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PERTURBATION ANALYSIS OF ANALYTIC BISEMIGROUPS AND APPLICATIONS TO LINEAR TRANSPORT THEORY

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A perturbation theorem is derived for bounded analytic bisemigroups on Banach spaces with the compact approximation property. The technique utilizes an abstract formulation of the Bochner-Phillips Theorem. The perturbation theorem is then applied to study uniqueness and existence of solutions of a boundary value problem in kinetic theory.

1. INTRODUCTION

In this article we derive a perturbation theorem for bounded analytic bisemigroups and discuss its applications to stationary transport equations. Here by a *bisemigroup* on a complex Banach space X we mean a function $E : \mathbf{R} \rightarrow L(X)$, where L(X) is the Banach algebra of bounded linear operators on X, such that

(i) $E(t)E(s) = E(t + s)$ for all real t, s with $ts > 0$

(ii) E is strongly continuous except for a strong jump discontinuity at $t = 0$.

(iii) $E(0^+) - E(0^-) = I$, the identity operator.

Then clearly

$$\Pi_{\pm} = \pm E(0^{\pm})$$

are bounded complementary projections on X, which are called the *separating projections* of the bisemigroup. Using the ± direct sum of the generators of the strongly continuous semigroups obtained by restricting the bisemigroup to $\mathbf{Ran}\,\Pi_{\pm}$, one may write the bisemigroup as

$$E(t) = \begin{cases} e^{-tS}\Pi_+, & t > 0 \\ -e^{-tS}\Pi_-, & t < 0, \end{cases}$$

where S is called its *infinitesimal generator*. We also write $E(t) = E(t,S)$. A bisemigroup is called bounded, (exponentially or strongly) decaying or analytic if both of the constituent semigroups have this property for a pair of separating projections. It should be noted that the separating projections need not be uniquely defined.

Bisemigroups arise in a natural way in linear transport theory, since the solutions of linear transport equations in homogeneous plane parallel media can be expressed as bisemigroups acting on a vector (cf.[22], [23], [24], [15]) and the kernels of the corresponding convolution equations involve derivatives of bisemigroups (cf. [21], [8], [22], [15]). In all these cases the bisemigroups are analytic and strongly decaying. A systematic study of bisemigroups was made in [2], where a complete characterization of the operators that generate an exponentially decaying (but not necessarily analytic) bisemigroup was obtained. Inspired by the need to define certain separating projections for linear transport models with non selfadjoint collision operators, some perturbation results for analytic bisemigroups have been developed (see [24], [15], [11], [10], [12]). Here we shall generalize the latter results.

Let us give a short outline of the linear transport problem that has motivated the present study. Let H be a complex Hilbert space, T an injective selfadjoint operator on H with positive/negative spectral projections $Q_\pm$ and A some compact perturbation of the identity. Write $B = I - A$. Then the abstract version of the half space problem in linear transport theory is the forward-backward boundary value problem

$$(d/dx)T\psi(x) = -A\psi(x), \quad 0 < x < \infty, \tag{1.1}$$

$$Q_+\psi(0) = \phi_+, \tag{1.2}$$

$$\|\psi(x)\|_H = o(x^n), \quad x \to \infty, \tag{1.3}$$

where $n = 0, 1, 2$. For this abstract problem there exists a plethora of applications in neutron transport theory, radiative transfer and rarefied gas dynamics. In most applications T is scalar type spectral, possibly in a Banach space setting. In other applications (such as gas dynamics for soft intermolecular potentials, or for Fokker-Planck type diffusion equations) one may even have to drop the boundedness of A. In all of these cases the solution can be written in bisemigroup form. For the sake of simplicity and also to have an equivalent Wiener-Hopf equation, we shall assume that A is a compact perturbation of the identity. Now suppose that $T^{-1}A$ does not have zero or purely imaginary eigenvalues. Then under mild hypotheses it can be shown that $T^{-1}A$ generates a strongly decaying analytic bisemigroup $E(\cdot, T^{-1}A)$ with separating projections $P_\pm$. The solutions of the equations (1.1) - (1.3) are precisely the functions

$$\psi(x) = E(x,T^{-1}A)g_+, \quad 0 \leq x < \infty,$$

where $g_+ \in \mathbf{Ran}\, P_+$ and $Q_+g_+ = \phi_+$, while the unique solvability of this problem is equivalent to the decomposition

$$\mathbf{Ran}\, P_+ \oplus \mathbf{Ran}\, Q_- = H.$$

It can be shown rigorously that under certain regularity conditions the boundary value problem(1.1) - (1.3) is equivalent to the vector-valued Wiener-Hopf equation

$$\psi(x) - \int_0^\infty H(x-y)B\psi(y)dy = E(x,T^{-1})\phi_+, \qquad 0 < x < \infty, \tag{1.4}$$

where $H(x) = -(d/dx)E(x,T^{-1})$. Moreover, the unique solution of the full line convolution equation

$$\psi(x) - \int_{-\infty}^\infty H(x-y)B\psi(y)dy = E(x,T^{-1})\phi, \qquad x \in \mathbf{R}, \tag{1.5}$$

is given by

$$\psi(x) = E(x,T^{-1}A)\phi, \qquad x \in \mathbf{R}.$$

Our proof of the perturbation result will be based on a study of the equation (1.5) on the Banach space of bounded continuous fuction from $\mathbf{R}$ to H with a possible jump discontinuity at zero.

