, we may drop the Hilbert space setting of our problem by assuming T to be an injective scalar-type spectral operator with real spectrum and resolution of the identity $\sigma(\cdot)$, and B to be a compact operator, both defined on the same Banach space, which satisfy the regularity assumption (1.10). (For the theory of spectral operators we refer to [10, 11].) As explained in [12], one may obtain an equivalent vector-valued integral equation of convolution type and a similar derivation of the formulae (2.1)–(2.4) as in a Hilbert space setting, provided Eqs. (1.1)–(1.3) are known to be uniquely solvable. Thus all examples treated in Section 5 are also solved in the corresponding L_p-spaces with $1 \leqslant p < \infty$ by the same expressions. In particular, for neutron transport with isotropic scattering the integral operators with kernels $F(x, \pm\mu, v)$ are compact operators on $L_p[0, 1]$ for $p \in [1, \infty)$ and $x \in [0, \infty)$, as a result of the Hölder continuity of the functions $\Lambda^{\pm}(\cdot)$ and $H(\cdot)$ on $[0, 1]$ with arbitrary Hölder index $\alpha \in (0, 1)$.

In the second place, it is worthwhile noting that Theorem 2.3 already provides a closed form expression for the solution of Eqs. (1.1)–(1.3). Additional assumptions on T and B are only required for the deformation of the integration over the imaginary line appearing in the expression for $F(x, \mu, v)$. We have implemented this deformation for the most common case of absolutely continuous spectrum along (part of) the real line and finite eigenvalue spectrum. A problem of an entirely different scope, however, was studied by Cercignani [8] (see [15, Chap. 8] for a related problem). In this problem the spectrum of the operator T has a nonempty interior. In this case the theory of generalized analytic functions has to be applied in order to obtain the necessary deformation of the integration curve.

In a future publication we hope to extend the analysis of the present article to boundary value problems of the type (1.1)–(1.3) on a finite interval $x \in (0, \tau)$. Here we expect to be able to draw on the representations for the reflection and transmission operators derived in [12].

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