In Section 2 we shall discuss preliminary results, which involve tensor products and invertibility in noncommutative Banach algebras. In Section 3 we shall derive the main bisemigroup perturbation result. Section 4 will be devoted to applications in stationary transport theory.

2. TENSOR PRODUCTS AND CONVOLUTION OPERATORS

We begin by introducing some standard notation. Given $(a,b) \subset \mathbf{R}$ and a complex Banach space Y, we denote by $L_p(Y)_a^b$ the Banach space of all strongly measurable functions $\psi(a,b) \rightarrow Y$ that are bounded with respect to the norm

$$\|\psi\|_{L(Y)_a^b} = \| \, \|\psi(\cdot)\|_Y \, \|_{L_p(a,b)},$$

where $1 \leq p \leq \infty$. Here strong measurability is meant with respect to the Lebesgue measure (cf. [29], [7], [20]). By $C(Y)_a^b$ we mean the Banach space of all bounded continuous functions $\psi : [a,b] \cap \mathbf{R} \rightarrow Y$ with supremum norm. Throughout this article T will be a closed operator on a complex Banach space X with the following properties:

(i) $\lambda = 0$ belongs either to the resolvent set of T or to its continuous spectrum,

(ii) T^{-1} generates a bounded analytic bisemigroup on X with separating projections $Q_\pm$,

(iii) there exists $\delta \in (0, \frac{1}{2}\pi)$ such that

$$\sigma(T \mid \mathbf{Ran}\, Q_{\pm}) \subset \{\pm\lambda : |\arg(\lambda)| < \delta\} \cup \{0\}.$$

We first prove a simple result (cf. [10]).

PROPOSITION 1. *The bisemigroup* $E(\cdot, T^{-1})$ *is strongly decaying.*

PROOF. By assumption, $\lambda = 0$ is in the continuous spectrum of T. If $T_{\pm}$ denotes the restriction of T to $\mathbf{Ran}\, Q_{\pm}$, then $\lambda = 0$ is either in the continuous spectrum or in the resolvent set of $T_{\pm}$. Now take $x \in \mathbf{Ran}(T_{+}^{-1})$, which is dense by assumption. Then $x = T^{-1}y$ for some $y \in \mathbf{Ran}\, Q_{+}$, whence $\|\exp(-tT^{-1})x\| = \|T^{-1}\exp(-tT^{-1})y\| \leq \text{const.} \times \|y\|/t \to 0$ as $t \to \infty$. Because the bisemigroup is bounded and $\mathbf{Ran}(T_{+}^{-1})$ is dense in $\mathbf{Ran}\, Q_{+}$, strong decay holds for all $x \in \mathbf{Ran}\, Q_{+}$. The proof of strong decay on $\mathbf{Ran}\, Q_{-}$ is similar.

For later use we observe that strong decay implies that $\lambda = 0$ is in either the continuous spectrum or the resolvent set of T^{-1}. Indeed, if $\lambda = 0$ is an eigenvalue of one of $T_{\pm}^{-1}$ and, for instance, $T_{+}x = 0$, then $\exp(-tT^{-1})x = \lim_{n\to\infty} (I - tT^{-1}/n)^{n}x = x$ for $t > 0$, contradicting the strong decay of the bisemigroup. If $\lambda = 0$ is in the residual spectrum of T^{-1}, we consider the adjoint bisemigroup, which is a bounded analytic bisemigroup (see [24], Corollary I.10.6) with zero as an eigenvalue of its generator. As a result, the adjoint as well as the original bisemigroup cannot be strongly decaying, which proves our assertion. □

Let us write $H(x)$ for $-(d/dx)E(x,T^{-1})$. Assuming B compact on X and writing $A = I - B$, we shall give sufficient conditions on T and B in order that $T^{-1}A$ generates a bounded analytic bisemigroup on X satisfying (i)–(iii), possibly for a different δ.

As it will turn out, one such condition is that

$$\int_{-\infty}^{\infty} \|H(x)B\|_{L(X)}dx < \infty, \tag{2.1}$$

which is easily seen to hold true if B satisfies the regularity condition

$$\mathbf{Ran}\, B \subset \mathbf{Ran}\, |T|^{\alpha} \cap D(|T|^{\beta}) \tag{2.2}$$

for certain $\alpha, \beta > 0$. Here it should be observed that a generator of a bounded analytic semigroup has positive and negative fractional powers [19]. Under (2.1) the kernel $H(\cdot)B$ of the convolution operator

$$(L\psi)(x) = \int_{-\infty}^{\infty} H(x-y)B\psi(y)dy \tag{2.3}$$

is Bochner integrable (cf. [8], [22], [15]) and consequently L is bounded on $C(X)_{-\infty}^{\infty}$, $C(X)_{-\infty}^{0} \oplus C(X)_{0}^{\infty}$ and $L_p(X)_{-\infty}^{\infty}$, where $1 \leq p \leq \infty$. Here a jump discontinuity at $x = 0$ is allowed for functions belonging to the second one of these spaces.

We shall now generalize these results by using cross norms (see [17], [7], [20]). Let Y and Z be two complex Banach spaces. One calls a norm α on the algebraic tensor product $Y \otimes Z$ a *reasonable cross norm* if $\alpha(y \otimes z) = \|y\|_Y \|z\|_Z$ for all $y \in Y$ and $z \in Z$, and if, for all $y^* \in Y^*$ and $z^* \in Z^*$, $y^* \times z^* \in (Y \times Z, \alpha)^*$ with functional norm $\leq \|y^*\| \cdot \|z^*\|$. Then $\|y \otimes z\|_\alpha = \|y\| \|z\|$ and $\|y^* \otimes z^*\|_\alpha = \|y^*\| \|z^*\|$ (cf.[7], [20]). The completion of $Y \otimes Z$ with respect to α is denoted by $Y \otimes_\alpha Z$.

A reasonable cross norm α on the algebraic tensor product $Y \otimes Z$ is called *uniform* if for all $A \in L(Y)$, $B \in L(Z)$ and $v \in Y \otimes Z$

$$\alpha((A \otimes B)v) \leq \|A\| \|B\| \alpha(v).$$

Here $A \otimes B$ is defined from $(A \otimes B)(y \otimes z) = (Ay) \otimes (Bz)$ by linear extension. The norm α induces a cross norm $\bar{\alpha}$ on $L(Y) \otimes L(Z)$ via

$$\bar{\alpha}(C) = \sup_{0 \neq v \in Y \otimes Z} (\alpha(Cv)/\alpha(v)), \qquad C \in L(Y) \otimes L(Z).$$

It is easily seen that $\bar{\alpha}(C)$ is the norm of C as a linear operator on $Y \otimes_\alpha Z$. As a result we have $\bar{\alpha}(C_1 C_2) \leq \bar{\alpha}(C_1)\bar{\alpha}(C_2)$, where the product $C_1 C_2$ is defined in the natural way. Thus $L(Y) \otimes_{\bar{\alpha}} L(Z)$ is a Banach algebra.

Two cross norms play a special role. The projective (π-) cross norm defined by

$$\|v\|_\pi = \inf\{ \sum_i \|y_i\|_Y \|z_i\|_Z : v = \sum_i (y_i \otimes z_i)\}$$

is the largest reasonable cross norm, while the injective (ε-) cross norm given by

$$\|v\|_\varepsilon = \sup\{ \sum_i \phi(y_i)\chi(z_i) : v = \sum_i (y_i \otimes z_i), \ \ \|\phi\|_{Y^*} = 1, \ \ \|\chi\|_{Z^*} = 1\}$$

is the smallest reasonable cross norm. It is straightforward to show that the cross norms ε and π are uniform, which implies that $\bar{\varepsilon}$ and $\bar{\pi}$ are Banach algebra norms (cf.[20], Lemma 1.12).

Next, let Z be the Banach algebra obtained from $L_1(\mathbf{R})$ by adjoining a unit, with convolution as its product operation and endowed with the norm

$$\|\xi \oplus u\|_Z = |\xi| + \|u\|_1.$$

Since $\|u\|_1$ also is the norm of the convolution operator

$$(C_u v)(t) = \int_{-\infty}^{\infty} u(t-s)v(s)ds$$

on $L_\infty(\mathbf{R})$, we may identify $L_1(\mathbf{R})$ isometrically with a closed subalgebra of $L(L_\infty(\mathbf{R}))$.

We shall need the generalization of Wiener's theorem on inverses of Fourier series [28] to noncommutative Banach algebras. Let A be an arbitrary Banach algebra with unit element e. An operator $\Phi \in L(A)$ is called a multiplicative projection on A, if $\Phi^2 = \Phi$, $\Phi e = e$ and $\Phi(a_1 a_2) = \Phi(a_1)\Phi(a_2)$ for every pair of elements $\{a_1, a_2\} \subset A$. The range of a multiplicative projection is a closed subalgebra of A with unit element e. An important example of a multiplicative projection is the mapping

$$\Phi_{\zeta_0}(a) = a(\zeta_0)$$

on $L_1(T)$, where $T = \{\zeta \in \mathbf{C} : |\zeta| = 1\}$ and

$$a(\zeta) = \sum_{j=-\infty}^{\infty} \zeta^j a_j, \quad |\zeta| = 1, \quad \{a_j\}_{j=-\infty}^{\infty} \in \ell_1.$$

Now let Z be a closed subalgebra of the center of A containing e, and let F be a subalgebra of A such that the finite sums

(2.4) $$\sum_j f_j z_j, \quad f_j \in F, \quad z_j \in Z,$$

form a dense set in A. For every non trivial multiplicative linear functional ϕ on the commutative Banach algebra Z we define

(2.5) $$\Phi(\sum_j f_j z_j) = \sum_j \phi(z_j) f_j.$$

Supposing Φ to be bounded on its domain, we can extend Φ in a unique way to a multiplicative projection on A. We call Φ the multiplicative projection induced by ϕ. We quote the following result, which was proved by Bochner and Phillips [6] for function algebras and extended to abstract algebras by Allan [1] and Gohberg and Leiterer [13].

THEOREM 2. *Let F be a subalgebra of A such that the elements* (2.4) *form a dense set of A, and all multiplicative linear functionals on Z induce (by* (2.5)*) bounded operators Φ on A. In order that* a $\in A$ *be left, right or two-sided invertible, it is necessary and sufficient that* Φ(a) *be left, right or two-sided invertible for all multiplicative projections Φ induced by non trivial multiplicative linear functionals on Z.*

Let us now consider the Banach algebra

$$A = Z \otimes_{\overline{\varepsilon}} L(X),$$

which is the completion of the algebra tensor product $Z \otimes L(X)$ with respect to the Banach algebra norm $\overline{\varepsilon}$. Obviously, A has a unit and Z is a closed commutative subalgebra of A. By a well-known result of Gelfand the nontrivial multiplicative functionals of Z are the evaluation mappings

$$\{\alpha \oplus \phi\}I \to \alpha + \int_{-\infty}^{\infty} e^{i\lambda x}\phi(x)dx,$$

where $\lambda \in \mathbf{R}$, and the mapping

$$\{\alpha + \phi\}I \to \alpha.$$

We see that the multiplicative projections on A induced by the nontrivial multiplicative functionals on Z are precisely the evaluation maps (for $\lambda \in \mathbf{R} \cup \{\infty\}$) on the Fourier transforms. Therefore, an operator function in A is left/right/two-sided invertible in A if and only if all values of its Fourier transform are invertible operators on X.

Let K(X) denote the Banach algebra of compact operators on X. The algebra $L_1(\mathbf{R}) \otimes_{\overline{\varepsilon}} K(X)$ is a Banach subalgebra of A, for which the corresponding convolution operator is bounded on $C(X)_{-\infty}^{\infty}$. This is clear from the fact that $C(X)_{-\infty}^{\infty}$ coincides with $C(\mathbf{R}) \otimes_{\varepsilon} X$ (cf. [20], Theorem 1.13). If one would know that in this case $\|\cdot\|_{\overline{\varepsilon}} = \|\cdot\|_{\varepsilon}$, then it is not necessary to give a separate proof of the boundedness of L on $C(X)_{-\infty}^{\infty}$ to prove that $H(\cdot)B \in L_1(\mathbf{R}) \otimes_{\overline{\varepsilon}} K(X)$.

One may also prove that the Fourier transforms of all $v \in L_1(\mathbf{R}) \otimes_{\varepsilon} K(X)$ are continuous (in the norm) functional on the extended real line that vanishes at infinity, under the hypothesis that X has the approximation property, i.e. that for every compact set $K \subset X$ and every $\varepsilon > 0$ there is a finite rank operator A in L(X) such that $\|Ax - x\| < \varepsilon$ for all $x \in K$ (cf. [17], [7]). Indeed, let $u = \sum_{i=1}^{n} f_i A_i$ belong to $L_1(\mathbf{R}) \otimes F(X)$, where F(X) is the ideal of approximable operators on X, i.e. $F(X) = X^* \otimes_{\varepsilon} X$. For every $\lambda \in \mathbf{R}$ the function $t \to e^{i\lambda t}u(t)$ obviously belongs to $L_1(\mathbf{R}) \otimes F(X)$. Let us denote the Pettis integral of this function by $\hat{u}(\lambda) = \sum_{i=1}^{n} \hat{f}_i(\lambda)A_i$. By the Riemann-Lebesgue lemma, we obtain $\hat{u} \in C_0(\mathbf{R}) \otimes F(X)$, where $C_0(\mathbf{R})$ is the space of continuous functions on $\mathbf{R}$ that decay at infinity. Here one notices that $C_0(\mathbf{R}, F(X)) = C_0(\mathbf{R}) \otimes_{\varepsilon} F(X)$. Now let C be a nuclear operator of trace norm one: $C = \sum_{i=1}^{\infty} \gamma_i e_i(\cdot, e_i')$, where $\|e_i\| = \|e_i'\| = 1$ and

$\sum_{i=1}^{\infty} \gamma_i = 1$. Using that the Banach space N(X) of nuclear operators is the dual of K(X) (= F(X)), via $C \to \mathbf{tr}(CT)$ for some $T \in N(X)$ (cf. [18], Section 4.a), we have

$$\sup_\lambda |\mathbf{tr}(C\hat{u}(\lambda))| = \sup_\lambda |\sum_{i=1}^{\infty} \gamma_i(\hat{u}(\lambda)e_i, e_i')| \le \sup_\lambda \sum_{i=1}^{\infty} \gamma_i |\hat{u}(\lambda)e_i, e_i')| \le$$

$$\le \sup_\lambda \sup_{||\phi||, ||\psi||, ||C|| \le 1} |(\hat{u}(\lambda)\phi, \psi)| \sum_{i=1}^{\infty} \gamma_i \le \sup_{||\phi||, ||\psi|| \le 1} ||(u(\cdot)\phi, \psi)||_1 \le$$

$$\le \sup\{||\mathbf{tr}(u(\cdot)C)||_1 : ||C||_{N(X)} \le 1\} = ||u||_\varepsilon .$$

Taking the supremum over C on the left we conclude that

$$||\hat{u}|| \le ||u||_\varepsilon.$$

Thus, if X has the approximation property, then F(X) = K(X) and the Fourier transform takes simple functions from $L_1(\mathbf{R}) \otimes_\varepsilon K(X)$ into simple functions in $C_0(\mathbf{R}, K(X))$. As it is a bounded operator, it may be extended by continuity to all of $L_1(\mathbf{R}) \otimes_\varepsilon K(X)$.

In the sequel an important role is played by the assumption

$$\phi(H(\cdot)x) \in L_1(\mathbf{R}), \qquad x \in X, \ \phi \in X^*. \tag{2.6}$$

This assumption is satisfied in particular if X is a Hilbert space and T is a normal operator on X satisfying the conditions (i) – (iii) at the beginning of Section 2. Indeed, if $\sigma(\cdot)$ is the resolution of the identity of T, there is a unitary equivalence between X and a direct integral of L_2 - spaces which turns T into a direct integral of operators of multiplication by the independent variable (cf. [3]). Then for all $x, y \in X$ there exist $\hat{x}, \hat{y} \in X$ such that $||\hat{x}|| = ||x||$, $||\hat{y}|| = ||y||$ and $|(\sigma(\cdot)x, y)| \le (\sigma(\cdot)\hat{x}, \hat{y})$. But then one easily estimates

$$\int_{-\infty}^{\infty} |(H(t)x, y)| \, dt \le \int_{-\infty}^{\infty} (H(t)\hat{x}, \hat{y}) dt = (\hat{x}, \hat{y}) \le ||x|| \, ||y|| ,$$

which proves the assertion. Another case where (2.6) is satisfied occurs when X is a Banach lattice, $|T|$ is positive on its domain and the bisemigroup $E(\cdot, T^{-1})$ consists of positive operators. Then X^* is a Banach lattice also and

$$\int_{-\infty}^{\infty} |\phi(H(t)x)| \, dt \le \int_{-\infty}^{\infty} |\phi|(H(t)|x|) dt =$$

$$= \int_{-\infty}^{\infty} -(d/dt)|\phi|(E(t,T^{-1})|x|) dt = |\phi|(|x|) < \infty,$$

because $E(\cdot, T^{-1})$ is strongly decaying (cf. Proposition 1).

We now easily prove the following result.

THEOREM 3. *Suppose* X *has the approximation property. Let the assumption* (2.6) *be satisfied,* B *be compact on* X *and* L *be bounded on* $C(X)_{-\infty}^{\infty}$. *Then the operator* $I - L$ *is invertible on* $C(X)_{-\infty}^{\infty}$ *if and only if* $T^{-1}A$ *does not have zero or purely imaginary eigenvalues.*

PROOF. In view of the above, $H(\cdot)B$ belongs to the $\bar{\varepsilon}$ - tensor product of $\mathbf{C} \oplus L_1(\mathbf{R})$ and F(X). Its Fourier transform $\hat{\mathcal{H}}(\cdot)$ is continuous on $\mathbf{R}$ and vanishes at $\pm\infty$ in the norm. Indeed, allowing for a trivial change of variable, we obtain

$$I - \int_{-\infty}^{\infty} e^{x/\lambda} H(x) B dx = (T - \lambda)^{-1}(T - \lambda A), \tag{2.7}$$

which is continuous in the norm on the extended imaginary line with value A at ∞ and I at zero. Hence, (2.7) is invertible for all extended imaginary λ if and only if $T^{-1}A$ does not have zero or purely imaginary eigenvalues, which proves the assertion. □

If X has the approximation property, B is compact and B satisfies (2.1), then $H(\cdot)B$ belongs to the π-tensor product of $\mathbf{C} \oplus L_1(\mathbf{R})$ and K(X) and the result is known (cf. [24] , [15]).

We remark that the boundedness of L on $C(X)_{-\infty}^{\infty}$ implies its boundedness on $C(X)_{-\infty}^{0} \oplus C(X)_{0}^{\infty}$. In order to prove this statement, it is sufficient to consider the case where $\psi(x) = 0$ for $x < 0$ and $\psi(x) = \xi$ = constant for $x > 0$. In this case one has

$$(L\psi)(x) = \begin{cases} [I - E(x,T^{-1})]\xi, & x > 0 \\ -E(x,T^{-1})\xi, & x < 0, \end{cases}$$

which belongs to $C(X)_{-\infty}^{0} \oplus C(X)_{0}^{\infty}$ with norm $\leq N\|\xi\|$ for some constant N.

Let us indicate one more case where L is bounded on $C(X)_{-\infty}^{\infty}$. As known, even if X does not have the approximation property, the operators on X of finite rank can be identified with the algebraic tensor product $X^* \otimes X$. If one completes this space in the π-norm, one thus obtains the nuclear operators and the π-norm is the nuclear norm (see [7], [26]). If X is a Hilbert space and X^* is identified with X, one obtains the trace class operators.

THEOREM 4. *Let the assumption* (2.6) *be satisfied, and let* B *be a nuclear operator on* X. *Then* L *is bounded on* $C(X)_{-\infty}^{\infty}$.

PROOF. Suppose first that

$$Bx = \phi(x)y, \quad x \in X,$$

where $y \in X$ and $\phi \in X^*$. Then for all $\chi \in X^*$ we have

$$\chi((L\psi)(t)) = \int_{-\infty}^{\infty} \chi(H(t-s)y)\phi(\psi(s))ds, \quad t \in \mathbf{R}.$$

Then a simple estimate yields

$$\sup_{t\in\mathbf{R}} |\chi((L\psi)(t))| \leq \|\chi(H(\cdot)y)\|_{L_1(\mathbf{R})} \sup_{\|\phi\|=1,\, s\in\mathbf{R}} |\phi(\psi(s))|.$$

However, a closed graph argument yields the existence of a finite constant M such that

$$\|\chi(H(\cdot)y)\|_{L_1(\mathbf{R})} \leq M\|\chi\|\,\|y\|.$$

Tàking the supremum over χ with $\|\chi\| = 1$, we obtain

$$\|L\psi\|_{C(X)_{-\infty}^{\infty}} \leq M\|y\|_X\,\|\psi\|_{C(X)_{-\infty}^{\infty}},$$

from which the result is immediate. □

If the assumption (2.6) is satisfied and X is a Hilbert space, one easily proves by Fourier transformation of the operator L that L is bounded on $L_2(X)_{-\infty}^{\infty}$ for any bounded B (see [9]). However, the L_2 -setting is not appropriate for getting bisemigroup perturbation results.

3. THE BISEMIGROUP PERTURBATION THEOREM

Let us state the main result of this article.

THEOREM 5. *Suppose* X *has the approximation property. Let the assumption* (2.6) *be satisfied, and let* B *be a compact operator such that* L *is bounded on* $C(X)_{-\infty}^{\infty}$. *Assume that* $T^{-1}A$ *does not have zero or purely imaginary eigenvalues. Then* $T^{-1}A$ *generates a strongly decaying bounded analytic semigroup on* X.

PROOF. From Theorem 3 it is clear that $I - L$ is boundedly invertible on $C(X)_{-\infty}^{0} \oplus C(X)_{0}^{\infty}$. For every vector $x \in X$ we now define

$$E(\cdot, T^{-1}A)x = (I - L)^{-1}E(\cdot, T^{-1})x.$$

Then the boundedness of $(I - L)^{-1}$ implies that $t \to E(t, T^{-1}A)$ is bounded and strongly continuous except possibly for a jump discontinuity at zero. Since L maps $L_\infty(X)_{-\infty}^{\infty}$ into $C(X)_{-\infty}^{\infty}$, the jump at zero does not change when applying $(I - L)^{-1}$ and therefore

$$E(0^+,T^{-1}A) - E(0^-,T^{-1}A) = I.$$

The bisemigroup property can be proved as follows. Putting $F(t) = E(t,T^{-1}A)$ and $\psi(t) = F(t)x$ we obtain for $t > 0$

$$((I - L)\psi)(t) = F(t + s)x - \int_s^\infty H(t + s - r)BF(r)xdr =$$

$$= (I - L)F(t + s)x + \int_{-\infty}^s H(t + s - r)F(r)xdr,$$

since for $r < s$ we have $t + s - r > 0$ and $H(t + s - r) = E(t,T^{-1})H(s - r)B$, while for $r > s$ we have $0 = E(t,T^{-1})H(s - r)B$. We can rewrite the above as

$$((I - L)\psi)(t) = E(t + s,T^{-1})x + E(t,T^{-1})LF(s)x = E(t,T^{-1})F(s)x.$$

Therefore $\psi(t) = F(t)F(s)x$ for $t > 0$. For $t < 0$ we have

$$((I - L)\psi)(t) = -\int_s^\infty H(t + s - r)BF(r)xdr =$$

$$= E(t,T^{-1}) \int_{-\infty}^\infty H(s - r)BF(r)xdr = E(t,T^{-1})F(s)x.$$

Combining the two cases and recalling the definition of $\psi(t)$, we get $F(t)F(s) = F(t + s)$ for $t, s > 0$ and $F(t)F(s) = 0$ for $t < 0$ and $s > 0$.

Applying Laplace transformation to the equation (1.4), one easily sees that

$$(T^{-1}A - \lambda)^{-1}x = \int_{-\infty}^\infty e^{\lambda t}E(t,T^{-1}A)xdt,$$

where a Weyl type of argument yields that every nonzero λ in $\sigma(T^{-1}A)$ satisfies $|\arg(\lambda)| < \delta$ for some $\delta \in (0, \frac{1}{2}\pi)$. Hence $T^{-1}A$ is indeed the generator of the bisemigroup $F(t)_{t\in\mathbf{R}}$. Repeating the above with T^{-1} and $T^{-1}A$ replaced by $e^{i\phi}T^{-1}$ and $e^{i\phi}T^{-1}A$ for sufficiently small $|\phi|$ and applying [18], Theorem IX 1.23, we obtain that $E(\cdot,T^{-1}A)$ is an analytic bisemigroup. Finally, since $\lambda = 0$ is either in the continuous spectrum or in the resolvent set of $T^{-1}A$, the bisemigroup $E(\cdot, T^{-1}A)$ is strongly decaying. □

If the hypotheses of Theorem 5 are satisfied except for the occurrence of finitely many imaginary eigenvalues of $T^{-1}A$, there is a partial result. Put

$$Z_\lambda(T^{-1}A) = \bigcup_{n=1}^{\infty} \mathbf{Ker}(T^{-1}A - \lambda)^n.$$

Then, according to a Weyl type of argument, $Z_\lambda(T^{-1}A)$ has finite dimension for all purely imaginary λ, and there are at most countably many such λ with nonzero $Z_\lambda(T^{-1}A)$ and ∞ as their only possible accumulation point. If T is bounded, there are only finitely many such λ, and $Z_0(T^{-1}A)$ is finite dimensional also. Now assume that $Z_\lambda(T^{-1}A)$ is nonzero for only finitely many zero or purely imaginary λ, and that these subspaces have finite dimension. Let Z be their direct sum and let β be an operator on Z without zero or purely imaginary eigenvalues. Then Z has a unique closed complement Z_1 such that the restriction of $T^{-1}A$ does not have an eigenvalue at zero and has its nonzero spectrum within a double sector of the form $\{\pm\lambda : |\mathbf{arg}(\lambda)| < \delta\}$ for some $\delta \in (0, \frac{1}{2}\pi)$. Let P_0 be the projection of X onto Z along Z_1, and put

$$A_\beta = T\beta^{-1}P_0 \oplus A(I - P_0), \quad B_\beta = I - A_\beta.$$

Then B_β is a finite rank perturbation of B, A_β is invertible and $T^{-1}A_\beta$ does not have imaginary eigenvalues. Moreover, the operator

$$(L_{corr}\psi)(t) = \int_{-\infty}^{\infty} H(t - s)[B - B_\beta]\psi(s)ds$$

is bounded, since $B - B_\beta$ has finite rank. Hence, under the hypotheses of Theorem 5, $T^{-1}A_\beta$ generates a bounded analytic bisemigroup on X with separating projections $P_\pm$. We then define

$$E(t,T^{-1}A) = e^{-tT^{-1}A}P_0 + E(t,T^{-1}A_\beta)(I - P_0). \tag{3.1}$$

In the right hand side of (3.1) the first term is a uniformly continuous group whose norm is $O(t^n)$ as $t \to \pm\infty$ for some $n \in \mathbf{N}$. When restricted to Z_1 the second term is a bounded analytic bisemigroup. For later use we write $P_{1,\pm} = (I - P_0)P_\pm$. We remark that some constructions of this type appeared before in [4], [22], [16] and [5].

4. RESULTS ON HALF LINES AND FINITE INTERVALS

In this section we derive some results on the convolution operator restricted to the positive half line and the finite interval $(0, \tau)$. An important role in the subsequent derivation will be played by the compactness of the difference $E(t,T^{-1}A) - E(t,T^{-1})$.

THEOREM 7. *Suppose* X *has the approximation property. Let the assumption*

(2.6) *be satisfied,* B *be compact and* L *be bounded on* $C(X)_{-\infty}^{\infty}$. *Suppose* $T^{-1}A$ *does not have zero or purely imaginary eigenvalues. Then for all* $t \in \mathbf{R}$ *(including* $t = 0^{\pm}$*) the operator* $E(t,T^{-1}A) - E(t,T^{-1})$ *is compact.*

PROOF. For extended imaginary λ one may define

$$\hat{k} = \int_{-\infty}^{\infty} e^{\lambda t} k(t)dt$$

as a continuous function in the norm that vanishes at infinity, whenever $k(\cdot)$ belongs to $L_1(\mathbf{R}) \otimes_{\bar{\varepsilon}} K(X)$, where we have utilized the fact that $\bar{\varepsilon}$ is a reasonable cross norm. Consider the Banach algebra

$$A = \mathbf{C} I_{L(X)} \oplus L_1(\mathbf{R}) \otimes_{\bar{\varepsilon}} K(X).$$

Then for every $u = \alpha \oplus (-k) \in A$ the Fourier transform $\alpha - \hat{k}(\cdot)$ is continuous in the norm on the extended real line and

$$\sup_{\lambda \in \mathbf{R}} \|\hat{u}(\lambda)\|_{L(X)} \leq \|u\|_A.$$

We may then apply the Bochner-Phillips theorem to prove that every $u = \alpha \oplus (-k) \in A$, such that its Fourier transform $\hat{u}(\lambda) = \alpha - \hat{k}(\lambda)$ is invertible for all extended real λ, is invertible in A. Moreover, for all such u the Wiener-Hopf operators

$$((I - L_{\pm})\psi)(t) = \psi(t) \mp \int_0^{\pm\infty} k(t-s)\psi(s)ds, \quad \pm t > 0, \tag{4.1}$$

are Fredholm operators on $C(X)_0^{\infty}$ or $C(X)_{-\infty}^{0}$, respectively. Indeed, by [14], Theorem 4.3 and 4.4, there is a Wiener-Hopf factorization of $\alpha - \hat{k}(\cdot)$ with respect to the natural "splitting" of the algebra A. Following the classical Wiener-Hopf method for solving the half-line convolution equation, one obtains that the operators (4.1) are Fredholm, which proves the assertion. By first approximating $u \equiv \alpha \oplus (-k)$ by $u_n \equiv \alpha \oplus (-k_n)$ in the algebraic tensor product of $L_1(\mathbf{R})$ and $K(X)$ and then approximating the k_n obtained by k_n in the algebraic tensor product of $L_1(\mathbf{R})$ and the operators on X of finite rank, we obtain an approximation of $L_{\pm}$ in the operator norm. Using the obvious result that $L -$ $- (L_- \oplus L_+)$ is compact for $k(\cdot) = H(\cdot)B$ and B of finite rank, we get its compactness in the general case. From its compactness on $C(X)_{-\infty}^{0} \oplus C(X)_0^{\infty}$ we easily obtain the compactness of $E(t,T^{-1}A) - E(t,T^{-1})$ for all t. □

If we only assume that $Z_0(T^{-1}A)$ is finite dimensional and $T^{-1}A$ has only finitely many imaginary eigenvalues, we may use (3.1) to prove that, under the remaining hypotheses of Theorem 7, $E(t,T^{-1}A) - E(t,T^{-1})$ is compact. In fact, by a finite rank perturbation one may reduce the problem to the situation of Theorem 7.

The above theorem has the following ramifications. Suppose X has the approximation property, and let condition (2.6) be satisfied, B be compact and L be bounded on $C(X)^{\infty}_{-\infty}$. Then the Wiener-Hopf equation (1.4) is a Fredholm equation on $C(X)^{\infty}_{0}$ if $T^{-1}A$ does not have zero or purely imaginary eigenvalues. Under the assumption $\mathbf{Ran}\, B \subset D(T)$ it can be proved that for every $\phi_+ \in \mathbf{Ran}\, Q_+$ the boundary value problem (1.1) - (1.3) with n = 0 and Equation (1.4) are equivalent in the following sense: If $\psi \in C(X)^{\infty}_{0}$, $\psi(x) \in D(T)$ for all $x \in (0,\infty)$, $T\psi$ is strongly differentiable on $(0,\infty)$ and Equations (1.1) - (1.3) are satisfied for n = 0, then ψ is a solution of Equation (1.4) in $C(X)^{\infty}_{0}$. Conversely, if ψ is a solution of Equation (1.4) in $C(X)^{\infty}_{0}$, then $\psi(x) \in D(T)$ for all $x \in (0,\infty)$ and Equations (1.1) - (1.3) are satisfied for n = 0. In the equivalence proof the assumption $\mathbf{Ran}\, B \subset D(T)$ is essential, but the absence of imaginary eigenvalues of $T^{-1}A$ is not. Equivalence proofs in situations where (2.1) holds true can be found in [22], [23], [24] and [15].

One method of proving the unique solvability of Equations (1.1) - (1.3) is to prove that (i) Equations (1.1) - (1.3) with $\phi_+ = 0$ have the zero solution only, and (ii) L_+ is a Fredholm operator of index 0. The equivalence of boundary value problem and Wiener-Hopf equation then yields the unique solvability of both. This situation occurs for $\mathbf{Re}\, A = \frac{1}{2}(A + A^*) \gg 0$. In this case $T^{-1}A$ does not have zero or purely imaginary eigenvalues [11]. By a series of approximations we may then derive the vanishing of the Fredholm index of $I - L_+$ from the corresponding property for A satisfying $\mathbf{Re}\, A \gg 0$, condition (2.2), $\mathbf{Ran}\, B \subset D(T)$ and B of finite rank. A second method of proving unique solvability is to assume that T is an injective selfadjoint operator on a Hilbert space and $\|B\| < 1$. The result then follows from the equivalence of boundary value problem and Wiener-Hopf equation in combination with the main result of [13]. In fact, it is sufficient to assume that X is a Hilbert space and the norm estimate

$$\sup_{\mathbf{Re}\,\lambda=0} \|\lambda(\lambda - T)^{-1}B\|_{L(X)} < 1$$

is satisfied.

Let us now consider the boundary value problem (1.1) - (1.3), or the equivalent Wiener-Hopf equation (1.4), for the case when $Z_0(T^{-1}A)$ is finite dimensional and $T^{-1}A$ has only finitely many imaginary eigenvalues. Then every solution of Equations (1.1) - - (1.3) has the form

$$\psi(x) = E(x,T^{-1}A)g_+ + E(x,T^{-1}A)g_0, \quad 0 \leq x < \infty,$$

where $g_+ \in \mathbf{Ran}\, P_{1,+}$, $Q_+(g_+ + g_0) = \phi_+$, and $g_0 \in N \subset Z$. Here $N = \bigoplus_{\mathbf{Re}\lambda=0} \mathbf{Ker}(T^{-1}A - \lambda)^n$. Hence, if $T^{-1}A$ does not have zero or purely imaginary eigenvalues, it is immaterial to the solvability of Equations (1.1) - (1.3) whether n = 0, 1 or otherwise. If there are imaginary eigenvalues, unique solvability is easily seen to be equivalent to the decomposition

$$\mathbf{Ran}\, P_{1,+} \oplus N \oplus \mathbf{Ker} Q_- = X.$$

For $\mathbf{Re}\, A \geq 0$ and $\mathbf{Ker}\, A = \mathbf{Ker}(\mathbf{Re}\, A)$ finite dimensional, it can be shown that $Z_0(T^{-1}A) = \mathbf{Ker}(T^{-1}A)^2$. Unique solvability of either of the problems (1.1)–(1.3) may then be linked to the structure of **Ker** A as an indefinite inner product space with respect to $[\cdot,\cdot] = (T\cdot,\cdot)$ (see [11], [12]).

In order to produce results on the abstract boundary value problem

$$(T\psi)(x) = -A\psi(x), \quad 0 < x < \tau, \tag{4.2}$$

$$Q_+\psi(0) = \phi_+, \tag{4.3}$$

$$Q_-\psi(\tau) = \phi_-, \tag{4.4}$$

where τ is finite, one has to study the operator

$$(L_\tau\psi)(t) = \int_0^\tau H(t-s)B\psi(s)ds$$

on $C(X)_0^\tau$. We have

THEOREM 8. *Suppose* X *has the approximation property. Let condition* (2.6) *be satisfied,* B *be compact and* L *be bounded on* $C(X)_{-\infty}^{\infty}$. *Then the operator* L_τ *is compact on* $C(X)_0^\tau$.

PROOF. Under the above assumptions on T and B, we have the boundedness of the full-line convolution operator L on $C(X)_{-\infty}^{\infty}$. Again by a similar series of approximations as in the proof of Theorem 7, we may reduce the problem to the case when B has finite rank and satisfies condition (2.2), for which the result is known (cf. [15]). □

It may be proved that under the assumption $\mathbf{Ran}\, B \subset D(T)$ the boundary value problem (4.2) - (4.4) is equivalent to the convolution equation

$$\psi(x) - \int_0^\tau H(x-y)B\psi(y)dy = E(x,T^{-1}A)\phi_+ - E(x-\tau,T^{-1}A)\phi_- \tag{4.5}$$

on $C(X)_0^\tau$. A proof of the equivalence under the condition (2.2) may be found in [22],

[23], [24] and [15]. As a result we may treat Equations (4.2) - (4.4), or alternatively Equation (4.5), as a Fredholm problem where uniqueness of a solution implies its existence. In this way we will get their unique solvability for X a Hilbert space, **Re** $A \geq 0$ and **Ker** A = **Ker**(**Re** A).

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